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Risk-Based Indifference Pricing in Jump Diffusion Markets with Regime-Switching

Torben Bielert
University of Wisconsin-Milwaukee

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Torben Bielert

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ABSTRACT

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Torben Bielert

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This paper is concerned with risk indifference pricing of a European type contingent claim in an incomplete market, where the evolution of the price of the underlying stock is modeled by a regime-switching jump diffusion. The rationale of using such a model is that it can naturally capture the inherent randomness of a prototypical stock market by incorporating both small and big jumps of the prices as well as the qualitative changes of the market. While the model provides a realistic description of the real market, it does introduces substantial difficulty in the analysis. In particular, in contrast with the classical Black-Scholes model, there are infinitely many equivalent martingale measures and hence the price is not unique in our incomplete market. In particular, there exists a big gap between the commonly used sub- and super-hedging prices.

We approach this problem using the framework of risk-indifference pricing. By transforming the pricing problem to an equivalent stochastic game problem, we solve this problem via the associated Hamilton-Jacobi-Bellman-Issac equations. Consequently

we obtain a new interval which is smaller than the interval from super- and sub-hedging.

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Chapter 1

Introduction

1973 counts as the birthyear of the famous Black-Scholes-Model which is still the current basis for pricing derivatives in our financial economy. The BS-Model is a simple model with a lot of reasonable and some "critical" assumptions which lead inside of this model to unique prices of derivatives and for some derivatives, for example European Options, we end up in an analytical price function (Black and Scholes (1973)).

One well known critical assumption is that the volatility in the Black-Scholes-Model is constant - in reality this is not the case. There were many research projects about the volatility of options, e.g. Derman and Kani (1994) or Elliot and Siu (2010). It turned out that the volatility depends on the current market situation and changes in the strucuture of the market implicate changes in our volatility. To model the state of the underlying economy we use a continuous-time Markov chain with a finite state space. The simplest case would consist of two states, namely bull and bear.

We include this idea in our model by letting our parameters depend on the Markov chain. This method is called regime switching. The idea of regime switching was mentioned for the first time in Hamilton (1989) and since then it is an often used extension.

Another critical assumption is that the returns are normally distributed which leads to the fact that the stock price is modeled by a geometric Brownian motion. In reality this is not the case. History has shown that extreme sudden events (called black swans (Taleb (2001))), for example a natural disasters, a war declaration etc. can have such a big impact to the stock price. Consequently, we can no longer assume that the price of the underlying stock is continuous. To this end, jump diffusion market was introduced as early as Merton (Merton (1976)).

These extensions lead to the jump diffusion markets with regime-switching, which is the model examined in this paper. In fact this is a model which represents the reality more in detail, but the most important related question is, if one is able to derive unique prices of derivatives inside this model. In the basic BS-Model, one uses a measure change to get inside of a risk-neutral modeled world (Girsanov (1960)). Then the discounted price process becomes a martingale and thus the price of the derivative is "just" its discounted expectation under the risk neutral measure. This measure is called an Equivalent Martingale Measure. With at least a Monte-Carlo-Simulation (Boyle et al. (1997)), one can easily derive the price. This is due to the fact that in the BS-Model we have exactly one risk neutral measure (Øksendal and Sulem (2007)). Markets with this property are called complete markets.

In our case, due to the random jumps and regime switching, we have an incomplete market. In jump diffusion markets the existence is, under specified conditions (Øksendal and Sulem (2007)), given, but in the general case the uniqueness isn't clear - it's even worse, as it is known that there are infinite many such probability measure.

Thus, the goal is to find an interval for the price. The most obvious approach is to use super- and subhedging to get the maximum and minimum over all possible prices as in Kramkov (1996). But in reality this leads to a big interval. The main goal of this paper is to improve this result and shorten the interval. We will use risk-indifference pricing to transform the problem of pricing a derivative into aa stochastic game which we are able to solve with two different methods. In the first approach we will transform our problem into HJBI equations which solution is known. In the second one we will use a viscosity solutions approach. Both methods will lead to the same interval for the price but the assumptions are different. The so found intverval is smaller or at least of equal size than the interval getting by super- and subhedging.

For the special case when we assume that we only have jumps of size 0, we still get the unique price for derivatives that we would get inside of the BS-Model.

The paper is motivated by the recent paper Øksendal and Sulem (2009) which investigate the risk-indifference pricing for a jump-diffusion market. Our paper extends the spectrum of application of risk-indifference pricing principle to regime switching jump-diffusion market.

The rest of thr thesis is organized as follows: In Chapter 2 we formulate our stock price and wealth process of the jump diffusion market with regime switching mathematically. Furthermore, we characterize all risk neutral measures in our market and explain the idea of measure change. In Chapter 3 we give a short introduction to utility- and risk-indifference pricing in general. In Chapter 4 we explain how to use risk-indifference pricing in detail for our problem and it shows how we transform our problem to become solvable for us. The Sections 4.3 and 4.4 include the main mathematical part of this thesis. We solve our transformed problem with the two different methods. In Chapter 5 we apply the results from the earlier chapters to derive an interval for the price by reversing the transformation and we compare it to the interval derived by super- and subhedging.

Chapter 2

Formulation

2.1 Market

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual condition on which is defined a one-dimensional standard $\mathfrak{F}_t$-adapted Brownian motion W and an $\mathfrak{F}_t$-adapted Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}_0$, where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Denote the intensity measure of N by $\nu(\cdot)$, which is assumed to be a σ-finite Lévy measure satisfying

$$\int_{\mathbb{R}_0} (1 \wedge |y|^2)\nu(dy) < \infty.$$

Consequently, the compensator $\tilde{N}$ of N is

$$\tilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt.$$

Suppose also that on $(\Omega, \mathfrak{F}, \{\mathfrak{F}\}_t, \mathbb{P})$ we define a continuous-time Markov chain

$\alpha = \{\alpha_t, t \geq 0\}$ with a finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$ and infinitesimal generator $Q = (q_{ij})$ with initial state i_0. Assume throughout the paper that W, N, and α are independent.

Suppose a market consists of two assets, a bond and a stock. The price of the bond evolves according to the equation

$$dB_t = r(t, \alpha_t) B_t dt. \tag{2.1}$$

This gives the discounting factor

$$\beta(t) = \frac{1}{B(t)} = \exp\{-\int_0^t r(s, \alpha_s) ds\}, \qquad 0 \leq t < \infty. \tag{2.2}$$

For simplicity we assume that $r(t, \alpha_t) \equiv 0$. For more details of how to set up the market with discounting, see Appendix (A). The price of the stock is modeled by the stochastic differential equation

$$dS_t = \mu(t, \alpha_t) S_{t-} dt + \sigma(t, \alpha_t) S_{t-} dW_t + S_{t-} \int_{\mathbb{R}_0} \gamma(t, \alpha_t, z) \tilde{N}(dt, dz). \tag{2.3}$$

The initial value $S(0)$ of the stock (which is equal to the initial value of the discounted stock) is denoted by s throughout the paper. In (2.1) and (2.3), $r(t, i)$, $\mu(t, i)$, $\sigma(t, i)$ are constants for each $i \in \mathcal{M}$ and $\gamma(t, i, z)$ is a constant in t for a given $i \in \mathcal{M}$. Furthermore, it satisfies $1 + \gamma(t, i, z) > 0$ for all $i \in \mathcal{M}$ and that

$$\int_{\mathbb{R}_0} |\gamma(t, i, z)|^2 \nu(dz) < \infty \quad \forall i \in \mathcal{M}, \forall t \in \ [0, T]$$

which guarantees that the price of the stock is real-valued and well-defined (see Appendix (B)). The rationale of modeling the evolution of the price of the stock through a regime-switching jump diffusion model such as (2.3) is that it can naturally capture the inherent randomness of a prototypical stock market: the Lévy jumps are well-known to incorporate both small and big jumps (Applebaum (2009), Cont and Tankov (2004)) while the regime switching mechanisms provide the qualitative changes of the market (Mao and Yuan (2006), Yin and Zhu (2010)). In the stock market, there is day-to-day jitter that causes minor fluctuations as well as big jumps caused by rare events arising from natural disasters, certain political events, terrorist atrocities, etc. Therefore the evolution of the price of the stock are usually not continuous. On the other hand, in the simplest case, the underlying market may be considered to have two distinct "regimes," namely bull and bear, which could reflect the state of the underlying economy, the general mood of investors in the market, and so on. The volatility and return rates can be quite different in the two regimes.

2.2 Wealth

Let us now consider an agent, with an initial endowment $x \geq 0$, who invests in the two assets of the market. Let X_t be the wealth of the agent at time t. Suppose $\pi(t)$ is the number of shares of stocks owns by the agent. Then, under the self-financing law, we have

$$\begin{aligned} dX_t = dX_t^\pi = & \pi(t)S_{t-}\mu(t,\alpha_t)dt \qquad +\sigma(t,\alpha_t)\pi(t)S_{t-}dW_t \\ & + \pi(t)S_{t-}\int_{\mathbb{R}_0}\gamma(t,\alpha_t,z)\tilde{N}(dt,dz). \end{aligned} \tag{2.4}$$

One can solve (2.4) to obtain

$$\begin{aligned} X^\pi(t) = & \Big[x + \int_0^t \pi(s)S_s\mu(s,\alpha_s)ds + \int_0^t \pi(s)S_s\sigma(s,\alpha_s)dW_s \\ & + \int_0^t\int_{\mathbb{R}_0}\pi(s)S_s\gamma(s,\alpha_s,z)\tilde{N}(ds,dz)\Big]. \end{aligned} \tag{2.5}$$

In this model the wealth process is uniquely defined by π - a given portfolio process. For a fixed finite time-horizon $T > 0$ and a fixed initial endowment x, we say that a wealth process or analog a portfolio process π is admissible on $[0,T]$, if $X_t^\pi \geq 0$ for all $t \in [0,T]$ holds almost surely, π is an $\mathfrak{F}_t$-process and furthermore

$$\begin{aligned} & \int_0^t \pi(s)S_s|\mu(s,i)|ds + \int_0^t \pi(s)^2S_s^2\sigma(s,i)^2ds \\ & + \int_0^t\int_{\mathbb{R}_0}\pi(s)^2S_s^2|\gamma(s,i,z)|^2\nu(dz)ds < \infty \end{aligned} \tag{2.6}$$

for all $t \in [0,T]$ and for all $i \in \mathcal{M}$ holds almost surely. In such a case, we denote $\pi \in \mathcal{A}(T,x)$.

2.3 Transformation into Martingales

Let us define two sets of measures $\mathscr{U}$, $\mathscr{V}$. For given $\mathfrak{F}_t$-predictable processes $\theta_0(t) = \theta_0(t, \alpha_t)$ and $\theta_1(t, z) = \theta_1(t, \alpha_t, z)$; $t \geq 0$, $z \in \mathbb{R}$ such that

$$\mathbb{E}\Big[\exp\{\frac{1}{2}\int_0^T \theta_0^2(s)ds + \int_0^T \int_{\mathbb{R}} \theta_1^2(s,z))N(ds,dz)\}\Big] < \infty, \tag{2.7}$$

or that

$$\mathbb{E}\Big[\exp\{\frac{1}{2}\int_0^T \theta_0^2(s)ds + \int_0^T \int_{\mathbb{R}}((1+\theta_1(s,z))\log(1+\theta_1(s,z)) - \theta_1(s,z))\nu(dz)ds\}\Big] \tag{2.8}$$

is smaller than infinity (Novikov condition), we define the process $Z_\theta(t) = Z_{(\theta_0,\theta_1)}(t)$ as

$$\begin{aligned} Z_\theta(t) =& k\exp\Big(\int_0^t \theta_0(s)dW(s) - \frac{1}{2}\int_0^t \theta_0(s)^2 ds + \int_0^t \int_{\mathbb{R}_0} \log(1+\theta_1(s,z))\tilde{N}(ds,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \log(1+\theta_1(s,z)) - \theta_1(s,z)\nu(dz)ds\Big), \quad \forall t \in [0,\infty). \end{aligned} \tag{2.9}$$

Thus we can describe the dynamics of Z_θ by:

$$\begin{aligned} dZ_\theta(t) &= Z_\theta(t-)\Big[\theta_0(t)dW_t + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{N}(dt,dz)\Big], \quad t \in [0,T] \\ Z_\theta(0) &= k > 0 \end{aligned} \tag{2.10}$$

Next we define the measure $\mathbb{Q}_\theta$ by

$$d\mathbb{Q}_\theta(\omega) = Z_\theta(T)d\mathbb{P}(\omega) \quad \text{on} \quad \mathfrak{F}_T \tag{2.11}$$

Now we want to transform our problems into a Markovian framework, we define the process $Y(t) = Y^{\theta,\pi}(t) \in \mathbb{R}^3$ as follows:

$$dY(t) = \begin{pmatrix} dZ_\theta(t) \\ dS(t) \\ dX^\pi(t) \end{pmatrix} \quad Y(0) = y = \begin{pmatrix} k \\ s \\ x \end{pmatrix}. \tag{2.12}$$

Similarly, we define $\tilde{Y}(t)$ by deleting the third component of the process $Y(t)$. Furthermore we assume that all our coefficients $\mu(t,\alpha_t)$, $\sigma(t,\alpha_t)$ and $\gamma(t,\alpha_t,z)$ are Markovian with respect to $\tilde{Y}(t)$ and α_t. Thus there exist functions $\bar{\mu}$, $\bar{\sigma}$ and $\bar{\gamma}$ such that:

$$\mu(t,\alpha_t) = \bar{\mu}(\alpha_t,\tilde{Y}(t)) \quad \sigma(t,\alpha_t) = \bar{\sigma}(\alpha_t,\tilde{Y}(t)) \quad \gamma(t,\alpha_t,z) = \bar{\gamma}(\alpha_t,\tilde{Y}(t),z)$$

Let $\mathbb{U}$ be the set of all Markovian controls

$$\theta(t,\alpha_t,z) = (\theta_0(t,\alpha_t),\theta_1(t,\alpha_t,z)) = (\bar{\theta}_0(\alpha_t,\tilde{Y}(t)),\bar{\theta}_1(\alpha_t,\tilde{Y}(t),z)).$$

satisfying (2.7) or (2.8) such that

$$\mathbb{E}[Z_\theta(T)] = Z_\theta(0) = k > 0. \tag{2.13}$$

Note that under (2.13) $Z_\theta(t)$ is a martingale.

Let $\mathbb{V}$ defined as follows:

$$\mathbb{V} = \{\theta \in \mathbb{U}; V\theta(t) = 0 \ \forall t \in [0,T]\} \tag{2.14}$$

where the operator V is defined as:

$$V\theta(t) = \mu(t, \alpha_t) + \sigma(t, \alpha_t)\theta_0(t, \alpha_t) + \int_{\mathbb{R}_0} \gamma(t, \alpha_t, z)\theta_1(t, \alpha_t, z)\nu(dz). \tag{2.15}$$

We now define the set of measures as follows:

$$\mathscr{U} = \{\mathbb{Q}_\theta; \theta \in \mathbb{U}\} \text{ and } \mathscr{V} = \{\mathbb{Q}_\theta; \theta \in \mathbb{V}\}.$$

Then by the Girsanov Theorem (Girsanov (1960)) we have that all measures $\mathbb{Q}_\theta \in \mathscr{V}$ with $Z_\theta(0) = k = 1$ are equivalent local martingale measures (EMM for short) (Øksendal and Sulem (2007)) and thus the process

$$W^{Q_\theta}(t) = W(t) - \int_0^t \theta_0(s)ds, \qquad \forall t \in [0, \infty) \tag{2.16}$$

is a Brownian Motion and

$$\tilde{N}^{Q_\theta}(dt, dz) = \tilde{N}(dt, dz) - \theta_1(t, z)v(dz)dt \tag{2.17}$$

is the $(\mathfrak{F}_t, Q_\theta)$-compensated Poison random measure of $N(.,.)$ on $(\Omega, \mathfrak{F}, \{\mathfrak{F}\}_t, \mathbb{Q}_\theta)$. Now (2.4) and (2.5) can be respectively written as

$$dX_t^\pi = X_t^\pi dt + \sigma(t, \alpha_t)\pi(t)dW_t^{Q_\theta} + \pi(t)\int_{\mathbb{R}_0} \gamma(t, \alpha_t, z)\tilde{N}^{Q_\theta}(dt, dz), \tag{2.18}$$

$$\begin{aligned} \beta(t)X^\pi(t) = x &+ \int_0^t \beta(s)\pi(s)\sigma(s, \alpha_s)dW_t^{Q_\theta}(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \beta(s)\pi(s)\gamma(s, \alpha_s, z)\tilde{N}^{Q_\theta}(ds, dz). \end{aligned} \tag{2.19}$$

In this set-up our (discounted) stockprice process and our (discounted) wealth process (in the case where it is admissible) are local martingales, see Karatzas (1988) and Øksendal and Sulem (2007).

Chapter 3

Utility and Risk Indifference Pricing in General

In a complete market, there exists for every bounded $\mathscr{F}_T$-measuruable claim G an inital value $x \in \mathbb{R}$ and a portfolio process π such that: $X^\pi(T; x) = G$ a.s. In this case the EMM $\mathbb{Q}_\theta$ is unique and thus the price of a contract with payoff G at time T is: $p(G) = \mathbb{E}_{\mathbb{Q}_\theta}[G]$.

In incomplete markets, there are infinitely many EMM's $\mathbb{Q}_\theta$. Thus it is not clear which one to use for pricing the claim. Since $\mathbb{V}$ is infinite in our model we have infinitely many EMM's. Thus our market is incomplete. In general we can find an upper and lower bound for the price of our claim by super-/subhedging:

$$\begin{aligned} p_{up}(G) &= \inf \{x;\ \text{there exist } \pi \in \mathcal{A}(T,x) \text{ such that } X^{\pi}(T) \geq G \ a.s.\} \\ &= \sup_{\mathbb{Q}\in\mathscr{V}} \ \mathbb{E}_{\mathbb{Q}}[G], \end{aligned}$$

$$\begin{aligned} p_{low}(G) &= \sup \{x;\ \text{there exist } \pi \in \mathcal{A}(T,x) \text{ such that } X^{\pi}(T) \leq G \ a.s.\} \\ &= \inf_{\mathbb{Q}\in\mathscr{V}} \ \mathbb{E}_{\mathbb{Q}}[G]. \end{aligned}$$

Usually p_{low} and p_{up} are quite different.

One way to shorten this gap is to use the utility indifference principle for pricing. For this end, we need to choose a particular utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. If a person is short in a contract, he receives an inital payment p for the contract. Thus the maximal expected utility for the seller is:

$$V_G(x+p) = \sup_{\pi\in\mathcal{A}(T,x+p)} \mathbb{E}[U(X^{\pi}(T;x+p) - G)], \tag{3.1}$$

where x is the seller's wealth before the contract is being made. Without the contract the seller's maximal expected utiliy is:

$$V_0(x) = \sup_{\pi\in\mathcal{A}(T,x)} \mathbb{E}[U(X^{\pi}(T;x))]. \tag{3.2}$$

The (seller's) utility indifference price $p = p^{utility}$ is then defined as the value of the initial payment that makes the seller utility indifferent to whether to sell the contract or not. Thus p is the solution of:

$$V_G(x+p) = V_0(x). \tag{3.3}$$

To find p we need to solve two stochastic control problems. A good introduction to utility function and some basic properties as well as some appliactions can be found in Henderson and Hobson (2009). There are several papers which cover this approach including: Musiela and Zariphopoulou (2004) or Benth and Meyer-Brandis (2005).

Another way of solving our pricing problem is via risk indifference pricing. In this case we substitute our utility function by a convex risk measure.

Definition 3.0.1. A mapping $\rho : \mathbb{F} \to \mathbb{R}$,where $\mathbb{F}$ is the set of $\mathfrak{F}_T$-measurable random variables, is called a convex risk measure if it is

(i) convex: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any $X, Y \in \mathbb{F}$ and $\lambda \in [0,1]$,

(ii) monotone: $\rho(X) \geq \rho(Y)$ if $X \leq Y$ and $X, Y \in \mathbb{F}$,

(iii) translation invariant: $\rho(X+m) = \rho(X) - m$ for $m \in \mathbb{R}$ and $X \in \mathbb{F}$.

A convex risk measure ρ is called coherent if in addition it is a positive homogeneous function, that is, $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in \mathbb{F}$ and $\lambda \in [0,1]$.

An example for a coherent risk measure is the Conditional Value-at-Risk (CVAR). More information about CVAR can be found in Chi and Tan (2010)).

Now we can set up equations in the same way as we did for utility indifference pricing:

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{A}(T,x+p)} \rho(X^\pi(T;x+p) - G) \tag{3.4}$$

and

$$\Phi_0(x) = \inf_{\pi \in \mathcal{A}(T,x)} \rho(X^\pi(T;x)). \tag{3.5}$$

The (seller's) risk indifference price $p = p^{risk}$ of the claim G, which has to be an element of $\mathbb{F}$, is defined as the price such that the seller is risk indifferent to whether sell or not. Thus p is the solution of:

$$\Phi_G(x+p) = \Phi_0(x). \tag{3.6}$$

This will lead us to two different prices: a price p^S_{risk} for the seller and a price p^B_{risk} for the buyer. We will prove that the following inequality is always true:

$$p_{low} \leq p^B_{risk} \leq p^S_{risk} \leq p_{up}.$$

Chapter 4

Risk Indifference Pricing in Detail

4.1 Formulation of the Problem

In this Chapter we will minimize our risk which comes from our negative wealth process $-X^{\pi}$ by choosing π. The main idea follows a paper Øksendal and Sulem (2009), where they prove a similar result for a jump diffusion market, but without regime switching.

To find a risk indifference price, we will use the following theorem:

Theorem 4.1.1. *(Föllmer and Schied (2002); Elliot and Siu (2010)) A map* $\rho : \mathbb{F} \to \mathbb{R}$ *is a convex risk measure if and only if there exists a family* $\mathscr{U}$ *of probability measures* $\mathbb{Q} \ll \mathbb{P}$ *on* $\mathcal{F}_T$ *and a convex "penalty function"* $\xi : \mathscr{U} \to \mathbb{R}_0 \cup \{+\infty\}$ *with* $\inf_{\mathbb{Q}\in\mathscr{U}} \xi(\mathbb{Q}) = 0$ *such that*

$$\rho(F) = \sup_{\mathbb{Q}\in\mathscr{U}} \{\mathbb{E}_{\mathbb{Q}}[-F] - \xi(\mathbb{Q})\} \quad F \in \mathbb{F}. \tag{4.1}$$

Thanks to this theorem, choosing a risk measure ρ is equivalent to choosing the family $\mathscr{U}$ of measures and the penalty function ξ. ρ becomes a coherent risk measure if we choose $\xi = 0$, see Artzner et al. (1999) and Delbaen (2000). For a given family $\mathscr{U}$ and for a given penalty function ξ using this theorem our Problems (3.4) and (3.5) become:

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{A}(T,x+p)} \Big[\sup_{\mathbb{Q} \in \mathscr{U}} \{ \mathbb{E}_{\mathbb{Q}}[-X^{\pi}(T; x+p) + G] - \xi(\mathbb{Q}) \} \Big], \tag{4.2}$$

$$\Phi_0(x) = \inf_{\pi \in \mathcal{A}(T,x)} \Big[\sup_{\mathbb{Q} \in \mathscr{U}} \{ \mathbb{E}_{\mathbb{Q}}[-X^{\pi}(T; x)] - \xi(\mathbb{Q}) \} \Big]. \tag{4.3}$$

For our purposes we will assume that the ξ has the form:

$$\begin{aligned} \xi(\mathbb{Q}_\theta) =& \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} \lambda(t, \alpha_t, \theta_0(t, \alpha_t, \tilde{Y}(t)), \theta_1(t, \alpha_t, \tilde{Y}(t), z), \tilde{Y}(t), z) \nu(dz) dt \\ & + h(\alpha_T, \tilde{Y}(T)) \Big] \end{aligned} \tag{4.4}$$

for some convex functions $\lambda \in C^1(\mathbb{R}^1 \times \mathcal{M} \times \mathbb{R}^4 \times \mathbb{R}_0)$, $h \in C^1(\mathcal{M} \times \mathbb{R}^2)$ such that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} |\lambda(t, \alpha_t, \theta_0(t, \alpha_t, \tilde{Y}(t), \theta_1(t, \alpha_t, \tilde{Y}(t), z), \tilde{Y}(t), z)| \nu(dz) dt \\ &+ |h(\alpha_T, \tilde{Y}(T))| \Big] < \infty. \end{aligned}$$

for all $(\theta, \pi) \in \mathbb{U} \times \mathscr{A}(T, x)$. Moreover, we assume that the claim G is of the form:

$$G = g(S(T)).$$

for some $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}_{Q_\theta}[|g(S(T))|] < \infty \quad \forall \theta \in \mathbb{L}.$$

Using this notation and our Markovian setting, we can rewrite Problem (3.4) as follows:

Problem 1. *Find $\Phi_G(t,i,y)$ and $(\theta^*, \pi^*) \in \mathbb{U} \times \mathscr{A}(T, x+p)$ (called an optimal triple) such that*

$$\Phi_G(t,i,y) := \inf_{\pi \in \mathscr{A}(T,x)} \left(\sup_{\theta \in \mathbb{U}} J^{\theta,\pi}(t,i,y) \right) = J^{\theta^*,\pi^*}(t,i,y), \tag{4.5}$$

where

$$\begin{aligned} J^{\theta,\pi}(t,i,y) =& \mathbb{E}^{t,i,y}\Big[- \int_t^T \Lambda(\theta(u,\alpha_u,\tilde{Y}(u)))du - h(\alpha_T, \tilde{Y}(T)) \\ & + Z_\theta(T) g(S(T)) - Z_\theta(T) X^\pi(T) \Big] \end{aligned} \tag{4.6}$$

and

$$\Lambda(\theta) = \Lambda(\theta(t,i,\tilde{y})) = \int_{\mathbb{R}_0} \lambda(t,i,\theta_0(t,i,\tilde{y}), \theta_1(t,i,\tilde{y},z), \tilde{y}, z)\nu(dz) \tag{4.7}$$

where $\tilde{y} = (k,s)$, $\forall i \in \mathcal{M}$. If we consider the case without investing in the claim G, we get:

Problem 2. *Find $\Phi_0(t,i,y)$ and $(\theta^*, \pi^*) \in \mathbb{U} \times \mathscr{A}(T, x)$ (called an optimal triple) such that*

$$\Phi_0(t,i,y) := \inf_{\pi \in \mathscr{A}(T,x)} \left(\sup_{\theta \in \mathbb{U}} J_0^{\theta,\pi}(t,i,y) \right) = J_0^{\theta^*,\pi^*}(t,i,y), \tag{4.8}$$

where

$$J_0^{\theta,\pi}(t,i,y) = \mathbb{E}^{t,i,y}\Big[-\int_t^T \Lambda(\theta(u,\alpha_u,\tilde{Y}(u)))du - h(\alpha_T,\tilde{Y}(T)) - Z_\theta(T)X^\pi(T)\Big]. \tag{4.9}$$

Beside these problems, we will treat related stochastic control problems:

$$\Psi_G(t,i,\tilde{y}) = \sup_{\mathbb{Q}\in\mathscr{V}}\{\mathbb{E}_\mathbb{Q}[G] - \xi(\mathbb{Q})\}.$$

and

$$\Psi_0(t,i,\tilde{y}) = \sup_{\mathbb{Q}\in\mathscr{V}}\{-\xi(\mathbb{Q})\}.$$

These can we rewrite with our Markovian setting as:

Problem 3. *Find $\Psi_G(t,i,\tilde{y})$ and $\theta^* \in \mathbb{V}$ such that*

$$\Psi_G(t,i,\tilde{y}) := \sup_{\theta\in\mathbb{V}} I^\theta(t,i,\tilde{y}) = I^{\theta^*}(t,i,\tilde{y}),$$

where

$$I^\theta(t,i,\tilde{y}) = \mathbb{E}^{t,i,\tilde{y}}\Big[-\int_t^T \Lambda(\theta(u,\alpha_u,\tilde{Y}(u)))du - h(\alpha_T,\tilde{Y}(T)) + Z_\theta(T)g(S(T))\Big]$$

and

Problem 4. *Find $\Psi_0(t,i,\tilde{y})$ and $\theta^* \in \mathbb{V}$ such that*

$$\Psi_0(t,i,\tilde{y}) := \sup_{\theta\in\mathbb{V}} I_0^\theta(t,i,\tilde{y}) = I_0^{\theta^*}(t,i,\tilde{y}),$$

where

$$I_0^\theta(t,i,\tilde{y}) = \mathbb{E}^{t,i,\tilde{y}}\Big[-\int_t^T \Lambda(\theta(u,\alpha_u,\tilde{Y}(u)))du - h(\alpha_T,\tilde{Y}(T))\Big].$$

Note that:

$$J^{\theta,\pi}(t,i,y) = I^\theta(t,i,\tilde{y}) - \mathbb{E}^{t,i,y}[Z_\theta(T)X^\pi(T)].$$

4.2 Generators for our Markovian Processes

For given $(\theta,\pi) \in \mathbb{U} \times \mathscr{A}(T,x)$ the process $Y^{\theta,\pi}(t)$ is Markovian with generator $A^{\theta,\pi}$ given by

$$\begin{aligned} A^{\theta,\pi}\phi(t,i,y) =& \frac{\partial\phi}{\partial t} + \mu s\frac{\partial\phi}{\partial s} + s\mu\pi\frac{\partial\phi}{\partial x} + \frac{1}{2}\theta_0^2 k^2\frac{\partial^2\phi}{\partial k^2} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2\phi}{\partial s^2} \\ &+ \frac{1}{2}s^2\sigma^2\pi^2\frac{\partial^2\phi}{\partial x^2} + \theta_0\sigma ks\frac{\partial^2\phi}{\partial k\partial s} + \theta_0\pi\sigma ks\frac{\partial^2\phi}{\partial k\partial x} + \pi\sigma^2 s^2\frac{\partial^2\phi}{\partial s\partial x} \\ &+ \int_{\mathbb{R}_0}\{\phi(t,i,k+k\theta_1,s+s\gamma,x+s\pi\gamma) - \phi(t,i,k,s,x) - k\theta_1\frac{\partial\phi}{\partial k} \\ &- s\gamma\frac{\partial\phi}{\partial s} - s\pi\gamma\frac{\partial\phi}{\partial x}\}\nu(dz) + \sum_{j=1}^m q_{ij}[\phi(t,j,y) - \phi(t,i,y)] \end{aligned} \tag{4.10}$$

for all $\phi = \phi(t,i,k,s,x) \in C^{1,2}([0,T]\times\mathcal{M}\times\mathbb{R}_+^3)$. Note that $\mu = \mu(i,y)$, $\sigma = \sigma(i,y)$, etc.

As before we consider now the process $\tilde{Y}^\theta(t)$ by deleting the third component of

$Y(t)$. Its generator is given by

$$
\begin{aligned}
A^{\theta}\psi(t,i,\tilde{y}) =& \frac{\partial \psi}{\partial t} + \mu s \frac{\partial \psi}{\partial s} + \frac{1}{2}\theta_0^2 k^2 \frac{\partial^2 \psi}{\partial k^2} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \psi}{\partial s^2} + \theta_0 \sigma k s \frac{\partial^2 \psi}{\partial k \partial s} \\
&+ \int_{\mathbb{R}_0} \{\psi(t,i,k+k\theta_1, s+s\gamma) - \psi(t,i,k,s) - k\theta_1 \frac{\partial \psi}{\partial k} - s\gamma \frac{\partial \psi}{\partial s}\}\nu(dz) \\
&+ \sum_{j=1}^{m} q_{ij}[\psi(t,j,\tilde{y}) - \psi(t,i,\tilde{y})]
\end{aligned}
\tag{4.11}
$$

for all $\psi = \psi(t,i,k,s) \in C^{1,2}([0,T] \times \mathcal{M} \times \mathbb{R}_+^2)$. From this we obtain the following simple result:

Lemma 4.2.1. *Let $\psi \in C^{1,2}([0,T] \times \mathcal{M} \times \mathbb{R}_+^2)$ and define*

$$\phi(t,i,k,s,x) := \psi(t,i,k,s) - kx.$$

Then, with $\tilde{y} = (k,s)$ as before,

$$
\begin{aligned}
A^{\theta,\pi}\phi(t,i,y) =& A^{\theta}\psi(t,i,\tilde{y}) - ks\pi(i,\tilde{y})\Big[\mu(i,\tilde{y}) + \theta_0(t,i,\tilde{y})\sigma(i,\tilde{y}) \\
&+ \int_{\mathbb{R}_0} \theta_1(t,i,\tilde{y},z)\gamma(i,\tilde{y},z)\nu(dz)\Big].
\end{aligned}
$$

Proof. From (4.10) and (4.11) we obtain:

$$A^{\theta,\pi}\psi(t,i,\tilde{y}) = A^{\theta}\psi(t,i,\tilde{y}).$$

Thus it only remains to compute:

$$\begin{aligned}A^{\theta,\pi}(kx) =& s\mu\pi k + s\theta_0\pi\sigma k \\ &+ \int_{\mathbb{R}_0} \{(k + k\theta_1)(x + sx\pi\gamma) - kx - k\theta_1 x - s\pi\gamma k\}\nu(dz) \\ &+ \sum_{j=1}^{m} q_{ij}[kx - kx] \\ &= sk\pi\Big[\mu + \theta_0\sigma + \int_{\mathbb{R}_0} \theta_1\gamma\nu(dz)\Big].\end{aligned}$$

□

From now on, we put $Ł = (0,T) \times \mathcal{M} \times \mathbb{R}^3_+$ and $\tilde{Ł} = (0,T) \times \mathcal{M} \times \mathbb{R}^2_+$ (called the solvency region).

Lemma 4.2.2. *Let ψ and ϕ be as in Lemma* (4.2.1). *We put $\Theta = \{(\theta_0, \theta_1); \theta_0 \in \mathbb{R}$ and θ_1 is a function from $\mathbb{R}_0$ to $\mathbb{R}\}$. Suppose that for all $\pi \in \mathbb{R}, (t, i, k, s) \in \tilde{L}$ there exists a maximum point $\hat{\theta} = \hat{\theta}(\pi)$ of the function*

$$\theta \to A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta; \quad \theta \in \Theta$$

and that $\pi \to \hat{\theta}(\pi)$ is a C^1- function. Moreover, suppose the map

$$\pi \to A^{\hat{\theta}(\pi)}\psi - \Lambda(\hat{\theta}(\pi)) - ks\pi V\hat{\theta}(\pi); \quad \pi \in \mathbb{R}$$

has a minimum point $\tilde{\pi} \in \mathbb{R}$. Define

$$\theta_{opt} := \hat{\theta}(\hat{\pi}).$$

Then

$$V\theta_{opt} = 0$$

and

$$\inf_{\pi}(\sup_{\theta}\{A^{\theta,\pi}\phi - \Lambda(\theta)\}) = A^{\theta_{opt}}\psi - \Lambda(\theta_{opt}) = \sup_{\theta:V\theta=0}\{A^{\theta}\psi - \Lambda(\theta)\}.$$

Proof. The first-order condition for a maximum point $\hat{\theta} = \hat{\theta}(\pi)$ for our given map for fixed t, i, k, s and π is

$$\nabla_{\theta}(A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta)_{\theta=\hat{\theta}} = 0,$$

where $\nabla_{\theta} = (\frac{\partial}{\partial\theta_0}, \frac{\partial}{\partial\theta_1})$ denotes the gradient operator. For the minimum point $\hat{\pi}$ of our given map, by the chain rule, we get:

$$\nabla_{\theta}(A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta)_{\theta=\hat{\theta}(\hat{\pi})}\Big(\frac{d\hat{\theta}(\pi)}{d\pi}\Big)_{\pi=\hat{\pi}} - ksV\hat{\theta}(\hat{\pi}) = 0.$$

Hence we can conclude that

$$V\hat{\theta}(\hat{\pi}) = 0.$$

Therefore, θ_{opt} statisfies the constraint $V\theta_{opt} = 0$. Thus:

$$\begin{aligned}
&\inf_{\pi}(\sup_{\theta}\{A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta\}) \\
&= \inf_{\pi}(A^{\hat{\theta}(\pi)}\psi - \Lambda(\hat{\theta}(\pi)) - ks\pi V\hat{\theta}(\pi)) \\
&= A^{\theta_{opt}}\psi - \Lambda(\theta_{opt}) \leq \sup_{\theta:V\theta=0}\{A^{\theta}\psi - \Lambda(\theta)\}.
\end{aligned}$$

On the other hand, we always have

$$
\begin{aligned}
&\inf_{\pi}(\sup_{\theta}\{A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta\}) \\
&\geq \inf_{\pi}(\sup_{\theta: V\theta=0}\{A^{\theta}\psi - \Lambda(\theta) - ks\pi V\theta\}) \\
&= \sup_{\theta: V\theta=0}\{A^{\theta}\psi - \Lambda(\theta)\}.
\end{aligned}
$$

Combining the two results we get our claim. □

4.3 Related HJBI equations

Since our Problem 1 is related to a well known class of stochastic differential games, we can apply Theorem 3.2 in Mataramvura and Øksendal (2008) to get the following theorem:

Theorem 4.3.1 (HJBI equations)**.** *Suppose* $\phi \in C^{1,2}(Ł) \cap C(\tilde{Ł})$ *and* $(\hat{\theta}, \hat{\pi}) \in \mathbb{U} \times \mathscr{A}(T, x)$ *satisfy the following conditions:*

(i) $A^{\theta,\hat{\pi}}\phi(t, i, y) - \Lambda(\theta(t, i, \tilde{y})) \leq 0 \quad \forall \theta \in \Theta, (t, i, y) \in Ł.$

(ii) $A^{\hat{\theta},\pi}\phi(t, i, y) - \Lambda(\hat{\theta}(t, i, \tilde{y})) \geq 0 \quad \forall \pi \in \mathbb{R}, (t, i, y) \in Ł.$

(iii) $A^{\hat{\theta},\hat{\pi}}\phi(t, i, y) - \Lambda(\hat{\theta}(t, i, \tilde{y})) = 0 \quad \forall (t, i, y) \in Ł.$

(iv) $\phi(T, i, k, s, x) = kg(s) - h(i, k, s) - kx \quad \forall (k, i, s, x) \in \mathbb{R}_+ \times \mathcal{M} \times \mathbb{R}^2_+.$

(v) The family $\{\phi(\tau, \alpha_\tau, Y^{\theta,\pi}(\tau))\}_{\tau \in \mathscr{T}}$ *is uniformly integrable for all* $(\theta, \pi) \in \mathbb{U} \times \mathscr{A}(T, x)$*, where* $\mathscr{T}$ *is the set of all* $\mathscr{F}_t$*-stopping times* $\tau \leq T$*.*

Then

$$\phi(t,i,y) = \Phi_G(t,i,y) = \inf_{\pi\in\mathscr{A}(T,x)}\Big(\sup_{\theta\in\mathbb{U}} J^{\theta,\pi}(t,i,y)\Big) = \sup_{\theta\in\mathbb{U}}\Big(\inf_{\pi\in\mathscr{A}(T,x)} J^{\theta,\pi}(t,i,y)\Big)$$
$$= \sup_{\theta\in\mathbb{U}} J^{\theta,\hat{\pi}}(t,i,y) = \inf_{\pi\in\mathscr{A}(T,x)} J^{\hat{\theta},\pi}(t,i,y) = J^{\hat{\theta},\hat{\pi}}(t,i,y); \quad \forall (t,i,y)\in Ł. \tag{4.12}$$

Proof. Choose $(\theta,\pi)\in\mathbb{U}\times\mathscr{A}(T,x)$. Then by the Dynkin formula (Øksendal and Sulem (2007)) for jump diffusion processes we have:

$$\mathbb{E}^{t,i,y}[\phi(\tau,\alpha_\tau,Y(\tau_L^{(N)}))] = \phi(t,i,y) + \mathbb{E}^{t,i,y}\Big[\int_0^{\tau_L(N)} A^{\theta,\pi}\phi(t,\alpha_t,Y(t))dt\Big] \tag{4.13}$$

where $Y(t) = Y^{\theta,\pi}(t)$ and

$$\tau_L^{(N)} = T\wedge \inf\{t>0 : |Y(t)|\geq N\}, \quad N=1,2,...$$

(I) If we apply (4.13) to $\theta,\hat{\pi}$ and use 1. for all $y=Y(t)$, we get

$$\mathbb{E}^{t,i,y}[\phi(\tau,\alpha_\tau,Y(\tau_L^{(N)}))] \geq \phi(t,i,y) - \mathbb{E}^{t,i,y}\Big[\int_0^{\tau_L^{(N)}} \Lambda(\theta(t,\alpha_t,\tilde{Y}(t)))dt\Big]$$

or

$$\phi(t,i,y) \leq \mathbb{E}^{t,i,y}\Big[\int_0^{\tau_L^{(N)}} \Lambda(\theta(t,\alpha_t,\tilde{Y}(t)))dt + \phi(\tau,\alpha_\tau,Y(\tau_L^{(N)}))\Big].$$

By letting $N\to\infty$ and using *(iv.)* and *(v.)* we obtain

$$\phi(t,i,y) \leq J^{\theta,\hat{\pi}}(t,i,y). \tag{4.14}$$

Since this holds for all $\theta \in \mathbb{U}$ we deduce that

$$\phi(t, i, y) \leq \inf_{\theta \in \mathbb{U}} J^{\theta, \hat{\pi}}(t, i, y). \tag{4.15}$$

Hence

$$\phi(t, i, y) \leq \sup_{\pi \in \mathscr{A}(T,x)} \Big(\inf_{\theta \in \mathbb{U}} J^{\theta, \pi}(t, i, y) \Big) = \Phi(t, i, y). \tag{4.16}$$

(II) Now we apply (4.13) to $\hat{\theta}, \pi$ with $\pi \in \mathscr{A}(T, x)$ and use *(ii.)* for all $y = Y(t)$. Then we get

$$\mathbb{E}^{t,i,y}[\phi(\tau, \alpha_\tau, Y(\tau_L^{(N)}))] \leq \phi(t, i, y) - \mathbb{E}^{t,i,y}\Big[\int_t^{\tau_L^{(N)}} \Lambda(\hat{\theta}(t, \alpha_t, \tilde{Y}(t)))dt \Big]$$

or

$$\phi(t, i, y) \geq \mathbb{E}^{t,i,y}\Big[\int_0^{\tau_L^{(N)}} \Lambda(\tilde{\theta}(t, \alpha_t, \tilde{Y}(t)))dt + \phi(\tau, \alpha_\tau, Y(\tau_L^{(N)})) \Big].$$

Letting $N \to \infty$ and using *(iv.)* and *(v.)* we obtain

$$\phi(t, i, y) \geq J^{\hat{\theta}, \pi}(t, i, y) \geq \inf_{\theta \in \mathbb{U}} J^{\theta, \pi}(t, i, y). \tag{4.17}$$

Since this hold for all $\pi \in \mathscr{A}(T, x)$ we deduce that

$$\phi(t, i, y) \geq \sup_{\pi \in \mathscr{A}(T,x)} \Big(\inf_{\theta \in \mathbb{U}} J^{\theta, \pi}(t, i, y) \Big) = \Phi(t, i, y). \tag{4.18}$$

(III) Finally, we apply (4.13) to $\hat{\theta}, \hat{\pi}$ and proceed as above. Thus will give us:

$$\phi(t, i, y) = J^{\hat{\theta}, \hat{\pi}}(t, i, y). \tag{4.19}$$

Combining (4.16), (4.18) and (4.19) we get

$$\Phi(t,i,y) \le \phi(t,i,y) = J^{\hat{\theta},\hat{\pi}}(t,i,y) \le \Phi(t,i,y). \tag{4.20}$$

Combining (4.17) and (4.14) we get:

$$\begin{aligned}\inf_{\theta\in\mathbb{U}}\Big(\sup_{\pi\in\mathscr{A}(T,x)} J^{\theta,\pi}(t,i,y)\Big) &\le \sup_{\pi\in\mathscr{A}(T,x)} J^{\hat{\theta},\pi}(t,i,y) \le \phi(t,i,y) \le \inf_{\theta\in\mathbb{U}} J^{\theta,\hat{\pi}}(t,i,y) \\ &\le \sup_{\pi\in\mathscr{A}(T,x)}\Big(\inf_{\theta\in\mathbb{U}} J^{\theta,\pi}(t,i,y)\Big) = \Phi(t,i,y).\end{aligned} \tag{4.21}$$

But on the other hand, we always have

$$\sup_{\pi\in\mathscr{A}(T,x)}\Big(\inf_{\theta\in\mathbb{U}} J^{\theta,\pi}(t,i,y)\Big) \le \inf_{\theta\in\mathbb{U}}\Big(\sup_{\pi\in\mathscr{A}(T,x)} J^{\theta,\pi}(t,i,y)\Big),$$

together with the others inequalities we get our final claim. □

From this we obtain the following theorem:

Theorem 4.3.2. *Suppose the value function $\Psi_G(t,i,\tilde{y})$ for Problem 3 satisfies the conditions of Lemma (4.2.2). Then the value function for Problem 1 is*

$$\Phi_G(t,i,y) = \Psi_G(t,i,\tilde{y}) - kx$$

and there exists an optimal $\theta_{opt} \in \mathbb{V}$ for Problem 3 such that for all $\pi \in \mathscr{A}(T,x)$ the pair

$$(\theta^*,\pi^*) = (\theta_{opt},\pi)$$

is an optimal pair for Problem 1.

Proof. By the HJBI equation for Problem 3 we know that $\forall t \in (0, T)$:

$$\sup_{\theta: V\theta=0} \{A^{\theta}\Psi_G(t,i,\tilde{y}) - \Lambda(\theta(t,i,\tilde{y}))\} = A^{\theta_{opt}(t,i,\tilde{y})}\Psi_G(t,i,\tilde{y}) - \Lambda(\theta_{opt}(t,i,\tilde{y})) = 0 \tag{4.22}$$

with terminal value

$$\Psi_G(t,i,\tilde{y}) = \Psi_G(t,i,k,s) = kg(s) - h(i,k,s). \tag{4.23}$$

Define

$$\phi(t,i,y) = \Psi_G(t,i,\tilde{y}) - kx \quad \forall (t,i,y) \in \text{Ł}. \tag{4.24}$$

We will show that ϕ satisfies all conditions of Theorem (4.3.1) and hence is the value function of Problem 1. Then by Lemma (4.2.1) we have

$$A^{\theta,\pi}\phi(t,i,y) - \Lambda(\theta) = A^{\theta}\Psi_G(t,i,\tilde{y}) - \Lambda(\theta) - ks\pi V\theta.$$

where $V\theta = V\theta(t,i,\tilde{y})$ is defined in (2.15). Therefore, condition *(i.)* - *(iii.)* of Theorem (4.3.1) get the form

(i) $A^{\theta}\Psi_G(t,i,k,s) - \Lambda(\theta) - ks\hat{\pi}V\theta(t,i,k,s) \leq 0 \quad \forall \theta \in \mathbb{R}^2$,

(ii) $A^{\tilde{\theta}}\Psi_G(t,i,k,s) - \Lambda(\hat{\theta}) - ks\pi V\hat{\theta}(t,i,k,s) \geq 0 \quad \forall \pi \in \mathbb{R}$,

(iii) $A^{\tilde{\theta}}\Psi_G(t,i,k,s) - \Lambda(\hat{\theta}) - ks\hat{\pi}V\hat{\theta}(t,i,k,s) = 0 \quad \forall (t,i,k,s) \in \tilde{\text{Ł}}$.

Choose $\hat{\pi}$ and $\theta_{opt} = \hat{\theta}(\hat{\pi})$ as in Lemma (4.2.2). Combining (4.22) with Lemma

(4.2.2) we get

$$A^{\theta}\Psi_G - \Lambda(\theta) - ks\hat{\pi}V\theta \le \sup_{\theta}\{A^{\theta}\Psi_G - \Lambda(\theta) - ks\hat{\pi}V\theta\}$$
$$= A^{\hat{\theta}(\hat{\pi})}\Psi_G - \Lambda(\hat{\theta}(\hat{\pi})) - ks\hat{\pi}V\hat{\theta}(\hat{\pi}) = \sup_{\theta:V\theta=0}\{A^{\theta}\Psi_G - \Lambda(\theta)\} = 0,$$

which proves *(i.)*. Moreover, since $V\theta_{opt} = 0$, we get by (4.22)

$$A^{\theta_{opt}}\Psi_G - \Lambda(\theta_{opt}) - ks\pi V\theta_{opt} = A^{\theta_{opt}}\Psi_G - \Lambda(\theta_{opt}) = 0 \quad \forall \pi \in \mathbb{R},$$

which proves *(ii.)* and *(iii.)*. Finally, we have to check that *(iv.)* holds: By (4.24) and (4.23) we have

$$\phi(T, i, k, s, x) = \Psi_G(T, i, k, s) - kx = kg(s) - h(i, s, x) - kx.$$

We conclude that ϕ, $\hat{\theta}(\hat{\pi})$ and $\hat{\pi}$ satisfy all the requirement of Theorem (4.3.1). Then

$$\phi(t, i, k, s, x) = \Phi_G(t, i, k, s, x) = \Psi_G(t, i, k, s) - kx.$$

Moreover, $\theta^* := \hat{\theta}(\hat{\pi})$ and $\pi^* := \hat{\pi}$ constitute an optimal pair. Now let $\pi \in \mathscr{A}(T, x)$ be arbitrary. Note that:

$$\mathbb{E}^{t,i,y}[Z_{\theta^*}(T)X^{\pi}(T)] = \mathbb{E}^{t,i,y}[Z_{\hat{\theta}}(T)X^{\pi}(T)] = k\mathbb{E}^{t,i,y}_{\frac{1}{k}Q_{\theta_{opt}}}[X^{\pi}(T)] = kx,$$

since $\frac{1}{k}Q_{\theta_{opt}}$ is an equivalent martingale measure. Therefore, going back to the definition of Ψ_G, we then have (4.5) - (4.7) with $Y^* = Y^{\theta^*,\pi^*}, Y = Y^{\theta_{opt},\pi}$:

$$\begin{aligned}\Phi_G(t,i,y) &= \inf_{\pi\in\mathscr{A}(T,x)} \Big(\sup_{\theta\in\mathbb{U}} J^{\theta,\pi}(t,i,y)\Big) = J^{\hat{\theta}(\hat{\pi}),\hat{\pi}}(t,i,y) \\ &= \mathbb{E}^{t,i,y}\Big[-\int_t^T \Lambda(\hat{\theta}(s,\alpha_s,\tilde{Y}^*(s)))ds + Z_{\theta^*}(T)g(S(T)) \\ &- h(\alpha_T, Z_{\theta^*}(T), S(T)) - Z_{\theta^*}(T)X^{\pi^*}(T)\Big] \\ &= \mathbb{E}^{t,i,y}\Big[-\int_t^T \Lambda(\hat{\theta}(s,\alpha_s,\tilde{Y}(s)))ds + Z_{\theta_{opt}}(T)g(S(T)) \\ &- h(\alpha_T, Z_{\theta_{opt}}(T), S(T))\Big] - kx = J^{\hat{\theta}(\hat{\pi}),\pi}(t,i,y).\end{aligned}$$

We conclude that for all $\pi \in \mathscr{A}(T,x)$ the pair

$$(\theta^*,\pi) = (\theta_{opt},\pi) \in \mathbb{V} \times \mathscr{A}(T,x)$$

is optimal for Problem 1, as claimed. □

4.4 Viscosity Solutions

We will use now a different approach to get the same result but in this case with weaker conditions, since the condition of Lemma (4.2.2) are very strong. The following definition is based on Barles and Imbert (2008). For further information see also Jakobsen and Karlsen (2006) and Crandall et al. (1992).

Definition 4.4.1 (Viscosity solutions)**.** Let C denote the set of functions $u : \tilde{\mathbb{L}} \to \mathbb{R}$ with at most linear growth.

- An upper semi continuous function $u \in C$ is a viscosity subsolution of the

HJBI equation for Problem 3, i.e.,

$$\sup_{\theta:V\theta=0}\{A^{\theta}u-\Lambda(\theta)\}=0 \quad \text{in } \tilde{Ł} \tag{4.25}$$

$$u(T,i,\tilde{y})=kg(s)-h(i,\tilde{y}), \tag{4.26}$$

if u satisfies (4.26) and for any $\phi\in C^2(\mathbb{R}\times\mathcal{M}\times\mathbb{R}^2)\cap C$ and $(t_0,i_0,\tilde{y}_0)\in\tilde{Ł}$ such that $\phi\geq u$ everywhere on $\tilde{Ł}$ and $\phi(t_0,i_0,\tilde{y}_0)=u(t_0,i_0,\tilde{y}_0)$, we have

$$\sup_{\theta:V\theta=0}\{A^{\theta}\phi-\Lambda(\theta)\}(t_0,i_0,\tilde{y}_0)\geq 0.$$

- An lower semi continuous function $u\in C$ is a viscosity supersolution of the HJBI equation for Problem 3, if u satisfies (4.26) and for any $\phi\in C^2(\mathbb{R}\times\mathcal{M}\times\mathbb{R}^2)\cap C$ and $(t_0,i_0,\tilde{y}_0)\in\tilde{Ł}$ such that $\phi\leq u$ everywhere on $\tilde{Ł}$ and $\phi(t_0,i_0,\tilde{y}_0)=u(t_0,i_0,\tilde{y}_0)$, we have

$$\sup_{\theta:V\theta=0}\{A^{\theta}\phi-\Lambda(\theta)\}(t_0,i_0,\tilde{y}_0)\leq 0.$$

- A continous function $u\in C$ is a viscosity solution of the HJBI equation for Problem 3, if u is both a viscosity subsolution and a viscosity supersolution of (4.25) and (4.26).

Similar, we define the expression viscosity (sub-/super) solutions u of the HJBI equation

$$\inf_{\pi\in\mathscr{A}(T,x)}\left(\sup_{\theta\in\Theta}\{A^{\theta,\pi}u-\Lambda(\theta)\}\right)=0 \quad \text{in } Ł \tag{4.27}$$

$$u(T,i,y)=kg(s)-h(i,\tilde{y})-kx \tag{4.28}$$

for Problem 1. We say that a function $u \in C(\mathbb{R}\times\mathcal{M}\times\mathbb{R}^2)\cap C$ satisfies the *dynamic programming principle* if

$$u(t_0, i_0, \tilde{y}_0) \geq \mathbb{E}^{t_0,i_0,\tilde{y}_0}\Big[u(\tau, \alpha_\tau, \tilde{Y}^\theta(\tau)) - \int_0^\tau \Lambda(\theta(s))ds\Big] \tag{4.29}$$

for all bounded stopping time τ, all $\theta \in \Theta$ and all $(t_0, i_0, \tilde{y}_0) \in \mathbb{R}\times\mathcal{M}\times\mathbb{R}^2$. For getting general conditions that the dynamic programming principle holds, see Ishikawa (2004) and Bouchard and Touzi (2011).

Theorem 4.4.2. *Under the dynamic programming principle the following statements are true:*

- *Suppose u is a viscosity subsolution of the HJBI equation* (4.25) *and* (4.26) *of Problem 3. Then*

  $$w(t, i, y) := u(t, i, \tilde{y}) - kx$$

  is a viscosity subsolution of the HJBI equation of Problem 1.

- *Suppose u satisfies* (4.29) *and* (4.28). *Then*

  $$w(t, i, y) := u(t, i, \tilde{y}) - kx$$

  is a viscosity supersolution of the HJBI equation for Problem 1.

- *Suppose u satisfies* (4.29) *and u is a viscosity solution of the HJBI equation of Problem 3. Then*

  $$w(t, i, y) := u(t, i, \tilde{y}) - kx$$

is a viscosity solution of the HJBI equation of Problem 1.

Proof. It suffices to prove the first two parts. Then the third part follows immediately.

Proof of the first part: Suppose u is a viscosity subsolution of (4.25). We want to prove that

$$w(t,i,y) := u(t,i,\tilde{y}) - kx$$

is a viscosity subsolution of (4.27). To this end, suppose $\phi \in C^2 \cap C$, $\phi \geq w$ and $\phi(t_0,i_0,y_0) = w(t_0,i_0,y_0)$ at some point $(t_0,i_0,y_0) \in \L$. Put

$$\Psi(t,i,y) := \phi(t,i,y) + kx; \quad (t,i,y) \in \L.$$

Then

$$\Psi \in C^2 \cap C, \Psi \geq \phi \text{ and } \Psi(t_0,i_0,y_0) = u(t_0,i_0,y_0).$$

Therefore, since u is a viscosity subsolution of HJBI, we have:

$$\sup_{\theta:V\theta=0} \{A^\theta \Psi - \Lambda(\theta)\}(t_0,i_0,y_0) \geq 0.$$

But then, by Lemma (4.2.1),

$$\begin{aligned}\inf_\pi \Big(\sup_\theta \{A^{\theta,\pi}\phi - \Lambda(\theta)\}\Big) &= \inf_\pi \Big(\sup_\theta \{A^\theta \phi - \Lambda(\theta) + k_0 s_0 \pi V\theta\}\Big) \\ &\geq \inf_\pi \Big(\sup_{\theta:V\theta=0} \{A^\theta \phi - \Lambda(\theta) + k_0 s_0 \pi V\theta\}\Big) \\ &= \sup_{\theta:V\theta=0} \{A^\theta \Psi - \Lambda(\theta)\} \geq 0 \quad \text{at } (t_0,i_0,y_0).\end{aligned}$$

This proves w is a subsolution of HJBI.

Proof of the second part: Suppose u satisfies (4.29). We want to prove that

$$w(t, i_0, y) := u(t, i_0, \tilde{y}) - kx$$

is a viscosity supersolution of (4.27). To this end, suppose $\phi \in C^2 \cap C$, $\phi \leq w$ and $\phi(t_0, i_0, y_0) = w(t_0, i_0, y_0)$ at some point $(t_0, i_0, y_0) \in Ł$. Put

$$\Psi(t, i, y) := \phi(t, i, y) + kx; \quad (t, i, y) \in Ł.$$

Then

$$\Psi \leq u \quad \text{and } \Psi(t_0, i_0, y_0) = u(t_0, i_0, y_0).$$

Therefore, since u satisfies (4.29), we have:

$$\begin{aligned}\Psi(t_0, i_0, \tilde{y}_0) = u(t_0, i_0, \tilde{y}_0) &\geq \mathbb{E}^{t_0, i_0, \tilde{y}_0}\Big[u(\tau, \alpha_\tau, \tilde{Y}^\theta(\tau)) - \int_0^\tau \Lambda(\theta) ds\Big] \\ &\geq \mathbb{E}^{t_0, i_0, \tilde{y}_0}\Big[\Psi(\tau, \alpha_\tau, \tilde{Y}^\theta(\tau)) - \int_0^\tau \Lambda(\theta) ds\Big].\end{aligned}$$

By the Dynkin formula we have:

$$\mathbb{E}^{t_0, i_0, \tilde{y}_0}[\Psi(\tau, \alpha_\tau, \tilde{Y}^\theta(\tau))] = \Psi(t_0, i_0, \tilde{y}_0) + \mathbb{E}^{t_0, i_0, \tilde{y}_0}\Big[\int_0^\tau A^\theta \Psi(s, \alpha_s, \tilde{Y}^\theta(s)) ds\Big].$$

Combining these two inequalities we get:

$$\mathbb{E}^{t_0, i_0, \tilde{y}_0}\Big[\int_0^\tau \{A^\theta \Psi(s, \alpha_s, \tilde{Y}^\theta(s)) - \Lambda(\theta)\} ds\Big] \leq 0.$$

Since this holds for all bounded stopping time τ, we conclude that

$$A^{\theta}\Psi - \Lambda(\theta) \leq 0 \quad \text{at } (t_0, i_0, \tilde{y}_0) \quad \forall \theta \in \Theta.$$

Hence

$$\sup_{\theta}\{A^{\theta}\Psi - \Lambda(\theta)\} \leq 0 \quad \text{at} \quad (t_0, i_0, \tilde{y}_0).$$

Therefore

$$\inf_{\pi}\left(\sup_{\theta}\{A^{\theta}\Psi - \Lambda(\theta) - k_0 s_0 \pi V \theta\}\right) \leq 0 \quad \text{at} \quad (t_0, i_0, \tilde{y}_0).$$

This proves thath w is a supersolution of HJBI, and hence completes the proof of the second part. □

Using this theorem we can now state the following viscosity solution version:

Theorem 4.4.3. *As before let* $\Phi_G(t,i,y) = \Phi_G(t,i,k,s,x)$ *and* $\Psi_G(t,i,\tilde{y}) = \Psi_G(t,i,k,s)$ *be the value functions of Problem 1 and Problem 3. Suppose that* $\Phi_G(t,i,k,s,x)$ *is the unique viscosity solution of the HJBI equation for Problem 1. Then*

$$\Phi_G(t,i,k,s,x) = \Psi_G(t,i,k,s) - kx. \tag{4.30}$$

Proof. By Pham (1998) Theorem 3.1 we know that $\Psi_G(t,i,k,s)$ is a viscosity solution of the HJBI equation for Problem 3. Moreover, $\Psi_G(t,i,k,s)$ satisfies the dynamic programming principle. Hence by our previous Theorem we get that

$$u(t,i,k,s,x) := \Psi_G(t,i,k,s) - kx.$$

is a viscosity solution of the HJBI equation for Problem 1. By uniqueness we get our claim. □

Sufficient conditions for the uniqueness of the viscosity solutionof the HJBI equation are given by Jakobsen and Karlsen (2006) and Pham (1998).

Chapter 5

Conclusion

We now apply Theorems (4.3.2) and (4.4.3) to find the risk indifference price $p = p_{risk}$ given in our introduction, given as the solution p of the equation:

$$\Phi_G(t, i, k, s, x + p) = \Phi_0(t, i, k, s, x)$$

where Φ_G is the solution of Problem 1. By both Theorems this equation becomes:

$$\Psi_G(t, i, k, s) - k(x + p) = \Psi_0(t, i, k, s) - kx,$$

which has the solution

$$p = p_{risk} = k^{-1}(\Psi_G(t, i, k, s) - \Psi_0(t, i, k, s)).$$

In particular, when we choose $k = 1$ (This makes the measures $\mathbb{Q}_\theta \in \mathscr{V}$ into a probability measures), we get:

Theorem 5.0.4 (Risk indifference pricing theorem - seller's price)**.** *Suppose that either the conditions of Theorem* (4.3.2) *or Theorem* (4.4.3) *hold. Then the seller's risk indifference price of* G*,* $p_{risk}^{seller}(G)$*, is given by*

$$p_{risk}^{seller}(G) = \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] - \xi(\mathbb{Q}_\theta)\} - \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{-\xi(\mathbb{Q}_\theta)\},$$

where $\mathscr{V}$ *is the set of equivalent martingale measures defined in chapter* (2.3)*.*

Note that:

$$p_{risk}^{seller}(G) \leq \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \mathbb{E}_{\mathbb{Q}_\theta}[G] + \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{-\xi(\mathbb{Q}_\theta)\} - \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{-\xi(\mathbb{Q}_\theta)\} \leq \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \mathbb{E}_{\mathbb{Q}_\theta}[G] = p_{up}(G),$$

with equality only if $\xi(\mathbb{Q}_\theta) = 0$ for all $\mathbb{Q}_\theta$. Similarly, we get:

Theorem 5.0.5 (Risk indifference pricing theorem - buyers's price)**.** *Suppose that either the conditions of Theorem* (4.3.2) *or Theorem* (4.4.3) *hold. Then the seller's risk indifference price of* G*,* $p_{risk}^{buyer}(G)$*, is given by*

$$p_{risk}^{buyer}(G) = \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] + \xi(\mathbb{Q}_\theta)\} - \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\xi(\mathbb{Q}_\theta)\},$$

where $\mathscr{V}$ *is again the set of equivalent martingale measures defined in Chapter* (2.3)*.*

Note again that:

$$p_{risk}^{buyer}(G) \geq \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \mathbb{E}_{\mathbb{Q}_\theta}[G] = p_{low}(G),$$

with equality only if $\xi(\mathbb{Q}_\theta) = 0$ for all $\mathbb{Q}_\theta$.

If we combine this two inequalities, we obtain the following chain of inequalities

Corollary 5.0.6. *We have:*

$$p_{low}(G) \leq p_{risk}^{buyer}(G) \leq p_{risk}^{seller}(G) \leq p_{up}(G).$$

Proof. It remains to prove the second inequality, namely that:

$$\inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] + \xi(\mathbb{Q}_\theta)\} - \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\xi(\mathbb{Q}_\theta)\} \leq \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] - \xi(\mathbb{Q}_\theta)\} + \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\xi(\mathbb{Q}_\theta)\}. \tag{5.1}$$

We know

$$\begin{aligned}
&\sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] - \xi(\mathbb{Q}_\theta)\} - \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] + \xi(\mathbb{Q}_\theta)\} \\
&\quad \geq \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] - \xi(\mathbb{Q}_\theta) - (\mathbb{E}_{\mathbb{Q}_\theta}[G] + \xi(\mathbb{Q}_\theta))\} \qquad (5.2) \\
&\quad = \sup_{\mathbb{Q}_\theta \in \mathscr{V}} \{-2\xi(\mathbb{Q}_\theta)\} = -2 \inf_{\mathbb{Q}_\theta \in \mathscr{V}} \xi(\mathbb{Q}_\theta),
\end{aligned}$$

from which (5.1) follows. □

From (5.2) we deduce the following:

Corollary 5.0.7. *If*

$$\underset{\mathbb{Q}_\theta \in \mathscr{V}}{\operatorname{argmax}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] - \xi(\mathbb{Q}_\theta)\} \cap \underset{\mathbb{Q}_\theta \in \mathscr{V}}{\operatorname{argmin}} \{\mathbb{E}_{\mathbb{Q}_\theta}[G] + \xi(\mathbb{Q}_\theta)\} \neq \emptyset, \tag{5.3}$$

then

$$p_{risk}^{seller}(G) = p_{risk}^{buyer}(G).$$

Note that (5.3) holds trivially if $\mathscr{V}$ consists of just one measure, which is the case if the market is complete. Thus our results agree to the well known results for uniqueness of the price in complete markets.

Chapter 6

Future Directions

We have found an alternative way to price derivatives in jump-diffusion markets with regime switching by using risk-indifference pricing. It turned out that the so found interval is more accurate than the interval that we get by using super- and subhedging. To use our approach the market needs either to satisfy the assumptions from Lemma (4.2.2) or that the viscosity solution for Problem (1) has a unique solution. Mainly inside the jump-diffusion market with regime switching it is not clear when the uniqueness is satisfied. Sufficient conditions for the uniqueness of the viscosity solution of the HJBI equation inside the jump-diffusion market without regime switching are given by Jakobsen and Karlsen (2006) and Pham (1998). These results could give some ideas and/or approaches to develop a result for the uniqueness with regime switching.

The quality of our approach depends extremely on the choice for the penalty function ξ. So far it is just a theoretical result which shows that the interval between the risk-indifference price for the buyer and the seller lies inside of the interval between the prices established by super- and subhedging. It is not verified that the intervals are different. For example if we choose the penalty function to be a constant we will get always the same interval with both methods.

Besides this it is possible that for different claims different penalty functions are optimal. Especially, if we take the point of view as an practitioner, we want to minimize the interval for the price of a derivative as best as possible to get closer to the "‘real"’ price. A detailed research study for several penalty functions for different common used derivatives is necessary to establish a better understanding of the role of ξ for the price interval.

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Appendix A

Market Set Up including Discounting

In this work we assume that $r(t, \alpha_t) \equiv 0$. For setting up the market in the general case we neet to model the discounted stock price $\bar{S}_t = \beta(t)S(t)$. For this we have the dynamics:

$$
\begin{aligned}
d\bar{S}_t &= \beta(t)[dS_t - r(t, \alpha_t)S(t)dt] \\
&= [\mu(t, \alpha_t) - r(t, \alpha_t)]\bar{S}_{t-}dt + \sigma(t, \alpha_t)\bar{S}_{t-}dW_t \\
&+ \bar{S}_{t-} \int_{\mathbb{R}_0} \gamma(t, \alpha_t, z)\tilde{N}(dt, dz) \\
&= \hat{\mu}(t, \alpha_t)\bar{S}_{t-}dt + \sigma(t, \alpha_t)\bar{S}_{t-}dW_t + \bar{S}_{t-} \int_{\mathbb{R}_0} \gamma(t, \alpha_t, z)\tilde{N}(dt, dz),
\end{aligned}
\tag{A.1}
$$

where we set $\hat{\mu}(t,\alpha_t) = \mu(t,\alpha_t) - r(t,\alpha_t)$ for simplicity. The dynamics of the wealth process are given by

$$\begin{aligned} dX_t = dX_t^\pi =& \left(r(t,\alpha_t)X_t^\pi + \pi(t)S_{t-}[\mu(t,\alpha_t) - r(t,\alpha_t)]\right) dt \\ &+ \sigma(t,\alpha_t)\pi(t)S_{t-}dW_t + \pi(t)S_{t-}\int_{\mathbb{R}_0} \gamma(t,\alpha_t,z)\tilde{N}(dt,dz). \end{aligned} \tag{A.2}$$

One can solve (A.2) to obtain

$$\begin{aligned} X^\pi(t) =& B(t)\Big[x + \int_0^t \beta(s)\pi(s)S_s(\mu(s,\alpha_s) - r(s,\alpha_s))ds \\ &+ \int_0^t \beta(s)\pi(s)S_s\sigma(s,\alpha_s)dW_s \\ &+ \int_0^t\int_{\mathbb{R}_0} \beta(s)\pi(s)S_s\gamma(s,\alpha_s,z)\tilde{N}(ds,dz)\Big]. \end{aligned} \tag{A.3}$$

Furhtermore, one is interested in the discountted wealth process:

$$\begin{aligned} d\bar{X}_t^\pi &= \beta(t)[dX_t^\pi - r(t,\alpha_t)X_t^\pi dt] \\ &= \beta(t)\Big[\pi(t)S_{t-}[\mu(t,\alpha_t) - r(t,\alpha_t)]dt + \sigma(t,\alpha_t)\pi(t)S_{t-}dW_t \\ &+ \pi(t)S_{t-}\int_{\mathbb{R}_0} \gamma(t,\alpha_t,z)\tilde{N}(dt,dz) - dC_t\Big] \\ &= \pi(t)\bar{S}_{t-}\hat{\mu}(t,\alpha_t)dt + \sigma(t,\alpha_t)\pi(t)\bar{S}_{t-}dW_t \\ &+ \pi(t)\bar{S}_{t-}\int_{\mathbb{R}_0} \gamma(t,\alpha_t,z)\tilde{N}(dt,dz). \end{aligned} \tag{A.4}$$

One can solve (A.4) to obtain

$$\begin{aligned} \bar{X}^\pi(t) =& x + \int_0^t \pi(s)\bar{S}_s(\mu(s,\alpha_s) - r(s,\alpha_s))ds + \int_0^t \pi(s)\bar{S}_s\sigma(s,\alpha_s)dW_s \\ &+ \int_0^t\int_{\mathbb{R}_0} \pi(s)\bar{S}_s\gamma(s,\alpha_s,z)\tilde{N}(ds,dz). \end{aligned} \tag{A.5}$$

In this model the wealth process and the discounted wealth process are uniquely defined by π - a given portfolio process. For a fixed finite time-horizon $T > 0$ and a fixed initial endowment x, we say that a (discounted) wealth process or analog a portfolio process π is admissible on $[0, T]$, if $X_t^\pi \geq 0$ for all $t \in [0, T]$ holds almost surely, π is an $\mathfrak{F}_t$-process and furthermore

$$\begin{aligned} &\int_0^t \beta(s,i)\pi_s S_s |\mu(s,i) - r(s,i)| ds + \int_0^t \beta(s,i)^2 \pi_s^2 S_s^2 \sigma(s,i)^2 ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \beta(s,i)^2 \pi_s^2 S_s^2 |\gamma(s,i,z)|^2 \nu(dz) ds < \infty \end{aligned} \tag{A.6}$$

for all $t \in [0, T]$ and for all $i \in \mathcal{M}$ holds almost surely. In such a case, we denote $\pi \in \mathcal{A}(T, x)$.

Appendix B

Condition on the Given Jump Function

To verify that our stock price process is real valued and well-defined, we use Ito's formula in one dimension for Lévy-processes under the condtition of $R = \infty$. For the function $f(t, x, i) = ln(x)$ we obtain:

$$
\begin{aligned}
&f(t, S(t), \alpha_t) \\
&= f(0, X_0, \alpha_0) + \int_0^t (\mu(\alpha_t) - \frac{1}{2}\sigma(\alpha_t)^2)dt + \int_0^t \sigma(\alpha_s)dW_s \\
&+ \int_0^t \int_{\mathbb{R}_0} (ln(1 + \gamma(s, z, \alpha_s)) - \gamma(s, z, \alpha_s))\nu(dz)ds + \int_0^t \int_{\mathbb{R}_0} ln(1 + \gamma(s, z, \alpha_s))\tilde{N}(ds, dz) \\
&= f(0, X_0, \alpha_0) + \int_0^t \mathcal{L}f(s, S(s), \alpha_s)ds + M_1^f(t) + M_2^f(t)
\end{aligned}
\tag{B.1}
$$

where:

$$\mathcal{L}f(s,S(s),\alpha_s) = \mu(\alpha_s) - \frac{1}{2}\sigma(\alpha_s)^2) + \int_{\mathbb{R}_0} (ln(1+\gamma(s,z,\alpha_s)) - \gamma(s,z,\alpha_s))\nu(dz)$$
$$M_1^f(t) = \int_0^t \sigma(\alpha_s)dW_s \text{ and}$$
$$M_2^f(t) = \int_0^t \int_{\mathbb{R}_0} ln(1+\gamma(s,z,\alpha_s))\tilde{N}(ds,dz). \tag{B.2}$$

At first we want to verify that $\mathcal{L}f(s,S(s),\alpha_s)$ is well-defined. For this, note that by Taylor expansion we have

$$\begin{aligned} ln(1+\gamma(s,z,\alpha_s)) - \gamma(s,z,\alpha_s) &= \gamma(s,z,\alpha_s) + \frac{1}{2}\frac{1}{(1+\omega\cdot\gamma(s,z,\alpha_s))}\gamma(s,z,\alpha_s)^2 \\ &- \gamma(s,z,\alpha_s) \\ &= \frac{1}{2}\frac{\gamma(s,z,\alpha_s)^2}{(1+\omega\cdot\gamma(s,z,\alpha_s))} \quad \text{where } \omega\in[0,1]. \end{aligned} \tag{B.3}$$

For satisfying the well-defined condition we need two assumptions:

Assumption A1. $\exists\, \delta > 0$ such that $\gamma(s,z,i) > -1+\delta\ \forall z \in \mathbb{R}_0,\ i \in M$

Assumption A2. $\int_{\mathbb{R}_0} |\gamma(s,z,i)^2|\nu(dz) < \infty \quad \forall i \in M$

Under this assumptions we have:

$$1+\omega\cdot\gamma(s,z,i) > 1+\omega\cdot(-1+\delta) \geq \min(1,\delta) > 0 \tag{B.4}$$

Hence,

$$|ln(1+\gamma(s,z,i)) - \gamma(s,z,i)| \leq \frac{1}{2\cdot\min(1,\delta)^2}|\gamma(s,z,i)|^2. \tag{B.5}$$

Since $\mu(i)$ and $\sigma(i)$ are well-defined for all $i \in M$ and under our assumptions

$$\int_{\mathbb{R}_0} |ln(1+\gamma(s,z,\alpha_s)) - \gamma(s,z,\alpha_s)|\nu(dz) < \infty. \tag{B.6}$$

Thus the whole operator $\mathcal{L}f(s,S(s),\alpha_s)$ is well-defined.
It remains to show that $M_1^f(t)$ and $M_2^f(t)$ are martingales. It is clear that the first expression is a martingale so we only have to consider the second one.

By virtue of Øksendal and Sulem (2007) we have to show that

$$\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}_0} |ln(1+\gamma(s,z,\alpha_s))|^2 \tilde{N}(ds,dz)\Big] < \infty. \tag{B.7}$$

Using the Meanvalue-Theorem gives us $ln(1+\gamma(s,z,i)) - ln(1) = \frac{1}{1+\theta\cdot\gamma(s,z,i)}\gamma(s,z,i)$ and by the same arguments as before there exists a $\delta\ > 0$ sucht that the expression is smaller or equal than $\min(1,\delta)\cdot\gamma(s,z,i)$ for ervery $i \in M$. Thus we have:

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}_0} |ln(1+\gamma(s,z,\alpha_s))|^2 \tilde{N}(ds,dz)\Big] \\
&< \min(1,\delta)^2 \cdot \mathbb{E}\int_0^t \int_{\mathbb{R}_0} |\gamma(s,z,\alpha_s)|^2 \nu(dz)ds \\
&< \sum_{i\in M} \min(1,\delta)^2 \cdot \mathbb{E}\int_0^t \int_{\mathbb{R}_0} |\gamma(s,z,i)|^2 \nu(dz)ds < \infty \text{ with } Assumption A2.
\end{aligned} \tag{B.8}$$

Thus we have shown that $M_2^f(t)$ is a martingale and espacially well-defined. This completes the proof that the process $f(t,S(t),\alpha_t)$ is well-defined and therefore $S(t)$.