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PBW Deformations of Artin-Schelter Regular Algebras and Their Homogenizations

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ABSTRACT

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A central object in the study of noncommutative projective geometry is the (Artin-Schelter) regular algebra, which may be considered as a noncommutative version of a polynomial ring. We extend these ideas to algebras which are not necessarily graded. In particular, we define an algebra to be essentially regular of dimension d if its homogenization is regular of dimension $d + 1$. Essentially regular algebras are described and it is shown that that they are equivalent to PBW deformations of regular algebras. In order to classify essentially regular algebras we introduce a modified version of matrix congruence, called sf-congruence, which is equivalent to affine maps between non-homogeneous quadratic polynomials. We then apply sf-congruence to classify homogenizations of 2-dimensional essentially regular algebras. We study the representation theory of essentially regular algebras and their homogenizations, as well as some peripheral algebras.

For Laura and Sophia

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Chapter 1

Introduction

Lying at the intersection of algebraic geometry and noncommutative algebra is the field of noncommutative projective geometry. This field has grown out of the seminal work of Artin and Schelter [9], wherein the original motivation was to define a notion of a noncommutative polynomial ring. They defined a connected graded algebra to be (Artin-Schelter) *regular* if it has finite global and GK dimension and satisfies a certain homological symmetry condition know as AS-Gorenstein (Definition 2.2.1). The study of these algebras has inspired the development of a vast array of new techniques and connections between algebra and other fields.

Much of this work is motivated by the following question: Can one enlarge the class of regular algebras to include algebras which are not graded?

We approach this problem much as one would in algebraic geometry. Given an algebra A which is presented by generators and (not-necessarily homogeneous) relations, we homogenize the relations to produce a new, graded algebra, $H(A)$, called the *homogenization* of A. In case $H(A)$ is regular, we say A is *essentially regular* (Definition 2.3.1). These algebras are also referred to as *central extensions of a regular algebra* in [41] and [23]. This follows from the standard fact that $H(A)$ is regular if and only if $\mathrm{gr}(A)$ is regular (Proposition 2.3.7). Thus, the study of regular homogenizations is equivalent to the study of *PBW deformations* of regular algebras.

Artin, Tate, and Van den Bergh classified regular algebras of dimension three by associating each to a point scheme and automorphism in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ [6]. This geometry was exploited in [7] to study the module structure of regular algebras. They developed a notion of *point modules*, which are modules of GK-dimension one and in bijective correspondence to the points of the noncommutative curve associated to the algebra. Information on the point modules allows one, in many cases, to determine the finite-dimensional simple modules of a regular algebra. The analysis of this, in the case of the so-called Sklyanin algebras, was completed by Walton in [58]. We ask whether the study of simple modules of a deformation can be computed using geometric tools. By considering the geometry associated to the homogenization of an essentially regular algebras, we prove the following.

Theorem 1 (Theorem 2.4.10)**.** *Suppose A is 2-dimensional essentially regular and not a finite module over its center. Then all finite-dimensional simple modules of A are 1-dimensional.*

Such a theorem does not seem accessible (or even true) in higher dimensions. However, we can generalize one part of this theorem to partially answer a conjecture by Walton regarding deformed Skylanin algebras.

Suppose R is a quadratic geometric algebra (Definition 2.4.2). If A is a PBW deformation of R, then $H(A)$ is geometric and so there exists a surjection $\phi : H(A) \to \mathcal{B}$, where $\mathcal{B}$ is a *twisted homogeneous ring.* The problem of determining the finite-dimensional simple modules of $H(A)$ then splits into two cases: those that are torsion over $\ker\phi$ and those that are torsionfree.

Theorem 2 (Theorem 2.4.12)**.** *Suppose A is a PBW deformation of a quadratic geometric algebra and M is a finite-dimensional simple module of $H(A)$ with homogenizing element x_0. If M is torsion over $\ker\phi$, then either $x_0M = 0$ and M is a finite-dimensional simple module over $\mathrm{gr}(A)$ or else M is 1-dimensional.*

A key initial result in the study of regular algebras is that there are exactly two types of regular algebras in dimension two. One might then hope to have a similar classification for essentially regular algebras. In fact, we are able to do a bit better.

Theorem 3 (Theorem 4.0.7)**.** *Suppose $A \cong K\langle x, y \mid f\rangle$ where f is a polynomial of degree two. Then A is isomorphic to one of the following algebras:*

$\mathcal{O}_q(K^2), f = xy - qyx \ (q \in K^\times)$,	$A_1^q(K), f = xy - qyx - 1 \ (q \in K^\times)$,
$\mathcal{J}, f = yx - xy + y^2$,	$\mathcal{J}_1, f = yx - xy + y^2 + 1$,
$\mathfrak{U}, f = yx - xy + y$,	$K[x], f = x^2 - y$,
$R_{x^2}, f = x^2$,	$R_{x^2-1}, f = x^2 - 1$,
$R_{yx}, f = yx$,	$\mathcal{S}, f = yx - 1$.

Furthermore, the above algebras are pairwise non-isomorphic, except

$$\mathcal{O}_q(K^2) \cong \mathcal{O}_{q^{-1}}(K^2) \text{ and } A_1^q(K) \cong A_1^{q^{-1}}(K).$$

Many of these algebras are well-known. The algebras $\mathcal{O}_q(K^2)$ are the *quantum planes* and $A_1^q(K)$ the *quantum Weyl algebras*. The algebra $\mathcal{J}$ is the *Jordan plane* and $\mathcal{J}_1$ is the *deformed Jordan plane*. If L is the two-dimensional solvable Lie algebra, then $\mathfrak{U} = \mathfrak{U}(L)$ is its enveloping algebra. This list slightly contradicts that given in [54] since $\mathcal{S}$ and $\mathcal{J}_1$ both have GK-dimension two. An easy consequence of this theorem is the following.

Corollary 1 (Corollary 2.3.14)**.** *The 2-dimensional essentially regular algebras are* $\mathcal{O}_q(K^2)$, $A_1^q(K)$, $\mathcal{J}$, $\mathcal{J}_1$ *and* $\mathfrak{U}$.

The proof of Theorem 4.0.7 is split between Chapters 3 and 4. In the former, we determine a maximal list of forms for f. This is done by introducing a modified form of matrix congruence. Let $M \in \mathcal{M}_n(K)$ and write M in block form as,

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix},$$

where $M_1 \in M_{n-1}(K)$, $M_2, M_3 \in K^{n-1}$, $m \in K$. We define the K-linear map $\mathrm{sf} : \mathcal{M}_n(K) \to \mathcal{M}_n(K)$ via the rule

$$\begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix} \mapsto \begin{pmatrix} M_1 & M_2 + M_3 \\ 0 & m \end{pmatrix}.$$

One may regard the matrix M as a (non-homogeneous) quadratic relation in $n-1$ variables. Hence, the map sf may be thought of as a matrix equivalent of combining like linear terms. We say a matrix is in *standard form* if $\mathrm{sf}(M) = M$. We define the following group,

$$\mathcal{P} = \left\{ \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_n(K) \mid P_1 \in \mathrm{GL}_{n-1}(K), P_2 \in K^{n-1} \right\}.$$

Definition 1 (Definition 3.3.2)**.** *We say* $M, M' \in \mathcal{M}_n(K)$ *are* ***standard-form congruent*** *(sf-congruent) and write* $M \sim_{sf} M'$ *if there exist* $P \in \mathcal{P}$ *and* $\alpha \in K^\times$ *such that* $sf(M) = \alpha \cdot sf(P^T M' P)$.

Canonical forms under standard-form congruence for matrices in $\mathcal{M}_3(K)$ are shown to be in near 1-1 correspondence algebras of the form $K\langle x, y \mid f\rangle$ with $\deg(f) = 2$. These forms are given in Table 3.1. The second step is to determine whether any of the algebras corresponding to these forms are isomorphic. We consider isomorphism problems for various families of algebras including quantum affine spaces (Theorem 4.2.11) and quantum Weyl algebras (Propositions 4.3.5 and

4.3.6). For the remainder, we consider ring-theoretic properties of the algebras. In particular, we classify the prime and primitive ideals in order to determine the finite-dimensional simple modules.

In Chapter 5 we study the homogenizations of 2-dimensional essentially regular algebras. In particular, sf-congruence can also be used to classify algebras of the form $K\langle x, y, z \mid f, xz - zx, yz - zy\rangle$, where $\deg(f) = 2$ and f is homogeneous. Using results of Irving ([32], [33]), we determine the prime ideals, which in turn allows us to determine the primitive ideals and the finite-dimensional simple modules.

Theorem 4 (Theorem 5.2.4)**.** *Let A be 2-dimensional essentially regular with generators x and y and $H = H(A)$ with homogenizing element z. If P is a nonzero prime ideal in H, then one of the following holds:*

1. *$z \in P$ and P corresponds to a prime of $H/(z)$;*
2. *$z - \alpha \in P$, $\alpha \in K^{\times}$, and P corresponds to a prime of $H/(z - 1)$;*
3. *$xy - yx \in P$;*
4. *$P \cap K[y, z] = (g_1 \cdots g_n)$ where the g_i are irreducible polynomials in $K[y, z]$ disjoint from $K[z]$;*
5. *$P \cap K[x, z] = (g_1 \cdots g_n)$ where the g_i are irreducible polynomials in $K[x, z]$ disjoint from $K[z]$.*

We conclude this work by generalizing the notion of homogenization by introducing skew homogenizations in Chapter 6. In this case, the homogenizing element is no longer assumed to be central, but instead normal such that it commutes with the original generators via a quantum-plane like relation. These algebras may be regarded as *Zhang twists* of standard homogenizations.

As with the homogenizations, we attempt to study the representation theory by first computing the prime ideals. Also, as with the homogenizations, these algebras can be constructed as skew polynomial rings over quantum planes. Our primary tools in this chapter come from results by Goodearl and Letzter ([28]) and results by Leroy and Matczuk ([43]). Much of the work will come in showing when δ is an inner derivation (resp. inner σ-derivation) in the skew polynomial ring $\mathcal{O}_q(K^2)[x; \delta]$ (resp. $\mathcal{O}_q(K^2)[x; \sigma, \delta]$). Thus, the following result is essential to our study.

Proposition 1 (Proposition 6.1.1)**.** *Let $R = \mathcal{O}_p(K^2)[x;\delta]$ with $\delta(z) = 0$. If p is not a root of unity, then δ is inner if and only if $\delta(y)$ has no constant term. If p is a primitive nth root of unity, then δ is inner if and only if z does not appear to a power dividing n in $\delta(y)$.*

Special attention is paid to skew homogenizations of the quantum Weyl algebras. These algebras may be considered as two-parameter analogs of the Heisenberg enveloping algebra. The single parameter case was studied by Kirkman and Small [38]. They showed that such an algebra has a primitive factor ring isomorphic to the Hayashi-Weyl algebra [30]. In our case, we show that if there exists $r, s \in \mathbb{Z}$ such that $p^r = q^s$, then the two parameter Heisenberg enveloping algebra contains a primitive central element Ω.

Theorem 5 (Theorem 6.3.6)**.** *Let $p, q \in K^\times$ be nonroots of unity. Suppose there exist $r, s \in \mathbb{Z}$ such that $p^r = q^s$. For all $\alpha \in K^\times$, $\mathcal{A} = H_{p,q}/(\Omega - \alpha)$ is a simple noetherian domain of GK-dimension two and global dimension one.*

Unless otherwise specified, we assume K is an uncountable, algebraically closed field with $\mathrm{char}(K) = 0$. All algebras may be regarded as K-algebras and all isomorphisms should be read as 'isomorphisms as K-algebras'. All algebras are assumed to be generated in degree one unless otherwise specified.

Chapter 2

Essentially regular algebras

The motivation for studying essentially regular algebras is to determine a sort of ungraded version of regularity. There are two graded algebras one may associate to an essentially regular algebra, the associated graded algebra $\mathrm{gr}(A)$ and the homogenization $H(A)$. We show that regularity of one implies regularity of the other (Proposition 2.3.7). Moreover, we show that, in the case A is noetherian essentially regular, both the global dimension and GK dimension of A are finite (Corollary 2.3.9). One proposed generalization of the notion of regular algebras is that of a Calabi-Yau algebra. In particular, Reyes, Rogalski and Zhang have shown that regular algebras are the same as connected graded skew Calabi-Yau algebras (Theorem 2.2.6). We show that essentially regular algebras are also skew Calabi-Yau (Proposition 2.3.11).

We begin this section with some background on filtered and graded algebras (Section 2.1) and then proceed to give some background on regular algebras (Section 2.2). In Section 2.3 we give the definition of an essentially regular algebra and explore its basic properties. In addition, we use Theorem 4.0.7 to classify all dimension two essentially regular and give several examples in dimension three.

In the generic case, one may associate a set of geometric data to a regular algebra. This is one of the most powerful facts about regular algebras. We exploit this geometry to study the finite-dimensional simple modules of essentially regular algebras. In particular, we show that the finite-dimensional simple modules of a 2-dimensional non-PI essentially regular algebra are 1-dimensional (Theorem 2.4.10). This is generalized in Theorem 2.4.12, which makes progress toward a conjecture of Walton regarding finite-dimensional simple modules of deformed Sklyanin algebras.

2.1 Filtered and graded algebras

Given a (noncommutative) polynomial $f \in K\langle x_1, \ldots, x_n\rangle$ with $\alpha = \deg(f)$, write

$$f = \sum_{1 \le i_1 \le \cdots \le i_\ell \le n} c_i x_{i_1}^{\alpha_{i_1}} \cdots x_{i_\ell}^{\alpha_{i_\ell}}, c_i \in K, \alpha_{i_k} \in \mathbb{N}, \sum_{k=1}^{\ell} \alpha_{i_k} \le \alpha, \tag{2.1}$$

where all but finitely many of the c_i are zero. The homogenization of f is then

$$\hat{f} = \sum_{1 \le i_1 \le \cdots \le i_\ell \le n} c_i x_{i_1}^{\alpha_{i_1}} \cdots x_{i_\ell}^{\alpha_{i_\ell}} x_0^{\alpha_{i_0}},$$

where x_0 is a new, central indeterminate and α_{i_0} is chosen such that $\sum_{k=0}^{\ell} \alpha_{i_k} = \alpha$. Then $\hat{f}$ is homogeneous.

Definition 2.1.1. *Let A be of form*

$$A = K\langle x_1, \ldots, x_n \mid f_1, \ldots, f_m\rangle. \tag{2.2}$$

The ***homogenization*** *$H(A)$ of A is the K-algebra on the generators $x_0, x_1, \ldots, x_n$ subject to the homogenized relations $\hat{f}_i$, $i \in \{1, \ldots, m\}$, as well as the additional relations $x_0 x_j - x_j x_0$ for all $j \in \{1, \ldots, n\}$.*

A *filtration* $\mathcal{F}$ on an algebra A is a collection of vector spaces $\{\mathcal{F}_n(A)\}$ such that $\mathcal{F}_n(A) \subset \mathcal{F}_{n+1}(A)$, $\mathcal{F}_n(A) \cdot \mathcal{F}_m(A) \subset \mathcal{F}_{n+m}(A)$, and $\bigcup \mathcal{F}_n(A) = A$. The filtration $\mathcal{F}$ is said to be *connected* if $\mathcal{F}_0(A) = K$ and $\mathcal{F}_\ell(A) = 0$ for all $\ell < 0$. The *associated graded algebra* of A is $\mathrm{gr}_{\mathcal{F}}(A) := \bigoplus_{i \ge 0} \mathcal{F}_i(A)/\mathcal{F}_{i-1}(A)$. The algebra $\mathrm{gr}_{\mathcal{F}}(A)$ is said to be connected if the filtration $\mathcal{F}$ is connected.

Associated to the pair $(A, \mathcal{F})$ is also the *Rees ring* of A,

$$\mathcal{R}_{\mathcal{F}}(A) := \bigoplus_{n \ge 0} \mathcal{F}_n(A) x_0^n.$$

For an algebra A of the form (2.2), there is a standard connected filtration wherein $\mathcal{F}_d(A)$ is the span of all monomials of degree at most d. Since this filtration is the only one we consider, we drop the subscript on $\mathrm{gr}(A)$ and $\mathcal{R}(A)$. One can recover A and $\mathrm{gr}(A)$ from $H(A)$ via $A \cong H(A)/(x_0 - 1)$ and $\mathrm{gr}(A) \cong H(A)/(x_0)$, respectively. An algebra is said to be *graded* if $\mathrm{gr}(A) = A$. In this case, we write A_d for the vector space spanned by homogeneous elements of degree d.

Let $F = (f_1, \ldots, f_m)$ be the set of relations of A. We can filter F in much the same way we filtered A. For each $i \in \{1, \ldots, m\}$, let r_i be the highest homogeneous degree component of f_i. Let $R = (r_1, \ldots, r_m)$. There is a canonical surjection

$$K\langle x_1, \ldots, x_n \mid R\rangle \to \mathrm{gr}(A). \tag{2.3}$$

Definition 2.1.2. *We say A is a **Poincaré-Birkhoff-Witt (PBW) deformation** of* $\mathrm{gr}(A)$ *if the map* (2.3) *is an isomorphism.*

Definition 2.1.2 may be considered as a generalization of the well-known PBW theorem for Lie algebras. Recall that when L is a Lie algebra and $U(L)$ its enveloping algebra, then $U(L)$ is a PBW deformation of $S(L)$, the symmetric algebra on L.

The *projective dimension* of a right A-module M is the minimum length of a projective resolution of M. The *right global dimension of* A, $\mathrm{rgld}(A)$, is the supremum of the projective dimensions of the right A-modules. *Left global dimension*, $\mathrm{lgld}(A)$, is defined similarly. When A is noetherian, the right and left global dimension coincide, and we write $\mathrm{gld}(A)$ for the *global dimension* of A. Let A_+ be the augmentation ideal generated by all degree one elements. If A is graded, then the global dimension of A is equal to the projective dimension of the *trivial module* $K_A = A/A_+$.

Let V be a K-algebra generating set for A and V^n the set of degree n monomials in A. If $\dim V^n \geq t^n$ for some $t > 1$, then A is said to have *exponential growth*. Otherwise, A has *subexponential growth*. If there exists $c, t \in \mathbb{N}$ such that $V^n \leq ct^n$ for all n, then A is said to have *polynomial growth*. The *Gelfand-Kirilov dimension* (GK dimension) of A is defined as

$$\mathrm{gk}(A) := \limsup_{n\to\infty} \log_n(\dim V^n).$$

Hence, if A has polynomial growth as defined above, then $\mathrm{gk}(A) = t$.

The algebra A is said to be *AS-Gorenstein* if $\mathrm{Ext}^i_A(K_A, A) \cong \delta_{i,d} \cdot {}_AK$ where $\delta_{i,d}$ is the Kronecker delta and $d = \mathrm{gld}(A)$. The AS-Gorenstein property may be thought of as a sort of homological symmetry condition. That is, the condition ensures that the length of a minimal projective resolution of the trivial module K_A is equal to that of the dual resolution obtained by taking $\mathrm{Hom}(\square, A)$ of each term in the resolution.

Of special importance to our analysis throughout this work is the concept of a *skew polynomial ring* (or Ore extension). Let R be a ring, $\sigma \in Aut(R)$ and δ a σ-derivation, that is, $\delta : R \to R$ satisfies the skew Leibniz rule, $\delta(rs) = \sigma(r)\delta(s)+\delta(r)s$ for all $r, s \in R$. This reduces to the usual Leibniz rule when $\sigma = \mathrm{id}_R$.

Definition 2.1.3. *Let R be a ring, $\sigma \in$ $\mathrm{Aut}(R)$ and δ a σ-derivation. The **skew polynomial ring** $R[\xi; \sigma, \delta]$ is defined as the overring of R with commutation defined by $\xi r = \sigma(r)\xi + \delta(r)$ for all $r \in R$. In case $\delta = 0$ we write $R[\xi; \sigma]$, and in case*

$\sigma = \mathrm{id}_R$ *we write* $R[\xi;\delta]$. *The latter are referred to as* ***differential operator rings****.*

By a generalization of the Hilbert Basis Theorem, $R[\xi;\sigma,\delta]$ is noetherian when R is noetherian ([29], Theorem 2.6).

If there exists a unit $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$, then σ is said to be an *inner automorphism* and we have $R[\xi;\sigma,\delta] = R[u\xi;u\delta]$. On the other hand, if there exists $a \in R$ such that $\delta(r) = ar - \sigma(r)a$ for all $r \in R$, then δ is said to be an *inner σ-derivation* and $R[\xi;\sigma,\delta] = R[\xi - a;\sigma]$.

Finally, a ring R is said to be a *polynomial identity ring* (PI ring) if there exists a polynomial $P \in \mathbb{Z}\langle x_1,\ldots,x_n\rangle$ such that $P(r_1,\ldots,r_n)$ for all $r_i \in R$. By [49], Corollary 13.1.13 (iii), if R is finitely generated as a right module over its center $\mathcal{Z}$, then R is PI. In this case, $\dim_{\mathcal{Z}}(R) = n^2 < \infty$. The number n is the *PI-degree* of R and we write $\mathrm{pid}(R) = n$.

2.2 Artin-Schelter regular algebras

The basic object of study in this thesis is the *Artin-Schelter regular algebra*. In this section, we define these algebras using notions developed in the previous section and outline the approach taken by Artin, Tate, and Van den Bergh ([6],[7]).

Definition 2.2.1. *A connected graded algebra H is said to be (Artin-Schelter)* ***regular*** *of dimension d if H has finite global dimension d, finite GK dimension, and is AS-Gorenstein.*

In all known cases, the GK dimension of a regular algebra coincides with its global dimension. We now recall some standard facts regarding regular algebras of low dimension. In dimension two, there are only two types, the quantum planes $\mathcal{O}_q(K^2)$, and the Jordan plane $\mathcal{J}$. In dimension three, every regular algebra surjects onto a *twisted homogeneous coordinate ring* $\mathcal{B}$. Details on these rings are given in Section 2.4 and for now we simply state that they are defined by a projective scheme E, a line bundle $\mathcal{L}$, and an automorphism σ of E. This is a natural generalization of a homogeneous coordinate ring in (commutative) algebraic geometry and there is an analog of ampleness called *σ-ampleness.* Artin and Van den Bergh [8] showed that a twisted homogeneous coordinate ring is noetherian when $\mathcal{L}$ is a σ-ample line bundle.

Theorem 2.2.2 (Artin, Tate, Van den Bergh). *A regular algebra of dimension* ≤ 3 *is noetherian.*

All known regular algebras are domains. In dimension two, the result follows by standard facts on skew polynomial rings. The most significant general result in this direction is recalled next, and the proof relies on knowledge of the *point modules* of a regular algebra.

Theorem 2.2.3 (Artin, Tate, Van den Bergh). *A regular noetherian algebras of dimension* ≤ 4 *is a domain.*

Closely related to regular algebras is the more recent notion of a Calabi-Yau and skew Calabi-Yau algebra, which we define now.

The *enveloping algebra* of A is defined as $A^e := A \otimes A^{op}$. If M is both a left and right A-module, then M is an A^e-module with the action given by $(a \otimes b) \cdot x = axb$ for all $x \in M$, $a, b \in A$. Correspondingly, given automorphisms $\sigma, \tau \in \mathrm{Aut}(A)$, we can define the twisted A^e-module ${}^{\sigma}M^{\tau}$ via the rule $(a \otimes b) \cdot x = \sigma(a)x\tau(b)$ for all $x \in M$, $a, b \in A$. When σ is the identity, we omit it.

Definition 2.2.4. *An algebra A is said to be* ***homologically smooth*** *if it has a finite resolution by finitely generated projectives as an A^e-module. The length of this resolution is the* ***Hochschild dimension*** *of A.*

The Hochschild dimension of a Calabi-Yau algebra is known to coincide with the global dimension ([17], Remark 2.8).

Definition 2.2.5. *An algebra A is said to be* ***skew Calabi-Yau*** *of dimension d if it is homologically smooth and there exists an automorphism $\tau \in \mathrm{Aut}(A)$ such that there are isomorphisms*

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \textit{if } i \neq d \\ A^{\tau} & \textit{if } i = d. \end{cases}$$

If τ is the identity, then A is said to be ***Calabi-Yau****.*

The condition on Ext in Definition 2.2.5 is sometimes referred to as the *rigid Gorenstein* condition [20].

PBW deformations of Calabi-Yau algebras were studied by Berger and Taillefer [17]. Their main result is that PBW deformations of Calabi-Yau algebras defined by quivers and potentials are again Calabi-Yau. More recently, Wu and Zhu proved that a PBW deformation of a noetherian Koszul Calabi-Yau algebra is Calabi-Yau if and only if its Rees ring is [59]. The next result is essential to our analysis.

Theorem 2.2.6 (Reyes, Rogalski, Zhang)**.** *An algebra is connected graded Calabi-Yau if and only if it is regular.*

We prove in the next section that PBW deformations of regular algebras are again skew Calabi-Yau (Proposition 2.3.11). This is then used to show that certain skew polynomial extensions of essentially regular algebras are again essentially regular.

2.3 Essentially regular algebras

Definition 2.3.1. *We say an algebra A is* ***essentially regular*** *of dimension d if $H(A)$ is regular of dimension $d+1$.*

The following lemma is useful in passing properties between A and $H(A)$.

Lemma 2.3.2. *Suppose x_0 is not a zero divisor. Then $H(A)[x_0^{-1}] \cong A[x_0^{\pm 1}]$.*

Proof. Let f be a defining relation for A and $\hat{f}$ the homogenized relation in H. Let $\alpha = \deg(f)$ and write f as in (2.1). Then

$$0 = x_0^{-\alpha}\hat{f} = \sum_{1\leq i_1\leq\cdots\leq i_\ell\leq n} c_i(x_0^{-1}x_{i_1})^{\alpha_{i_1}}\cdots(x_0^{-1}x_{i_\ell})^{\alpha_{i_\ell}}.$$

If we let $X_0 = x_0$ and $X_i = x_0^{-1}x_i$ for $i > 0$, then the $\{X_i\}_{i\geq 0}$ generate $A[x_0^{\pm 1}]$ in $H[x_0^{-1}]$. Conversely, in $A[x_0^{\pm 1}]$ we have

$$0 = x_0^{\alpha} f = \sum_{1\leq i_1\leq\cdots\leq i_\ell\leq n} c_i(x_0x_{i_1})^{\alpha_{i_1}}\cdots(x_0x_{i_\ell})^{\alpha_{i_\ell}}.$$

If we let $X_0 = x_0$ and $X_i = x_0x_i$ for $i > 0$, then the $\{X_i\}_{i\geq 0}$ generate $H[x_0^{-1}]$ in $A[x_0^{\pm 1}]$. □

In the case that x_0 is not a zero divisor, we have $H(A) \cong \mathcal{R}(A)$ ([59], Proposition 2.6) and $H(A)$ becomes a regular central extension of $\mathrm{gr}(A)$ (see [23], [41]). However, this need not always be the case, as the next example illustrates, and so we choose to use $H(A)$ instead of $\mathcal{R}(A)$ in Definition 2.3.1.

Example 2.3.3. *Let $A = K\langle x_1, x_2 \mid x_1^2 - x_2\rangle$. Then $A \cong K[x]$. However, the algebra $H(A)$ is not regular. Indeed,*

$$x_1x_2x_0 = x_1x_1^2 = x_1^2x_1 = x_2x_0x_1 = x_2x_1x_0 \Rightarrow (x_1x_2 - x_2x_1)x_0 = 0.$$

Thus, either x_0 is a zero divisor or else $H(A)$ is commutative. The latter cannot hold because $H(A)/(x_0) \cong K\langle x_1, x_2 \mid x_1^2\rangle$ is not commutative. By [7], Theorem 3.9, all regular algebras of dimension at most four are domains. Hence, in this case, $H(A) \ncong \mathcal{R}(A)$.

In light of Theorem 2.2.3 and the previous example, we hereafter assume that $H(A)$ is a domain whenever A is essentially regular.

Example 2.3.4. *The dimension of an essentially regular algebra is not the same as its global dimension in all cases. The first Weyl algebra, $A_1 = K\langle x, y \mid yx - xy + 1\rangle$, is dimension two essentially regular but has global dimension one.*

Let $\mathcal{Z}(A)$ denote the center of A. One would expect a natural equivalence between the center of a homogenization and the homogenization of a center. The next proposition formalizes that idea.

Proposition 2.3.5. *Suppose x_0 is not a zero divisor. By identifying generators, we have $\mathcal{Z}(H(A)) = H(\mathcal{Z}(A))$.*

Proof. By [53], Propositions 1.2.20 (ii) and 1.10.13, along with Lemma 2.3.2,

$$H(\mathcal{Z}(A))[x_0^{-1}] \cong \mathcal{Z}(A)[x_0^{\pm 1}] = \mathcal{Z}(A[x_0^{\pm 1}]) \cong \mathcal{Z}(H(A)[x_0^{-1}]) = \mathcal{Z}(H(A))[x_0^{-1}].$$

Thus, $H(\mathcal{Z}(A))[x_0^{-1}] \cong \mathcal{Z}(H(A))[x_0^{-1}]$. It remains to be shown that the subalgebras $H(\mathcal{Z}(A))$ and $\mathcal{Z}(H(A))$ are isomorphic and, moreover, the elements can be identified by generators.

Let $\hat{f} \in H(\mathcal{Z}(A))$, then $\hat{f} \in H(\mathcal{Z}(A))[x_0^{-1}]$ and, by Lemma 2.3.2, $f \in \mathcal{Z}(A)[x_0^{\pm 1}]$. Since only positive powers of x_0 appear in f, then $f \in \mathcal{Z}(A)[x_0]$. Therefore, $f \in \mathcal{Z}(A)[x_0^{\pm 1}] = \mathcal{Z}(A[x_0^{\pm 1}])$ and so $\hat{f} \in \mathcal{Z}(H(A)[x_0^{-1}]) = \mathcal{Z}(H(A))[x_0^{-1}]$. Again, since only positive powers of x_0 appear in $\hat{f}$, then $\hat{f} \in \mathcal{Z}(H(A))$. The converse is similar. □

We now consider properties that pass between an essentially regular algebra and its homogenization.

Proposition 2.3.6. *Suppose x_0 is not a zero divisor and $H = H(A)$.*

1. *A is prime if and only if H is prime;*

2. *A is PI if and only if H is PI;*

3. *A is noetherian if H is noetherian;*

4. *H is noetherian if* $\mathrm{gr}(A)$ *is noetherian;*

5. *H is not primitive.*

Proof. (1) is well-known since x_0 is central and not a zero divisor. (2) is a consequence of Proposition 2.3.5. (3) is clear because A is a factor algebra of H and (4) follows from [6], Lemma 8.2. The algebra H is affine over the uncountable, algebraically closed field K, so (5) follows from [37], Proposition 3.2. □

The next proposition shows that essentially regular algebras are equivalent to PBW deformations of regular algebras.

Proposition 2.3.7. *An algebra A is essentially regular if and only if* $\mathrm{gr}(A)$ *is regular. Moreover, if A is essentially regular, then it is a PBW deformation of* $\mathrm{gr}(A)$.

Proof. Let $B = \mathrm{gr}(A)$ and $H = H(A)$. Since $x_0 \in H$ is central and not a zero divisor, then by the Rees Lemma ([52], Theorem 8.34), $\mathrm{Ext}^n_B(K_B, B) \cong \mathrm{Ext}^{n+1}_H(K_H, H)$. Hence, B is AS-Gorenstein if and only if H is. Moreover, since B (resp. H) is graded, then $\mathrm{gld}(B) = \mathrm{pd}(K_B)$ (resp. $\mathrm{gld}(H) = \mathrm{pd}(K_H)$). By the Rees Lemma, $\mathrm{gld}(B) = d$ if and only if $\mathrm{gld}(H) = d + 1$. The sequence $0 \to x_0H \to H \to B \to 0$ is exact, so $\mathrm{gk}(B) \leq \mathrm{gk}(H) - 1 < \infty$ when H is regular. Conversely, if B is regular, then $\mathrm{gk}(A) = \mathrm{gk}(B) < \infty$. Localization at the central regular element x_0 in H and in $A[x_0]$ preserves GK dimension ([49], Proposition 8.2.13). This, combined with Lemma 2.3.2, gives,

$$\mathrm{gk}(H) = \mathrm{gk}(H[x_0^{-1}]) = \mathrm{gk}(A[x_0^{\pm 1}]) = \mathrm{gk}(A) + 1 < \infty.$$

That A is a PBW deformation now follows from [22], Theorem 1.3. □

Corollary 2.3.8. *If A is regular, then A is essentially regular.*

Corollary 2.3.9. *If A is a noetherian essentially regular algebra, then A has finite global and GK dimension.*

Proof. By [49], Corollary 6.18, and because $\mathrm{gr}(A)$ is regular, $\mathrm{gld}\, A \leq \mathrm{gld}\, \mathrm{gr}(A) < \infty$. The statement on GK dimension follows from the proof of Proposition 2.3.7. □

Corollary 2.3.10. *An algebra A is essentially regular of dimension d if and only if $A[\xi]$ is essentially regular of dimension $d + 1$.*

Proof. We need only observe that $H(A[\xi]) = H(A)[\xi]$ and that regularity is preserved under polynomial extensions. $\square$

We now show that the class of essentially regular algebras lives within the class of skew Calabi-Yau algebras. This will enable us to prove that the property of essential regularity is preserved under certain skew polynomial extensions.

Proposition 2.3.11. *If A is noetherian essentially regular, then A is skew Calabi-Yau.*

Proof. By Proposition 2.3.7, A is a PBW deformation of the regular algebra $\mathrm{gr}(A)$. By [60], A has a rigid dualizing complex $R = A^{\sigma}[n]$ for some integer n and some $\sigma \in \mathrm{Aut}(A)$. This is precisely the condition for A to be rigid Gorenstein. The filtration on A is noetherian and connected, so A has a finitely generated A^e-projective resolution ([49], Theorem 7.6.17). Thus, A is homologically smooth. $\square$

Suppose A is of the form (2.2). If $\sigma \in \mathrm{Aut}(A)$ with $\deg(\sigma(x_i)) = 1$, then σ lifts to an automorphism $\hat{\sigma} \in \mathrm{Aut}(H(A))$ defined by $\hat{\sigma}(x_0) = x_0$ and $\hat{\sigma}(x_i) = \widehat{\sigma(x_i)}$ for $i > 0$. To see this, let g be a defining relation for A and $\hat{g}$ the corresponding relation in $H(A)$. For a generator x_i of A, $\sigma(x_i) = y_{i,1} + y_{i,0}$ for some $y_{i,1} \in A_1$ and $y_{i,0} \in A_0 = K$. Thus, $\hat{\sigma}(x_i) = y_{i,1} + y_{i,0}x_0$. We must show that this rule implies $\hat{\sigma}(g) = \widehat{\sigma(g)}$. Suppose $\deg(g) = d$ and write $g = \sum_{i=0}^{d} g_i$ with $\deg(g_i) = i$. Then $\sigma(g_i) = \sum_{j=0}^{i} g_{i,j}$ where $\deg(g_{i,j}) = j$. Thus,

$$\widehat{\sigma(g)} = \widehat{\sum_{i=0}^{d} \sigma(g_i)} = \sum_{i=0}^{d} \widehat{\sum_{j=0}^{i} g_{i,j}} = \sum_{i=0}^{d} \sum_{j=0}^{i} g_{i,j} x_0^{d-j}.$$

Now $\hat{g} = \sum_{i=0}^{d} g_i x_0^{d-i}$ and a similar computation shows

$$\hat{\sigma}(\hat{g}) = \sum_{i=0}^{d} \hat{\sigma}(g_i) x_0^{d-i} = \sum_{i=0}^{d} \sum_{j=0}^{i} (g_{i,j} x_0^{i-j}) x_0^{d-i} = \sum_{i=0}^{d} \sum_{j=0}^{i} g_{i,j} x_0^{d-j}.$$

Similarly, if δ is a σ-derivation of A with $\deg \delta(x_i) \leq 2$ for all i, then $\hat{\delta}$ is a $\hat{\sigma}$-derivation of $H(A)$ with $\hat{\delta}(x_0) = 0$ and $\hat{\delta}(x_i) = \widehat{\delta(x_i)}$ for $i > 0$.

Lemma 2.3.12. *Let A, σ, and δ be as above. Then $H(A[\xi; \sigma, \delta]) = H(A)[\xi; \hat{\sigma}, \hat{\delta}]$.*

Proof. The defining relations for $A[\xi; \sigma, \delta]$ are $f_1, \ldots, f_m$ along with $e_1, \ldots, e_n$ where $e_i = x_i\xi - \sigma(\xi)x_i - g_i$. The defining relations for $H(A[\xi; \sigma, \delta])$ are then $\hat{f}_i, \ldots, \hat{f}_m$ along with $\hat{e}_i = x_i\xi - \hat{\sigma}(\xi)x_i - \hat{g}_i$. Defining $\hat{\delta}(x_i) = \hat{g}_i$ we see that $\hat{\delta}$ is a $\hat{\sigma}$-derivation of $H(A)$. $\square$

Proposition 2.3.13. *Let A be essentially regular. If σ and δ are as above, then $A[\xi;\sigma,\delta]$ is essentially regular.*

Proof. Let $R = H(A)[\xi;\hat{\sigma},\hat{\delta}]$. By Lemma 2.3.12, it suffices to prove that R is regular. Since $H(A)$ is regular, then it is Calabi-Yau. By [45], Theorem 3.3, skew polynomial extensions of Calabi-Yau algebras are Calabi-Yau and so R is Calabi-Yau. Moreover, $\hat{\sigma}$ and $\hat{\delta}$ preserve the grading on $H(A)$ and so R is graded. Thus, by Theorem 2.2.6, R is regular. □

We end this section with a classification of essentially regular algebras in dimension two and several examples of those of dimension three. If A is essentially regular of dimension two, then $H(A)$ is dimension three regular. Hence, $H(A)$ either has three generators subject to three quadratic relations, or else it has two generators subject to two cubic relations. Since $H(A)$ is a homogenization, then the commutation relations for x_0 give two quadratic relations, so there must be some presentation in the first form. Since $A \cong H(A)/(x_0 - 1)$, then the commutation relations drop off and we are left with one quadratic relation. Those algebras of the form $K\langle x, y \mid f\rangle$ where f is quadratic are classified in Theorem 4.0.7. An easy consequence of this is the next corollary.

Corollary 2.3.14. *The dimension two essentially regular algebras are*

$$\mathcal{O}_q(K^2), A_1^q(K), \mathcal{J}, \mathcal{J}_1, \mathfrak{U}.$$

Proof. The algebras $\mathcal{O}_q(K^2)$ and $\mathcal{J}$ are 2-dimensional regular [9]. On the other hand, R_{x^2} and R_{yx} are not domains and therefore not regular [7]. Therefore, $A_1^q(K)$, $\mathcal{J}_1$ and $\mathfrak{U}$ are essentially regular of dimension two, whereas $K[x]$, R_{x^2-1} and $\mathcal{S}$ are not by Proposition 2.3.7. □

We collect a few examples of 3-dimensional essentially regular algebras. We have already observed that if H is 3-dimensional regular, then H is 3-dimensional essentially regular (Corollary 2.3.8) and if A is 2-dimensional regular, then $A[\xi]$ is 3-dimensional essentially regular (Corollary 2.3.10).

Example 2.3.15 ([42])**.** *Let L be at 3-dimensional Lie algebra over K. The enveloping algebra $U(L)$ is 3-dimensional essentially regular. A special case of interest is when $L = \mathfrak{sl}_2(\mathbb{C})$ (see [40]).*

Example 2.3.16 ([22]). *Essentially regular algebras need not be skew polynomial rings. The down-up algebra $A(\alpha, \beta, \gamma)$ for $\alpha, \beta, \gamma \in K$ is defined as the K-algebra on generators d, u subject to the relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$, $du^2 = \alpha udu + \beta u^2 d + \gamma u$. The algebra $A(\alpha, \beta, \gamma)$ is 3-dimensional essentially regular if and only if $\beta \neq 0$.*

Example 2.3.17. *Let $(a : b : c) \in \mathbb{P}^2 \backslash D$ where*

$$D = \{(0:0:1), (0:1:0), (1:0:0)\} \cup \{(a:b:c) \mid a^3 = b^3 = c^3 = 1\}$$

such that $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$. The (3-dimensional) Sklyanin algebra has presentation

$$K\langle x, y, z \mid axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2 \rangle.$$

The deformed Sklyanin algebra S has generators x, y, z and relations

$$\begin{aligned} axy + byx + cz^2 + d_1 + e_1 &= 0, \\ ayz + bzy + cx^2 + d_2 + e_2 &= 0, \\ azx + bxz + cy^2 + d_3 + e_3 &= 0, \end{aligned}$$

where, for all $i \in \{1, 2, 3\}$, d_i is a linear term in x, y, z and $e_i \in K$. By Proposition 2.3.7, S is 3-dimensional essentially regular.

2.4 Geometry of homogenized algebras

In [6], Artin, Tate, and Van den Bergh showed that every dimension two and dimension three regular algebra surjects onto a twisted homogeneous coordinate ring. We begin this section by defining a twisted homogeneous coordinate ring following the exposition in [36]. We then go on to define the related concept of a *geometric algebra*, which was originally called an *algebra defined by geometric data* by Vancliff and Van Rompay [57].

While one would not expect such a construction for deformations of regular algebras, we might hope to recover information about the deformed algebra from the geometry associated to the homogenization of a deformation. We show that, in certain cases, this geometry allows us to classify all finite-dimensional simple modules of a deformed regular algebra.

Let $\mathcal{L}$ be a line bundle, E a projective scheme, and $\sigma \in \operatorname{Aut}(E)$. Set $\mathcal{L}_0 = \mathcal{O}_E$ (the structure sheaf on E) and, for $d \geq 1$, $\mathcal{L}_d = \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{L}^{\sigma} \otimes_{\mathcal{O}_E} \cdots \otimes_{\mathcal{O}_E} \mathcal{L}^{\sigma^{d-1}}$. Define the (graded) vector spaces $\mathcal{B}_m = H^0(E, \mathcal{L}_m)$. Taking global sections of the natural isomorphism $\mathcal{L}_d \otimes_{\mathcal{O}_E} \mathcal{L}_e^{\sigma^d} \cong \mathcal{L}_{d+e}$ gives a multiplication defined by $a \cdot b = a\sigma^m(b) \in \mathcal{B}_{m+n}$ for $a \in \mathcal{B}_m$, $b \in \mathcal{B}_n$. If $\sigma = \mathrm{id}_E$, then this construction defines the (commutative) homogeneous coordinate ring of E.

Definition 2.4.1. *The **twisted homogeneous coordinate ring** of E with respect to $\mathcal{L}$ and σ is the $\mathbb{N}$-graded ring $\mathcal{B} = \mathcal{B}(E, \mathcal{L}, \sigma) := \bigoplus_{d \geq 0} H^0(E, \mathcal{L}_d)$ with multiplication defined as above.*

Artin and Stafford have shown that every domain of GK dimension two is isomorphic to a twisted homogeneous coordinate ring [5]. Hence, if H is 3-dimensional regular and $g \in H_3$ is a normal element, then $\operatorname{gk}(H/(g)) \leq 2$ and therefore $H/(g)$ must be isomorphic to some $\mathcal{B}$. By [6], Theorem 6.8, every 3-dimensional regular algebra contains such an element.

To define a geometric algebra, we make a slight change of notation to conform to convention. In addition, we specialize to the case of quadratic algebras. These algebras were originally defined by Vancliff and Van Rompay. They were renamed *geometric algebras* by Mori [50] and we use his definition here.

The free algebra $K\langle x_0, x_1, \ldots, x_n \rangle$ is equivalent to the tensor algebra $T(V)$ where $V = \{x_0, \ldots, x_n\}$. If H is a quadratic homogeneous algebra, then we write $H = T(V)/(R)$ where R is the set of defining polynomials of H. Any defining polynomial may be regarded as a bilinear form $f : V \otimes_K V \to K$. Write $f = \sum \alpha_{ij} x_i \otimes x_j$, $\alpha_{ij} \in K$. If $p, q \in \mathbb{P}(V^*)$, written as $p = (a_0 : a_1 : \cdots : a_n)$ and $q = (b_0 : b_1 : \cdots : b_n)$, then $f(p, q) = \sum \alpha_{ij} a_i b_j$ (we drop the tensor product for convenience). Define the *vanishing set* of R as

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p, q) = 0 \text{ for all } f \in R\}.$$

Definition 2.4.2. *A homogeneous quadratic algebra $H = T(V)/(R)$ is called **geometric** if there exists a scheme $E \subset \mathbb{P}(V^*)$ and $\sigma \in \operatorname{Aut} E$ such that*

G1 $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$,

G2 $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \textit{ for all } p \in E\}$.

*The pair (E, σ) is called the **geometric data** corresponding to H.*

All regular algebras of dimension at most three are geometric. The classification of quadratic regular algebras given in [6] shows that either $E = \mathbb{P}^2$ or else E is a cubic divisor in $\mathbb{P}^2$. The projective scheme E is referred to as the point scheme of H. It is not true that every 4-dimensional regular algebra is geometric [56]. However, it seems that they are in the generic case.

Theorem 2.4.3. *If A is a PBW deformation of a quadratic geometric algebra, then $H(A)$ is geometric.*

Proof. Let $V = \{x_1, \dots, x_n\}$ and $W = \{x_0\} \cup V$. Write $H(A) = T(W)/(R)$ and $\text{gr}(A) = T(V)/(S)$. Choose $p, q \in \mathcal{V}(R)$ and write

$$p = (a_0 : a_1 : \cdots : a_n), \quad q = (b_0 : b_1 : \cdots : b_n).$$

Let $e_i = x_0x_i - x_ix_0$, $i = 1, \dots, n$, be the commutation relations of x_0 in H. Suppose $a_0 = 0$, then $e_i(p, q) = 0$ implies $a_ib_0 = 0$ for $i = 1, \dots, n$. Since q is not identically zero, then $b_0 = 0$. Reversing the argument, we see that $a_0 = 0$ if and only if $b_0 = 0$. Let $E_0 \subset \mathcal{V}(R)$ be those points with the first coordinate zero and define $\sigma|_{E_0}$ to be the automorphism corresponding to $\text{gr}(A)$. Then E_0 is σ-invariant and the restriction $(E_0, \sigma|_{E_0})$ is the geometric pair for $\text{gr}(A)$.

Let $E_1 \subset E$ be those points with first coordinate nonzero. If $a_0 \neq 0$, then $b_0 \neq 0$ and so there is no loss in letting $a_0 = b_0 = 1$. Hence, $e_i(p, q) = 0$ implies $a_i = b_i$ and so we define $\sigma|_{E_1} = \text{id}_{E_1}$.

We define the scheme $E := E_0 \cup E_1 \subset \mathbb{P}(V^*)$ where E_0 corresponds to the point scheme of $\text{gr}(A)$ and E_1 corresponds to the diagonal of $\mathcal{V}(R)$. That is, $p \in E_1$ if $(p, p) \in \mathcal{V}(R)$. Define the automorphism σ where $\sigma|_{E_0}$ is the automorphism corresponding to $\text{gr}(A)$ and $\sigma|_{E_1} = \text{id}_{E_1}$. Thus, E_0 and E_1 are σ-invariant and $\mathcal{V}(R)$ is the graph of E. It is left to check that **G2** holds.

Let F be R reduced to E_0. If $f \in F$, then $f(p, p) = 0$ for all $p \in E_1$, so f corresponds to a relation in commutative affine space. Define $C = K[x_1, \dots, x_n]/(F)$. If $g \in V \otimes_k V$ such that $g(p, \sigma(p)) = 0$ for all $p \in E_1$, then $\hat{g} := g|_{E_1} \in \mathcal{V}(F)$. By the Nullstellensatz, $\hat{g}^n \in F$ for some n. On the other hand, if $p \in E_0$, then either g is a commutation relation or else $\text{gr}(g) \in S$. Thus, g is quadratic and so $n = 1$. Hence, $g \in R$. $\square$

The algebra $\mathfrak{V} = K\langle x, y \mid yx - xy + y^2 + x\rangle$ is isomorphic to $\mathfrak{U}$ (Proposition 3.4.6). However, $H(\mathfrak{U}) \not\cong H(\mathfrak{V})$ by the next corollary. It is not known whether

$H(A) \cong H(B)$ implies $A \cong B$ in general. The following corollary provides a similar result for graded algebras.

Corollary 2.4.4. *If $H(A) \cong H(B)$ where A and B are essentially regular, then $\mathrm{gr}(A) \cong \mathrm{gr}(B)$ as graded algebras.*

Proof. Suppose $\Phi : H(A) \to H(B)$ is an isomorphism and let (E, σ), (F, τ) be the corresponding geometric pairs. The point schemes E and F decompose as $E = E_0 \cup E_1$ and $F = F_0 \cup F_1$, where E_0 corresponds to the point scheme of $\mathrm{gr}(A)$ and F_0 to $\mathrm{gr}(B)$. The automorphisms σ and τ fix E_0 and F_0, respectively. Hence, Φ restricts to an isomorphism $(E_0, \sigma|_{E_0}) \cong (F_0, \tau|_{F_0})$. These point schemes are in 1-1 correspondence with $\mathrm{gr}(A)$ and $\mathrm{gr}(B)$, and so induce an isomorphism as graded algebras [50]. □

If H is geometric, then $H_1 \approx \mathcal{B}_1$ and so there is a surjection $\mu : H \to \mathcal{B}$. In particular, $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}(V^*)}(1)$ where $j : E \to \mathbb{P}(V^*)$ is the natural embedding. When H is noetherian, $I = \ker \mu$ is finitely generated by homogeneous elements and so there is hope of pulling information about H back from $\mathcal{B}$. Let M be a finite-dimensional simple module of H. Then M is either I-torsion or it is I-torsionfree. Those of the first type may be regarded as modules over $H/I \cong \mathcal{B}$. Those of the second type are not as tractable, though results from [7] give us a complete picture in the case that H is regular of dimension three. Our goal is to generalize the following example to homogenizations of 2-dimensional essentially regular algebras that are not PI.

Example 2.4.5. *If $H = H(\mathcal{J}_1)$, then $E_1 = \{(1 : a : \pm i)\}$. The finite-dimensional simple modules of $\mathcal{J}_1$ are exactly of the form $\mathcal{J}_1/((x_1 - a)\mathcal{J}_1 + (x_2 \pm i)\mathcal{J}_1)$. They are all non-isomorphic.*

Closely related to these results is a conjecture by Walton.

Conjecture 2.4.6 (Walton)**.** *Let S be a PBW deformation of a Sklyanin algebra that is not PI. Then all finite-dimensional simple modules of S are 1-dimensional.*

While we cannot solve this conjecture in its entirety, we make progress towards an affirmative answer by showing that the modules which are torsion over the canonical map $H(S) \to \mathcal{B}$ are 1-dimensional.

Let A be essentially regular and $H = H(A)$ geometric with geometric pair (E, σ). If f is a defining relation of H and $p = (p_0 : p_1 : \cdots : p_n) \in E_1$, then $f(p, \sigma(p)) = 0$

implies $\sigma(p) = p$. Thus, we write $f(p,p) = 0$ or, more simply, $f(p) = 0$. This is equivalent to defining the module,

$$M_p = H/((x_0 - 1)H + (x_1 - p_1)H + \cdots + (x_n - p_n)H).$$

Since H acts on M_p via scalars, then M is 1-dimensional. Since $x_0 - 1 \in \operatorname{Ann}(M_p)$, then M_p may be identified with the A-module $A/((x_1 - p_1)A + \cdots + (x_n - p_n)A)$. By an abuse of notation, we also call this A-module M_p. Conversely, if $M = \{v\}$ is a 1-dimensional (simple) A-module, then A acts on M via scalars, say $x_i.v = c_i v$, $c_i \in K$, $i = 1, \dots, n$. By setting $x_0.v = v$, M becomes an H-module. This action must satisfy the defining relations of H and so setting $p_i = c_i$ gives $f(p) = 0$. We have now proved the next proposition.

Proposition 2.4.7. *Let A be essentially regular. The A-module M is 1-dimensional if and only if $M \cong M_p$ for some $p \in E_1$.*

Corollary 2.4.8. *Let M_p and M_q be 1-dimensional (simple) modules of an essentially regular algebra A. Then $M_p \cong M_q$ if and only if $p = q$.*

For A essentially regular, we believe that certain conditions will imply that these are all of the finite-dimensional simple modules. In the following, we will show that this is the case when A is 2-dimensional essentially regular and not PI.

Lemma 2.4.9. *If A is essentially regular and M is a finite-dimensional simple module of A, then* $\operatorname{Ann}(M) \neq 0$.

Proof. Let $H = H(A)$. Write $M = M_A$ (resp. $M = M_H$) when M is regarded as an A-module (resp. H-module). If $N_H \subset M_H$ as an H-module, then $N_A \subset M_A$, so $N_H = 0$ or $N_H = M_H$. Thus, M_H is a simple module and, moreover, $\dim_A(M_A) = \dim_H(M_H)$. By [58], Lemma 3.1, if M_H is a finite-dimensional simple module and P is the largest graded ideal contained in $\operatorname{Ann}(M_H)$, then $\operatorname{gk}(H/P) \in \{0, 1\}$. If $P = 0$, then $\operatorname{gk}(H/P) = \operatorname{gk}(H) > 1$ when H is regular of dimension greater than one. Hence, if $\dim(M_H) > 1$, then $\operatorname{Ann}(M_H) \neq 0$. Since $x_0.m = m$, then $x_0 - 1 \in \operatorname{Ann}(M_H)$, but $x_0 - 1$ is not a homogeneous element so $x_0 - 1 \notin P$. Let $r \in P \subset \operatorname{Ann}(M_H)$ with $r \neq 0$. If $r \in K[x_0]$ with $r \neq x_0 - 1$, then $1 \in P$ so $\operatorname{Ann}(M_H) = H$. Hence, $r \notin K[x_0]$ and so $r \not\equiv 0 \bmod (x_0 - 1)$. Thus, $\operatorname{Ann}(M_A) \neq 0$. $\square$

Every finite-dimensional simple module of $\mathcal{J}$ or $\mathcal{O}_q(K^2)$, $q \in K^\times$ a nonroot of unity, is 1-dimensional (see Chapter 4). This fact, along with the above results, implies the following result for homogenizations of 2-dimensional essentially regular algebras. However, in light of Proposition 2.4.3, it seems reasonable that it may apply to certain higher dimensional algebras as well.

Theorem 2.4.10. *Let A be an essentially regular algebra of dimension two that is not PI. If M is a finite-dimensional simple A-module, then M is 1-dimensional.*

Proof. Let $g \in H = H(A)$ be the canonical element such that $H/(g) \cong \mathcal{B} = \mathcal{B}(E, \mathcal{L}, \sigma)$ and let $Q = \operatorname{Ann} M$. Because $|\sigma| = \infty$, the set of g-torsionfree simple modules of H is empty ([7], Theorem 7.3). Hence, we may assume M is g-torsion and therefore M corresponds to a finite-dimensional simple module of $\mathcal{B}$.

Since H is a homogenization, then $g = g_0 g_1$ where $g_i \notin K$ for $i = 1, 2$. It is clear that $x_0 \mid g$ so set $g_0 = x_0$. Hence, $g_0 \in Q$ or $g_1 \in Q$ because Q is prime.

If g_0 and g_1 are irreducible, then the point scheme decomposes as $E = E_0 \cup E_1$. Thus, M corresponds to a finite-dimensional simple module of $\mathcal{B}(E_0, \mathcal{L}, \sigma|_{E_0})$ or $\mathcal{B}(E_1, \mathcal{L}, \sigma|_{E_1})$. In the first case, we have that $\mathcal{B}(E_0, \mathcal{L}, \sigma|_{E_0})$ is isomorphic to the twisted homogeneous coordinate ring of $\mathcal{O}_q(K^2)$ or $\mathcal{J}$. Since $\sigma|_{E_1} = \mathrm{id}$, then $\mathcal{B}(E_1, \mathcal{L}, \sigma|_{E_1})$ is commutative. Hence, H/Q is commutative and Q contains $x_0 - 1$ so M is a 1-dimensional simple module of A.

If g divides into three linear factors g_i, $i = 1, 2, 3$, then $\mathcal{B}/g_i\mathcal{B}$ is isomorphic to the twisted homogeneous coordinate ring of $\mathcal{O}_q(K^2)$ or $\mathcal{J}$ for each i. □

As a corollary, we recover a well-known result regarding the Weyl algebra.

Corollary 2.4.11. *The first Weyl algebra A_1 has no finite-dimensional simple modules.*

Proof. If $p \in E_1$, then $p = (1, a, b)$. The defining relation $f = x_1x_2 - x_2x_1 - 1$ gives $f(p, p) = ab - ba - 1 = 1 \neq 0$. Hence, $E_1 = \emptyset$. □

More generally, suppose A is a PBW deformation of a noetherian geometric algebra. By Proposition 2.4.3, $H(A)$ is geometric. Let (E, σ) be the geometric data associated to $H(A)$ and let I be the kernel of the canonical map $H(A) \to \mathcal{B}(E, \mathcal{L}, \sigma)$. Let E_1 be the fixed points of E and $E_0 = E \backslash E_1$. We say $F \subset E_0$ is *reducible* if there exists F', F'' such that $F = F' \cup F''$ and $\sigma(F') \subset F'$, $\sigma(F'') \subset F''$. We say F is *reduced* if it is not reducible.

Theorem 2.4.12. *With the above notation, if M is a finite-dimensional simple module of $H(A)$ that is I-torsion, then M is either a module over* $\operatorname{gr}(A)$ *or else M is 1-dimensional.*

Proof. Let M be an I-torsion simple module of H, so we may regard M as a simple module of $\mathcal{B}$. Let $Q = \mathrm{Ann}(M)$ and so $Q \neq 0$ by Lemma 2.4.9. If P is the largest homogeneous prime ideal contained in Q, then P corresponds to a reduced closed subscheme of $F \subset E$ and $\mathcal{B}/P \cong \mathcal{B}(F, \mathcal{O}_F(1), \sigma|_F)$ ([14], Lemma 3.3). These subschemes are well-understood in this case, and so either F corresponds to a subscheme in the twisted homogeneous coordinate ring associated to $\mathrm{gr}(A)$, or else F is a (set of) singletons, in which case $\mathcal{B}/P$ is commutative. □

Let S be a deformed Sklyanin algebra that is not PI. By Theorem 2.4.12 and [58], Theorem 1.3, the only finite-dimensional simple module over $\mathrm{gr}(S)$ is the trivial one. By [41], there are exactly eight fixed points in E. Hence, all I-torsion, finite-dimensional simple modules are 1-dimensional.

The algebra $U(\mathfrak{sl}_2(K))$ is essentially regular of dimension three, is not PI, but does have finite-dimensional simple modules of every dimension n. There are other examples of essentially regular algebras exhibiting the same behavior (see [51], [16]). This leads to the following conjecture.

Conjecture 2.4.13. *Let A be essentially regular of dimension three that is not PI. Then either all finite-dimensional simple modules are 1-dimensional or else A has finite dimensional simple modules of arbitrarily large dimension.*

We end this section with a brief foray into the PI case. Suppose A is prime PI and essentially regular. By Proposition 2.3.6, $H = H(A)$ is also prime PI. Moreover, if we let Q_A be the quotient division ring of A and Q_H that of H, then

$$\mathrm{pid}(H) = \mathrm{pid}(Q_H) = \mathrm{pid}(Q_A(x_0)) = \mathrm{pid}(A[x_0]) = \mathrm{pid}(A).$$

Suppose A is 2-dimensional essentially regular and PI. Then $H = H(A)$ is PI and, in particular, $H = H(A_1^q(K))$ or $H(\mathcal{O}_q(K^2))$ with q a primitive nth root of unity. In each case, the PI-degree is exactly n. One can also show that $n = |\sigma|$ where σ is the automorphism of the geometric pair (E, σ) corresponding to H. Of course, the g-torsion finite-dimensional simple modules of either algebra correspond to the finite-dimensional simple modules of $\mathcal{O}_q(K^2)$.

The g-torsionfree simple modules of H are in 1-1 correspondence with those of $H[g^{-1}]$. Let Λ_0 be its degree 0 component. Since H contains a central homogeneous element of degree 1, then $\mathrm{pid}(\Lambda_0) = \mathrm{pid}(H) = n$ ([39], page 149). Thus, by [58], Theorem 3.5, H has a g-torsionfree simple module of dimension n.

2.5 A 5-dimensional family of regular algebras

Suppose A and B are regular. In terms of generators and relations, the algebra $C = A \otimes B$ is easy to describe. Let $\{x_i\}$ be the generators for A and $\{y_i\}$ those for B. Let $\{f_i\}$ be the relations for A and $\{g_i\}$ those for B. Associate $x_i \in A$ with $x_i \otimes 1 \in A \otimes B$, and similarly for the y_i. Then $A \otimes B$ is the algebra on generators $\{x_i, y_i\}$ satisfying the relations $\{f_i, g_i\}$ along with the relations $x_i y_j - y_j x_i = 0$ for all i, j.

A similar description holds when A and B are essentially regular. By comparing global dimension, one sees that $H(A \otimes B) \ncong H(A) \otimes H(B)$. However, a related identity will be used to prove the following proposition.

Proposition 2.5.1. *Let A and B be essentially regular algebras. Then $A \otimes B$ is essentially regular.*

Proof. We must show that $H(A \otimes B)$ is regular given that $H(A)$ and $H(B)$ are. Suppose z_0 is the homogenizing element in $H(A \otimes B)$ and x_0, y_0 those in $H(A)$ and $H(B)$, respectively. By Proposition 2.3.7 and [47], it suffices to prove the following:

$$H(A \otimes B)/z_0 H(A \otimes B) \cong H(A)/x_0 H(A) \otimes H(B)/y_0 H(B). \tag{2.4}$$

This is clear from the defining relations for the given algebras. $\square$

Corollary 2.5.2. *If A and B are 2-dimensional essentially regular, then $H(A \otimes B)$ is 5-dimensional regular.*

Using the techniques developed above, we hope to understand the module structure of algebras of the form $H(A \otimes B)$.

Example 2.5.3. *Let $A = B = \mathcal{J}$ with generating sets $\{x_1, x_2\}$ and $\{y_1, y_2\}$, respectively. Let $\hat{x}_i = x_i \otimes 1$ and $\hat{y}_i = 1 \otimes y_i$ for $i = 1, 2$. Then $C = A \otimes B$ is generated by $\{\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2\}$ and the defining relations are*

$$f = \hat{x}_1\hat{x}_2 - \hat{x}_2\hat{x}_1 + \hat{x}_1^2, \quad g = \hat{y}_1\hat{y}_2 - \hat{y}_2\hat{y}_1 + \hat{y}_1^2,$$
$$h_{ij} = \hat{x}_i\hat{y}_j - \hat{y}_j\hat{x}_i \text{ for } i, j \in \{1, 2\}.$$

Let E^A, E^B, E^C be the point schemes of A, B and C, respectively. We claim that $E \cong E^A \cup E^B$. Let $p = (a_1 : a_2 : a_3 : a_4) \in \mathbb{P}^3$. Then $p \in E^C$ if there exists $q = (b_1 : b_2 : b_3 : b_4) \in \mathbb{P}^3$ such that (p, q) is a zero for the above defining relations.

The relation f_1 *gives* $\frac{a_1}{a_2} = \frac{b_1}{b_1+b_2}$ *and* f_2 *gives* $\frac{a_3}{a_4} = \frac{b_3}{b_3+b_4}$. *Substituting into the additional relations gives* $a_3 = a_4 = 0$ *or else* $a_1 = a_2 = 0$. *In the first case the points correspond to* E_A *and otherwise to* E_B.

The following proposition generalizes the above example.

Proposition 2.5.4. *Suppose* A *and* B *are regular and* $C = A \otimes B$ *is noncommutative. Then* $E^C = E^A \cup E^B$.

Proof. Let $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ be the generators of C subject to relations $\{f_i, g_i, h_{ij}\}$ such that the subalgebra generated by the x_i (resp. y_i) subject to the relations f_i (resp. g_i) is isomorphic to A (resp. B). Identify A and B with their respective images in C. Let $h_{ij} = x_i y_j - y_j x_i$ for $1 \leq i \leq n, 1 \leq j \leq m$. Let E be the point scheme of C and σ the corresponding automorphism. Choose $p \in E_A \times E_B$ and let $q = \sigma(p)$. Write

$$p = (a_1 : \cdots : a_n : b_1 : \cdots : b_m), \text{ and } q = (c_1 : \cdots : c_n : d_1 : \cdots : d_m).$$

We claim either $a_i = 0$ for all $i \in \{1, \dots, n\}$ or else $b_j = 0$ for all $j \in \{1, \dots, m\}$.

Let $\sigma_A = \sigma|_A$ and $\sigma_B = \sigma|_B$. Suppose there exists l, k such that $a_l \neq 0$ and $b_k \neq 0$. There is no loss in letting $a_l = 1$. Hence, $0 = h_{lj}(p, q) = d_j - b_j c_l$. If $c_l = 0$, then $d_j = 0$ for all j. Hence, $\sigma_B(b_1 : \dots : b_m) = 0$, a contradiction, so $c_l \neq 0$. Then $b_j = d_j$ for all j. Thus, either $a_i = 0$ for all i or else σ_B is constant, so B is commutative. An identical argument shows that either $b_i = 0$ for all i or else σ_A is constant, so A is commutative. If A and B are commutative, then so is C. $\square$

If A and B are essentially regular, then the point scheme of $H(A \otimes B)$ has two components, E_0 and E_1, and E_0 corresponds to that of $H(A \otimes B)/z_0 H(A \otimes B)$ (see (2.4)). An argument similar to that from the previous proposition shows that E_1 corresponds to $E_1^A \cup E_1^B$. Consequently, if M is a 1-dimensional simple module of $A \otimes B$, then M is isomorphic to a 1-dimensional simple module of A or B.

Chapter 3

Standard form congruence

Suppose the algebra A is defined as

$$A = K\langle x, y \mid f\rangle, \quad \deg(f) = 2. \tag{3.1}$$

In case f is homogeneous, the classification of such algebras is well-known. The polynomial f can be represented by a matrix $M \in \mathcal{M}_2(K)$ and matrix congruence corresponds to linear isomorphisms between homogeneous algebras. Hence, canonical forms for matrices in $\mathcal{M}_2(K)$ give a maximal list of algebras to consider. One must verify that there are no non-linear isomorphisms between the remaining algebras. The details of this are given in Section 3.2

In Section 3.3, we give a method for extending this idea to algebras (3.2) in which f is not necessarily homogeneous. In particular, we develop a modified version of matrix congruence called *standard-form congruence*. We compute canonical forms in $\mathcal{M}_3(K)$ under standard-form congruence and these are listed in Table 3.1.

In Chapter 4 the classification of algebras of form (3.1) is completed by considering ring-theoretic properties of these algebras. The end result is that the canonical forms in Table 3.1 are in near 1-1 correspondence with isomorphism classes of algebras of the form (3.1).

3.1 Canonical forms for matrix congruence

We say two matrices $M, M' \in \mathcal{M}_n(K)$ are *congruent* and write $M \sim M'$ if there exists $P \in \mathrm{GL}_n(K)$ such that $M = P^T M' P$. Canonical forms for congruent matrices date back (at least) to the work on Turnbull and Aitken in 1932 [55]. More recently, they were studied by Horn and Sergeichuk [31]. The interested reader is directed to the expository article by Terán [24], which also explains the relationship between the two forms.

The Horn-Sergeichuk forms depend on three block-types which we henceforth refer to as *HS-blocks*,

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, J_1 = \begin{pmatrix} 0 \end{pmatrix},$$

$$\Gamma_n = \begin{pmatrix} 0 & & & & & (-1)^{n+1} \\ & & & & \iddots & (-1)^n \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 1 \end{pmatrix},$$

$$H_{2n}(\mu) = \begin{pmatrix} 0 & I_n \\ J_n(\mu) & 0 \end{pmatrix}, H_2(\mu) = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}.$$

Theorem 3.1.1 (Horn, Sergeichuk [31])**.** *Each square complex matrix is congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the three types $J_n(0)$, Γ_n, and $H_{2n}(\mu)$, $\mu \neq 0, (-1)^{n+1}$. Moreover, $H_{2n}(\mu)$ is determined up to replacement of μ by μ^{-1}.*

Thus, in $\mathcal{M}_2(K)$ there are precisely four forms,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, q \in K^\times. \tag{3.2}$$

It follows from Theorem 3.1.1 that the matrix $\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$ is congruent to $\begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$. It furthermore follows that two such matrices are congruent only if the parameters are equal or inverses of each other (Corollary 3.4.2).

3.2 The homogeneous case

Let $f = ax^2 + bxy + cyx + dy^2, a, b, c, d \in K$. By a slight abuse of notation,

$$f = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, we can represent any homogeneous quadratic polynomial by an element of $\mathcal{M}_2(K)$. If $A = K\langle x, y \mid f \rangle$, then f is called a *defining polynomial* for A and the matrix corresponding to f is called a *defining matrix* for A. The map ϕ given by $x \mapsto p_{11}x + p_{12}y$ and $y \mapsto p_{21}x + p_{22}y$, $p_{ij} \in K$, with $p_{11}p_{22} - p_{12}p_{21} \neq 0$, corresponds to a linear isomorphism between the algebras with defining polynomials f and $\phi(f)$. Moreover, if $P = (p_{ij})$, then $P^T M P$ is the matrix for $\phi(f)$. Thus, matrix congruence is equivalent to linear isomorphisms of quadratic algebras.

The matrix forms (3.2) correspond to four algebras, R_{x^2}, R_{yx}, $\mathcal{J}$, and $\mathcal{O}_q(K^2)$ ($q \in K^\times$). Moreover, $\mathcal{O}_p(K^2) \cong \mathcal{O}_q(K^2)$ if and only if $p = q^{\pm 1}$ (Corollary 4.2.12). It is left to show that there are no additional isomorphisms between the algebras. In this case, it is not difficult. In particular, $\mathcal{J}$ and $\mathcal{O}_q(K^2)$ are domains while R_{x^2} and R_{yx} are not. The Jordan plane has one height one prime ideal and a quantum plane has (at least) two height one prime ideals. The algebra R_{x^2} is prime while R_{yx} is not. Further details may be found in Chapter 4.

3.3 The general case

In the non-homogeneous case, we write $f = ax^2 + bxy + cyx + dy^2 + \alpha x + \beta y + \gamma$, $a, b, c, d, \alpha, \beta, \gamma \in K$. We can represent f by a $3x3$ matrix via the rule

$$f = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

We extend the terms *defining polynomial* and *defining matrix* from Section 3.2 as one would expect. However, our choice of defining matrix is not unique. One could choose to define f by

$$f = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Hence, it is necessary to fix a *standard form* for the defining matrices of non-homogeneous polynomials. We restrict our attention to the following set,

$$G_3 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \end{pmatrix} \,\middle|\, \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0 \right\} \subset \mathcal{M}_3(K).$$

Every degree two polynomial has a unique corresponding matrix in G_3. Consider the matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \in \mathcal{M}_3(K).$$

This corresponds to the polynomial

$$\begin{aligned} f &= m_{11}x^2 + m_{12}xy + m_{13}x + m_{21}yx + m_{22}y^2 + m_{23}y + m_{31}x + m_{32}y + m_{33} \\ &= m_{11}x^2 + m_{12}xy + m_{21}yx + m_{22}y^2 + (m_{13} + m_{31})x + (m_{23} + m_{32})y + m_{33}, \end{aligned}$$

which in turn corresponds to the matrix

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} + m_{31} \\ m_{21} & m_{22} & m_{23} + m_{32} \\ 0 & 0 & m_{33} \end{pmatrix}.$$

Hence, we define a K-linear map $\mathrm{sf} : \mathcal{M}_3(K) \to G_3$ by

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \mapsto \begin{pmatrix} m_{11} & m_{12} & m_{13} + m_{31} \\ m_{21} & m_{22} & m_{23} + m_{32} \\ 0 & 0 & m_{33} \end{pmatrix}.$$

Let $p_{ij} \in K$ and define a K-linear map by

$$\phi(x) = p_{11}x + p_{12}y + p_{13}, \quad \phi(y) = p_{21}x + p_{22}y + p_{23}, \quad \phi(1) = 1.$$

If $p_{11}p_{22} - p_{12}p_{21} \neq 0$, then ϕ defines an affine isomorphism between $K\langle x, y \mid f\rangle$ and $K\langle x, y \mid \phi(f)\rangle$. Thus, the matrices corresponding to affine isomorphisms of these algebras should be contained in the set

$$\mathcal{P}_3 = \left\{ \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_3(K) \mid P_1 \in \mathrm{GL}_2(K), P_2 \in K^2 \right\}.$$

In general, we want a map that fixes the degree two part of a quadratic polynomial and adds the linear parts. We write $M \in \mathcal{M}_n(K)$ in block form

$$M = \left\{ \begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix} \mid M_1 \in \mathcal{M}_{n-1}(K), M_2, M_3 \in K^{n-1}, m \in K \right\}. \tag{3.3}$$

We call M_1 the *homogeneous block* of M. Define the set

$$G_n = \left\{ \begin{pmatrix} M_1 & M_2 \\ 0 & m \end{pmatrix} \in \mathcal{M}_n(K) \mid M_1 \in \mathcal{M}_{n-1}(K), M_2 \in K^{n-1}, m \in K \right\}.$$

Then define the map $\mathrm{sf} : \mathcal{M}_n \to G_n$ by

$$\begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix} \mapsto \begin{pmatrix} M_1 & M_2 + M_3 \\ 0 & m \end{pmatrix}, \tag{3.4}$$

where the matrix is written according to (3.3). The matrices corresponding to affine isomorphisms of these algebras should be contained in the set

$$\mathcal{P}_n = \left\{ \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_n(K) \mid P_1 \in \mathrm{GL}_{n-1}(K), P_2 \in K^{n-1} \right\}.$$

Proposition 3.3.1. *$\mathcal{P}_n$ is a group.*

Proof. That $\mathcal{P}_n$ contains the identity matrix is clear. Let $P, P' \in \mathcal{P}_n$. Then

$$PP' = \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P_1' & P_2' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_1P_1' & P_1P_2' + P_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{P}_n.$$

Since $P_1 \in \mathrm{GL}_{n-1}(K)$, then we can set $P_1' = P_1^{-1} \in \mathrm{GL}_{n-1}(K)$ and $P_2' = -P_1^{-1}P_2$. It is now clear from the above that $P' = P^{-1}$. □

Under ordinary matrix congruence, two matrices which are scalar multiples of each other are always congruent. However, if we restrict to $\mathcal{P}_n$, that is no longer the case. Hence, in our modified definition of congruence, we set scalar multiple matrices to be congruent to each other.

Definition 3.3.2. *We say $M, M' \in \mathcal{M}_n(K)$ are **standard-form congruent** (sf-congruent) and write $M \sim_{sf} M'$ if there exist $P \in \mathcal{P}_n$ and $\alpha \in K^\times$ such that $sf(M) = \alpha \cdot sf(P^T M' P)$.*

It turns out that $\mathcal{P}_n$ is bigger than is necessary. Indeed, it is possible to restrict to those elements $\mathcal{P}_n$ which are of determinant one. To see this, suppose $M \sim_{\mathrm{sf}} N$. By definition, there exists $P \in \mathcal{P}_n$ and $\alpha \in K^\times$ such that $M = \alpha \cdot \mathrm{sf}(P^T NP)$. Let $\beta = \det(P)^{\frac{1}{n}} \neq 0$. Then

$$\mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T NP) = \alpha \cdot \frac{\beta^2}{\beta^2} \cdot \mathrm{sf}(P^T NP) = (\alpha\beta^2) \cdot \mathrm{sf}\left((\beta^{-1}P)^T N (\beta^{-1}P)\right).$$

If we let $\gamma = \alpha\beta^2$ and $Q = \beta^{-1}P$, then $\mathrm{sf}(M) = \gamma \cdot \mathrm{sf}(Q^T NQ)$ and

$$\det(Q) = \det(\beta^{-1}P) = \beta^{-n}\det(P) = \left(\det(P)^{\frac{1}{n}}\right)^{-n}\det(P) = 1.$$

The next proposition shows that sf-congruence is a true extension of congruence.

Proposition 3.3.3. *Let $M, N \in \mathcal{M}_n(K)$ with homogeneous blocks M_1, N_1, respectively. If $M \sim_{sf} N$, then $M_1 \sim N_1$.*

Proof. By hypothesis, $\mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T NP)$ for some $P \in \mathcal{P}$, $\alpha \in K^\times$. Then

$$\begin{aligned} \begin{pmatrix} M_1 & M_2 \\ 0 & m \end{pmatrix} &= \mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T NP) \\ &= \alpha \cdot \mathrm{sf}\left(\begin{pmatrix} P_1^T & 0 \\ P_2^T & 1 \end{pmatrix} \begin{pmatrix} N_1 & N_2 \\ 0 & m \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \alpha \cdot \mathrm{sf}\left(\begin{pmatrix} P_1^T N_1 P_1 & * \\ * & * \end{pmatrix} \right) = \begin{pmatrix} \alpha \cdot P_1^T N_1 P_1 & * \\ 0 & * \end{pmatrix}. \end{aligned}$$

Thus, $M_1 = \alpha \cdot P_1^T N_1 P_1$, so $M_1 \sim N_1$. □

Proving that standard-form congruence is an equivalence relation requires the following technical lemmas.

Lemma 3.3.4. *If $M \in \mathcal{M}_n(K)$ and $P \in \mathcal{P}_n$, then $sf(P^T MP) = sf(P^T sf(M)P)$.*

Proof. We have,

$$\begin{aligned}
\mathrm{sf}(P^T MP) &= \mathrm{sf}\left(\begin{pmatrix} P_1^T & 0 \\ P_2^T & 1 \end{pmatrix}\begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix}\right) \\
&= \mathrm{sf}\left(\begin{pmatrix} P_1^T M_1 P_1 & P_1^T M_1 P_2 + P_1^T M_2 \\ P_2^T M_1 P_1 + M_3^T P_1 & P_2^T M_1 P_2 + P_2^T M_2 + M_3^T P_2 + m \end{pmatrix}\right) \\
&= \begin{pmatrix} P_1^T M_1 P_1 & P_1^T M_1 P_2 + P_1^T M_2 + (P_2^T M_1 P_1 + M_3^T P_1)^T \\ 0 & P_2^T M_1 P_2 + P_2^T M_2 + M_3^T P_2 + m \end{pmatrix} \\
&= \begin{pmatrix} P_1^T M_1 P_1 & P_1^T M_1 P_2 + P_1^T M_2 + P_1^T M_1^T P_2 + P_1^T M_3 \\ 0 & P_2^T M_1 P_2 + P_2^T M_2 + M_3^T P_2 + m \end{pmatrix} \\
&= \mathrm{sf}\left(\begin{pmatrix} P_1^T & 0 \\ P_2^T & 1 \end{pmatrix}\begin{pmatrix} M_1 & M_2 + M_3 \\ 0 & m \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix}\right) \\
&= \mathrm{sf}(P^T \mathrm{sf}(M)P).
\end{aligned}$$

□

The next to last step in the above proof uses the fact that M_3 and P_3 are vectors and therefore $M_3^T P_2 = P_2^T M_3$. We now have the tools we need to prove that sf-congruence is indeed an equivalence relation.

Proposition 3.3.5. *Standard-form congruence defines an equivalence relation.*

Proof. Reflexivity is obvious. Now suppose $M \sim_{\mathrm{sf}} M'$, so $\mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T M' P)$ for some $\alpha \in K^\times, P \in \mathcal{P}_n$. By Lemma 3.3.4,

$$\begin{aligned}
(P^{-1})^T \mathrm{sf}(M)(P^{-1}) &= \alpha \cdot (P^{-1})^T \mathrm{sf}(P^T M' P)(P^{-1}) \\
\mathrm{sf}\left((P^{-1})^T \mathrm{sf}(M)(P^{-1})\right) &= \alpha \cdot \mathrm{sf}\left((P^{-1})^T \mathrm{sf}(P^T M' P)(P^{-1})\right) \\
\alpha^{-1} \cdot \mathrm{sf}\left((P^{-1})^T M(P^{-1})\right) &= \mathrm{sf}\left((P^{-1})^T P^T M' P(P^{-1})\right) \\
\alpha^{-1} \cdot \mathrm{sf}\left((P^{-1})^T M(P^{-1})\right) &= \mathrm{sf}(M').
\end{aligned}$$

Hence, $M' \sim_{\mathrm{sf}} M$, so symmetry holds. Finally, suppose $M \sim_{\mathrm{sf}} M'$ and $M' \sim_{\mathrm{sf}} M''$. Then there exists $\alpha, \beta \in K^\times$ and $P, Q \in \mathcal{P}_n$ such that $\mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T M' P)$ and $\mathrm{sf}(M') = \beta \cdot \mathrm{sf}(Q^T M'' Q)$. By two additional applications of Lemma 3.3.4,

$$\begin{aligned}
\mathrm{sf}(M) &= \alpha \cdot \mathrm{sf}(P^T M' P) = \alpha \cdot \mathrm{sf}\left(P^T \mathrm{sf}(M') P\right) \\
&= \alpha \cdot \mathrm{sf}\left(P^T \left(\beta \cdot \mathrm{sf}(Q^T M'' Q)\right) P\right) = (\alpha\beta) \cdot \mathrm{sf}((QP)^T M''(QP)).
\end{aligned}$$

Thus, $M \sim_{\mathrm{sf}} M''$, so transitivity holds as well. □

Alg	M	[M]		Alg	M	[M]
R_{x^2}	$\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&\mu\\0&0&0\\0&0&\frac{\mu^2}{4} \end{pmatrix}$		R_{yx}	$\begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&\mu\\1&0&\nu\\0&0&\mu\nu \end{pmatrix}$
R_{x^2-1}	$\begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{pmatrix}$	$\begin{pmatrix} 1&0&\mu\\0&0&0\\0&0&\kappa \end{pmatrix}$ $\kappa \neq \mu^2/4$		$\mathcal{S}$	$\begin{pmatrix} 0&0&0\\1&0&0\\0&0&-1 \end{pmatrix}$	$\begin{pmatrix} 0&0&\mu\\1&0&\nu\\0&0&\kappa \end{pmatrix}$ $\kappa \neq \mu\nu$
$\mathcal{J}$	$\begin{pmatrix} 0&-1&0\\1&1&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&-1&0\\1&1&\nu\\0&0&\frac{\nu^2}{4} \end{pmatrix}$		$\mathcal{J}_1$	$\begin{pmatrix} 0&-1&0\\1&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&-1&0\\1&1&\nu\\0&0&\kappa \end{pmatrix}$ $\kappa \neq \nu^2/4$
$K[x,y]$	$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}$		$\mathcal{O}_q(K^2)$	$\begin{pmatrix} 0&1&0\\-q&0&0\\0&0&0 \end{pmatrix}$ $q \neq 1$	$\begin{pmatrix} 0&1&\mu\\-q&0&\nu\\0&0&\frac{\mu\nu}{1-q} \end{pmatrix}$
A_1	$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&-1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&\kappa \end{pmatrix}$ $\kappa \neq 0$		$A_1^q(K)$	$\begin{pmatrix} 0&1&0\\-q&0&0\\0&0&-1 \end{pmatrix}$ $q \neq 1$	$\begin{pmatrix} 0&1&\mu\\-q&0&\nu\\0&0&\kappa \end{pmatrix}$ $\kappa \neq \frac{\mu\nu}{1-q}$
$\mathfrak{V}$	$\begin{pmatrix} 0&-1&1\\1&1&0\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&-1&\mu\\1&1&\nu\\0&0&\kappa \end{pmatrix}$ $\mu \neq 0$		$\mathfrak{U}$	$\begin{pmatrix} 0&1&0\\-1&0&-1\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&\mu\\-1&0&\nu\\0&0&\kappa \end{pmatrix}$ $(\mu,\nu) \neq (0,0)$
$K[x]$	$\begin{pmatrix} 1&0&0\\0&0&1\\0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&\mu\\0&0&\nu\\0&0&\kappa \end{pmatrix}$ $\nu \neq 0$				

Table 3.1: Canonical forms for $\mathcal{M}_3(K)$ under sf-congruence

3.4 Canonical forms under sf-congruence

Table 3.1 lists canonical forms for matrices in $\mathcal{M}_3(K)$ under sf-congruence. The column Alg lists the algebras. The column M gives a defining matrix for the algebra and $[M]$ gives the general form of M under sf-congruence. Throughout, assume $\mu, \nu, \kappa \in K$ are arbitrary unless otherwise stated.

In this section, we show that this list is complete (Theorem 3.4.5). If $M \sim_{\text{sf}} N$, then $M_1 \sim N_1$ by Proposition 3.3.3. To determine the canonical form of $M \in \mathcal{M}_3(K)$ under sf-congruence, we first perform the necessary congruence to put M_1 into one of the canonical forms (3.2).

Assume $M \in G_3$. Our next step is to determine the stabilizer

$$\text{Stab}(M) = \{P \in \text{GL}_n(K) \mid P^T M P = M\}$$

when M is one of the canonical forms (3.2). In general, these stabilizer groups correspond to some orthosymplectic group but, because some of the forms are degenerate, there are shifts in the dimensions. Once computed, this will allow us to determine which pairs (M_2, m) determine distinct forms. The groups $\text{Stab}(M)$ below are computed below directly, though we will make some remarks on stabilizers for arbitrary HS-blocks subsequently.

Proposition 3.4.1. *The following are the stabilizers for the matrices in* (3.2) *relative to matrix congruence. Suppose throughout that $r, s \in K^\times$ are arbitrary.*

$$\operatorname{Stab}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = \left\{\begin{pmatrix}\pm 1 & 0\\ r & s\end{pmatrix}\right\}, \operatorname{Stab}\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right) = \left\{\begin{pmatrix}r & 0\\ 0 & r^{-1}\end{pmatrix}\right\},$$
$$\operatorname{Stab}\left(\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\right) = \left\{\pm\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}\right\}, \operatorname{Stab}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) = \mathrm{SL}_2(K),$$
$$\operatorname{Stab}\left(\begin{pmatrix}0 & 1\\ -q & 0\end{pmatrix}\right) = \left\{\begin{pmatrix}r & 0\\ 0 & r^{-1}\end{pmatrix}\right\} \quad (q \in K^\times, q \neq 1).$$

Proof. Throughout, let $P \in \operatorname{Stab}(M)$ and write $P = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

Let $M_{K[x,y]} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, the matrix corresponding to the standard basis nondegenerate dimension two alternating form. Thus, $\operatorname{Stab}(M_{K[x,y]}) = \mathrm{Sp}(2) \cong \mathrm{SL}_2(K)$.

Let $M_{R_{x^2}} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Then, $P^T M_{R_{x^2}} P = \begin{pmatrix}a^2 & ab\\ ab & b^2\end{pmatrix}$. If $P^T M_{R_{x^2}} P = M_{R_{x^2}}$, then $a = \pm 1$ and $b = 0$.

Let $M_{\mathcal{J}} = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$. Then, $M = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. The first matrix corresponds to a degenerate symmetric bilinear form and therefore the stabilizer group is analogous to $\operatorname{Stab}(M_{R_{x^2}})$. The second matrix is anti-symmetric and nonsingular and therefore the stabilizer group is $\mathrm{SL}_2(K)$. Thus,

$$\operatorname{Stab}(M_{\mathcal{J}}) = \operatorname{Stab}(M_{R_{x^2}}) \cap \mathrm{SL}_2(K).$$

Let $M_{R_{yx}} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Then, $P^T M_{R_{yx}} P = \begin{pmatrix}ac & bc\\ ad & bd\end{pmatrix}$. If $P^T M_{R_{yx}} P = M_{R_{yx}}$, then $a = d^{-1}$. Furthermore, $b = c = 0$.

Let $M_{\mathcal{O}_q(K^2)} = \begin{pmatrix}0 & 1\\ -q & 0\end{pmatrix}$, $q \neq 1$. Then,

$$M_{\mathcal{O}_q(K^2)} = \frac{1-q}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{1+q}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Then $\operatorname{Stab}(M_{\mathcal{O}_q(K^2)}) = O(2) \cap \mathrm{Sp}(2) \cong K^\times$. □

Corollary 3.4.2. *For $p, q \in K^\times$, let $M_p, M_q \in M_2(K)$ matrices corresponding to $\mathcal{O}_p(K^2)$ and $\mathcal{O}_q(K^2)$, respectively. Then $M_q \sim M_p$ if and only if $p = q^{\pm 1}$.*

Proof. Sufficiency is provided by Theorem 3.1.1. Suppose $M_q \sim M_p$ and choose $P = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_n(K)$ such that $M_q = P^T M_p P$. Then

$$P^T M_p P = \begin{pmatrix}(1-p)ac & ad - qbc\\ bc - pad & (1-q)bd\end{pmatrix}.$$

Comparing entries of M_q and $P^T M_p P$, we see that $ac = bd = 0$. Thus, either $b = c = 0$ or $a = d = 0$. In the first case, $P^T M_p P = (ad) M_p$, and so $p = q$. In the second case, $P^T M_p P = (pbc) M_{p^{-1}}$, and so $p = q^{-1}$. □

Corollary 3.4.3. *Let $p, q \in K^\times$. The defining matrices corresponding to $A_1^p(K)$ and $A_1^q(K)$ are sf-congruent if and only if $p = q^{\pm 1}$.*

Proof. Let $M_p, M_q \in G_3$ be the corresponding matrices. That $M_p \sim_{\mathrm{sf}} M_q$ if $p = q^{\pm 1}$ is an easy check and we omit it. The converse now follows by Corollary 3.4.2 and Proposition 3.3.3. □

Lemma 3.4.4. *Let L be one of the forms* (3.2). *For all $\gamma \in K^\times$,*

$$\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \sim_{sf} \begin{pmatrix} L & 0 \\ 0 & \gamma \end{pmatrix}.$$

Proof. Let $P_1 \in \mathrm{Stab}(L)$. Then

$$\begin{pmatrix} \sqrt{\gamma} P_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \sqrt{\gamma} P_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma L & 0 \\ 0 & \gamma \end{pmatrix} = \gamma \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Our last step is to determine, for each canonical form in $\mathcal{M}_2(K)$, which pairs (M_2, m) give sf-congruent matrices.

Theorem 3.4.5. *The canonical forms presented in Table 3.1 are complete.*

Proof. Suppose $M \in \mathcal{M}_3(K)$ such that $M_1 \sim_{\mathrm{sf}} L$. We perform necessary congruence operations to put M_1 in canonical form. Thus, M is sf-congruent to a block matrix of the form

$$N = \begin{pmatrix} L & N_2 \\ 0 & n \end{pmatrix}$$

where L is one of (3.2), $N_2 = \begin{pmatrix} u & v \end{pmatrix}^T \in K^2$, and $n \in K$. Let $P = \begin{pmatrix} P_1 & P_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{P}_3$ such that $P_1 \in \mathrm{Stab}(L)$ and $P_2 = \begin{pmatrix} e & f \end{pmatrix}^T \in K^2$. Write P_1 as in Proposition 3.4.1.

First, suppose $L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. There are two cases for the stabilizer of L corresponding to ± 1. Both cases are similar and we only consider the positive case below,

$$\mathrm{sf}(P^T N P) = \begin{pmatrix} 1 & 0 & 2e + u + rv \\ 0 & 0 & sv \\ 0 & 0 & e^2 + eu + fv + n \end{pmatrix}.$$

Because $\det(P) \neq 0$, then $s \neq 0$. Thus, $sv = 0$ if and only if $v = 0$. In case $v \neq 0$ we set $e = 0$, $s = v^{-1}$, $r = -\frac{u}{v}$, and $f = -\frac{n}{v}$. This is the canonical form corresponding to $K[x]$. In case $v = 0$, then we set $e = -\frac{1}{2}u$. The bottom right entry becomes $-\frac{1}{4}u^2 + 4$. Thus, if $n = \frac{1}{4}u^2$, we have the canonical form corresponding to R_{x^2} and otherwise, by Lemma 3.4.4, the form corresponds to that of R_{x^2-1}.

Suppose $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have

$$\operatorname{sf}(P^T N P) = \begin{pmatrix} 0 & 0 & r(u+f) \\ 0 & 0 & r^{-1}(e+v) \\ 0 & 0 & fe + eu + fv + n \end{pmatrix}.$$

Setting $f = -u$ and $e = -v$ gives a bottom right entry of $n - uv$. Thus, there are two cases corresponding to $n = uv$ and $n \neq uv$. In the former case we arrive at the canonical form of R_{yx} and in the other case, by Lemma 3.4.4, that of $\mathcal{S}$.

Suppose $L = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. There are two cases for the stabilizer. We consider only the positive case, which gives,

$$\operatorname{sf}(P^T N P) = \begin{pmatrix} 0 & -1 & u \\ 1 & 1 & 2f + ru + v \\ 0 & 0 & f^2 + eu + fv + n \end{pmatrix}.$$

Setting $f = -\frac{1}{2}(ru + v)$ allows us to make the (2,3)-entry zero. If $u = 0$, then the (3,3)-entry becomes $n - \frac{1}{4}v^2$. Thus, in case $n = \frac{1}{4}v^2$ we have the canonical form for $\mathcal{J}$ and otherwise we have the form for $\mathcal{J}_1$. If $u \neq 0$, then we can make $u = 1$ and set $e = u^{-1}(\frac{1}{4}v^2 - n) + 1$ so that this is the canonical form for $\mathfrak{V}$.

Let $L = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$. Then

$$\operatorname{sf}(P^T N P) = \begin{pmatrix} 0 & 1 & -r((q-1)f - u) \\ -q & 0 & -r^{-1}(q-1)e - v \\ 0 & 0 & -fe(q-1) + eu + fv + n \end{pmatrix}.$$

Set $f = u(q-1)^{-1}$ and $e = v(q-1)^{-1}$. Then the bottom right entry becomes $n + uv(q-1)^{-1}$. Thus, if $n = uv(q-1)^{-1}$ then this form corresponds to that of $\mathcal{O}_q(K^2)$ and otherwise, by Lemma 3.4.4, it corresponds to that of $A_1^q(K)$.

Let $L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\operatorname{sf}(P^T N P) = \begin{pmatrix} 0 & 1 & au + cv \\ -1 & 0 & bu + dv \\ 0 & 0 & eu + fv + n \end{pmatrix}.$$

Suppose $u = v = 0$. If $n = 0$, then we have the canonical form for $K[x, y]$ and otherwise we have the form for A_1. Suppose $u = 0$ and $v \neq 0$. Then set $b = c = 1$, $d = 0$, and $f = -nv^{-1}$. This gives the canonical form for $\mathfrak{U}$. Similarly for the case $v = 0$ and $u \neq 0$. Finally, suppose $u, v \neq 0$. Then we can choose e, f such that $eu + fv = -n$. Similarly, we can choose b, d such that $bu + dv = 0$. If we choose a, c such that $au + dv = 0$, then $\det(P) = 0$, so $au + cv \neq 0$ and we have the canonical form again for $\mathfrak{U}$. $\square$

The observant reader may have noticed a discrepancy in the Table 3.1 and Theorem 4.0.7. There is an additional canonical form corresponding to the algebra $\mathfrak{V}$, which is not included in Theorem 4.0.7. This is explained by the following result.

Proposition 3.4.6. *The algebras $\mathfrak{U}$ and $\mathfrak{V}$ are isomorphic.*

Proof. Let X, Y be the generators for $\mathfrak{U}$ with defining polynomial $YX - XY + Y$ and let x, y be the generators for $\mathfrak{V}$ with defining polynomial $yx - xy + x + y^2$. Define a map $\Phi : \mathfrak{U} \to \mathfrak{V}$ by $\Phi(X) = -y$, $\Phi(Y) = x + y^2$. This map extends to an algebra homomorphism since

$$\begin{aligned}\Phi(Y)\Phi(X) - \Phi(X)\Phi(Y) + \Phi(Y) &= (x + y^2)(-y) - (-y)(x + y^2) + (x + y^2)\\ &= yx - xy + x + y^2.\end{aligned}$$

We also define $\Psi : \mathfrak{V} \to \mathfrak{U}$ by $\Psi(x) = Y - X^2$, $\Psi(y) = -X$. This map also extends to an algebra homomorphism since

$$\begin{aligned}&\Psi(y)\Psi(x) - \Psi(x)\Psi(y) + \Psi(x) + \Psi(y)\Psi(y)\\ &\qquad = (-X)(Y - X^2) - (Y - X^2)(-X) + (Y - X^2) - (-X)^2 = 0.\end{aligned}$$

It is readily checked that $\Psi(\Phi(X)) = X$ and $\Psi(\Phi(Y)) = Y$ so that $\Psi = \Phi^{-1}$. $\square$

This is the one case considered here where two algebras are isomorphic even though their defining matrices are not sf-congruent. This makes sense as the map Φ constructed above is not an affine isomorphism. Moreover, $\mathfrak{U}$ is a PBW deformation of $K[x, y]$ while $\mathfrak{V}$ is a PBW deformation of $\mathcal{J}$. The relationship between these algebras is explored further in Chapter 5. In particular, we show that isomorphism classes of homogenizations of the algebras in Table 3.1 are in 1-1 correspondence with canonical forms of matrices in $\mathcal{M}_n(K)$ under sf-congruence.

Chapter 4

Two-generated algebras

Our goal in this chapter is to complete the proof of the following theorem.

Theorem 4.0.7. *Suppose $A \cong K\langle x, y \mid f\rangle$ where f is a polynomial of degree two. Then A is isomorphic to one of the following algebras:*

$$\begin{array}{ll}
\mathcal{O}_q(K^2), f = xy - qyx \ (q \in K^\times), & A_1^q(K), f = xy - qyx - 1 \ (q \in K^\times), \\
\mathcal{J}, f = yx - xy + y^2, & \mathcal{J}_1, f = yx - xy + y^2 + 1, \\
\mathfrak{U}, f = yx - xy + y, & K[x], f = x^2 - y, \\
R_{x^2}, f = x^2, & R_{x^2-1}, f = x^2 - 1, \\
R_{yx}, f = yx, & \mathcal{S}, f = yx - 1.
\end{array}$$

Furthermore, the above algebras are pairwise non-isomorphic, except

$$\mathcal{O}_q(K^2) \cong \mathcal{O}_{q^{-1}}(K^2) \text{ and } A_1^q(K) \cong A_1^{q^{-1}}(K).$$

In Chapter 3, we showed that there are a minimal number of canonical forms for relations up to affine isomorphism. In addition, we showed that $\mathfrak{U} \cong \mathfrak{V}$ (Proposition 3.4.6). It is left to show that the remaining algebras are non-isomorphic.

Immediately, one can divide the algebras into two classes: the domains and non-domains. The domains can be further subdivided into quantum planes (Section 4.2), quantum Weyl algebras (Section 4.3), and differential operator rings (Section 4.4). Proving that an algebra belongs to exactly one of these classes requires a study of their automorphism groups. We consider isomorphisms within each class, which requires different techniques in each case.

An ideal P of a ring R is said to be *prime* if for any two ideals I and J in P, $IJ \subset P$ implies $I \subset P$ or $J \subset P$. The ring R is said to be prime if (0) is a prime ideal. The set of all prime ideals in a ring R is the *prime spectrum*, $\mathrm{Spec}(R)$. In the case of the non-domains we are not able to give a full description of the prime spectrum, specifically in the case of R_{x^2} and R_{x^2-1}. However, we do make partial progress (Proposition 4.1.4). In the case of the domains we give a full description by utilizing the tools of skew polynomial rings.

An ideal P is *primitive* if there exists a faithful simple module M such that $P = \mathrm{Ann}(M)$. A primitive ideal is prime but the converse is not true. To determine the primitive ideals in these cases, we follow Dixmier's programme [21].

Let R be a ring. For all ideals I of R, we define the sets $V(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$. Then $\mathrm{Spec}(R)$ may be regarded as a topological space wherein the $V(I)$ are the closed sets (*Zariski topology*). We say a prime ideal P of R is *locally closed* in $\mathrm{Spec}(R)$ if there exists an open set $U \supseteq \{P\}$ such that $\{P\}$ is closed in U.

The *Jacobson radical* of a ring R, $J(R)$, is the intersection of all maximal right ideals in R. The ring R is said to be *semiprimtive* if $J(R) = 0$ and it is *Jacobson* if $J(R/P) = 0$ for all prime ideals P of R. By [21], Proposition II.7.12, an affine noetherian K-algebra over an uncountable field K is Jacobson. In this case, we have a pleasant description of the primitive ideals.

Lemma 4.0.8. *A prime ideal P in a ring R is locally closed in* $\mathrm{Spec}(R)$ *if and only if the intersection of all prime ideals properly containing P is an ideal properly containing P. Moreover, if R is a Jacobson ring, every locally closed prime is primitive.*

Proof. See [21], Lemmas II.7.7 and II.7.11. □

An element $a \in R$ is said to be *normal* if $aR = Ra$. When R is a domain, $a \in R$ is normal if and only if there exists an automorphism $\sigma \in \mathrm{Aut}(R)$ such that $ar = \sigma(r)a$ for all $r \in R$. As with the prime and primitive ideals, we give a full description of the automorphism group and normal elements in the case of the domains. Partial results are given for the non-domains.

Portions of this chapter are to appear in [26].

4.1 The non-domains

That the algebras R_{x^2}, R_{x^2-1} and R_{yx} are not domains is clear from the defining relations. In $\mathcal{S}$, since $yx = 1$, we have $(xy - 1)x = x - x = 0$. Our goal in this section is to prove that the algebras R_{x^2}, R_{x^2-1}, R_{yx} and $\mathcal{S}$ are all distinct. We begin with $\mathcal{S}$, which was studied extensively, and in greater generality, by Bavula [12]. We compile some of his results here.

Proposition 4.1.1. *The algebra $\mathcal{S}$ has the following properties:*

(1) The algebra is not left or right noetherian;

(2) $\operatorname{gk}(\mathcal{S}) = 2$ *and* $\operatorname{gld}(\mathcal{S}) = 1$*;*

(3) The prime ideals can be found explicitly;

(4) The algebra $\mathcal{S}$ is primitive.

Proposition 4.1.2. *The algebra R_{yx} has the following properties:*

(1) The algebra is not left or right noetherian;

(2) $\operatorname{gk}(R_{yx}) = \operatorname{gld}(R_{yx}) = 2$*;*

(3) $\operatorname{Spec} R_{yx} = \{(x), (x, y-a), (y), (y, x-b) \mid a \in K, b \in K^\times\}$*;*

(4) The primitive ideals are $\{(x, y-a), (y, x-b) \mid a \in K, b \in K^\times\}$*;*

Proof. The sum $\sum_{i\geq 0} x^i y R_{yx}$ is direct and therefore R_{yx} is not right noetherian. Likewise, R_{yx} is not left noetherian, proving (1).

The set $\{x^i y^j\}$ is a K-algebra basis of R_{yx} and so $\operatorname{gk} R_{yx} = \operatorname{gk} K[x,y] = 2$. The right (resp. left) global dimension of a graded K-algebra is equal to the projective dimension of K_A (resp. ${}_AK$) [6]. Consider the sequence of left modules,

$$0 \longrightarrow R_{yx} \xrightarrow{M} R_{yx}^2 \xrightarrow{\mathbf{x}} R_{yx} \xrightarrow{\varepsilon} K \longrightarrow 0,$$

where the maps are defined below. We claim this sequence is exact. Let $\varepsilon : R_{yx} \to K$ be the projection defined by $\varepsilon(x) = \varepsilon(y) = 0$. Then $p \in \ker \varepsilon$ if p has nonzero constant term. Let $\mathbf{x}$ be right multiplication by $\begin{pmatrix} x \\ y \end{pmatrix}$. If $(r,s) \in R_{yx}^2$, then $\mathbf{x}(r,s) = rx + sy$. Hence, $\operatorname{im} \mathbf{x} = \ker \varepsilon$ and $\ker \mathbf{x} = \{(r,0) \in R_{yx}^2 \mid r \in R_{yx} y\}$. Finally, let M be right multiplication by the row vector $(y, 0)$. Notice that $M\mathbf{x} = (yx)$, the generating relation for R_{yx}. Then $\operatorname{im} M = \ker \mathbf{x}$ and M is injective. Hence, $\operatorname{lgld} R_{yx} = 2$. Likewise, $\operatorname{rgld} R_{yx} = 2$. This proves (2).

Let P be a prime ideal in R_{yx}. Since $y R_{yx} x = 0 \in P$, then $x \in P$ or $y \in P$. Moreover, $R_{yx}/(x) \cong K[y]$ and $R_{yx}/(y) \cong K[x]$. This proves (3). However, $(x) = \bigcap (x, y-\alpha)$. Hence, (x) is not locally closed and a similar argument shows (y) is also not locally closed. The ideals $(x, y-\alpha)$ and $(y, x-\beta)$ are maximal and therefore locally closed, proving (4). $\square$

Let A, B be two rings and denote by $A * B$ their free product. Then R_{x^2} (resp. R_{x^2-1}) can be constructed as a free product with $A = K[x]/(x^2)$ (resp. $A = K[x]/(x^2-1)$) and $B = K[y]$.

Theorem 4.1.3 (Bergman, [18])**.** *Let A and B be two rings,*

$$\operatorname{gld}(A * B) = \max\{\operatorname{gld} A, \operatorname{gld} B\}.$$

A full analysis of the prime spectrum of R_{x^2} and R_{x^2-1} seems unattainable at this time. However, we can locate a number of prime ideals in both algebras. For any $\alpha \in K$, $R_{x^2}/(y-\alpha) \cong K[x]/(x^2)$, *the ring of dual numbers*. This is not a prime ring since $xRx = 0$. However, it is a local ring with a unique maximal ideal (x). Since every prime ideal must then live inside of (x), it is not difficult to see that it is the unique prime ideal of $K[x]/(x^2)$. Similarly, for any $\alpha \in K$, $R_{x^2-1}/(y-\alpha) \cong K[x]/(x^2-1)$ and this has two maximal ideals, $(x+1)$ and $(x-1)$.

Proposition 4.1.4. *Let $R = R_{x^2}$ or R_{x^2-1}. Let $q(y) \in K[y]$ with $\deg q(y) \geq 2$. Then the ideal generated by $q(y)$ is prime in R.*

Proof. Write $R = A * K[y]$ for $A = K[x]/(x^2)$ or $K[x]/(x^2-1)$. We have $R/(q(y)) \cong A * B$ where $B = K[y]/(q(y))$. If $\deg q(y) \geq 3$, then B has dimension at least three over K. Thus, by [44], Theorem 2, $R/(q(y))$ is a primitive ring, that is, Q is a primitive ideal in R. In case $\deg q(y) = 2$, then A and B both have dimension two over K and the afformentioned theorem does not apply. Let T_{12} be the free algebra generated by $\{xy\}$. Suppose I, J are nonzero ideals in R. By [44], Theorem 1, there exists $a \in I \cap T_{12}$ and $b \in J \cap T_{12}$ such that $a, b \neq 0$. Since T_{12} is a domain, $ab \neq 0$ so $AB \neq 0$ and therefore 0 is a prime ideal in R. Thus, Q is a prime ideal in R. $\square$

Proposition 4.1.5. *The algebra R_{x^2} has the following properties:*

(1) The algebra is not left or right noetherian;

(2) $\operatorname{gk}(R_{x^2}) = \operatorname{gld}(R_{x^2}) = \infty$*;*

(3) A partial description of the prime ideals is given in Proposition 4.1.4;

(4) The algebra R_{x^2} is not primitive.

Proof. Let $a, b \in R_{x^2}$ with $a, b \neq 0$. Then $ayb \neq 0$, so $aR_{x^2}b = 0$ implies $a = 0$ or $b = 0$. Hence, R_{x^2} is prime. The sum $\sum_{i \geq 0} y^i xy R_{x^2}$ is direct so R_{x^2} is not right noetherian. Likewise, R_{x^2} is not left noetherian proving (1).

Let R_n be the nth homogeneous component of R_{x^2}. The claim on GK dimension will follow easily once we show that $\dim R_n = F(n+2)$, where $F(n)$ is the nth Fibonnaci number. We have $R_0 = K$ and $R_1 = \{x, y\}$ so $\dim R_0 = 1 = F(2)$ and $\dim R_1 = 2 = F(3)$. Proceeding inductively, it suffices to show $\dim R_{n+1} = \dim R_n + \dim R_{n-1}$. It is clear that $R_{n+1} = yR_n \sqcup xR_n$ (disjoint union). Since $\mathrm{r.ann}(y) = 0$, then $\dim yR_n = \dim R_n$. Also, $\mathrm{r.ann}(xy) = 0$, so $\dim xyR_{n-1} = \dim R_{n-1}$. Since $xR_n = xyR_{n-1} \sqcup x^2R_{n-1}$ and $x^2R_{n-1} = 0$, then $\dim xR_n = \dim R_{n-1}$.

Let $A = K[x]/(x^2)$. If $\mathrm{gld}\, A < \infty$, then by [10], Theorem 1.10, A is a domain. Clearly A is not a domain so $\mathrm{gld}\, A = \infty$. By 4.1.3, $\mathrm{gld}\, R_{x^2} = \infty$, proving (2). $\square$

Proposition 4.1.6. *The properties for R_{x^2-1} are identical to those listed in Proposition 4.1.5 for R_{x^2} except that* $\mathrm{gld}\, R_{x^2-1} = 1$.

Proof. The proof to most statements is identical to those for R_{x^2}. In this case, let $A = K[x]/(x^2 - 1)$. Since $A \cong K \times K$ is semisimple, then $\mathrm{gld}\, A = 0$ and so by Theorem 4.1.3, $\mathrm{rgld}\, R_{x^2-1} = \max\{\mathrm{rgld}\, A, \mathrm{rgld}\, K[y]\} = 1$. $\square$

Proposition 4.1.7. *The algebras R_{yx}, R_{x^2}, R_{x^2-1} and $\mathcal{S}$ are all non-isomorphic.*

Proof. By the above results, the algebras R_{x^2}, R_{x^2-1} and $\mathcal{S}$ are prime while R_{yx} is not. We have $\mathrm{gld}\, R_{x^2-1} = \mathrm{gld}\, \mathcal{S} = 1$ whereas $\mathrm{gld}\, R_{x^2} = \infty$. Finally, $\mathrm{gk}\, \mathcal{S} = 2$ whereas $\mathrm{gk}\, R_{x^2-1} = \infty$. $\square$

4.2 Quantum planes

We begin with basic properties of the quantum planes $\mathcal{O}_q(K^2)$, $q \in K^\times$. While this will be sufficient to show that the quantum planes are non-isomorphic to the quantum Weyl algebras and the differential operator rings, the ring-theoretic properties are not sufficient to show that $\mathcal{O}_p(K^2) \cong \mathcal{O}_q(K^2)$ if and only if $p = q^{\pm 1}$. We do this in Section 4.2.1 using linear algebraic techniques of graded algebras.

Recall that, for $q \in K^\times$, $\mathcal{O}_q(K^2) \cong K\langle x, y \mid xy - qyx\rangle$. Hence, $\mathcal{O}_q(K^2)$ is the skew polynomial ring $K[y][x; \sigma]$ where the $\sigma \in \mathrm{Aut}(K[y])$ is defined by $\sigma(y) = qy$. When q is a primitive nth root of unity, σ is of finite order n.

To determine the prime ideals of $\mathcal{O}_q(K^2)$ when $q \in K^\times$ is a non-root of unity is an easy exercise. One need only check that the localization $\mathcal{O}_q(K^2)[x^{-1}, y^{-1}]$ is simple, implying that each nonzero prime ideal contains x or y. Since $\mathcal{O}_q(K^2)/(x) \cong$

$K[y] \cong \mathcal{O}_q(K^2)/(y)$, then the prime ideals of $\mathcal{O}_q(K^2)$ are

$$\{(0), (x), (x, y-\alpha), (y), (y, x-\beta) \mid \alpha \in K, \beta \in K^\times\}. \tag{4.1}$$

Describing the primitive ideals is now a relatively simple matter. Since (0) is contained in the ideals (x) and (y) and $(0) \subsetneq (xy) = (x) \cap (y)$, then (0) is locally closed. The remainder follows as in the proof of Proposition 4.1.2. Thus, the primitive ideals are

$$\{(0), (x, y-\alpha), (y, x-\beta) \mid \alpha \in K, \beta \in K^\times\}.$$

The root of unity case is considerably more complex. The prime ideal structure of skew polynomials rings $R[x;\sigma]$ was considered by Irving [32] in the case that R is commutative. We appeal to more general results that will be useful later.

Theorem 4.2.1 (Leroy, Matczuk [43])**.** *Let I be a nonzero ideal of $R = Q[x;\sigma,\delta]$ such that $I \cap Q = 0$. If some power of σ is inner, then there is a 1-1 correspondence between* $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(\mathcal{Z}(R))$.

If $q \in K^\times$ is a primitive nth root of unity, then a straightforward computation shows that $\mathcal{Z}(\mathcal{O}_q(K^2)) = K[x^n, y^n]$. Hence, in this case we have the additional prime ideals $(x^n - \xi, y^n - \psi)$, $\xi, \psi \in K^\times$. We can now deduce the finite-dimensional simple modules.

Proposition 4.2.2. *Let $q \in K^\times$, $q \neq 1$. The 1-dimensional modules of $\mathcal{O}_q(K^2)$ are all of the form* $\mathrm{Span}\{v\}$ *and either $x.v = 0$ and $y.v = \alpha v$ for some $\alpha \in K$ or else $x.v = \beta v$ and $y.v = 0$ for some $\beta \in K$. If q is not a root of unity, then these are all of the finite-dimensional simple modules. If q is a primitive nth root of unity, then $\mathcal{O}_q(K^2)$ is PI and all simple modules are finite dimensional.*

To determine the normal elements we first recall results regarding the automorphism groups. This will also be critical to proving Theorem 4.0.7.

Proposition 4.2.3 (Alev, Chamarie, [1])**.** *If $q \neq \pm 1$, then* $\mathrm{Aut}(\mathcal{O}_q(K^2)) \cong (K^\times)^2$. *If $q = -1$, then* $\mathrm{Aut}(\mathcal{O}_q(K^2)) \cong (K^\times)^2 \rtimes \{\omega\}$ *where ω is the involution switching the generators x and y.*

Proposition 4.2.4. *Suppose $q \neq 1$ and let $0 \neq g \in \mathcal{O}_q(K^2)$ be a normal element. If q is not a root of unity, then $g = cy^r x^s$ for some $c \in K^\times$, $r, s \in \mathbb{N}$. If q is a primitive nth root of unity, then $g = \sum c_i y^{r_i} x^{s_i}$ where $c_i \in K^\times$ for all i and $r_i \equiv r_j$ mod n, $s_i \equiv s_j$ mod n for all i, j.*

Proof. (Case 1: $q \neq -1$) Let $\rho \in \mathrm{Aut}(\mathcal{O}_q(K^2)) = (K^\times)^2$ be the automorphism corresponding to g. Then $\rho(y) = \varepsilon y$ and $\rho(x) = \varepsilon' x$ for some $\varepsilon, \varepsilon' \in K^\times$. Write $g = \sum \gamma_{ij} y^i x^j$. Then,

$$\varepsilon \sum \gamma_{ij} y^{i+1} x^j = \rho(y) g = gy = \sum \gamma_{ij} y^i (x^j y) = \sum \gamma_{ij} q^j y^{i+1} x^j.$$

Choose u, v maximal such that $\gamma_{uv} \neq 0$. Then $\varepsilon = q^v$ and if there exists u', v' with $v' \neq v$ such that $\gamma_{u'v'} \neq 0$, this gives $\varepsilon = q^{v'}$, a contradiction unless $q^{v'} = q^v$. This happens if and only if q is a root of unity and $v \equiv v' \bmod n$. Furthermore,

$$\varepsilon' \sum \gamma_{iv} q^i y^i x^{v+1} = \rho(x) g = gx = \sum \gamma_{ij} y^i x^{v+1}.$$

In a similar manner as above, $\varepsilon' = q^{-u}$.

(Case 2: $q = -1$) In this case, there is the additional possibility that the automorphism ρ may interchange x and y. Then,

$$\varepsilon \sum \gamma_{ij} (-1)^i y^i x^{j+1} = \varepsilon x \sum \gamma_{ij} y^i x^j = \rho(y) g = gy = \sum \gamma_{ij} (-1)^j y^{i+1} x^j.$$

It is clear that the two sides are inequivalent. Hence, the normal elements expressed in the statement of the proposition comprise all for $\mathcal{O}_q(K^2)$. $\square$

4.2.1 Isomorphisms of quantum affine space

We now wish to show that $\mathcal{O}_p(K^2) \cong \mathcal{O}_q(K^2)$ if and only if $p = q^{\pm 1}$. We will, in fact, solve a more general problem on isomorphisms of quantum affine space.

We say $\mathbf{q} = (q_{ij}) \in \mathcal{M}_n(K^\times)$ is *multiplicatively antisymmetric* if $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for all $i \neq j$. Let $\mathcal{A}_n \subset \mathcal{M}_n(K^\times)$ be the subset of multiplicatively antisymmetric matrices. The symmetric group on n letters, $\mathcal{S}_n$, acts on $\mathcal{A}_n$ by $\sigma.A = [a_{\sigma(i)\sigma(j)}]$ for $A \in \mathcal{A}_n$. We say $\mathbf{p}$ is a permutation of $\mathbf{q}$ if there exists $\sigma \in \mathcal{S}_n$ such that $\mathbf{p} = \sigma.\mathbf{q}$.

For $\mathbf{q} \in \mathcal{A}_n$, the quantum affine n-space $\mathcal{O}_{\mathbf{q}}(K^n)$ is defined as the algebra with generating basis $\{x_i\}$, $1 \leq i \leq n$, subject to the relations $x_i x_j = q_{ij} x_j x_i$ for all $1 \leq i, j \leq n$. The algebra $\mathcal{O}_{\mathbf{q}}(K^n)$ is affine connected graded. By [21], Lemma II.9.7, $\mathrm{gk}(\mathcal{O}_{\mathbf{p}}(K^n)) = n$. Hence, if $\mathcal{O}_{\mathbf{p}}(K^n) \cong \mathcal{O}_{\mathbf{q}}(K^m)$, then $n = m$. We prove that two quantum affine spaces, $\mathcal{O}_{\mathbf{p}}(K^n)$ and $\mathcal{O}_{\mathbf{q}}(K^n)$, are isomorphic if and only if $\mathbf{p}$ is a permutation of $\mathbf{q}$ (Theorem 4.2.11).

If R is an affine connected graded algebra and $a \in R$, then we can decompose a into its homogeneous components, $a = a_0 + \cdots + a_n$, $a_d \in A_d$. If $\Phi : R \to S$ is a map between affine connected graded algebras and x_i a generating element of R, we denote by $\Phi_d(x_i)$ the homogeneous degree d component of the image of x_i under Φ.

Throughout this section, let $\Phi : R \to S$ be a (not necessarily graded) isomorphism between affine connected graded algebras. Let $\{x_i\}$ (resp. $\{y_i\}$) be a generating basis for R (resp. S) and suppose $1 \le i \le n$ in both cases.

Lemma 4.2.5. *The degree one components of $\Phi(x_1), \ldots, \Phi(x_n)$ are all K-linearly independent. Moreover, Φ_1 maps R_1 isomorphically onto S_1.*

Proof. The isomorphism Φ is completely determined by its action on the x_i. Hence, the elements $\{\Phi(x_i)\}$ generate all of S. Let $f_i \in R$ such that $y_i = \Phi(f_i)$, $i \in \{1, \ldots, n\}$. Since $\deg(y_i) = 1$, then $y_i = \Phi_1(f_i)$.

Because S is graded, then $\Phi_2(x_i) \cdot \Phi_d(x_j) \in S_{d+2}$. Moreover, since S is connected graded, then $\Phi_0(x_i) \in S_0 = K$. Let $r = x_{i_1} \cdots x_{i_m}$ be an arbitrary monomial in R. Then

$$\Phi_1(r) = \left(\prod_{k=1}^{m} \Phi(x_{i_k}) \right)_1 = \left(\prod_{k=1}^{m} \Phi_0(x_{i_k}) + \Phi_1(x_{i_k}) \right)_1 .$$

Thus, we can write,

$$y_i = \sum_{j=1}^{n} \alpha_{ij} \Phi_1(x_j),\ \ \alpha_{ij} \in K.$$

Hence $\Phi_1 : R_1 \to S_1$ is onto. Moreover, $\dim_K(R_1) = \dim_K(S_1)$ and so Φ_1 is an isomorphism. □

The next step is to show that the constant term of the image of each generator is zero. This need not always hold, but it does in the generic case.

Lemma 4.2.6. *If $i, j \in \{1, \ldots, n\}$ such that $x_i x_j - p x_j x_i = 0$ for some $p \in K^\times$, $p \neq 1$, then $\Phi_0(x_i) = \Phi_0(x_j) = 0$.*

Proof. Without loss of generality, suppose $\Phi_0(x_i) \neq 0$. Let $T = \Phi(x_i)\Phi(x_j) - p\Phi(x_j)\Phi(x_i)$. Then $T_0 = \Phi_0(x_i)\Phi_0(x_j)(1-p) = 0$, so $\Phi_0(x_j) = 0$. Thus, $T_1 = \Phi_0(x_i)\Phi_1(x_j)(1-p) = 0$. Since $\Phi_1(x_j) \neq 0$ by Lemma 4.2.5, then $T_1 \neq 0$, a contradiction. □

We include one additional general result.

Lemma 4.2.7. *The isomorphism Φ determines a permutation $\tau \in \mathcal{S}_n$.*

Proof. Let $M = (\alpha_{ij})$. Then $\det(M) \neq 0$. By Lemma 4.2.5, $\Phi_1 : R_1 \to S_1$ is a vector space isomorphism, and so $\det(M) \neq 0$. We proceed by induction, with the case of $n = 1$ being trivial. Suppose this holds for some $n = k$. We will prove the result for $n = k + 1$. Let M_j be the minor of M corresponding to the entry α_{1j}. Then,

$$\det(M) = \sum_{j=1}^{k+1} (-1)^{j+1} \alpha_{1j} \det(M_j).$$

Because $\det(M) \neq 0$, there exists some $\tau(i) \in \{1, \dots, n\}$ such that $\alpha_{i\tau(i)} \det(M_{\tau(i)}) \neq 0$. We pass now to $M_{\tau(i)}$ and, because $\dim(M_{\tau(i)}) = k^2$, the result now follows by induction. □

The following result is not necessary for our analysis of the quantum affine spaces, but will be necessary in considering isomorphism problems in Chapter 5.

Lemma 4.2.8. *The subring $\mathcal{Z}(S)$ is graded.*

Proof. Let $a \in \mathcal{Z}(S)$ and write $a = a_0 + \cdots a_n$ according to the grading in S. Assume b is homogeneous of degree d. Then

$$0 = ab - ba = \sum_{i=0}^{d} (a_i b - b a_i).$$

Each component is homogeneous of degree $n + d$ and so a_i commutes with all homogeneous elements of S. It now follows easily that $a_i \in \mathcal{Z}(S)$ for all i. □

We now specialize to $R = \mathcal{O}_{\mathbf{p}}(K^n)$ and $S = \mathcal{O}_{\mathbf{q}}(K^n)$. By Lemma 4.2.7, the isomorphism $\Phi : R \to S$ gives a permutation $\tau \in \mathcal{S}_n$. It now suffices to show that $\mathbf{p} = \tau.\mathbf{q}$.

Lemma 4.2.9. *If $r, s \in \{1, \dots, n\}$ such that $p_{rs} \neq 1$, then $p_{rs} = q_{\tau(r)\tau(s)}$.*

Proof. By Lemma 4.2.6, $\Phi_0(x_r) = \Phi_0(x_s) = 0$. Write $\Phi_1(x_r) = \sum \alpha_i y_i$ and $\Phi_1(x_s) = \sum \beta_i y_i$. Let $T(r, s) = \Phi(x_r)\Phi(x_s) - p_{rs}\Phi(x_s)\Phi(x_r)$. Because $T(r, s) \in S$, then each graded component $T_d(r, s)$ is zero. In particular,

$$0 = T_2(r, s) = (1 - p_{rs}) \left(\sum_{d=1}^{n} \alpha_d \beta_d y_d^2 \right) + \sum_{1 \leq i \neq j \leq n} (\alpha_i \beta_j - p_{rs} \alpha_j \beta_i)\, y_i y_j.$$

Since $p_{rs} \neq 1$, then $\alpha_d = 0$ or $\beta_d = 0$ for each d. Thus,

$$\begin{aligned} T_2(r,s) &= \sum_{1\le i<j\le n} \left[(\alpha_i\beta_j - p_{rs}\alpha_j\beta_i) + q_{ji}(\alpha_j\beta_i - p_{rs}\alpha_i\beta_j)\right] y_i y_j \\ &= \sum_{1\le i<j\le n} \left[(\alpha_j\beta_i(q_{ji} - p_{rs}) + \alpha_i\beta_j(1 - q_{ji}p_{rs})\right] y_i y_j. \end{aligned} \tag{4.2}$$

By Lemma 4.2.7, $\alpha_{\tau(r)}$, $\beta_{\tau(s)} \neq 0$. Thus, $\alpha_{\tau(s)} = 0$ and $\beta_{\tau(r)} = 0$. If $\tau(r) > \tau(s)$, then by (4.2) the coefficient of $y_{\tau(s)}y_{\tau(r)}$ is $\alpha_{\tau(r)}\beta_{\tau(s)}(q_{\tau(r)\tau(s)} - p_{rs})$. Therefore, $p_{rs} = q_{\tau(r)\tau(s)}$. One the other hand, if $\tau(r) < \tau(s)$, then the coefficient of $y_{\tau(r)}y_{\tau(s)}$ is $\alpha_{\tau(r)}\beta_{\tau(s)}(1 - q_{\tau(s)\tau(r)}p_{rs})$. Therefore, $p_{rs} = q^{-1}_{\tau(s)\tau(r)} = q_{\tau(r)\tau(s)}$. Because $p_{rs} \neq 1$, then $r \neq s$ and so, because τ is a permutation, $\tau(r) \neq \tau(s)$ and so the result follows. □

Lemma 4.2.10. *If $r, s \in \{1, \dots, n\}$ such that $p_{rs} = 1$, then $p_{rs} = q_{\tau(r)\tau(s)}$.*

Proof. Let $\mathbf{p}^{\#}$ (resp. $\mathbf{q}^{\#}$) denote the number of entries in $\mathbf{p}$ (resp. $\mathbf{q}$) not equal to 1. By Lemma 4.2.9, $\mathbf{p}^{\#} \le \mathbf{q}^{\#}$. Because Φ is an isomorphism, then we can apply Lemma 4.2.9 to Φ^{-1} to get that $\mathbf{q}^{\#} \le \mathbf{p}^{\#}$. Thus, $\mathbf{p}^{\#} = \mathbf{q}^{\#}$. □

Theorem 4.2.11. *$\mathcal{O}_{\mathbf{p}}(K^n) \cong \mathcal{O}_{\mathbf{q}}(K^n)$ if and only $\mathbf{p}$ is a permutation of $\mathbf{q}$.*

Proof. Suppose $\mathbf{p}$ is a permutation of $\mathbf{q}$. Then there exists $\sigma \in \mathcal{S}_n$ such that $\mathbf{p} = \sigma.\mathbf{q}$. We wish to define a homomorphism $\mathcal{O}_{\mathbf{p}}(K^n) \to \mathcal{O}_{\mathbf{q}}(K^n)$ via the rule $\Psi(x_i) = y_{\sigma(i)}$. For all i, j, $1 \le i, j \le n$, this rule gives

$$\Psi(x_i)\Psi(x_j) - p_{ij}\Psi(x_j)\Psi(x_i) = y_{\sigma(i)}y_{\sigma(j)} - q_{\sigma(i)\sigma(j)}y_{\sigma(j)}y_{\sigma(i)} = 0.$$

Hence, Ψ extends to a homomorphism which is clearly bijective. Thus, $\mathcal{O}_{\mathbf{p}}(K^n) \cong \mathcal{O}_{\mathbf{q}}(K^n)$.

Conversely, suppose $\Phi : \mathcal{O}_{\mathbf{p}}(K^n) \to \mathcal{O}_{\mathbf{q}}(K^n)$ is an isomorphism. Lemma 4.2.7 gives a permutation $\tau \in \mathcal{S}_n$. By Lemmas 4.2.9 and 4.2.10, $\mathbf{p} = \tau.\mathbf{q}$. □

Corollary 4.2.12. *$\mathcal{O}_p(K^2) \cong \mathcal{O}_q(K^2)$ are isomorphic if and only if $p = q^{\pm 1}$.*

The methods developed in this section can be used for a variety of additional isomorphism problems. In particular, they are used to study isomorphisms between the algebras $H(\mathcal{O}_q(K^2))$ and $H(A_1^p(K))$ (Propositions 5.4.4 and 5.4.5). Further applications, including those to single-parameter and certain multi-parameter quantum matrix algebras, can be found in [26].

4.3 Quantum Weyl algebras

As with the previous section, we review results on prime ideals and automorphisms, compute the normal elements, and then proceed to proving $A_1^p(K) \cong A_1^q(K)$ if and only if $p = q^{\pm 1}$. Suppose $q \in K^\times$ is not a root of unity. Let $\theta = xy - yx \in A_1^q(K)$. Then θ is normal in $A_1^q(K)$ and in $A_1^q(K)/(\theta)$ we have

$$1 = xy - qyx = xy - yx + yx - qyx = (1-q)yx.$$

Thus, $A_1^q(K)/(\theta) \cong K[y, y^{-1}]$. By [27], Theorem 8.4 (a), the intersection of all nonzero prime ideals in $A_1^q(K)$ is (θ), and so prime ideals are of the form

$$\{(0), (\theta), (\theta, y - \gamma) \mid \gamma \in K^\times\}.$$

Now suppose $q \in K^\times$ is a primitive nth root of unity. If $q = 1$, then $A_1^q(K)$ is a simple ring so assume $n > 1$. Results in this case have been obtained independently by Irving, Goodearl, and Jordan (see, in particular, [34], Proposition 2.2). It can also be deduced from Theorem 4.2.1. The prime ideals are of the form

$$\{(0), (\theta), (\theta, y - \gamma), (p) \mid \gamma \in K^\times, p \text{ is an irreducible polynomial in } K[x^n, y^n]\}.$$

We observe

Proposition 4.3.1. *Let $q \in K^\times$, $q \neq 1$. The 1-dimensional modules of $A_1^q(K)$ are all of the form* $\mathrm{Span}\{v\}$ *where $x.v = \alpha v$ and $y.v = \alpha^{-1}(1-q)^{-1}v$ for some $\alpha \in K^\times$. If q is not a root of unity, then these are all of the finite-dimensional simple modules. If q is a primitive nth root of unity, then $A_1^q(K)$ is PI and all simple modules are finite dimensional.*

The automorphism group of $A_1^q(K)$ was considered in [3] and [1]. Recalling it here serves two purposes. The first is to prove $\mathcal{O}_p(K^2) \ncong A_1^q(K)$ for all $p, q \in K^\times$, and the second is to determine the degree one normal elements of $A_1^q(K)$.

Proposition 4.3.2 (Alev, Chamarie, Dumas)**.** *If $q \neq \pm 1$, then* $\mathrm{Aut}(A_1^q(K)) \cong (K^\times)$. *If $q = -1$, then* $\mathrm{Aut}(A_1^q(K)) \cong K^\times \rtimes \{\omega\}$ *where ω is the involution switching the generators x and y.*

We say a normal element g in a skew polynomial ring $R[x; \sigma, \delta]$ is *degree one* if $g = ax + b$ for some $a, b \in R$. Recall that $A_1^q(K)$ can be constructed as a skew polynomial ring $K[y][x; \sigma, \delta]$ where $\sigma(y) = qy$ and $\delta(y) = 1$.

Proposition 4.3.3. *Suppose $q \neq 1$. Degree one normal elements in $A_1^q(K)$ all have the form $c\theta$ where $c \in K^\times$.*

Proof. Checking that such an element is normal is an easy exercise and we omit it. Suppose $q \neq \pm 1$ and $ax + b \in A_1^q(K)$ is a degree one normal element, so $a, b \in K[y]$. Let $\rho \in \mathrm{Aut}(A_1^q(K)) = K^\times$ be the corresponding automorphism and let $\varepsilon \in K^\times$ such that $\rho(y) = \varepsilon y$ and $\rho(x) = \varepsilon^{-1} x$. Then,

$$(ax + b)y = axy + by = a(qyx + 1) + by = (qay)x + (a + by).$$

On the other hand,

$$(ax + b)y = \rho(y)(ax + b) = \varepsilon y(ax + b) = (\varepsilon ay)x + \varepsilon.$$

Then $\varepsilon = q$ and $a = (q - 1)yb$. Now $ax + b = b[(q - 1)yx + 1]$ and so

$$(ax + b)x = b((q - 1)yx^2 + x).$$

By the construction of $A_1^q(K)$ as a skew polynomial ring, we have,

$$\begin{aligned} (ax + b)x = \rho(x)(ax + b) &= q^{-1}(xb)((q - 1)yx + 1) \\ &= q^{-1}(\sigma(b)x + \delta(b))((q - 1)yx + 1) \\ &= q^{-1}\left(\sigma(b)(q - 1)yx^2 + q\sigma(b)x + \delta(b)(q - 1)yx + \delta(b)\right). \end{aligned}$$

But then $\delta(b) = 0$, so $b \in K^\times$.

Now suppose $q = -1$ and again suppose ρ interchanges x and y. Then,

$$ax^2 + bx = (ax + b)x = \rho(x)(ax + b) = \varepsilon y(ax + b) = \varepsilon ayx + \varepsilon yb.$$

Thus, $a = 0$, a contradiction. □

It remains to be shown that $A_1^p(K) \cong A_1^q(K)$ if and only if $p = q^{\pm 1}$. That these conditions are sufficient follows from Corollary 3.4.3. Because A_1 is simple and because the automorphism group of $A_1^q(K)$ for $q = -1$ is distinct (Proposition 4.3.2), we assume $p, q \notin \{-1, 1\}$.

Recall that $A_1^q(K)$ is PI if and only if q is a primitive root of unity of order ℓ, in which case $\mathcal{Z}(A_1^q(K)) = K[x^\ell, y^\ell]$, and otherwise $\mathcal{Z}(A_1^q(K)) = K$ ([11], Lemma 2.2). Hence, we consider the nonroot and root of unity cases separately (Propositions 4.3.5 and 4.3.6, respectively). The nonroot of unity case actually follows from [2], Proposition 3.11. However, the proof given here is more direct and is re-used in Proposition 4.3.6.

Let $\{X, Y\}$ (resp. $\{x, y\}$) be a generating basis for $A_1^p(K)$ (resp. $A_1^q(K)$) and define the normal elements $\Theta = XY - YX \in A_1^p(K)$ and $\theta = xy - yx \in A_1^q(K)$. Throughout the remainder of this section, assume $\Phi : A_1^p(K) \to A_1^q(K)$ is an isomorphism. By *degree* we mean total degree in X and Y in $A_1^p(K)$ (resp. x and y in $A_1^q(K)$). The next lemma can be thought of as an ungraded version of Lemma 4.2.5.

Lemma 4.3.4. *If* $A_1^p(K) \cong A_1^q(K)$, *then* $\deg(\Phi(X)), \deg(\Phi(Y)) \geq 1$.

Proof. Without loss, suppose $\deg(\Phi(X)) = 0$. Then $\Phi(X) \in \mathcal{Z}(A_1^q(K))$, implying $X \in \mathcal{Z}(A_1^p(K))$. This cannot hold by the above discussion. □

Proposition 4.3.5. *Let* $p, q \in K^\times$ *with* p, q *non-roots of unity. If* $A_1^p(K) \cong A_1^q(K)$, *then* $p = q^{\pm 1}$.

Proof. By [27], the intersection of all nonzero prime ideals in $A_1^p(K)$ (resp. $A_1^q(K)$) is $\Theta A_1^p(K)$ (resp. $\theta A_1^q(K)$). Hence, $\Phi(\Theta A_1^p(K)) = \Phi(\Theta)\Phi(A_1^p(K)) = \Phi(\Theta)A_1^q(K)$. Since $\Phi(\Theta) \in \theta A_1^q(K)$, then $\Phi(\Theta) = \lambda\theta$ for some $\lambda \in A_1^q(K)$. We claim $\lambda \in K^\times$. The ideal $\theta A_1^q(K)$ is generated by θ, so there exists $g \in A_1^q(K)$ such that $g \cdot \lambda\theta = \theta$. Hence, λ is a unit in $A_1^q(K)$ and therefore $\lambda \in K^\times$. This gives

$$\Phi(\Theta) = \lambda\theta = \lambda(xy - yx) = \lambda(q-1)yx + \lambda,$$

and so,

$$\Phi(X)\Phi(Y) = \Phi(Y)\Phi(X) + \lambda(q-1)yx + \lambda.$$

Since Φ is an isomorphism,

$$\begin{aligned} 0 = \Phi(XY - pYX - 1) &= \Phi(X)\Phi(Y) - p\Phi(Y)\Phi(X) - 1 \\ &= (\Phi(Y)\Phi(X) + \lambda(q-1)yx + \lambda) - p\Phi(Y)\Phi(X) - 1 \\ &= (1-p)\Phi(Y)\Phi(X) + \lambda(q-1)yx + (\lambda - 1), \end{aligned}$$

and so,

$$\Phi(Y)\Phi(X) = (p-1)^{-1}\left(\lambda(q-1)yx + (\lambda - 1)\right). \tag{4.3}$$

We claim the degrees of $\Phi(X)$ and $\Phi(Y)$ in $A_1^q(K)$ are both one. Write $\Phi(X) = a = a_0 + \cdots a_n$, $a_n \neq 0$, and $\theta(Y) = b = b_0 + \cdots b_m$, $b_m \neq 0$, wherein a_d is the sum of the monomomials of total degree d written according to the filtration $\{y^i x^j \mid i, j \in \mathbb{N}\}$ (and similarly for b_d). Because $A_1^q(K)$ is a domain, the highest degree component of $\Phi(Y)\Phi(X)$ is $b_m a_n \neq 0$. If n or m is greater than one, then the left hand side of (4.3) will have degree greater than two, a contradiction. This proves the claim. Thus, we can write $\Phi(X) = \alpha x + \beta y + \gamma$ and $\Phi(Y) = \alpha' x + \beta' y + \gamma'$. Substituting this into (4.3) gives

$$\begin{aligned}\alpha'\alpha x^2 + \alpha'\beta xy + \alpha'\gamma x + \beta'\alpha yx + \beta'\beta y^2 + \beta'\gamma y + \gamma'\alpha x + \gamma'\beta y + \gamma'\gamma \\ = \lambda\frac{q-1}{p-1}yx + \frac{\lambda-1}{p-1}.\end{aligned} \tag{4.4}$$

Thus, $\alpha'\alpha = \beta'\beta = 0$. If $\alpha = \beta = 0$, then $\Phi(X)$ is a constant and similarly for $\Phi(Y)$ if $\alpha' = \beta' = 0$. This contradicts Lemma 4.3.4.

If $\alpha' = \beta = 0$, then (4.4) reduces to

$$\beta'\alpha yx + \beta'\gamma y + \gamma'\alpha x + \gamma'\gamma = (p-1)^{-1}(\lambda(q-1)yx + (\lambda - 1)).$$

Thus, $\beta'\alpha \neq 0$ but $\beta'\gamma = \gamma'\alpha = 0$ so $\gamma = \gamma' = 0$. This holds only if $\lambda = 1$ so

$$\begin{aligned}0 &= \Phi(XY - pYX - 1) = \beta'\alpha(xy - pyx) - 1 \\ &= \beta'\alpha(qyx + 1 - pyx) - 1 = \beta'\alpha(q-p)yx + (\beta'\alpha - 1).\end{aligned}$$

Therefore, $p = q$.

Otherwise, $\alpha = \beta' = 0$ and (4.4) reduces to

$$\begin{aligned}\alpha'\beta xy + \alpha'\gamma x + \gamma'\beta y + \gamma'\gamma &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)) \\ \alpha'\beta(qyx+1) + \alpha'\gamma x + \gamma'\beta y + \gamma'\gamma &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)) \\ q\alpha'\beta yx + \alpha'\gamma x + \gamma'\beta y + (\alpha'\beta + \gamma'\gamma) &= (p-1)^{-1}(\lambda(q-1)yx + (\lambda-1)).\end{aligned}$$

As above, $\gamma = \gamma' = 0$ so

$$\begin{aligned}0 &= \Phi(XY - pYX - 1) = \alpha'\beta(yx - pxy) - 1 \\ &= \alpha'\beta(yx - p(qyx+1)) - 1 = \alpha'\beta(1-pq)yx - (p\alpha'\beta + 1).\end{aligned}$$

Therefore, $p = q^{-1}$. $\square$

Proposition 4.3.6. *Let $p, q \in K^\times$ with $p, q \neq \pm 1$ primitive roots of unity. If $A_1^p(K) \cong A_1^q(K)$, then $p = q^{\pm 1}$.*

Proof. As in Proposition 4.3.5, write $\Phi(X) = a = a_0 + \cdots + a_n$ and $\Phi(Y) = b = b_0 + \cdots + b_m$, $a_n, b_m \neq 0$. By Lemma 4.3.4, $m + n > 0$. We decompose a_n and b_m further as

$$a_n = \sum_{i=0}^{n} a_{n,i} y^{n-i} x^i, \quad b_m = \sum_{j=0}^{m} b_{m,j} y^{m-j} x^j,$$

where $a_{n,i}, b_{m,j} \in K$ for all i, j. Choose r, s minimal such that $a_{n,r}, b_{m,s} \neq 0$. As $0 = \theta(XY - pYX - 1) = ab - pba - 1$, the highest y-degree term in $a_n b_m - p b_m a_n$ is $a_{n,r} b_{m,s} \left[q^{r(m-s)} - pq^{s(n-r)}\right] y^{n+m-r-s} x^{r+s} = 0$. Hence,

$$q^{r(m-s)} - pq^{s(n-r)} = q^{r(m-s)}(1 - pq^{ns-mr}) = 0.$$

This implies that

$$p = q^{mr-ns}. \tag{4.5}$$

Likewise, $q = p^t$ for some $t \in \mathbb{N}$. Thus, p and q are roots of unity of the same order ℓ. Hence, $\mathcal{Z}(A_1^p(K)) = K[X^\ell, Y^\ell]$ and $\mathcal{Z}(A_1^q(K)) = K[x^\ell, y^\ell]$. Then $\Phi(X^\ell) = a^\ell = a'_{n\ell} + a'_{n\ell-1} + \cdots a'_0$ where a'_d is the term of a^ℓ of total degree d. Thus,

$$a'_{n\ell} = \alpha_{n,r}^\ell q^v y^{(n-r)\ell} x^{r\ell} + \sum_{j=0}^{r\ell-1} \alpha'_{n\ell,j} y^{n\ell-j} x^j, \tag{4.6}$$

with $v \in \mathbb{Z}$ and $\alpha'_{n\ell,j} \in K$. Similarly, $\Phi(Y^\ell) = b^\ell = b'_{m\ell} + b'_{m\ell-1} + \cdots b'_0$ where

$$b'_{m\ell} = \beta_{m,s}^\ell q^w y^{(m-s)\ell} x^{s\ell} + \sum_{j=0}^{s\ell-1} \beta'_{m\ell,j} y^{m\ell-j} x^j. \tag{4.7}$$

The restriction of Φ to the centers of the respective algebras determines an automorphism of the polynomial ring in two variables. The centrality of X^ℓ and Y^ℓ implies $\theta(X^\ell)$ and $\theta(Y^\ell)$ are central. Thus, $a'_e = b'_e = 0$ if $e \not\equiv 0$ modulo ℓ and $\alpha'_{n\ell,j} = \beta'_{m\ell,j} = 0$ if $j \not\equiv 0$ modulo ℓ. Lemma 2 of [46] shows that there are three possibilities for an automorphism of the polynomial ring in two variables (see also [3]).

Case 1: There exists $t \in \mathbb{N}$ and $\lambda \in K$ such that $a'_{n\ell} = \lambda(b'_{m\ell})^t$. Substituting into (4.6) and (4.7) shows that $r = st$ and $n = mt$, so $ns = mr$. Then (4.5) implies $p = 1$, a contradiction.

Case 2: There exists $t \in \mathbb{N}$ and $\lambda \in K$ such that $b'_{m\ell} = \lambda(a'_{n\ell})^t$. This gives the same contradiction as above.

Case 3: $\Phi(X^\ell) = \zeta x^\ell + \xi y^\ell + \omega$ and $\Phi(Y^\ell) = \zeta' x^\ell + \xi' y^\ell + \omega'$ with $\zeta, \xi, \omega, \zeta', \xi', \omega' \in K$. Hence, the $\deg \Phi(X) = \Phi(Y) = 1$ and we refer to the proof of Proposition 4.3.5. $\square$

4.4 Differential operator rings

The algebras $\mathfrak{U}$, $\mathcal{J}$, and $\mathcal{J}_1$ all appear as differential operator rings over $K[y]$. Denote the algebra $K[y][x;\delta]$ by R_f (resp. R_g) where $f = \delta(y)$ (resp. $g = \delta(y)$). Throughout this section, we assume $\deg(f), \deg(g) > 0$. We say an ideal I of $K[y]$ is *δ-invariant* if $\delta(I) \subset I$.

Proposition 4.4.1. *The ring R_f is prime. Moreover, if P is a non-zero prime of R_f then $I = (h)$ where $h \mid f$ is irreducible.*

Proof. This follows directly from [29], Theorem 3.22. $\square$

Computing the prime spectrum of the algebras $\mathfrak{U}$, $\mathcal{J}$, and $\mathcal{J}_1$ is now just a matter of factoring the corresponding polynomial f. Thus, we have

$$\begin{aligned}
\operatorname{Spec}\mathfrak{U} &= \{(0), (y), (y, x-\alpha) \mid \alpha \in K\},\\
\operatorname{Spec}\mathcal{J} &= \{(0), (y), (y, x-\alpha) \mid \alpha \in K\},\\
\operatorname{Spec}\mathcal{J}_1 &= \{(0), (y+i), (y-i), (y \pm i, x-\alpha) \mid \alpha, \beta \in K\}.
\end{aligned}$$

Let h be an irreducible factor of $f \in K[y]$. Since $R_f/(h) \cong K[y]$, then (h) is not locally closed in $\operatorname{Spec} R_f$. On the other hand, f has only finitely many irreducible factors and therefore (0) is locally closed. Thus, the primitive ideals of R_f are exactly (0) and the maximal ideals.

Similarly, since $\operatorname{Ann} M$ is prime for any finite-dimensional simple module M of R_f, then these are easily classified. In particular,

$$\begin{aligned}
\mathfrak{U} :& M_\alpha = R/(Ry + R(x-\alpha)), \alpha \in K;\\
\mathcal{J} :& M_\alpha = R/(Ry + R(x-\alpha)), \alpha \in K;\\
\mathcal{J}_1 :& M_{\pm,\alpha} = R/(R(y \pm i) + R(x-\alpha)).
\end{aligned}$$

Alev and Dumas studied the isomorphism problem for such algebras in [4]. We recall their result here.

Proposition 4.4.2 ([4], Proposition 3.6)**.** *The algebras $R_f \cong R_g$ are isomorphic if and only if there exists $\lambda, \alpha \in K^\times$ and $\beta \in K$ such that $f(y) = \lambda g(\alpha y + \beta)$.*

Corollary 4.4.3. *The algebras $\mathfrak{U}, \mathcal{J}$ and $\mathcal{J}_1$ are all non-isomorphic.*

Proof. Since $\deg(xy - yx) = 1$ in $\mathfrak{U}$, then $\mathfrak{U}$ is not isomorphic to $\mathcal{J}$ and $\mathcal{J}_1$. If $\mathcal{J} \cong \mathcal{J}_1$, then by Proposition 4.4.2 there exists α, β, λ such that

$$\alpha\lambda(y^2 + 1) = (\alpha y + \beta)^2 = \alpha^2 y^2 + 2\alpha\beta y + \beta^2.$$

Comparing coefficients of y we get that $\alpha = 0$ or $\beta = 0$, a contradiction. □

Corollary 4.4.4. *Automorphisms of R_f are triangular of the form $x \mapsto \lambda x + h$, $y \mapsto \alpha y + \beta$, for some $\alpha, \lambda \in K^\times$, $\beta \in K$, and $h \in K[y]$ such that*

$$f(\alpha y + \beta) = \alpha\lambda f(y). \tag{4.8}$$

If $p \in R_f$ is a polynomial in y dividing fp', then p is normal. Clearly, p commutes with y and if $\rho \in \mathrm{Aut}(R_f)$ is such that $\rho(x) = x + h$, $h \in K[y]$, then

$$px = \rho(x)p = (x + h)p = xp + hp = px + \delta(p) + hp = px + fp' + hp. \tag{4.9}$$

Because p divides fp', we can choose h such that $hg = -fp'$. We show below that the set of such p multiplicatively generate all of the normal elements in R_f.

Corollary 4.4.5. *Normal elements in R_f are of the form $p \in K[y]$ such that p divides fp'.*

Proof. That such an element is normal follows from the above discussion. Write $p = \sum \gamma_{ij} y^i x^j$. We order terms according to degree lexicographic ($x > y$). Let ρ be the automorphism corresponding to p, with form given in Corollary 4.4.4. Then,

$$\begin{aligned}
\sum \gamma_{ij} y^i x^{j+1} &= \left(\sum \gamma_{ij} y^i x^j\right) x = px = \rho(x)p = (\lambda x + h)\left(\sum \gamma_{ij} y^i x^j\right) \\
&= \sum \gamma_{ij}\left(\lambda(xy^i)x^j + h(y^i x^j)\right) = \sum \gamma_{ij}\left(\lambda(y^i x + \delta(y^i))x^j + h(y^i x^j)\right) \\
&= \sum \gamma_{ij}\lambda y^i x^{j+1} + \sum \gamma_{ij}\left(\lambda i f + hy\right) y^{i-1} x^j.
\end{aligned}$$

Comparing terms of highest degree, we see that $\lambda = 1$. Once we show that $\deg_x(p) = 0$, then (4.9) implies that p divides fp'.

Assume, by way of contradiction, that $\deg_x(p) \neq 0$. Choose u, v maximal such that $\gamma_{uv} \neq 0$. By assumption, $v \neq 0$. This implies that we must have $\sum \gamma_{ij}(if + hy)y^{i-1}x^j = 0$, which forces $hy = -uf$. If there exists another pair u', v', $(u' \neq u)$, such that $\gamma_{u'v'} \neq 0$, then $hy = -u'f$, a contradiction. Hence, $p = y^u \sum \gamma_{uj} x^j$. Then,

$$\begin{aligned}
y^u \sum (\alpha y + \beta)\gamma_{uj} x^j = (\alpha y + \beta)p = \rho(y)p = py = y^u \sum \gamma_{uj}(x^j y) \\
= y^u \sum \gamma_{uj}\left(yx^j + \sum_{l=1}^{j} \binom{j}{l} \delta^l(y) x^{j-l}\right).
\end{aligned}$$

Thus,

$$\sum(\alpha y+\beta)\gamma_{uj}x^j=\sum\gamma_{uj}\left(yx^j+\sum_{l=1}^{j}\binom{j}{l}\delta^l(y)x^{j-l}\right). \tag{4.10}$$

Since $\gamma_{uv}\neq 0$, then by comparing coefficients of x^v we see that $\alpha=1$ and $\beta=0$. Thus, (4.10) reduces to

$$\sum\gamma_{uj}\sum_{l=1}^{j}\binom{j}{l}\delta^l(y)x^{j-l}=0.$$

This implies that $\delta^v(y)=0$, which occurs only if $f\in K$, a contradiction. $\square$

It has been observed by Bell [13] that the above generalizes to the case where $R=S[x;\delta]$ for any commutative ring S. In particular, normal elements are of the form $p\in S$ such that $p\mid\delta(p)$.

Suppose that for every $\phi\in\mathrm{Aut}(R_f)$ there exists $h\in K[y]$ such that $\phi=\phi_h$ where $\phi_h(x)=x+h$ and $\phi_h(y)=y$. Then $\mathrm{Aut}(R_f)$ is isomorphic to the abelian group $(K[y],+)$ via the map $h\mapsto\phi_h$. This is clear by observing $(\phi_{h_1}\circ\phi_{h_2})(x)=\phi_{h_1}(x+h_2)=x+h_1+h_2=\phi_{h_1+h_2}(x)$. This occurs when the only solution to (4.8) is the trivial one, i.e., when $\alpha=\lambda=1$ and $\beta=0$. This is the only case in which $\mathrm{Aut}(R_f)$ is abelian.

Proposition 4.4.6. *If* $\mathrm{Aut}(R_f)$ *is abelian, then* $\mathrm{Aut}(R_f)\cong(K[y],+)$.

Proof. We claim the only solution to (4.8) is the trivial one. Let $\phi\in\mathrm{Aut}(R_f)$ be arbitrary and write $\phi(x)=\lambda x+h$ and $\phi(y)=\alpha y+\beta$, with $\alpha,\lambda\in K^\times$, $\beta\in K$ and $h\in K[y]$. Let $\psi\in\mathrm{Aut}(R_f)$ be defined by $\psi(x)=x+y$ and $\psi(y)=y$. Then

$$\begin{aligned}(\phi\circ\psi)(x)&=\phi(x+y)=\lambda x+h+\alpha y+\beta,\\(\psi\circ\phi)(x)&=\psi(\lambda x+h)=\lambda(x+y)+h=\lambda x+\lambda y+h.\end{aligned}$$

Since $\mathrm{Aut}(R_f)$ is abelian, then $\beta=0$. Let ϕ be as before with $\beta=0$ and $\psi'\in\mathrm{Aut}(R_f)$ defined by $\psi'(x)=x+y+1$ and $\psi'(y)=y$. Then

$$\begin{aligned}(\phi\circ\psi')(x)&=\phi(x+y+1)=\lambda x+h+\alpha y+1,\\(\psi'\circ\phi)(x)&=\psi'(\lambda x+h)=\lambda(x+y+1)+h=\lambda x+\lambda y+\lambda+h.\end{aligned}$$

Since $\mathrm{Aut}(R_f)$ is abelian, then $\alpha=\lambda=1$. $\square$

Corollary 4.4.7. *The groups* $\mathrm{Aut}(\mathfrak{U})$, $\mathrm{Aut}(\mathcal{J})$, *and* $\mathrm{Aut}(\mathcal{J}_1)$ *are non-abelian.*

Proof. In each case, we require α, β and λ satisfying (4.8). For $\mathfrak{U}$, we have $\alpha\lambda y = \alpha y + \beta$. This gives $\beta = 0$ and $\lambda = 1$. Hence, automorphisms are of the form, $x \mapsto x + h, y \mapsto \alpha y$, $\alpha \in K^\times, h \in K[y]$. For $\mathcal{J}$, we require $\alpha\lambda y^2 = (\alpha y + \beta)^2 = \alpha^2 y^2 + 2\alpha\beta y + \beta^2$. Hence, $\beta = 0$ and $\lambda = \alpha$. Therefore, automorphisms are of the form, $x \mapsto \alpha x + h, y \mapsto \alpha y$, $\alpha \in K^\times, h \in K[y]$. For $\mathcal{J}_1$ we require $\alpha\lambda(y^2 + 1) = (\alpha y + \beta)^2 + 1 = \alpha^2 y^2 + 2\alpha\beta y + (\beta^2 + 1)$. This gives that $\alpha\beta = 0$ so $\beta = 0$ and $\alpha^2 = \alpha\lambda = 1$ so $\alpha = \lambda = \pm 1$. Therefore, automorphisms are of the form $x \mapsto \alpha x + h, y \mapsto \alpha y$, $\alpha = \pm 1, h \in K[y]$. In each case, there exist non-trivial solutions to (4.8). Thus, each automorphism group is non-abelian. $\square$

4.5 Classification

We now have the tools we need to prove Theorem 4.0.7.

Proof. Let A and A' be of the form (3.1) with defining matrices $M, M' \in \mathcal{M}_3(K)$, respectively. If $M \sim_{\mathrm{sf}} M'$, then $A \cong A'$. By Theorem 3.4.5 and Proposition 3.4.6, we need only show that there are no additional isomorphisms between the algebras in the present theorem.

The non-domains R_{yx}, R_{x^2}, R_{x^2-1} and $\mathcal{S}$ are all non-isomorphic by Proposition 4.1.7. The algebra with defining polynomial $x^2 - y$ is isomorphic to $K[x]$ via the map $x \mapsto x$ and $y \mapsto x^2$. It is one of only two commutative algebras considered (the other being $\mathcal{O}_1(K^2) \cong K[x, y]$) and is therefore distinct. By Corollary 4.4.3, the algebras $\mathfrak{U}$, $\mathcal{J}$ and $\mathcal{J}_1$ are all non-isomorphic.

That $\mathcal{O}_p(K^2) \cong \mathcal{O}_q(K^2)$ if and only if $p = q^{-1}$ was proved in Corollary 4.2.12. The corresponding result for the quantum Weyl algebras follows from Propositions 4.3.5 and 4.3.6. Recall the automorphisms groups of $\mathcal{O}_q(K^2)$ and $A_1^q(K)$ from Propositions 4.2.3 and 4.3.2, respectively. By counting subgroups of order four, it follows that $A_1^p(K) \ncong \mathcal{O}_q(K^2)$ for all $p, q \in K^\times$. In particular, $K^\times$ has one subgroup of order four and $(K^\times)^2$ has four. On the other hand, in $K^\times \rtimes \{\omega\}$ there are two subgroups of order four and in $(K^\times)^2 \rtimes \{\omega\}$ there are eight. Hence, $\mathcal{O}_p(K^2) \ncong A_1^q(K)$ for all $p, q \in K^\times$.

Let $S = \mathcal{O}_q(K^2)$ or $A_1^q(K)$, $q \neq 1$, and let $R = \mathfrak{U}, \mathcal{J}$, or $\mathcal{J}_1$. If $q = -1$ then x^2 is central so S is not primitive by [37], Proposition 3.2. On the other hand, R is primitive. If $q \neq \pm 1$, then $\mathrm{Aut}(S)$ is abelian, whereas $\mathrm{Aut}(R)$ is non-abelian by Corollary 4.4.7. $\square$

Our results can be summed up succinctly in the following theorem.

Theorem 4.5.1. *Let A and A' be of the form* (3.1) *with defining matrices $M, M' \in \mathcal{M}_3(K)$, respectively. If $M \sim_{sf} M'$, then $A \cong A'$. Conversely, if $A \cong A'$, then $M \sim_{sf} M'$ unless A and A' represent the forms of $\mathfrak{U}$ and $\mathfrak{V}$.*

Chapter 5

Homogenizations

In this chapter, we consider algebras of the form

$$H = K\langle x, y, z \mid xz - zx, yz - zy, f\rangle \tag{5.1}$$

where $f \in K\langle x, y, z\rangle$ is homogeneous of degree two and $f \notin K[z]$. By the commutation relations for z, it is not hard to see that any algebra of the form (5.1) is a homogenization of an algebra of the form (3.1). We invoke the methods used in Chapters 3 and 4 to classify these algebras up to isomorphism. We show, in particular, that isomorphism classes of these algebras are in 1-1 correspondence with sf-congruence conjugacy classes in $\mathcal{M}_3(K)$.

Theorem 5.0.2. *Let H be for the form* (5.1). *Then H is isomorphic to one of the following algebras, with one representative of f given in each case:*

$H(\mathcal{O}_q(K^2)), f = xy - qyx \ (q \in K^\times)$,	$H(A_1^q(K)), f = xy - qyx - z^2 \ (q \in K^\times)$,
$H(\mathcal{J}), f = yx - xy + y^2$,	$H(\mathcal{J}_1), f = yx - xy + y^2 + z^2$,
$H(\mathfrak{U}), f = yx - xy + yz$,	$H(\mathfrak{V}), f = yx - xy + y^2 + xz$,
$H(R_{x^2}), f = x^2$,	$H(R_{x^2-1}), f = x^2 - z^2$,
$H(R_{yx}), f = yx$,	$H(\mathcal{S}), f = yx - z^2$,
$H(K[x]), f = x^2 - yz$.	

Furthermore, the above algebras are pairwise non-isomorphic, except

$$H(\mathcal{O}_q(K^2)) \cong H(\mathcal{O}_{q^{-1}}(K^2)) \text{ and } H(A_1^q(K)) \cong H(A_1^{q^{-1}}(K)).$$

The key difference in this situation, versus that in the case of two-generated algebras, is that the algebras $H(\mathfrak{U})$ and $H(\mathfrak{V})$ are non-isomorphic (see Proposition

5.4.1). That $H(\mathcal{O}_p(K^2)) \cong H(\mathcal{O}_q(K^2))$ if and only if $p = q^{\pm 1}$ follows immediately from Theorem 4.2.11. We can further adapt those methods from Section 4.2.1 to prove $H(\mathcal{O}_p(K^2)) \not\cong H(A_1^q(K))$ for all $p, q \in K$ (Proposition 5.4.4) and $H(A_1^p(K)) \cong H(A_1^q(K))$ if and only if $p = q^{\pm 1}$ (Proposition 5.4.5).

As before, we study the representation theory of algebras of the form (5.1) by analyzing their prime and primitive ideals. In the case of the domains, we fully classify the prime ideals in Theorem 5.2.4. For the non-domains we achieve partial results.

Along the way, we also consider a class of differential operator rings which contains the algebras $H(\mathfrak{U})$, $H(\mathcal{J})$, and $H(\mathcal{J}_1)$. These algebras are of the form $H_f = K[y, z][x; \delta]$ where $\delta(z) = 0$ and $\delta(y) = f \in K[y, z]$. The properties of H_f mimic those of the differential operator rings considered in Chapter 4.

5.1 Standard-form congruence

We now show that sf-congruence can be used to classify all algebras of the form (5.1) up to *linear isomorphism*. The remainder then is devoted to showing that there are no additional isomorphisms between the algebras.

Suppose H is of the form (5.1). Then H can be represented by a triple of matrices, (X, Y, M), of the form

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix}, \tag{5.2}$$

with $\left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right) \neq 0$. This representation follows by letting $\vec{x} = \begin{pmatrix} x & y & z \end{pmatrix}$ so that

$$xz - zx = \vec{x}X\vec{x}^T, \quad yz - zy = \vec{x}Y\vec{x}^T, \quad f = \vec{x}M\vec{x}^T.$$

As in Chapter 3, the matrix M is not uniquely determined for f unless we fix a standard form for M. Since z is central, then the map sf defined in (3.4) is well-defined in this case.

Let H' be another algebra of form (5.1). Then H' is also defined by a triple, say (X, Y, M'). Suppose $\phi : H \to H'$ is a linear isomorphism and let $P \in \mathcal{P}_3$ be the matrix of ϕ. It is too much to ask that $P^TXP = X$ and $P^TYP = Y$. We can still hope to preserve those relations up to linear combination. The following proposition shows that sf-congruence preserves the commutation relations for z.

Proposition 5.1.1. *Let $P \in \mathcal{P}_3$ and let X, Y be as in* (5.2)*. The matrices X and Y are linear combinations of $P^T X P$ and $P^T Y P$.*

Proof. Write

$$P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let,

$$U = P^T X P = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1 & -a_2 & 0 \end{pmatrix} \text{ and } V = P^T Y P = \begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ -b_1 & -b_2 & 0 \end{pmatrix}.$$

We require $r, s \in K$ such that $rU + sV = X$. That is, $ra_1 + sb_1 = 1$ and $ra_2 + sb_2 = 0$. This system has a solution since $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0$. Similarly, there exist $r', s' \in K$ such that $r'U + s'V = Y$. Hence, $\mathcal{P}$ fixes the commutation relations for z. $\square$

Thus, there is no loss in referring to M as the *defining matrix* of H. Moreover, if $M \sim_{\text{sf}} M'$, then the triples (X, Y, M) and (X, Y, M') define isomorphic algebras. Therefore, the canonical forms from Table 3.1 apply here as well, and the algebras in that table correspond to their respective homogenizations.

5.2 Prime ideals of homogenizations

If A is 2-dimensional essentially regular, then A is a skew polynomial ring. By Lemma 2.3.12, $H(A)$ is as well. Moreover, we can regard each as a skew polynomial ring over $K[y, z]$ (or $K[x, z]$). In these cases we can completely determine the prime ideals. In case A is not essentially regular, it is possible to at least partially describe the prime ideals of H. The following is an immediate corollary of Proposition 2.3.5.

Corollary 5.2.1.

$$Z(H(\mathcal{J})) = Z(H(\mathcal{J}_1)) = Z(H(\mathfrak{U})) = Z(H(\mathfrak{V})) = K[z].$$

$$Z(H(\mathcal{O}_q(K^2))) = \begin{cases} K[x^n, y^n, z] & \textit{if } q \textit{ is a primitive } n\textit{th root of unity} \\ K[z] & \textit{otherwise.} \end{cases}$$

$$Z(H(A_1^q(K))) = \begin{cases} K[x^n, y^n, z] & \textit{if } q \neq 1 \textit{ is a primitive } n\textit{th root of unity} \\ K[z] & \textit{otherwise.} \end{cases}$$

We say an algebra H of the form (5.1) has *trivial center* in case $\mathcal{Z}(H) = K[z]$. Proving Theorem 5.0.2 is relatively painless in the cases that the algebra has trivial center. Most of our work will concentrate on the algebras with non-trivial center.

Lemma 5.2.2. *Suppose H is of the form* (5.1). *If P is a prime ideal of H with $P \cap K[z] \neq 0$, then P contains $z - \alpha$ for some $\alpha \in K$.*

Proof. Let $g \in P \cap K[z]$ be nonzero. If g is not irreducible in $K[z]$, then $g = g_1 g_2$ for some $g_1, g_2 \in K[z]$. Because $K[z]$ is central, then $g_1 H g_2 = g_1 g_2 H \subset P$. The primeness of P implies $g_1 \in P$ or $g_2 \in P$. Hence, P contains $az - b$ for some $a, b \in K$, $a \neq 0$, and so contains $a^{-1}(az - b) = z - a^{-1}b$. □

Before proceeding to the main theorem, we need one additional definition.

Definition 5.2.3. *Let J be an ideal in a ring R and $\sigma \in \mathrm{Aut}(R)$. Then J is **σ-cyclic** if $J = J_1 \cap \cdots \cap J_n$ where the J_i are distinct prime ideals of R such that $\sigma^{-1}(J_{i+1}) = J_i$ and $\sigma^{-1}(J_1) = J_n$.*

Let I be an ideal of a commutative ring R. The *radical* of I in R is

$$\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n\}.$$

It is not difficult to see that the radical of an ideal is again an ideal in R. The ideal I is said to be *primary* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for all $a, b \in R$ and some $n \in \mathbb{N}$. If I is primary, then $\sqrt{I}$ is prime.

Suppose H is a domain of form (5.1). If z is not a zero divisor in H, then we can localize at the set $C = K[z] \backslash \{0\}$. We refer to this ring as H_C.

Theorem 5.2.4. *Let H be a domain of the form* (5.1). *If P is a nonzero prime ideal in H, then one of the following holds:*

1. *$z \in P$ and P corresponds to a prime of $H/(z)$;*
2. *$z - \alpha \in P$, $\alpha \in K^\times$, and P corresponds to a prime of $H/(z - \alpha)$;*
3. *$xy - yx \in P$;*
4. *$P \cap K[y, z] = (g_1 \cdots g_n)$ where the g_i are irreducible polynomials in $K[y, z]$ disjoint from $K[z]$;*
5. *$P \cap K[x, z] = (g_1 \cdots g_n)$ where the g_i are irreducible polynomials in $K[x, z]$ disjoint from $K[z]$.*

Proof. First, suppose $P' = P \cap K[z] \neq 0$. Then P' is a prime ideal of $K[z]$ and so, by Lemma 5.2.2, $z - \alpha \in P$ for some $\alpha \in K$. Now assume $P \cap K[z] = 0$. In this case, P extends to a prime ideal in H_C. Let $R = K(z)[y]$, then R has Krull dimension 1. Let $I = P \cap R$. By [33], Theorem 7.2, one of the following must hold:

- H_C/P is commutative;
- I is σ-cyclic;
- I is primary with $\sigma(\sqrt{I}) = \sqrt{I}$.

If H_C/P is commutative, then $xy - yx \in P$. If I is σ-cyclic, then $I = P_1 \cap \cdots \cap P_n$ for distinct prime ideals of R. But the prime ideals of R are exactly extensions of prime ideals of $K[y, z]$ disjoint from $K[z]$. Therefore, $I = (g\sigma(g) \cdots \sigma^{n-1}(g))$ for some irreducible $g \in K[y, z]$, $g \notin K[z]$, $\sigma^n(g) = g$. Otherwise, I is primary. Since $\sqrt{I}$ is prime, then $\sqrt{I} = (g)$ where $g \in R$ is irreducible. We claim $I = (g^n)$. Because R is a principal ideal domain, $I = (h)$ for some $h \in R$. Write $h = tg^n$ where n is maximal such that g does not divide t. We claim t is a constant. Suppose otherwise, then $g^m \notin I$ for any $m > 0$. This contradicts $g \in \sqrt{I}$, and so the claim holds and $I = (g^n)$. For the remaining case, we need only observe that we can rewrite H as a skew polynomial ring with base ring $K[x, z]$ and repeat. $\square$

Note that the (4) and (5) occur if and only if H is PI. Thus, H PI implies $H = H(\mathcal{O}_q(K^2))$ or $H = H(A_1^q(K))$ with q a primitive root of unity. The last piece in classifying the ideals of a homogenization is the case in (4) and (5) where the intersection of P with the base ring is (0). In this case, it suffices to localize the base ring $K[y, z]$ to $Q = K(y, z)$ and consider $Q[x; \sigma, \delta]$, where σ and δ have been extended to Q in the usual way. Then we can appeal to Theorem 4.2.1. Thus, the prime ideals lying over (0) in Q are of the form (g) where $g \in K[x^n, y^n, z]$ is irreducible and such that $g \notin K[y^n, z]$ and $g \notin K[x^n, z]$.

It is worth commenting on case (3). If H is a domain of the form (5.1) with defining relation p, then $H/(xy - yx) \cong K[x, y, z]/(p_0)$ where p_0 is the image of p under the canonical map $K\langle x, y, z\rangle \to K[x, y, z]$. Thus, $(xy-yx)$ is prime in H if and only if p_0 is irreducible. In the case of $H(\mathcal{O}_q(K^2))$ we have $H(\mathcal{O}_q(K^2))/(xy - yx) \cong K[x, y, z]/(yx)$ and so $(xy - yx)$ is not prime. However, we do have that (x) and (y) are prime. On the other hand, $H(A_1^q(K))/(xy - yx) \cong K[x, y, z]/((1 - q)yx - z^2)$. Since $(1-q)yx-z^2$ is irreducible, then $xy-yx$ does indeed generate a prime ideal. In the case of the algebras H_f we have $H_f/(xy - yx) \cong K[x, y, z]/(f)$. Thus, $(xy - yx)$ is prime in H_f if and only if f is irreducible. It is clear that $H(\mathcal{J})$ and $H(\mathcal{J}_1)$ do not satisfy this condition while $H(\mathfrak{U})$ does, whence $H(\mathfrak{U})/(xy - yx) \cong K[x, z]$.

We can now identify the primitive ideals when H is a domain of the form (5.1). By Proposition 2.3.6, (0) is not primitive. On the other hand, $H/(z)$ and $H/(z - 1)$

are primitive rings by results in Chapter 4. Moreover, the maximal ideals in $H/(z)$ and $H/(z-1)$ correspond bijectively to maximal (and therefore primitive) ideals in H. The following is a consequence of Theorem 2.4.10, but now also follows from Theorem 5.2.4.

Proposition 5.2.5. *Let H be a domain of the form* (5.1). *If H is not PI, then all finite-dimensional simple modules are 1-dimensional.*

In the case that $H(A)$ is not a domain, it is not always possible to give a full description of the prime ideals. However, we can still recover some information.

Proposition 5.2.6. *The algebras $H(K[x]), H(R_{x^2}), H(R_{yx}), H(R_{x^2-1})$, and $H(\mathcal{S})$ are not domains.*

Proof. In the case of $H(R_{x^2})$ and $H(R_{yx})$, this is obvious. The result for $H(K[x])$ follows by Example 2.3.3. In $H(\mathcal{S})$,

$$y(xy - z^2) = yxy - yz^2 = yxy - z^2y = (yx - z^2)y = 0.$$

Finally, in $H(R_{x^2-1})$,

$$(x + z)(x - z) = x^2 - xz + zx - z^2 = x^2 - z^2 = 0.$$

□

By Proposition 2.3.6, the algebras $H(R_{x^2}), H(R_{x^2-1})$, and $H(\mathcal{S})$ are prime and $H(R_{yx})$ is not. A consequence of Proposition 5.2.6 is that $H(R_{yx})$ is not prime. It is now possible to give a partial description of the ideals in the non-domain case.

Proposition 5.2.7. *Let A be one of $R_{yx}, R_{x^2}, \mathcal{S}$ or R_{x^2-1} and let $H = H(A)$. If J is a nonzero prime in H then one of the following holds:*

1. *$z \in J$, so J corresponds to a prime of $H/(z)$;*
2. *$z - \alpha \in J$ for some $\alpha \in K^\times$, so J corresponds to a prime in $H/(z-1)$;*
3. *J is a prime of H_C disjoint from $K[z]$.*

We cannot give a complete list of the prime ideals in $H(K[x])$ as we have in the previous cases, however, by considering its factor algebras we can list many of them.

Proposition 5.2.8. *Let J be a prime of $H(K[x])$. The following are possibilities for J:*

1. *$z \in J$, so J corresponds to a prime of R_{x^2};*

2. *$z - \alpha \in J$ or $y - \alpha \in J$ for some $\alpha \in K^\times$, so J corresponds to a prime of $K[x]$;*

3. *$x \in J$, so J corresponds to a prime of $K[y, z]/(yz)$;*

4. *$x - \alpha \in J$ for some $\alpha \in K^\times$, so J corresponds to a prime of $K[z, z^{-1}]$.*

5.3 Certain differential operator rings

The algebras $H(\mathfrak{U})$, $H(\mathcal{J})$, and $H(\mathcal{J}_1)$ can all be expressed as differential operator rings $H_f = K[y, z][x; \delta]$ where $\delta(z) = 0$ and $\delta(y) = f \in K[y, z]$. We consider this class of algebras in general and show that its properties mirror those of $K[y][x; \delta]$. Thus, many of the results below are adaptations of those in [4]. Our hope is that these results will serve as a starting point for a study of the more general case where $\delta(z) \neq 0$.

The following proposition overlaps Corollary 5.2.1 in the case of $H(\mathfrak{U})$, $H(\mathcal{J})$, and $H(\mathcal{J}_1)$. However, it also applies to any algebra of the form H_f.

Proposition 5.3.1. *If $f \neq 0$, then $\mathcal{Z}(H_f) = K[z]$.*

Proof. It is clear that $K[z] \subset H_f$. We must show the opposite inclusion. Let $r \in H_f$ and suppose $\deg_y(r) \neq 0$. Write $r = \sum \alpha_{ij} z^i y^j$. Then

$$\begin{aligned} xr &= \sum \alpha_{ij} x(z^i y^j) = \sum \alpha_{ij} \left(z^i y^j x + \delta(z^i y^j)\right) \\ &= \sum \alpha_{ij} \left(z^i y^j x + j f z^i y^{j-1}\right) = rx + f \sum \alpha_{ij} j z^i y^{j-1}. \end{aligned}$$

Because H_f is a domain, then $r \in \mathcal{Z}(H_f)$ if and only if $\sum \alpha_{ij} j z^i y^{j-1} = 0$. □

Theorem 5.3.2. *H_f is isomorphic to H_g if and only if there exists $\alpha, \varepsilon, \lambda \in K^\times$ and $\beta, \gamma \in K$ such that $f(\alpha y + \beta z + \gamma, \varepsilon z) = \lambda \alpha g(y, z)$.*

Proof. Let X, Y, Z be the standard generators for H_f and x, y, z those for H_g. Let $\theta : H_f \to H_g$ be an isomorphism. By Proposition 5.3.1, $\theta(Z) \in K[z]$. The ideal generated by f contains all commutators $[a, b]$ with $a, b \in H_f$. Similarly for g in H_g. Hence, if $u, v \in H_g$, then there exists $r, s \in H_f$ such that $\theta(r) = u$ and $\theta(s) = v$. Then $uv - vu = \theta(rs - sr) \in \theta(fH_f) = \theta(f)H_g$. But $uv - vu \in gH_g$. By comparing degrees, there exists a unit $\varepsilon \in H_g$ such that $\theta(f) = \varepsilon g$. All units in H_g lie in K

and so $\varepsilon \in K^\times$. Suppose $\deg_x \theta(Y) \neq 0$. By considering the highest degree term in f we have $\deg_x \theta(f) \neq 0$. Since $\theta(f) = \varepsilon g$, then $\deg_x g \neq 0$, a contradiction.

Because θ is an isomorphism, there exists $t \in H_f$ such that $\theta(t) = x$. Write $t = \sum \alpha_{lij} Z^l Y^i X^j$. Suppose $\deg_x \theta(X) > 1$. Since $\theta(Y) \in K[y, z]$ and $\theta(Z) \in K[z]$, then $\deg_x \theta(t) > 1$, a contradiction. Hence, $\theta(X) = \lambda x + h$ for some $\lambda, h \in K[y, z]$. Because all units in $K[y, z]$ lie in K, this further implies that $\lambda \in K$.

Write $\theta(Y) = \sum_{i=0}^{n} z^i p_i$ where $p_i \in K[y]$ for all i. Since $\theta(f) = \varepsilon g$, then

$$\begin{aligned}
\varepsilon(xy - yx) &= \theta(XY - YX) \\
&= (\lambda x + h)\left(\sum_{i=0}^{n} z^i p_i\right) - \left(\sum_{i=0}^{n} z^i p_i\right)(\lambda x + h) \\
&= \sum_{i=0}^{n} z^i (x p_i - p_i x) = g \sum_{i=0}^{n} z^i p_i'.
\end{aligned}$$

Since $xy - yx = g$, then $p_i' = 0$ if $i > 1$. In particular, $\theta(Y) = \alpha y + \beta x + \gamma$ where $\alpha, \beta \in K^\times$ and $\gamma \in K$. Thus,

$$(\lambda x + h)(\alpha y + \beta x + \gamma) - (\alpha y + \beta x + \gamma)(\lambda x + h) = \alpha\lambda(xy - yx) = \varepsilon g(y, z),$$

so $\varepsilon = \alpha\lambda$. Moreover, $H_f \cong H_g$ only if $\deg(f) = \deg(g)$. $\square$

Corollary 5.3.3. *Automorphisms of H_f are of the form*

$$x \mapsto x + h, y \mapsto \alpha y + \beta z + \gamma, z \mapsto \varepsilon z, \tag{5.3}$$

$\alpha, \varepsilon, \lambda \in K^\times$, $\beta, \gamma \in K$, and $h \in K[y, z]$, such that

$$f(\alpha y + \beta z + \gamma, \varepsilon z) = \lambda\alpha g(y, z).$$

Corollary 5.3.4. *The automorphism groups for $H(\mathfrak{U})$, $H(\mathcal{J})$, and $H(\mathcal{J}_1)$ are described below in terms of* (5.3)*:*

$$\begin{aligned}
\operatorname{Aut}(H(\mathfrak{U})) &: \alpha = \varepsilon^{-1}, \beta = \gamma = 0; \\
\operatorname{Aut}(H(\mathcal{J})) &: \alpha = \lambda, \beta = \gamma = 0; \\
\operatorname{Aut}(H(\mathcal{J}_1)) &: \alpha, \varepsilon = \pm 1, \beta = \gamma = 0.
\end{aligned}$$

5.4 Classification

We divide the algebras into domains and non-domains. Our key result is the following.

Proposition 5.4.1. *Let A and $\hat{A}$ be essentially regular of dimension two with $Z(A) = Z(A') = K$. Let $H = H(A)$ and $H' = H(A')$, respectively. If $H \cong H'$ then one of the following holds:*

- $A \cong A'$ *and* $\mathrm{gr}(A) \cong \mathrm{gr}(A')$
- $A \cong \mathrm{gr}(A')$ *and* $A' \cong \mathrm{gr}(A)$.

Proof. By the hypothesis on the center of A, we have $\mathcal{Z}(H) = \mathcal{Z}(H') = K[z]$. Suppose $\Phi : H \to H'$ is the given isomorphism. Let I and J be the ideal in H generated by z and $z - 1$, respectively, and let $\Phi(I) = I', \Phi(J) = J'$. Since I and J are generated by a central element in H, then $I' \cap K[z] \neq 0$ and similarly for J'. By Proposition 5.2.2, $I' = (z - \alpha), J' = (z - \beta)$, $\alpha, \beta \in K$. This gives, $\Phi(H/I) \cong \Phi(H)/\Phi(I) \cong H'/I'$. Similarly, $\Phi(H/J) \cong H'/J'$. □

We observe that the previous proposition holds so long as the homogenizing element is regular in H and $\hat{H}$. We can now show that the non-domains are non-isomorphic.

Proposition 5.4.2. *The algebras $H(R_{x^2}), H(R_{x^2-1})$ and $H(K[x])$ have infinite GK dimension while $H(R_{yx})$ and $H(\mathcal{S})$ have GK dimension 3.*

Proof. By [53], Proposition 6.2.22, $\mathrm{gk}\, R[z] = 1 + \mathrm{gk}\, R$. This proves the result for $H(R_{x^2})$ and $H(R_{yx})$. Since R_{x^2} is a homomorphic image of $H(R_{x^2-1})$, then by [49], Proposition 8.2.2, we have $\mathrm{gk}\, H(R_{x^2-1}) \geq \mathrm{gk}\, R_{x^2} = \infty$.

We claim $H(\mathcal{S})$ has a K-algebra basis $\{x^i y^j z^k\}$. Let $m \in H(\mathcal{S})$ be a monomial. Since z is central, we may always write $m = m'z$. If yx appears in m' then we replace yx with z^2 and move it to the right. This proves the claim about the basis. Hence, $\mathrm{gk}\, H(\mathcal{S}) = \mathrm{gk}\, K[x, y, z] = 3$.

For $H(K[x])$, we establish a basis by first replacing x^2 with yz in every instance. If z appears in a monomial, it may be used to move any instance of y to the right of any instance of x. Finally, we move any power of z to the right. Hence, $H(K[x])$ contains two types of monomials: those that contain z and those that do not. The latter set is in 1-1 correspondence with the monomials of R_{x^2}. The monomials containing z may further be divided into two families: those that contain x and those that do not. Both sets are in 1-1 correspondence with the monomials of $K[y, z]$. Thus, we see readily that $\mathrm{gk}\, H(K[x]) = \mathrm{gk}\, R_{x^2} = \infty$. □

Proposition 5.4.3. *The algebras $H(R_{x^2})$, $H(R_{x^2-1})$, $H(R_{yx})$, $H(\mathcal{S})$, and $H(K[x])$ are non-isomorphic.*

Proof. The algebras $H(R_{x^2}), H(R_{x^2-1})$ and $H(\mathcal{S})$ are prime whereas $H(K[x])$ and $H(R_{yx})$ are not. However, by Proposition 5.4.2, $\mathrm{gk}(H(R_{yx})) = 2$ while $H(K[x])$ has infinite GK dimension. Similarly, $\mathrm{gk}(H(\mathcal{S})) = 2$ whereas

$$\mathrm{gk}(H(R_{x^2})) = \mathrm{gk}(H(R_{x^2-1})) = \infty.$$

It is easy to see that $\mathcal{Z}(R_{x^2}) = \mathcal{Z}(R_{x^2-1}) = K$. Since z is regular in both algebras, then Proposition 2.3.5 shows that $\mathcal{Z}(R_{x^2}) = \mathcal{Z}(R_{x^2-1}) = K[z]$. In both rings, the ideals $(z-\alpha)$ for $\alpha \in K$ are prime. Hence, $H(R_{x^2})$ is nonisomorphic to $H(R_{x^2-1})$ by Proposition 5.4.1. $\square$

For the domains, virtually all cases are covered by Proposition 5.4.1. The only remaining case involves isomorphisms between $H(\mathcal{O}_p(K^2))$ and $H(A_1^q(K))$. This is accomplished using the same tools as in the previous chapter. Specifically, Lemmas 4.2.5 and 4.2.6.

Proposition 5.4.4. *We have $H(\mathcal{O}_p(K^2)) \not\cong H(A_1^q(K))$ for all $p, q \in K^\times$.*

Proof. In case $p = 1$, $H(\mathcal{O}_p(K^2))$ is commutative so assume $p \neq 1$. Let $\{X, Y, Z\}$ be the generators for $H(\mathcal{O}_p(K^2))$ and $\{x, y, z\}$ those for $H(A_1^q(K))$. Suppose we have an isomorphism $\Phi : H(\mathcal{O}_p(K^2)) \to H(A_1^q(K))$. Write $\Phi(X) = \sum \alpha_{ijk} x^i y^j z^k$ and $\Phi(Y) = \sum \beta_{ijk} x^i y^j z^k$. By Lemma 4.2.6, $\alpha_{000} = \beta_{000} = 0$. Let $T = \Phi(X)\Phi(Y) - p\Phi(Y)\Phi(X)$ and let T_d be as in Section 4.2.1. Then

$$\begin{aligned}
T_2 = &(\alpha_{100}\beta_{100}x^2 + \alpha_{010}\beta_{010}y^2)(1-p) \\
&+ (\alpha_{001}\beta_{001}(1-p) + \alpha_{100}\beta_{010} - p\alpha_{010}\beta_{100})z^2 \\
&+ (\alpha_{100}\beta_{001} + \alpha_{001}\beta_{100})(1-p)xz + (\alpha_{010}\beta_{001} + \alpha_{001}\beta_{010})(1-p)yz \\
&+ (\alpha_{100}\beta_{010}(q-p) + \alpha_{010}\beta_{100}(1-pq))yx.
\end{aligned}$$

The coefficients of x^2 and y^2 must be zero, so we have four cases. We show that all the cases lead to a contradiction.

(1) ($\alpha_{100} = \alpha_{010} = 0$) By Lemma 4.2.5, $\alpha_{001} \neq 0$. Then the coefficient of z^2 reduces to $\alpha_{001}\beta_{001}(1-p) = 0$, so $\beta_{001} = 0$. Similarly, the coefficients of xz and yz give $\beta_{100} = 0$ and $\beta_{001} = 0$, respectively. This contradicts Lemma 4.2.5.

(2) ($\beta_{100} = \beta_{010} = 0$) This is similar to case (1).

(3) ($\alpha_{100} = \beta_{010} = 0$) The coefficient of xz and yz reduce to $\alpha_{001}\beta_{100}(1-p) = 0$ and $\alpha_{010}\beta_{001}(1-p) = 0$, respectively. If $\alpha_{010} = 0$ or $\beta_{100} = 0$, then we return to the previous cases. Hence, we may assume $\alpha_{001} = \beta_{001} = 0$. The coefficient of z^2 reduces to $-p\alpha_{010}\beta_{100}$, so $\alpha_{010} = 0$ or $\beta_{100} = 0$, a contradiction.

(4) ($\alpha_{010} = \beta_{100} = 0$) This is similar to case (3). $\square$

By Proposition 5.4.1, $H(A_1^p(K)) \cong H(A_1^q(K))$ if and only if $p = q^{\pm 1}$ when $p, q \in K^\times$ are not roots of unity or one. If $p, q \neq 1$ are roots of unity, then a method similar to the previous proposition may be applied. However, there is a slight hitch that makes the computations trickier.

Proposition 5.4.5. *Let $p, q \in K^\times$. Then $H(A_1^p(K)) \cong H(A_1^q(K))$ if and only if $p = q^{\pm 1}$.*

Proof. That $p = q^{\pm 1}$ implies $H(A_1^p(K)) \cong H(A_1^q(K))$ follows from Corollary 3.4.3. By the above discussion, we may assume $p, q \neq 1$. Let X, Y, Z be the standard generators of $H(A_1^p(K))$ and x, y, z those of $H(A_1^q(K))$. Let Φ be the given isomorphism and $T = \Phi(X)\Phi(Y) - p\Phi(Y)\Phi(X) - \Phi(Z)^2$ with T_d as before. Let

$$\Phi(X) = \sum \alpha_{ijk} x^i y^j z^k, \Phi(Y) = \sum \beta_{ijk} x^i y^j z^k, \text{ and } \Phi(Z) = \sum \gamma_{ijk} x^i y^j z^k.$$

Since $Z \in \mathcal{Z}(H(A_1^p(K)))$, then $\Phi(Z) \in \mathcal{Z}(H(A_1^q(K)))$. Thus, by Lemma 4.2.8, $\gamma_{100} = \gamma_{010} = 0$. Moreover, by Lemma 4.2.5,

$$a_{100}b_{010} - a_{010}b_{100} \neq 0. \tag{5.4}$$

We have $T_0 = \alpha_{000}\beta_{000}(1-p) - \gamma_{000}^2$, and so

$$\alpha_{000}\beta_{000}(1-p) = \gamma_{000}^2. \tag{5.5}$$

Then

$$\begin{aligned} T_1 = {} & (\alpha_{000}\beta_{100} + \alpha_{100}\beta_{000})(1-p)x + (\alpha_{000}\beta_{010} + \alpha_{010}\beta_{000})(1-p)y \\ & - ((1-p)(\alpha_{000}\beta_{001} + \alpha_{001}\beta_{000}) - 2\gamma_{000}\gamma_{001})z. \end{aligned}$$

Suppose $\beta_{000} = 0$. By (5.5), $\gamma_{000} = 0$. Moreover, T_1 reduces to $\alpha_{000}(\beta_{100}x + \beta_{010}y)(1-p) = 0$, and so $\alpha_{000} = 0$. A similar argument shows $\alpha_{000} = 0$ implies $\beta_{000} = \gamma_{000} = 0$.

Suppose $\alpha_{000}, \beta_{000} \neq 0$. If $\beta_{100} = 0$, then the coefficient of x in T_1 being zero implies $\alpha_{100} = 0$. This contradicts (5.4).

Our last step is to consider T_2. We have

$$\begin{aligned}T_2 = (1-p)(\alpha_{100}\beta_{100}x^2 + \alpha_{010}\beta_{010}y^2) \\ + \left[(1-p)\alpha_{001}\beta_{001} - \gamma_{001}^2 + \alpha_{100}\beta_{010} - p\alpha_{010}\beta_{100}\right] z^2 \\ + \left[\alpha_{100}\beta_{010}(1-qp) + \alpha_{010}\beta_{100}(p-q)\right] yx \\ + (1-p)(\alpha_{100}\beta_{001} + \alpha_{001}\beta_{100})xz + (1-p)(\alpha_{010}\beta_{001} + \alpha_{001}\beta_{010})yz.\end{aligned}$$

Thus, by considering the coefficients of x^2 and y^2 we arrive at two cases. In the first case, $\alpha_{100} = \beta_{010} = 0$. Then the coefficient of yx reduces to $\alpha_{010}\beta_{100}(p-q)$, and so $p = q$. The second case is that $\alpha_{010} = \beta_{100} = 0$ and then the coefficient of yx reduces to $\alpha_{100}\beta_{010}(1-qp)$ and so $p = q^{-1}$. □

Proof of Theorem 5.0.2. By Proposition 5.4.4, $H(\mathcal{O}_p(K^2)) \not\cong H(A_1^q(K))$ for all $p, q \in K^\times$. It follows from Theorem 4.2.11 that $H(\mathcal{O}_p(K^2)) \cong H(\mathcal{O}_q(K^2))$ if and only if $p = q^{\pm 1}$. Similarly, $H(A_1^p(K)) \cong H(A_1^q(K))$ if and only if $p = q^{\pm 1}$ by Proposition 5.4.5. The remainder is covered by Propositions 5.4.3 and 5.4.1 along with Theorem 4.0.7. □

We now summarize our results in a similar manner as to Theorem 4.5.1. The statement, however, is slightly more satisfying.

Theorem 5.4.6. *Let H and H' be of the form* (5.1) *with defining matrices $M, M' \in \mathcal{M}_3(K)$, respectively. Then $M \sim_{sf} M'$ if and only if $H \cong H'$.*

Chapter 6

Skew homogenizations

This chapter is dedicated to generalizing the notion of homogenization. An alternative method for extending A to a regular algebra is via a *skew homogenization*. Here, the homogenizing element x_0 is not assumed to be central but instead normal.

Definition 6.0.7. *Let $A = K\langle x_1, \dots, x_n \mid f_1, \dots, f_m\rangle$ and let $\hat{f}_j$ be the homogenizations of f_j for each $j \in \{1, \dots, m\}$. For each $i \in \{1, \dots, n\}$, let $q_i \in K^\times$. The algebra $H_{\mathbf{q}}(A)$ is said to be a* ***skew homogenization*** *of A if it is a K-algebra presented by the $n+1$ generators $x_0, x_1, \dots, x_n$ subject to the relations $\hat{f}_j$ as well as the additional relations $x_0x_i - q_ix_ix_0$ for all i. In case $q_i = 1$ for all i, then we recover the* ***homogenization*** *of A and, as usual, write $H(A)$.*

As with homogenizations, we can always recover $\mathrm{gr}(A)$ from $H_{\mathbf{q}}(A)$ via $\mathrm{gr}(A) \cong H_{\mathbf{q}}(A)/(x_0)$. However, we cannot recover A unless $H_{\mathbf{q}}(A) = H(A)$. We are interested in skew homogenizations which are regular. In order for this to be the case, the choice of the q_i cannot be made arbitrarily. In the case of $n = 3$, we employ Bergman's Diamond Lemma [19] to show that a given skew homogenization has a K-algebra basis equivalent to that of a polynomial ring in three variables.

Let $A = K\langle x_1, \dots, x_n\rangle/I$ where $I = (f_1, \dots, f_m)$. Let w_i be the leading word of f_i for each $i \in \{1, \dots, m\}$ according to some ordering. Let $w' = f_i - w_i$. If w is some other word containing w_i, so $w = uw_iv$, then the corresponding reduction is $w \mapsto uw'v$. A word w is *reduced* if it contains no w_i as a subword. An *overlap ambiguity* is a case when $w = uw_i = w_jv$ for leading words w_i, w_j with $u, v \neq 1$. The case when $w_i = uw_jv$ for leading words w_i, w_j is an *inclusion ambiguity*. In both cases, the ambiguity is *resolvable* if reducing in all possible ways leads to the same result. Otherwise, the difference of the results must be added as an additional relation in I and the process repeats.

Algebra A	Restrictions	$H_{\mathbf{q}}(A)$
$\mathcal{O}_q(K^2)$	none	$O_{p,q,r}$
$A_1^q(K)$	$r = p^{-1}$	$H_{p,q}$
$\mathcal{J}$	$r = p$	$H_q\mathcal{J}$
$\mathcal{J}_1$	$r = p = \pm 1$	$H'(\mathcal{J}_1)$
$\mathfrak{U}$	$r = 1$	$H_p\mathfrak{U}$
$yx - xy + y^2 + xz$	$r = p = 1$	$H(\mathfrak{V})$

Table 6.1: Restrictions on $\mathbf{q}$ for skew homogenizations

Theorem 6.0.8 (Bergman's Diamond Lemma)**.** *The set of reduced words corresponding to $(f_1, \ldots, f_m)$ is a K-basis for A if all overlap and inclusion ambiguities are resolvable.*

For A essentially regular of dimension two, we determine all restrictions on the parameters q_i. Let $H_{\mathbf{q}}(A)$ have generators x, y, z subject to the relations $zx - rxz$, $zy - pyz$, and the homogenized relation $\hat{f}$. There are no inclusion ambiguities to check, so we check the only overlap ambiguity $z(yx) = (zy)x$. The restrictions on r and p in each case are listed in Table 6.1

If H is connected and $\mathbb{N}$-graded algebra, then a *twisting system* on H is a set of graded K-algebra automorphisms of H, $\tau = \{\tau_n \mid n \in \mathbb{N}\}$, such that $\tau_n(a\tau_m(b)) = \tau_n(a)\tau_{n+m}(b)$, for all $n, m, l \in \mathbb{N}$, $a \in H_m$, $b \in H_l$.

Definition 6.0.9. *Let τ be a twisting system of an algebra H. The* ***Zhang twist*** *H^τ of H has the same K-algebra basis as H but a new multiplication $*$ given by*

$$a * b = a\tau_n(b) \text{ for all } a \in H_n, b \in H_l.$$

By [61], Theorem 1.3, H^τ is regular if and only if H is regular. By [7], Theorem 8.16, regular skew homogenization is a Zhang twist of a regular homogenization.

Theorem 6.0.10 (Zhang)**.** *Let A and B be two $\mathbb{N}$-graded algebras. If B is isomorphic to a Zhang twist of A, then the category of graded A-modules is isomorphic to the category of graded B-modules.*

Thus, to study the graded module structure of a skew homogenization, it is sufficient to determine to which homogenization it is twisted-equivalent.

Example 6.0.11. *Let H be a skew homogenization of $A_1^q(K)$. Define a Zhang twist* *on H via $\tau_1(x) = p^{-1}x$, $\tau_1(y) = py$, and $\tau_1(z) = z$. Then we have*

$$\begin{aligned} z * x - p^{-1}x * z &= z\tau_1(x) - p^{-1}x\tau_1(z) = p^{-1}(zx - xz) \\ z * y - py * z &= z\tau_1(y) - py\tau_1(z) = p(zy - yz) \\ x * y - qy * x - z * z &= x\tau_1(y) - qy\tau_1(x) - z\tau_1(z) = p(xy - qp^{-2}yx - p^{-1}z^2). \end{aligned}$$

Let H^τ denote this new algebra. Then H^τ has defining relations

$$zx - xz, zy - yz, xy - qp^{-2}yx - p^{-1}z^2.$$

Define $\sigma \in \mathrm{Aut}(H^\tau)$ by $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = p^{1/2}z$. It is then clear that $H^\tau \cong H(A_1^{qp^{-2}}(K))$. Thus, $H^\tau \cong H(A_1)$ if and only if $q = p^2$.

A similar computation shows that $O_{p,q,r}$ is twisted-equivalent to $H(\mathcal{O}_{pqr^{-1}}(K^2))$. In all other cases of A 2-dimensional essentially regular we have that $H_{\mathbf{q}}(A)$ is twisted-equivalent to $H(A)$. Hence, in terms of module structure, there is little to say about these skew homogenizations that has not been said previously in this work. However, the direct study of these skew homogenizations leads to two generalizations which are worth considering.

The algebras $H_p\mathfrak{U}$, $H_q\mathcal{J}$ and $H'(\mathcal{J}_1)$ may be considered as differential operator rings over the quantum plane. In particular, we fix throughout the notation $R = K[z][y;\tau]$ with $\tau \in \mathrm{Aut}(K[z])$ defined by $\tau(z) = pz$. Technically, this is the ring $\mathcal{O}_{p^{-1}}(K^2)$, but it is isomorphic to $\mathcal{O}_p(K^2)$ by Corollary 4.2.12. Let $C = \{y^iz^j \mid i, j \in \mathbb{N}\} \subset R$ and denote by R_C the localization of R at C. If $H = R[x;\sigma,\delta]$ is a skew polynomial ring over R, then denote the localization of H at C by $H_C = R_C[x;\sigma,\delta]$, where σ and δ are extended to R_C. We consider two situations here. The first is the case in which $\sigma = \mathrm{id}_R$ and $\delta(y) \in K[y,z]$ (Section 6.1). The second is the case in which $\sigma(y) = y$, $\sigma(z) = \tau^{-1}(z)$, and $\delta(z) = 0$ (Section 6.2).

We finish this chapter with a modified skew homogenization of the quantum Weyl algebra. In the defining relation $xy - qyx - z^2$ we replace z^2 with a single degree two element z. These skew homogenizations are of particular interest because they can be regarded as two-parameter analogs of the Heisenberg enveloping algebras. We give conditions for such an algebra to have a distinguished central element which gives rise to a primitive factor ring. These primitive factors rings are analogous to the Hayashi-Weyl algebra [30].

6.1 Certain differential operator rings over quantum planes

We consider generalizations of the algebra $H_p\mathfrak{U}$. Suppose $H = R[x;\delta]$ with $\delta(z) = 0$ and $\delta(z) \in R$. If $\delta(y) \in K[z]$, then we are in the situation of [35]. In the case of $H_p\mathfrak{U}$, $\delta(y) = yz$.

Let $\delta(y) = \sum \alpha_{ij} y^i z^j$. Then we have

$$0 = \delta(zy - pyz) = z\delta(y) - p\delta(y)z = z\sum_{i,j} \alpha_{ij} y^i z^j - p\sum_{i,j} \alpha_{ij}(y^i z^j)z$$
$$= \sum_{i,j} \alpha_{ij}(zy^i)z^j - p\sum_{i,j} \alpha_{ij} y^i z^{j+1} = \sum_{i,j} \alpha_{ij}(p^i - p)y^i z^{j+1}.$$

If p is not a root of unity, then $\alpha_{ij} = 0$ whenever $i \neq 1$. Thus, in this case, $\delta(y) = yf(z)$ for some $f(z) \in K[z]$. If p is a primitive nth root of unity, $p \neq 1$, then $\alpha_{ij} = 0$ whenever $n \nmid i - 1$. Then $\delta(y) = y\sum y^{in} f_i(z)$ where $f_i(z) \in K[z]$ for each i.

We can now describe the prime ideals of H which have nonzero intersection with R. It is clear that $H/(y) \cong K[x,z]$ and $H/(z) \cong K[x,y]$. Hence, if p is not a root of unity and P is a prime ideal of H with $P \cap R \neq 0$, then by (4.1) P corresponds to a prime of the polynomial ring in two variables.

Our analysis of this case and the general case in the next section will depend heavily on determining when δ is an inner derivation.

Proposition 6.1.1. *Keep notation as above. If p is not a root of unity, then the derivation δ is inner if and only if $f(z)$ has no constant term. If p is a primitive nth root of unity, then δ is inner if and only if z does not appear to a power dividing n in $\delta(y)$.*

Proof. In both cases, the sufficient condition can be checked easily by choosing an appropriate element d such that $\delta(y) = dy - yd$ and $\delta(z) = dz - zd$. We prove that the indicated conditions are necessary. For an arbitrary element $d = \sum_{i,j} \alpha_{ij} y^i z^j \; inR$, we have

$$dy - yd = \sum_{i,j} \alpha_{ij} y^i (z^j y) - \sum_{i,j} \alpha_{ij} y^{i+1} z^j = y\sum_{i,j} \alpha_{ij}(p^j - 1)y^i z^j. \tag{6.1}$$

Suppose p is not a root of unity, then $\delta(y) = yf(z)$. Because H is a domain, then (6.1) implies $f(z) = \sum_{i,j} \alpha_{ij}(p^j - 1)y^i z^j$. The right-hand side lies in $K[z]$ only if $\alpha_{ij} = 0$ whenever $i \neq 0$. Moreover, the term on the right-hand side is 0 when $(i,j) = (0,0)$. Thus, we must have that $f(z)$ has zero constant term.

The case of p a primitive nth root of unity is similar. In this case, $\alpha_{ij} = 0$ whenever $i \mid n$. Moreover, if $j \mid n$, then $p^{-j} - 1 = 0$. Thus, we must have that z does not appear to a power dividing n in $\delta(y)$. □

Suppose p is not a root of unity. If $f(z)$ has a constant term, then δ is not inner and there are no additional primes by the above proposition. In particular, this holds if $f(z) = 1$. Otherwise, $H_C = R_C[x - d]$ where $d = \sum \alpha_j (p^j - 1)^{-1} z^j$. Now there is no loss in passing to the full quotient ring Q of R. Let $\mathcal{H} = Q[x - d]$. Since $Z(Q) = K$, then $Z(\mathcal{H}) = K[x - d]$. By Theorem 4.2.1, the primes of $\mathcal{H}$ which are disjoint from Q are in 1-1 correspondence with the primes of $K[x - d]$.

If p is a primitive nth root of unity, the ideals (y) and (z) remain prime. Denote the ideal $(y^n - a, z^n - b)$ in R by $I_{a,b}$. If $I_{a,b}$ is δ-invariant, then it extends to an ideal of H. Let $I = I_{a,b}$ for some fixed choice of $a, b \in K^\times$. If $r \in I$, then $r = r_1(y^n - a) + r_2(z^n - b)$ for some $r_1, r_2 \in H$ and

$$\delta(r) = \delta(r_1)(y^n - a) + r_1\delta(y^n - a) + \delta(r_2)(z^n - b) + r_2\delta(z^n - b) \equiv r_1\delta(y^n) \bmod I.$$

Hence, I is δ-prime if and only if $\delta(y^n) \in I$. Then

$$\delta(y) = y \sum y^{in} f_i(z) \equiv y \sum a^i f_i(z) \bmod I.$$

Let $f(z) = \sum a^i f_i(z)$. Then $\delta(y) = yf(z) \bmod I$ and the powers of z appearing in $f(z)$ are all relatively prime to n.

Proposition 6.1.2. *Keep the notation as above. Suppose p is a primitive nth root of unity. Then*

$$\delta(y^k) = y^k \sum_{i=0}^{k-1} f(p^i z).$$

Proof. The case of $k = 0$ follows from the discussion above. Assume this holds for some k. Then

$$\begin{aligned}\delta(y^{k+1}) = \delta(y^k)y + y^k\delta(y) &= \left[y^k \sum_{i=0}^{k-1} f(p^i z)\right] y + y^{k+1} f(z) \\ &= y^k \left[\sum_{i=0}^{k-1} f(p^i z)y\right] + y^{k+1} f(z) = y^{k+1} \left[\sum_{i=0}^{k-1} f(p^{i+1} z)\right] + y^{k+1} f(z) \\ &= y^{k+1} \left[\sum_{i=1}^{k} f(p^i z)\right] + y^{k+1} f(z) = y^{k+1} \sum_{i=0}^{k} f(p^i z).\end{aligned}$$

□

As an immediate consequence,

$$\delta(y^n) = y^n \sum_{i=0}^{n-1} f(p^i z) \equiv b \sum_{i=0}^{n-1} f(p^i z) \bmod I.$$

Write $f(z) = \sum_{j=0}^{m} \alpha_j z^j$ where $\alpha_j = 0$ if $j \equiv 0 \bmod n$. Let $[k]_p$ be the *p-number*,

$$[k]_p = \frac{p^k - 1}{p - 1} = 1 + p + \cdots + p^{k-1}.$$

Note that $[n]_p = 0$ when p is an nth root of unity. We then have

$$\delta(y^n) \equiv \sum_{i=0}^{n-1} f(p^i z) \bmod I \equiv n\alpha_0 + \sum_{j=1}^{m} [n]_p \alpha_j z^j \bmod I \equiv n\alpha_0 \bmod I.$$

Thus, $\delta(y^n) \equiv 0 \bmod I$ if and only if $\alpha_0 = 0$. We have now proved the following proposition.

Proposition 6.1.3. *If p is a primitive nth root of unity and $a, b \in K^\times$, then $I_{a,b}$ is δ-prime if and only if $\alpha_0 = 0$. Thus, in $H_p\mathfrak{U}$, $I_{a,b}$ extends to a prime ideal.*

Proof. The first statement follows by the above discussion. Let $H = H_p\mathfrak{U}$ and $I = I_{a,b}$ for some $a, b \in K$. By Proposition 6.1.3, I is a δ-prime in R. Then $H/IH = (R/I)[x; \delta]$, where δ is extended to R/I. Since I is prime in R, then R/I is prime and so H/IH is a skew polynomial extension of a prime ring. Hence, IH is prime. $\square$

There are two more things to consider in the root of unity case. One is, given an ideal I as above, what ideals lie over I? The other question is which ideals lie over (0)? The answer to the first question is surprisingly similar to Proposition 6.1.1. Suppose $I_{a,b}$ is δ-prime in R and let $\hat{H} = (R/I)[\hat{y}; \hat{\delta}]$. Now R/I is a simple ring and so by [49], Theorem 1.8.4, $\hat{H}$ is simple if and only if $\hat{\delta}$ is not an inner derivation on R/I. A computation similar to Proposition 6.1.1 shows the following.

Proposition 6.1.4. *Let p be a primitive nth root of unity and $I = I_{a,b}$ for some $a, b \in K^\times$. Write $\delta(y) \equiv yf(z) \bmod I$. The derivation $\hat{\delta}$ is inner on R/I if and only if $f(z)$ has no constant term.*

Proof. Let $f(z) = \sum_{j=0}^{n-1} \alpha_j z^j$ and let $d = \sum_{j=0}^{n-1} (p^{-j} - 1)^{-1} \alpha_j z^j$. One checks that $0 = \hat{\delta}(z) = dz - zd$ and $yf(z) = \hat{\delta}(y) = dy - yd$. $\square$

If $\hat{\delta}$ is inner on R/I, then $H \cong (R/I)[\hat{x} - d]$. Since $\mathcal{Z}(R/I) = K$, then by Theorem 4.2.1, the primes lying over I are in 1-1 correspondence with the primes of $K[\hat{x} - d]$.

It's left to consider those primes lying over (0) in R. Let Q the quotient division ring of R and let $\hat{H} = Q[x; \delta]$. Since Q is a simple ring (and a $\mathbb{Q}$-algebra), then by [49], Theorem 1.8.4, $\hat{H}$ is simple if and only if δ is not an inner derivation on Q. We now give a criterion for this when p is a primitive root of unity.

Proposition 6.1.5. *Let p be a nonzero primitive nth root of unity, $n > 1$. If P is a prime of $H_p\mathfrak{U}$, then one of the following holds:*

1. *$y \in P$ and P corresponds to a prime of $K[x, z]$;*

2. *$z \in P$ and P corresponds to a prime of $K[x, y]$;*

3. *$(x - (p-1)^{-1}z - \alpha) \in P$, $\alpha \in K$, and P corresponds to a prime of $\mathcal{O}_p(K^2)$.*

Proof. Keep notation as above. Suppose p is not a root of unity. By Proposition 6.1.1, δ is inner on R and so the primes of H lying over (0) in R are in 1-1 correspondence with the primes of $K[x - d]$ where $d = (p-1)^{-1}z$, not including $(x - d)$. Hence, the primes are generated by $\Omega_\alpha = x - (p-1)^{-1}z - \alpha$ with $\alpha \in K^\times$. Modulo the relation $\Omega_\alpha = 0$ we have

$$\begin{aligned} 0 = xy - yx + yz &= ((p-1)^{-1}z + \alpha)y - y((p-1)^{-1}z + \alpha) + yz \\ &= (p-1)^{-1}(yz - zy) + yz = (p-1)^{-1}(yz - zy + pyz - yz) \\ &= -(p-1)^{-1}(zy - pyz). \end{aligned}$$

Thus, $H_p\mathfrak{U}/\Omega_\alpha \cong \mathcal{O}_p(K^2)$.

Suppose p is a primitive nth root of unity. Proposition 6.1.1 implies that δ is inner on the full quotient ring $Q(R)$ and so we have the ideals in (4). There is an additional possibility that $P \cap R = I_{a,b}$ and P corresponds to a prime of $(R/I_{a,b})[x-d]$ as described above. It is easily checked that these ideals are exactly those in (4). □

6.2 Skew polynomial rings over quantum planes

We considered the mixed case $H = R[x; \sigma, \delta]$ in one situation where $\sigma(y) = y$, $\sigma(z) = \tau^{-1}(z)$, and $\delta(z) = 0$. In the case that $\delta(y) \in K[z]$, we are in the situation of [35]. Our methods include algebras such at $H'\mathcal{J}_1$. In H we have

$$0 = \delta(zy - pyz) = \sigma(z)\delta(y) + \delta(z)y - p(\sigma(y)\delta(z) + \delta(y)z) = p^{-1}z\delta(y) - p\delta(y)z.$$

Hence, we have the criterion that $z\delta(y) = p^2\delta(y)z$. This holds, in particular, in the algebras $H_q\mathcal{J}$ and $H'\mathcal{J}_1$. The defining relations are

$$zx - pxz = zy - pyz = yx - xy + \delta(y) = 0,$$

and in the case of $H'\mathcal{J}_1$, $p = \pm 1$. Let $\delta(y) = \sum \alpha_{ij} y^i z^j$. Then

$$0 = z\delta(y) - p^2\delta(y)z = \sum \alpha_{ij}(p^i - p^2) y^i z^{j+1}. \tag{6.2}$$

If p is not a root of unity, then (6.2) implies $\alpha_{ij} = 0$ whenever $i \neq 2$. Thus, in this case, $\delta(y) = y^2 f(z)$ for some $f(z) \in K[z]$. If p is a primitive nth root of unity, $p \neq \pm 1$, then (6.2) implies $\alpha_{ij} = 0$ whenever $n \nmid i - 1$. Thus, $\delta(y) = y^2 \sum y^{in} f_i(z)$ where $f_i(z) \in K[z]$ for each i.

It is clear that z generates a prime ideal and $H/(z) \cong K[x][y; d]$ where $d = \delta|_{z=0}$. If $p \neq \pm 1$, then y generates a prime ideal and $H/(y) \cong \mathcal{O}_p(K^2)$ with generators x and z. The automorphism σ is inner on H_C and hence, $H_C = R_C[y^{-1}x; y^{-1}\delta]$. To see this, we need only observe that

$$y(y)y^{-1} = y = \sigma(y) \text{ and } y(z)y^{-1} = p^{-1}z = \sigma(z). \tag{6.3}$$

From this point we can appeal to our results from the previous section.

Proposition 6.2.1. *If P is a nonzero prime of $H_q\mathcal{J}$, then either $z \in P$ and P corresponds to a prime ideal of $\mathcal{J}$ or else $y \in P$ and P corresponds to a prime ideal of $\mathcal{O}_p(K^2)$.*

Proof. Suppose p is not a primitive nth root of unity. We need only show that there are no ideals lying over (0) in R_C. In H_C we have $\hat{\delta}(y) = y^{-1}\delta(y) = y$. By Proposition 6.1.1, $\hat{\delta}$ is not inner on R_C and so there are no additional ideals lying over (0).

Now let p be a primitive nth root of unity, $n > 1$, and let $I = I_{a,b}$ for some $a, b \in K^\times$, as before. Since $\hat{\delta}(y) = y \bmod I$, then I is not $\hat{\delta}$-prime by Proposition 6.1.3. Let $Q = Q(R)$. By an abuse of notation, let $\hat{\delta}$ be the extended derivation on Q. Since $\hat{\delta}(y) = y$, then by Proposition 6.1.1, $\hat{\delta}$ is not inner on Q. Hence, there are no primes lying over (0) in Q. $\square$

Let $H = H'\mathcal{J}_1$ and note that (6.3) still holds, but it will be convenient for us to flip y and y^{-1}. There is no problem in doing this since $p = p^{-1} = -1$. Hence, $H_C = R_C[yx; \hat{\delta}]$ where $\hat{\delta} = y\delta$.

Proposition 6.2.2. *If P is a nonzero prime of $H'\mathcal{J}_1$, then one of the following holds:*

1. *$z \in P$ and P corresponds to a prime ideal of $\mathcal{J}$;*
2. *$P = (y^2 - a, z^2 + a)H$ with $a \in K^\times$;*
3. *$P = (y^2 - a, z^2 + a, yx - b)H$ with $a, b \in K^\times$.*

Proof. Let $I = (y^2 - a, z^2 - b)$ in H. By Proposition 6.1.3, I is prime if and only if $\delta(y^n) \in I$. Observe that $\delta(y^2) = y^2(y^2 + z^2) \equiv b(a + b) \bmod I$. Hence, I is prime if and only if $a = -b$. Since $\delta(y) \equiv 0 \bmod I$, then H_C/I corresponds to the polynomial ring $R_C/I[\hat{x}] = R_C/I[yx]$. Let $Q = Q(R)$. Since z^2 appears in $\hat{\delta}(y)$, then $\hat{\delta}$ is not inner on Q. Hence, there are no primes lying over 0 in R. □

6.3 Skew homogenizations of quantum Weyl algebras

Of particular interest are skew homogenization quantum Weyl algebras. Define

$$H_{p,q} = K\langle x, y, z \mid zx - p^{-1}xz, zy - pyz, xy - qyx - z\rangle, \; p, q \in K^\times. \tag{6.4}$$

Assigning degree one to the generators x and y and degree two to z gives $H_{p,q}$ the form of a graded algebra. We have shifted from our usual convention so that our results align with those of [38]. In particular, Kirkman and Small studied the algebra $H_{p,q}$ in the case $p = q^{-1}$. These algebras may be considered as a two-parameter analog of the Heisenberg enveloping algebra.

Define the (p, q)-*number* to be

$$[n]_{p,q} = \sum_{i=0}^{n} q^i p^{-(n-i)} = (q^n - p^{-n})/(q - p^{-1}). \tag{6.5}$$

Note that $[n]_{p,q} = 0$ if p and q are both primitive roots of unity and both orders divide n.

For the skew polynomial construction, we take the base ring to be $R = \mathcal{O}_{p^{-1}}(K^2)$ with generators y, z and commutation given by $yz = p^{-1}zy$. Then $H_{p,q} \cong R[x; \sigma, \delta]$ where σ is the R-automorphism given by $\sigma(y) = qy, \sigma(z) = pz$ and δ is a σ-derivation on R given by $\delta(y) = z, \delta(z) = 0$.

Though we do not include them here, $H_{p,q}$ can also be constructed as an ambiskew polynomial ring, a generalized Weyl algebra, and it is isomorphic to a subalgebra of a two-parameter analog of $U(\mathfrak{sl}_3)$. The reader is referred to [25] for details.

Lemma 6.3.1. *In $H_{p,q}$ the following identites hold for $n > 0$,*

$$x^n y = q^n y x^n + [n]_{p,q} z x^{n-1}, \tag{6.6}$$

$$x y^n = q^n y^n x + [n]_{p,q} y^{n-1} z. \tag{6.7}$$

Proof. We prove (6.6) and leave (6.7) for the reader. The statement for $n = 1$ is clear from (6.4). Assume true for $n = k$. For $n = k+1$ we have

$$\begin{aligned} x^{k+1}y &= x(q^k y x^k + [k]_{p,q} z x^{k-1}) = q^k(qyx + z)x^k + [k]_{p,q}(pzx)x^{k-1}) \\ &= q^{k+1} y x^{k+1} + (q^k + p[k]_{p,q}) z x^k = q^{k+1} y x^{k+1} + [k+1]_{p,q} z x^k. \end{aligned}$$

□

The ideal (z) is prime since $H_{p,q}/(z) \cong \mathcal{O}_q(K^2)$. The element $\theta = (1-qp^{-1})xy - z$ is normal in $H_{p,q}$ and $H_{p,q}/(\theta) \cong \mathcal{O}_p(K^2)$. Thus, (θ) is also a prime ideal in $H_{p,q}$. In the special case that $p = q$, $(z) = (\theta)$.

Proposition 6.3.2. *The algebra $H_{p,q}$ is PI if and only if p and q are primitive roots of unity (not necessarily of the same order).*

Proof. If H is PI, then so is every factor ring. However, $\mathcal{O}_q(K^2)$ is only PI if q is a root of unity. Similarly for p and $\mathcal{O}_p(K^2)$.

Suppose p is a primitive nth root of unity and q a primitive mth root of unity. Let $W = K[x^{mn}, y^{mn}, z^n]$. We claim $W \subset Z(H_{p,q})$. Once shown, the result follows easily. That z^n commutes with x and y is clear and similarly that x^{mn} and y^{mn} commute with z. It remains to show that x^{mn} and y^{mn} commute with y and x, respectively. This follows easily from (6.6) and (6.7). □

Let P be a prime ideal of H. As we have seen, if $z \in P$, then P corresponds to a prime ideal of $\mathcal{O}_q(K^2)$. Let $H = H_{p,q}$ and let H_C be as in the introduction to this chapter.

Proposition 6.3.3. *If $p \neq q$, then δ is an inner σ-derivation on H_C.*

Proof. Let $\beta = (1 - qp^{-1})^{-1}$ and $t = \beta zy^{-1}$. It suffices to show that $0 = \delta(z) = tz - \sigma(z)t$ and $z = \delta(x) = tx - \sigma(x)t$.

$$(\beta zy^{-1})z - \sigma(z)(\beta zy^{-1}) = \beta z(y^{-1}z - py^{-1}z) = 0,$$
$$(\beta zy^{-1})y - \sigma(y)(\beta zy^{-1}) = \beta(z - q(yz)y^{-1}) = \beta(1 - qp^{-1})z = z.$$

□

Let $\Theta = x - t$ with $t = \beta zy^{-1}$ as above. Then $H_C \cong R_C[\Theta; \sigma]$. If P is a prime of H_C and $\Theta \in P$, then $\theta = \beta\Theta y \in P \cap H$. Hence, the primes in H_C containing Θ are in 1-1 correspondence with the primes of H containing θ. These in turn are in 1-1 correspondence with the primes of $\mathcal{O}_p(K^2)$.

Proposition 6.3.4. *$H_C[\Theta^{-1}]$ is a simple ring if and only if there do not exist integers r, s such that $q^r = p^s$.*

Proof. One readily checks that $\Theta z = pz\Theta$ and $\Theta y = qy\Theta$ so that $H_C[\Theta^{-1}]$ is a quantum 3-space. The result now follows from [48], Proposition 1.3. □

Corollary 6.3.5. *If there do not exist $r, s \in \mathbb{Z}$ such that $q^r = p^s$, then H is a primitive ring and so $\mathcal{Z}(H) = K$.*

Proof. Since the only nonzero primes of H are (z) and (θ), then (0) is locally closed and therefore primitive. The second result now follows by [37], Proposition 3.2. □

Suppose there exist $r, s \in \mathbb{Z}$ such that $q^r = p^s$, then it will follow that σ^r is an inner automorphism. Let $a = z^{-s}y^r$. Then

$$a^{-1}ya = y^{-r}z^s yz^{-s}y^r = p^s y^{-r}(yz^s)z^{-s}y^r = q^r y = \sigma^r(y),$$
$$a^{-1}za = y^{-r}z^s zz^{-s}y^r = p^r y^{-r}z^s z^{-s}(y^r z) = p^r z = \sigma^r(z).$$

Thus, $H_C = R_C[a(x-t)^r]$. Let $\Omega = a(x-t)^r$. The primes lying over (0) in R are in 1-1 correspondence with the ideals of $\mathcal{Z}(R)[\Omega]$. These ideals are generated by $\Omega - \alpha$ for $\alpha \in K^\times$. We have now proved the following.

Theorem 6.3.6. *Let $p, q \in K^\times$ be nonroots of unity. Suppose there exist $r, s \in \mathbb{Z}$ such that $p^r = q^s$. For all $\alpha \in K^\times$, $\mathcal{A} = H_{p,q}/(\Omega - \alpha)$ is a simple noetherian domain of GK-dimension two and global dimension one.*

Proof. Most of this follows by the discussion above. Since the ideal $P = (\Omega - \alpha)$ is maximal in $H_{p,q}$, then $\mathcal{A}$ is a simple domain. Since P is generated by a central nonzero divisor, then the GK dimension of $\mathcal{A}$ is at most two. A ring of GK dimension one is necessarily PI. Hence, if p is not a root of unity, then $\text{gk}(\mathcal{A}) = 2$.

That $\text{gld}(\mathcal{A}) = 1$ follows analogously to the Weyl algebra. If $r = 1$, then this follows by [12], Theorem 1.6. As above, $H_C \cong R_C[\Theta; \sigma]$. By [48], Corollary 3.10, $\text{gld}(R_C) = 1$. Thus, by [49], Theorem 7.5.3, $\text{gld}(H_C) = 2$. Observe that $\mathcal{A}$ is free as a $K[x]$-module. Define $B_1 = \mathcal{A} \otimes_{K[x]} K(x)$ and $B_2 = \mathcal{A} \otimes_{K[y]} K(y)$. Then $B_1 \cong B_2$. Moreover, $B_1 \cong H_C/(\Omega - 1)H_C$. The element $\Omega - 1$ is central and regular, so $\text{gld}(B_1) = 1$. Let $B = B_1 \oplus B_2$. Since B is a faithfully free $\mathcal{A}$-module, then $\text{gld}(\mathcal{A}) \leq \text{gld}(B) = \max\{\text{gld}(B_1), \text{gld}(B_2)\} = 1$. $\square$

The algebra $\mathcal{A}$ may be regarded as a two-parameter version of the Hayashi-Weyl algebra. In particular, one can show furthermore that $\mathcal{A}$ is primitive and therefore a 'good analog' of the Weyl algebra. More details on this algebra may be found in [25] and, for an alternative approach, in [15].

We have deferred until now the case $H_q = H_{q,q}$. This algebra was studied in [38], Section 2. The authors showed that when q is not a root of unity, the algebra H_q is primitive, regular, and every nonzero prime ideal contains z. Our approach will arrive at the same result but also include the root of unity case.

Proposition 6.3.7. *The automorphism σ is inner on R_C.*

Proof. If $a = yz^{-1}$, then

$$\begin{aligned} a^{-1}ya &= zy^{-1}yyz^{-1} = (zy)z^{-1} = (pyz)z^{-1} = qy = \sigma(y), \\ a^{-1}za &= zy^{-1}zyz^{-1} = zy^{-1}(pyz)z^{-1} = pz = \sigma(z). \end{aligned}$$

$\square$

Let $\hat{\delta} = a\delta$. By Proposition 6.3.7, $H_C = R_C[ax; \hat{\delta}]$. By [29], Lemma 3.21, every prime ideal of H_C intersects R_C in a δ-stable prime ideal. In the case of q not a root of unity, R_C is a simple ring so we need only consider those ideals that intersect R_C in zero. Let P be a nonzero prime ideal of H_C with $P \cap R_C = 0$. Such an ideal exists only if $\hat{\delta}$ is inner on R_C.

Proposition 6.3.8. *The derivation $\hat{\delta}$ is not inner on R_C.*

Proof. Write $d = \sum \alpha_{ij} y^i z^j$. Then

$$y = az = \hat{\delta}(y) = dy - yd = \sum \alpha_{ij} y^i (z^j y) - \sum \alpha_{ij} y^{i+1} z^j$$
$$= \sum \alpha_{ij} p^j y^{i+1} z^j - \sum \alpha_{ij} y^{i+1} z^j = \sum \alpha_{ij} (p^j - 1) y^{i+1} z^j.$$

In order to have equality, we need the coefficient $(p^0 - 1)\alpha_{0,0} = 1$, but that is absurd. □

In case q is a primitive nth root of unity, it is left to check that the ideals $(y^n - a, z^n - b)$ are not $\hat{\delta}$-invariant. Since $\hat{\delta}(y) = y$, then clearly $\hat{\delta}(y^n) = y$. So $\hat{\delta}(y^n - a) = y$ and thus, when $n \neq 1$, these ideals are not $\hat{\delta}$-invariant.

6.3.1 Isomorphisms and automorphisms

Now that we have a handle on the prime ideals of $H_{p,q}$, it is reasonable to ask when two such algebras are isomorphic.

Proposition 6.3.9. *Let $p, q \in K^\times$. Then $H_{p,q} \cong H_{p^{-1},q^{-1}}$ and $H_{p,q} \cong H_{q,p}$.*

Proof. Let x, y, z be the standard generators for $H_{p,q}$ and X, Y, Z those for $H_{p^{-1},q^{-1}}$. Define a rule $\Phi : H_{p^{-1},q^{-1}} \to H_{p,q}$ by $X \mapsto y, Y \mapsto x, Z \mapsto -qz$. This rule satisfies the defining relations for $H_{p,q}$,

$$0 = \Phi(X)\Phi(Z) - p\Phi(Z)\Phi(X) = -q(yz - pzy),$$
$$0 = \Phi(Z)\Phi(Y) - p\Phi(Y)\Phi(Z) = -q(zx - pxz),$$
$$0 = \Phi(X)\Phi(Y) - q\Phi(Y)\Phi(X) - \Phi(Z) = yx - qxy + qz = -q(xy - q^{-1}yx - z).$$

Thus, Φ extends to an algebra homomorphism. It is clearly bijective and therefore an isomorphism.

Now let X, Y, Z be the generators for $H_{q,p}$. Define a rule $\Phi : H_{q,p} \to H_{p,q}$ by $X \mapsto x$, $Y \mapsto y$ and $Z \mapsto -q^{-1}p\theta = q^{-1}p[z - (1 - qp^{-1})xy]$. We check that the defining relations for $H_{p,q}$ are satisfied,

$$\begin{aligned}
0 &= \Phi(X)\Phi(Z) - q\Phi(Z)\Phi(X) \\
&= q^{-1}p\left(x[z-(1-qp^{-1})xy] - q[z-(1-qp^{-1})xy]x\right) \\
&= q^{-1}p\left(xz-(1-qp^{-1})x(qyx+z) - qzx + q(1-qp^{-1})xyx\right) \\
&= q^{-1}p(qp^{-1}xz - qzx) = xz - pzx, \\
0 &= \Phi(Z)\Phi(Y) - q\Phi(Y)\Phi(Z) \\
&= q^{-1}p\left([z-(1-qp^{-1})xy]y - qy[z-(1-qp^{-1})xy]\right) \\
&= q^{-1}p\left(zy-(1-qp^{-1})(qyx+z)y - qyz + q(1-qp^{-1})yxy\right) \\
&= q^{-1}p(qp^{-1}zy - qyz) = zy - pyz, \\
0 &= \Phi(X)\Phi(Y) - p\Phi(Y)\Phi(X) - \Phi(Z) = xy - pyx - q^{-1}p[z-(1-qp^{-1})xy] \\
&= xy - pyx - q^{-1}pz + q^{-1}pxy - xy = q^{-1}p(xy - qyx - z).
\end{aligned}$$

As before, Φ extends to an algebra homomorphism which is bijective and therefore an isomorphism. □

Proposition 6.3.10. *Let $\Psi : H_{p,q} \cong H_{p',q'}$ be an isomorphism and all parameters are not roots of unity. Then either $p = p^{\pm 1}$ and $q = q^{\pm 1}$ or else $p = q^{\pm 1}$ and $q = p^{\pm 1}$.*

Proof. Let P be a height one prime ideal of $H_{p,q}$. Then P is a principal ideal and if we consider only the case where P is generated by a noncentral element in P there are two possibilities: either $P = (z)$ or $P = (\theta)$. Thus, either Ψ fixes those two ideals or else it interchanges them. In the first case, the restriction of Ψ to $H_{p,q}/P$ induces an isomorphism $\mathcal{O}_q(K^2) \to \mathcal{O}_{q'}(K^2)$, so $q' = q^{\pm 1}$. In the second case, restricting Φ to $H_{p,q}/P$ induces an isomorphism $\mathcal{O}_p(K^2) \to \mathcal{O}_{p'}(K^2)$, so $p' = p^{\pm 1}$. Thus, $H_{p,q} \cong H_{p^{\pm 1},q^{\pm 1}}$. □

We can now consider automorphisms of the algebras $H_{p,q}$. Note that the map interchanging the ideals (z) and (θ) is an automorphism if and only if $p = q$. The automorphism group in the Kirkman-Small case was considered in in [3], wherein the authors prove that the automorphism group is $K^\times \rtimes \{\omega\}$ when the parameter is not a root of unity. In case there is no relation between the parameters, the automorphism group is smaller. This is yet another example of "quantum rigidity".

Proposition 6.3.11. *If $p, q \in K^\times$ and there do not exist $r, s \in \mathbb{Z}$ such that $p^r = q^s$, then*

$$\operatorname{Aut}(H_{p,q}) \cong K^\times.$$

Proof. Let $\rho \in \mathrm{Aut}(H_{p,q})$. By the above discussion, we can assume ρ fixes the two height one ideals, (z) and (θ). Then $\rho(z) = \varepsilon z$ and $\rho(\theta) = \lambda\theta$ for some $\varepsilon, \lambda \in K^{\times}$. Thus,

$$\begin{aligned} \rho(\theta) &= \lambda((1 - qp^{-1})xy - z) \\ (1 - pq^{-1})\rho(x)\rho(y) - \varepsilon(z) &= \lambda(1 - qp^{-1})xy - \lambda z \\ \rho(x)\rho(y) &= \lambda xy - (1 - qp^{-1})^{-1}(\varepsilon - \lambda)z. \end{aligned} \tag{6.8}$$

Thus, the total degree of $\rho(y)\rho(x)$ is two. In light of Lemma 4.2.5, we write, $\rho(x) = a_1x + a_2y + a_3$ and $\rho(y) = b_1x + b_2y + b_3$, with $a_i, b_i \in K$ for all $i \in \{1, 2, 3\}$. Then,

$$\begin{aligned} T &= \rho(x)\rho(y) \\ &= a_1b_1x^2 + a_2b_2y^2 + a_1b_2xy + a_2b_1yx + (a_1b_3 + a_3b_1)x + (a_2b_3 + a_3b_2)y + a_3b_3. \end{aligned}$$

In T we must have $a_3b_3 = 0$. If $a_3 = 0$, then the coefficient of x in T becomes $a_1b_3 = 0$ and the coefficient of y becomes $a_2b_3 = 0$. Since $a_1b_2 - a_2b_1 \neq 0$, then we cannot simultaneously have $a_1 = a_2 = 0$. Thus, $b_3 = 0$. Similarly, if we assume $b_3 = 0$, then we see that $a_3 = 0$. Thus, $a_3 = b_3 = 0$ and T reduces to

$$T = a_1b_1x^2 + a_2b_2y^2 + a_1b_2xy + a_2b_1yx.$$

Since the coefficients of x^2 and y^2 must be zero, we can now consider two cases.

Case 1 ($a_2 = b_1 = 0$) In this case, $T = a_1b_2xy$. Then (6.8) implies that $\lambda = a_1b_2$ and $\varepsilon = \lambda$.

Case 2 ($a_1 = b_2 = 0$) In this case, T reduces to $a_2b_1yx = a_2b_1q^{-1}(xy - z)$. Then (6.8) implies that $a_2b_1q^{-1} = \lambda$ and $a_2b_1q^{-1} = (1 - qp^{-1})^{-1}(\varepsilon - \lambda)$. Thus $\varepsilon - \lambda = (1 - qp^{-1})\lambda$ and so $\varepsilon = (2 - qp^{-1})\lambda$. □

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CURRICULUM VITAE

Education

- 2009 – 2013, PhD in Mathematics, University of Wisconsin-Milwaukee (UWM)

  Dissertation Title: PBW deformations of Artin-Schelter regular algebras and their homogenizations

  Advisor: Dr. Allen Bell

- 2007–2009, MS in Mathematics, UWM

- 1999–2003, BA in Journalism and Mathematics, Indiana University Bloomington

Selected Talks

Analogs of the harmonic oscillator problem

- (2013) University of Wisconsin-Madison Lie Theory Seminar, Madison, WI
- (2012) UWM Mathematics Colloquium, Milwaukee, WI

PBW deformations of Artin-Schelter regular algebras

- (2013) AMS Central Sectional Meeting, Iowa State University, Ames, IA
- (2012) Joint Mathematics Meeting, San Diego, CA

Noncommutative geometric algebras

- (2012) Midwest Algebraic Geometry Graduate Conference, University of Wisconsin-Madison, Madison, WI

Homogenizations and skew homogenizations of Artin-Schelter regular algebras

- (2012) DePaul Algebra Seminar, DePaul University, Chicago, IL

Essentially regular algebras

- (2012) Algebra, Combinatorics, Dynamics, and Applications, Queen's College, Belfast, Northern Ireland, UK
- (2012) Denison Conference 31, The Ohio State University, Columbus, OH

Honors

- (2013) AMS Sectional Meeting Graduate Student Travel Grant: Funds to travel to the AMS sectional meeting at Iowa State University
- (2013) JMM Graduate Student Travel Grant: Funds to travel to the annual meeting in San Diego, CA
- (2012) UWM Graduate Student Travel Award: Funds to present research at conferences
- (2011, 2012) Mark Lawrence Teply Award: Recognition of research potential
- (2010) Ernst Schwandt Teaching Award: Recognition of outstanding teaching
- (2008-2013) UWM Chancellor's Graduate Award: Award to retain graduate students with exceptional academic records and high promise of future success