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Category O Representations of the Lie Superalgebra osp(3,2)

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CATEGORY $\mathcal{O}$ REPRESENTATIONS OF THE LIE SUPERALGEBRA $\mathfrak{osp}(3, 2)$

by

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ABSTRACT

CATEGORY $\mathcal{O}$ REPRESENTATIONS OF THE LIE SUPERALGEBRA $\mathfrak{osp}(3,2)$

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Under the Supervision of Professor Ian M. Musson

In his seminal 1977 paper [Kac77], V. G. Kac classified the finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero. However, over thirty years later, the representation theory of these algebras is still not completely understood, nor is the structure of their enveloping algebras.

In this thesis, we consider a low-dimensional example, $\mathfrak{osp}(3,2)$. We compute the composition factors and Jantzen filtrations of Verma modules over $\mathfrak{osp}(3,2)$ in a variety of cases.

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CHAPTER 1

INTRODUCTION

The purpose of this thesis is to describe all category $\mathcal{O}$ representations of the Lie superalgebra $\mathfrak{osp}(3,2)$. This chapter provides an introduction to Lie superalgebras and a concrete description of $\mathfrak{osp}(3,2)$ as a matrix algebra. Chapter 2 includes several results from the representation theory of Lie superalgebras that will be used in the sequel. Chapters 3 to 5 describe the Verma modules over $\mathfrak{osp}(3,2)$ under various conditions on the highest weight.

Throughout, we work over a field which is algebraically closed of characteristic 0; for notational simplicity, $\mathbb{C}$.

We assume some familiarity with the study of Lie algebras, which much of this material generalizes. This introduction is adapted from [Mus12, Ch. 1-6].

1.1 LIE SUPERALGEBRAS

Lie superalgebras are a generalization of the well-studied Lie algebra (for a good treatment, see [Hum78]). However, while Lie algebras are typically developed initally the tangent spaces to Lie groups, Lie superalgebras are here developed axiomatically, independent of the underlying Lie supergroups. (Lie supergroups are group objects in the category of supermanifolds, as Lie groups are group objects in the category of manifolds. For a treatment of Lie supergroups, see for example [BV91].)

1.1.1 Axiomatic Development

A *Lie superalgebra* $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space (over a field $\mathbb{C}$) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with a bilinear bracket $[\cdot,\cdot]$. An element $x \in \mathfrak{g}_0 \cup \mathfrak{g}_1$ is called *homogeneous*, and $|x| = i$ if $x \in \mathfrak{g}_i$ is called the *degree* of x. By using the notation $|x|$, we implicitly assume that x is homogenous. For two homogeneous elements x and y, the notation $\operatorname{sgn}(x,y) = (-1)^{|x||y|}$ simplifies the following defintions greatly. The bracket satisfies the following axioms for homogeneous elements, extended to $\mathfrak{g}$ via bilinearity.

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, where addition is mod 2 ($\mathbb{Z}_2$ grading)
- $[x, y] + \mathrm{sgn}(x, y)[y, x] = 0$ (graded skew-symmetry)
- $\mathrm{sgn}(x, z)[x, [y, z]] + \mathrm{sgn}(x, y)[y, [z, x]] + \mathrm{sgn}(y, z)[z, [x, y]] = 0$ (graded Jacobi identity)

It is easy to see that with these axioms, $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$ module, where the action is the adjoint action.

We make extensive use of the $\mathbb{Z}_2$ grading. We shall refer to $\mathfrak{g}_0$ as the *even part* of $\mathfrak{g}$, and say that $x \in \mathfrak{g}_0$ is an *even element* of $\mathfrak{g}$. Similarly, $\mathfrak{g}_1$ is the *odd part* of $\mathfrak{g}$, and $x \in \mathfrak{g}_1$ is an *odd element*. Further, for $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, set $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ for $i = 0, 1$.

We say that a Lie superalgebra is *simple* if it is not abelian and the only $\mathbb{Z}_2$-graded ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$. We say that $\mathfrak{g}$ is *classical simple* if $\mathfrak{g}$ is simple and $\mathfrak{g}_1$ is a completely reducible $\mathfrak{g}_0$-module.

Assumption 1. *In the remainder, we make the assumption that $\mathfrak{g}_0$ is reductive and $\mathfrak{g}_1$ is semisimple as a $\mathfrak{g}_0$ module. (Both of these assumptions apply to our main focus of study, $\mathfrak{osp}(3, 2)$, described in section 1.6.) This implies, among other things, that if $\mathfrak{g}$ is simple, it is classical simple ([Mus12, Thm. 1.2.9]).*

1.1.2 Matrix Superalgebras

It is also easy to see from the definition that given a $\mathbb{Z}_2$-graded associative algebra $A = A_0 \oplus A_1$ we can construct a Lie superalgebra by defining $[x, y] = xy - \mathrm{sgn}(x, y)yx$ for homogeneous elements and extending via bilinearity. Thus, as in the Lie algebra case, we construct some initial examples of Lie superalgebras as matrix algebras. The easiest of these to describe is the general Lie superalgebra

$$\mathfrak{gl}(m, n) = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] : \begin{array}{c} A \in M_{m\times m},\ D \in M_{n\times n} \\ B \in M_{m\times n},\ C \in M_{n\times m} \end{array} \right\}$$

$$\mathfrak{gl}(m,n)_0 = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right] : \ A \in M_{m\times m}, \ D \in M_{n\times n} \right\} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$$
$$\mathfrak{gl}(m,n)_1 = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right] : \ B \in M_{m\times n}, \ C \in M_{n\times m} \right\},$$

where $M_{m\times n}$ denotes the algebra of $m \times n$ matrices. As in the Lie algebra case, further matrix superalgebras are obtained as subalgebras of $\mathfrak{gl}(m,n)$ which preserve bilinear forms.

1.2 CARTAN SUBALGEBRAS AND ROOT SPACES

It is useful to describe the root space of a Lie superalgebra. We begin with a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ (recall that $\mathfrak{g}_0$ is a Lie algebra). For $\alpha \in \mathfrak{h}_0^*$, set

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}_0\}$$

and let

$$\Delta = \{\alpha \in \mathfrak{h}_0^* : \alpha \neq 0, \mathfrak{g}^\alpha \neq 0\}$$

be the set of *roots* of $\mathfrak{g}$. Since the action of $\mathfrak{h}_0$ on any finite-dimensional simple $\mathfrak{g}_0$-module is diagonalizable, and $\mathfrak{g}_1$ is semisimple (by Assumption 1), the action of $\mathfrak{h}_0$ is also diagonalizable. Thus we have a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha \tag{1.2.1}$$

where $\mathfrak{h} = \mathfrak{g}^0$ is the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$.

Several properties of the roots of Lie algebras carry over to the superalgebra case, including the following important result.

Lemma 1.2.1 ([Mus12, Lem. 2.1.1])**.** *If* $\mathfrak{g}$ *is a classical simple Lie superalgebra and* $\alpha, \beta, \alpha+\beta$ *are roots of* $\mathfrak{g}$*, then* $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$*.*

The usual corollary, that if $\alpha + \beta \neq 0$ is not a root then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$, also holds.

1.2.1 Types of Roots

It is unsurprising to find that Δ is also $\mathbb{Z}_2$ graded. We define Δ_0 to be the roots of $\mathfrak{g}_0$, and $\Delta_1 = \Delta \backslash \Delta_0$. Elements of Δ_0 are called *even roots*, while elements of Δ_1 are called *odd roots*.

Unlike the Lie algebra case, it is possible that $\alpha, 2\alpha$ are roots for some $\alpha \in \mathfrak{h}^*$. If this occurs, then $\alpha \in \Delta_1$, but odd roots of this kind behave differently from other odd roots (most importantly with respect to reflection). If $\alpha \in \Delta_1$ is an odd root such that 2α is a root, we say that α is *odd non-isotropic*. Otherwise, we say α is *isotropic* (or *odd isotropic*).

In addition to the two sets above, we sometimes wish to consider the sets

$$\overline{\Delta_0} = \{\alpha \in \Delta : \alpha \neq 2\beta \text{ for any } \beta \in \Delta_1\}$$
$$\overline{\Delta_1} = \{\text{odd isotropic roots}\}.$$

1.3 BOREL AND PARABOLIC SUBALGEBRAS

A subalgebra $\mathfrak{b}$ of a Lie superalgebra $\mathfrak{g}$ is a *Borel subalgebra* if

- $\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_0$,
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+$ nilpotent in $\mathfrak{g}$, and
- $\mathfrak{b}$ is maximal with these properties.

In the Lie algebra case, all Borel subalgebras are conjugate. However, in the superalgebra case, instead we have the following theorem.

Theorem 1.3.1 ([Mus12, Thm. 3.1.2])**.** *If $\mathfrak{g}$ is a classical simple Lie superalgebra, then there are only a finite number of conjugacy classes of Borel subalgebras under the action of* $\operatorname{Aut}\mathfrak{g}$.

This creates some complications in the representation theory, in particular, as regards highest weights and highest weight vectors (see chapter 2). We consider a representative of each conjugacy class; among these, we call one (one representative of one class) *distinguished*. Each Borel subalgebra corresponds to a basis of simple roots. If $\mathfrak{b}$ and $\mathfrak{b}'$ are Borel subalgebras of $\mathfrak{g}$, we say that $\mathfrak{b}$ and $\mathfrak{b}'$ are *adjacent* if

- $\mathfrak{b}_0 = \mathfrak{b}'_0$, and
- $\mathfrak{b}_1 \cap \mathfrak{b}'_1$ is codimension one in both $\mathfrak{b}_1$ and $\mathfrak{b}'_1$.

In this case, there is an odd root α of $\mathfrak{g}$ such that

$$\mathfrak{g}^{\alpha} \subseteq \mathfrak{b} \quad \text{and} \quad \mathfrak{g}^{-\alpha} \subseteq \mathfrak{b}'.$$

Note that in this situation, α must be isotropic. Adjacent pairs of Borel algebras turn out to be extemely useful, since for $\mathfrak{g} \neq \mathfrak{p}(n)$ or $\mathfrak{psl}(2,2)$, the following holds.

Theorem 1.3.2 ([Mus12, Thm. 3.1.3])**.** *If $\mathfrak{b}$ and $\mathfrak{b}'$ are Borel subalgebras of $\mathfrak{g}$ such that $\mathfrak{b}_0 = \mathfrak{b}'_0$, then there is a sequence*

$$\mathfrak{b} = \mathfrak{b}^{(0)}, \mathfrak{b}^{(1)}, \ldots, \mathfrak{b}^{(m)} = \mathfrak{b}'$$

of Borel subalgebras such that $\mathfrak{b}^{(i-1)}$ and $\mathfrak{b}^{(i)}$ are adjacent for $1 \leqslant i \leqslant m$.

A Borel subalgebra $\mathfrak{b}$ has a nilpotent complement $\mathfrak{n}^-$ such that

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Such a decomposition is called a *triangular decomposition*.

Hereafter, we may refer to a Borel subalgebra simply as a Borel.

1.3.1 Parabolic Subalgebras

At times, it is helpful to work with a superset of a Borel subalgebra $\mathfrak{b}$, a so-called *parabolic subalgebra*. Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, and let $\mathfrak{r}$ be the complement of $\mathfrak{p}$. Then there exists a partition $\Delta = \Delta^{\mathfrak{r}} \dot{\cup} \Delta^{\mathfrak{p}}$, where $\Delta^{\mathfrak{r}} \subsetneq \Delta^-$ and $\Delta^{\mathfrak{p}} \supsetneq \Delta^+$, such that

$$\mathfrak{r} = \bigoplus_{\alpha \in \Delta^{\mathfrak{r}}} \mathfrak{g}^{\alpha} \qquad \mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^{\mathfrak{p}}} \mathfrak{g}^{\alpha}.$$

The subalgebra $\mathfrak{p}$ is called a parabolic subalgebra.

1.4 WEYL GROUP

The Weyl group $\mathcal{W}$ of a Lie superalgebra $\mathfrak{g}$ is the Weyl group of the underlying Lie superalgebra $\mathfrak{g}_0$. Hereafter we denote the reflection in the hyperplane orthogonal to a root α by σ_α. Note that if $\alpha = 2\beta$, where β is an odd non-isotropic root, we will often use the notation σ_β rather than σ_α.

In computation, we define a bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$. If α is a non-isotropic root, $\beta \in \mathfrak{h}^*$, define

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

To simplify this notation somewhat, we define $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, and write

$$\sigma_\alpha(\beta) = \beta - (\beta,\alpha^\vee)\alpha.$$

(Note that if $\beta = n\alpha$, $\beta^\vee = \frac{1}{n}\alpha^\vee$.)

1.4.1 The Dot Action

When studying modules, it will be useful to consider a shifted action of the Weyl group. We define

$$\rho = \tfrac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha - \tfrac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha.$$

Note that ρ depends on the choice of positive roots, and therefore on the choice of Borel. When we wish to make the choice of Borel explicit, we may write $\rho(\mathfrak{b})$. We now define the dot action of $\mathcal{W}$ on $\mathfrak{h}^*$ by

$$\sigma_\alpha.\lambda = \sigma_\alpha(\lambda+\rho) - \rho.$$

1.4.2 Odd Reflections

It is sometimes useful to consider an analog of reflections for roots $\alpha \in \overline{\Delta_1}$. Suppose B is a basis of simple roots for $\mathfrak{g}$ and $\alpha \in B \cap \overline{\Delta_1}$. Then for any $\beta \in B$ we define a root $r_\alpha(\beta)$ by

$$r_\alpha(\beta) = \begin{cases} -\alpha & \text{if } \beta = \alpha \\ \alpha + \beta & \text{if } \alpha + \beta \text{ is a root} \\ \beta & \text{otherwise} \end{cases} .$$

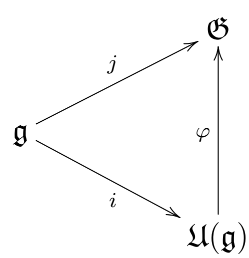
Then $r_\alpha(B) = \{r_\alpha(\beta) : \beta \in B\}$ is a basis of simple roots, said to be ([Ser11]) obtained from B by the *odd reflection* r_α.

1.5 ENVELOPING ALGEBRAS AND THE PBW THEOREM

Generalizing the Lie algebra case, we embed a Lie superalgebra $\mathfrak{g}$ in an associative $\mathbb{Z}_2$-graded algebra $\mathfrak{G}$, so that the multiplication respects the bracket, that is, for x, y homogeneous in $\mathfrak{g}$,

$$xy - \operatorname{sgn}(x, y)yx = [x, y],$$

where the multiplication takes place in $\mathfrak{G}$. Such an algebra is called an *enveloping algebra*, and the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ is universal among such algebras, in the sense that for any enveloping algebra $\mathfrak{G}$, with morphisms $i : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ and $j : \mathfrak{g} \to \mathfrak{G}$ which respect the bracket, there is a unique homomorphism φ making the following diagram commute:



(Henceforth, we will concern ourselves only with $\mathfrak{U}(\mathfrak{g})$.) Enveloping algebras are useful in representation theory, in that $\mathfrak{g}$ modules are identically $\mathfrak{U}(\mathfrak{g})$ modules (an easy consequence of universality). Also, $\mathfrak{U}(\mathfrak{g})$ is unique (up to isomorphism) because universal objects are unique; to prove existance, we construct a universal enveloping algebra as a quotient of a $\mathbb{Z}_2$-graded tensor algebra, similar to the method in the Lie algebra case (for details, see [Mus12, p. 131-132]). As in the Lie algebra case, the key results on universal enveloping algebras are the PBW theorem and its corollaries.

Theorem 1.5.1 (PBW Theorem, [Mus12, Thm. 6.1.1])**.** *Let* X_0 *be a basis for* $\mathfrak{g}_0$ *over* $\mathbb{C}$ *and* X_1 *a basis for* $\mathfrak{g}_1$ *over* $\mathbb{C}$*, and let* $\leqslant$ *be a total order on* $X = X_0 \cup X_1$*. Then the set of all monomials of the form*

$$x_1 x_2 \cdots x_n$$

with $x_i \in X$*,* $x_i \leqslant x_{i+1}$*, and* $x_i \neq x_{i+1}$ *if* $x \in X_1$ *is a basis for* $\mathfrak{U}(\mathfrak{g})$ *over* $\mathbb{C}$*.*

This admits several easy corollaries. First, we can rewrite in terms of monomials with exponents.

Corollary 1.5.2 ([Mus12, Thm. 6.1.2])**.** *Let* $x_1, \ldots, x_m$ *be a vector space basis for* $\mathfrak{g}$ *consisting of homogenous elements. Then the set of all monomials of the form*

$$x_1^{a_1} \ldots x_m^{a_m}$$

with $a_i \in \mathbb{N}$ *if* $|x_i| = 0$ *and* $a_i \in \{0, 1\}$ *if* $|x_i| = 1$ *is a basis for* $\mathfrak{U}(\mathfrak{g})$

Proof. By rewriting. □

Noting that odd basis vectors appear at most once in any such monomial, we can relate $\mathfrak{U}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g}_0)$.

Corollary 1.5.3 ([Mus12, Lem. 6.1.3])**.** *In the notation of Theorem 1.5.1, the algebra* $\mathfrak{U}(\mathfrak{g})$ *is a free left and right module over* $\mathfrak{U}(\mathfrak{g}_0)$ *with basis consisting of all monomials of the form*

$$x_1 x_2 \cdots x_n$$

with $x_i \in X_1$ *and* $x_i < x_{i+1}$*. In particular, if* $\mathfrak{g}_1$ *is finite dimensional, then* $\mathfrak{U}(\mathfrak{g})$ *is finitely generated and free as a left and right* $\mathfrak{U}(\mathfrak{g})$*-module, with* 1 *as part of a free basis.*

Proof. By proper choice of $\leqslant$, so that elements of X_1 come first or last as appropriate. $\square$

Finally, we relate the universal enveloping algebra to the enveloping algebras of subalgebras.

Corollary 1.5.4 ([Mus12, Lem. 6.1.4]). *If* $\mathfrak{g}$ *is a direct sum of Lie superalgebras* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$*, then*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}) \otimes \mathfrak{U}(\mathfrak{b})$$

as vector spaces.

Proof. Take a suitable basis for $\mathfrak{g}$, ordered so that elements of $\mathfrak{a}$ come before those of $\mathfrak{b}$. $\square$

1.6 PROPERTIES OF $\mathfrak{osp}(3, 2)$

1.6.1 A Note on Describing Matrices

In describing elements of $\mathfrak{osp}(3, 2)$, below, we will specify the entries in matrices as follows.

$$\begin{array}{c|ccc|cc|}
 & 0 & 1 & -1 & 2 & -2 \\
\hline
0 & * & * & * & * & * \\
1 & * & * & * & * & * \\
-1 & * & * & * & * & * \\
\hline
2 & * & * & * & * & * \\
-2 & * & * & * & * & *
\end{array}$$

(This labeling helps to highlight symmetry.) We will use the notation $e_{i,j}$ to specify the matrix with a 1 in the i, j entry and a 0 elsewhere.

1.6.2 A Matrix Algebra which Preserves a Bilinear Form

In chapters 3 to 5 we consider $\mathfrak{osp}(3, 2)$. In the classification of the classical simple Lie superalgebras (see for example [Mus12, Ch. 1–2]) this is $B(1, 1)$. In general,

$B(m,n) = \mathfrak{osp}(2m+1, 2n)$ is a subalgebra of $\mathfrak{gl}(2m+1, 2n)$ which preserves a bilinear form given by a matrix of the form

$$\left[\begin{array}{c|c} G & 0 \\ \hline 0 & H \end{array}\right]$$

where G is a non-degenerate symmetric $2m+1 \times 2m+1$ matrix and and H is a non-degenerate skew-symmetric $2n \times 2n$ matrix. Canonically, we take

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_m \\ 0 & \mathrm{Id}_m & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}$$

In this case, all of the assumptions made in the previous sections apply, namely

- $\mathfrak{g}_0$ is a semisimple Lie algebra; in this case, $\mathfrak{g}_0 = \mathfrak{o}(3) \oplus \mathfrak{sp}(2)$,
- $\mathfrak{g}_1$ is a finitely generated $\mathfrak{g}_0$-module, and
- $\mathfrak{h} = \mathfrak{h}_0$, with basis $\{h_\varepsilon = e_{1,1} - e_{-1,-1}, h_\delta = e_{2,2} - e_{-2,-2}\}$.

1.6.3 Root Space, Basis and Borel Subalgebras

The root space of $\mathfrak{osp}(3,2)$ is seen in fig. 1.1, where $\{\varepsilon, -\varepsilon\}$ is the root space of the copy of $\mathfrak{o}(3)$ in $\mathfrak{g}_0$ and $\{2\delta, -2\delta\}$ is the root space of the copy of $\mathfrak{sp}(2)$.

In fig. 1.1, and in diagrams to follow, open dots (○) indicate even roots, filled dots (●) indicate odd non-isotropic roots, and crossed dots (⊗) indicate isotropic roots.

Remark 1.6.1. *The Lie algebra $SL_2(\mathbb{C})$ has several real forms, including $SL_2(\mathbb{R})$ and $SO_3(\mathbb{R})$. The representation theory for the copy of $\mathfrak{o}(3)$ in $\mathfrak{g}_0$ resembles that of $SO_3(\mathbb{R})$ rather than that of $SL_2(\mathbb{R})$.*

We obtain a basis for $\mathfrak{osp}(3,2)$ according to [Mus12, Ex. 2.7.4]. A "mulitplication table" for $\mathfrak{osp}(3,2)$ with the basis in table 1.1 can be found in appendix A.

Remark 1.6.2. *Note that the basis in table 1.1 may differ in scalar multiple from the basis given in [Mus12, Ex. 2.7.4]; this simplifies later computations. However,*

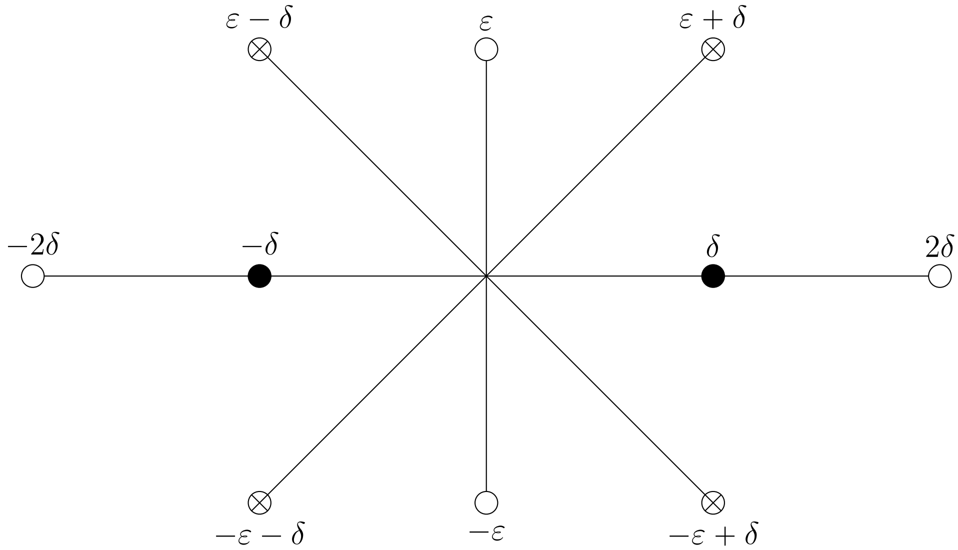


Figure 1.1: The root space of $\mathfrak{osp}(3, 2)$

$$
\begin{aligned}
h_\varepsilon &= e_{1,1} - e_{-1,-1} & h_\delta &= e_{2,2} - e_{-2,-2} \\
e_\varepsilon &= e_{1,0} - e_{0,-1} & e_{-\varepsilon} &= e_{0,1} - e_{-1,0} \\
e_{2\delta} &= e_{2,-2} & e_{-2\delta} &= e_{-2,2} \\
e_\delta &= e_{2,0} - e_{0,-2} & e_{-\delta} &= e_{-2,0} + e_{0,2} \\
e_{\varepsilon-\delta} &= e_{-2,-1} + e_{1,2} & e_{\varepsilon+\delta} &= e_{2,-1} - e_{1,-2} \\
e_{-\varepsilon-\delta} &= -e_{-2,1} - e_{-1,2} & e_{-\varepsilon+\delta} &= -e_{2,1} + e_{-1,-2}
\end{aligned}
$$

Table 1.1: A basis for $\mathfrak{osp}(3, 2)$

this interferes with "handedness" of the basis, and will have minor consequences for scalars appearing in Šapavolov elements (see section 2.6).

Recall that Borel subalgebras of Lie superalgebras are not unique up to conjugacy. Indeed, $\mathfrak{osp}(3,2)$ has two distinct conjugacy classes of Borel subalgebras, represented by the Dynkin diagrams in fig. 1.2.

Figure 1.2: Representatives of the Borel subalgebras for $\mathfrak{osp}(3,2)$.

In chapters 3 to 5, we take $\mathfrak{g} = \mathfrak{osp}(3,2)$, and use the notation $\alpha^{(i)}, \beta^{(i)}, \mathfrak{b}^{(i)}$ as in fig. 1.2. Note that these Borel subalgebras are adjacent, with $\mathfrak{g}^{-\varepsilon+\delta} \subseteq \mathfrak{b}^{(1)}$ and $\mathfrak{g}^{\varepsilon-\delta} \subseteq \mathfrak{b}^{(2)}$. When a Borel $\mathfrak{b}$ is specified without an index, α denotes the non-isotropic simple root, and β denotes the isotropic simple root. The symbol γ denotes the root $\varepsilon + \delta$.

CHAPTER 2
REPRESENTATION THEORY

We now turn our attention to the study of modules over $\mathfrak{g}$ (or equivalently, $\mathfrak{U}(\mathfrak{g})$) modules. In particular, we consider left modules, here and throughout, and the action of $\mathfrak{g}$ is the left action. Again, some familiarity is assumed with the representation theory of Lie algebras of the category $\mathcal{O}$. For a thorough treatment of this theory, see [Hum08]. The material in this chapter is adapted from [Mus12, Ch. 8–10].

Throughout this chapter we make the assumption that $\mathfrak{g}$ is either basic classical simple of type different from A or $\mathfrak{g} = \mathfrak{gl}(m, n)$. This is equivalent to the assumption that $\mathfrak{g}$ can be constructed as a contragradient Lie superalgebra $\mathfrak{g}(A, \tau)$, see [Mus12, Ex. 5.6.12]. This is required for several results in this chapter, and will pass without further comment.

We use the notation Π to denote a basis of simple roots, or $\Pi_{\mathfrak{b}}$ when we wish to call attention to the associated Borel, and set $Q^+ = \sum_{\zeta \in \Pi} \mathbb{N}\zeta$.

2.1 VERMA MODULES

Verma modules for semisimple Lie algebras were introduced in Verma's thesis; see [Ver68]. We proceed to define Verma modules for classical simple Lie superalgebras in an analogous way. The case $\mathfrak{g} = \mathfrak{q}(n)$ involves some complications, so we will not consider it here; the case is noted as excluded in relevant results.

Lemma 2.1.1 ([Mus12, Lem. 8.2.2])**.** *For $\lambda \in \mathfrak{h}_0^*$, there is a unique finite dimensional graded simple $\mathfrak{b}$-module V_λ such that $\mathfrak{n}^+ V_\lambda = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$ and $v \in V_\lambda$.*

If $\mathfrak{g} \neq \mathfrak{q}(n)$, then V_λ is one-dimensional. In this case, we write $V_\lambda = \mathbb{C}v_\lambda$.

Next, we fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let

$V_\lambda = \mathbb{C}v_\lambda$ be as in Lemma 2.1.1. Then we define the *Verma module* for $\mathfrak{g}$ by

$$M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} V_\lambda.$$

When we are considering more than one Borel, say $\mathfrak{b}^{(1)}, \mathfrak{b}^{(2)}$, we will denote the respective Verma modules by

$$M^{(i)}(\lambda) = \mathrm{Ind}_{\mathfrak{b}^{(i)}}^{\mathfrak{g}} V_\lambda.$$

We now give some of the most basic properties of Verma modules.

Lemma 2.1.2 ([Mus12, Lem. 8.2.3])**.** *Let $M_0(\lambda)$ denote the Lie algebra Verma module of weight λ for $\mathfrak{g}_0$. Then we have the following.*

- *The module $M(\lambda)$ has a unique maximal $\mathbb{Z}_2$-graded submodule.*
- *$M(\lambda) = \mathfrak{U}(\mathfrak{n}^-)V_\lambda$; this is a free $\mathfrak{U}(\mathfrak{n}^-)$-module with basis a vector space basis for V_λ. (For $\mathfrak{g} \neq \mathfrak{q}(n)$, $M(\lambda) = \mathfrak{U}(\mathfrak{n}^-)v_\lambda$ as $\mathfrak{g}$-modules.)*
- *There is a surjective map of $\mathfrak{U}(\mathfrak{g})$-modules*

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_0)} M_0(\lambda) \to M(\lambda).$$

- $\mathrm{End}_{\mathfrak{U}(\mathfrak{g})} M(\lambda) \cong \mathrm{End}_{\mathfrak{U}(\mathfrak{h})} V_\lambda$. *(For $\mathfrak{g} \neq \mathfrak{q}(n)$, $\mathrm{End}_{\mathfrak{U}(\mathfrak{g})} M(\lambda) \cong \mathbb{C}$.)*

By Lemma 2.1.2, the module $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ and a unique simple quotient $L(\lambda)$. Any nonzero factor module of $M(\lambda)$ is called a *module generated by a highest weight vector with weight λ.*

2.2 CATEGORY $\mathcal{O}$

The Verma modules are generalized in a convenient way by the category $\mathcal{O}$ introduced by Bernšteĭn, Gel′fand, and Gel′fand, [BGG71, BGG75, BGG76]. By definition, objects in the category $\mathcal{O}$ of $\mathfrak{U}(\mathfrak{g}_0)$ modules are those with the following properties.

(a) $M = \bigoplus_{\mu \in \mathfrak{h}_0^*} M^\mu$, where

$$M^\mu = \{v \in M : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

We call M^μ the *weight space* of weight μ, and say M is a direct sum of weight spaces, or that M is a *weight module*.

(b) For all $v \in M$, $\dim \mathfrak{U}(\mathfrak{n}_0^+)v < \infty$.

(c) M is a finitely generated $\mathfrak{U}(\mathfrak{g}_0)$-module.

This is a full subcategory of the category of $\mathfrak{U}(\mathfrak{g}_0)$-modules, that is, the morphisms are precisely $\mathfrak{U}(\mathfrak{g}_0)$-module morphisms. We consider the category $\widetilde{\mathcal{O}}$ of graded $\mathfrak{U}(\mathfrak{g})$-modules which belong to the category $\mathcal{O}$ when viewed as $\mathfrak{U}(\mathfrak{g}_0)$ modules by restriction.

We note that the definition of the category $\widetilde{\mathcal{O}}$ depends only on the triangular decomposition of $\mathfrak{g}_0$. However, when we refer to a highest weight, we implicitly fix a triangular decomposition of $\mathfrak{g}$. This is equivalent to fixing a choice of Borel subalgebras. When working in the $\mathfrak{osp}(3,2)$ case, if we are choosing one Borel specifically, we shall use the superscripts $^{(1)}$, $^{(2)}$ to refer to the Borels $\mathfrak{b}^{(1)}$, $\mathfrak{b}^{(2)}$ in fig. 1.2.

To further develop the theory of Verma modules, we will require the following. Denote by $\mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{U}(\mathfrak{g})$. Note that under the current assumption $\mathfrak{g} \neq \mathfrak{q}(n)$, the module V_λ in Lemma 2.1.1 is one-dimensional. Thus if $z \in \mathfrak{Z}(\mathfrak{g})$ and $v \in V_\lambda$, zv is a highest weight vector of weight λ and thus a scalar multiple of v. This allows us to define an algebra homomorphism $\chi_\lambda : \mathfrak{Z}(g) \to \mathbb{C}$ by

$$zv = \chi_\lambda(z)(v).$$

We call χ_λ the *central character of* $\mathfrak{U}(\mathfrak{g})$ *afforded by the* $\mathfrak{U}(\mathfrak{g})$*-module* $M(\lambda)$. It is easy to see that $z \in \mathfrak{Z}(\mathfrak{g})$ acts on $M(\lambda)$ as the scalar $\chi_\lambda(z)$.

It is easy to see that homomorphic images of an object in category $\mathcal{O}$ are again objects in $\mathcal{O}$. The following results, especially Lemma 2.2.4, motivate the study undertaken in chapters 3 to 5.

Lemma 2.2.1 ([Mus12, Lem. 8.2.5]). *If L is a simple object in $\widetilde{\mathcal{O}}$, then $L \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

Lemma 2.2.2 ([Mus12, Lem. 8.2.6]). *Let M be a module of category $\mathcal{O}$.*

(a) If M is nonzero, then M contains a highest weight vector.

(b) There is a finite series of submodules

$$M = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

such that each M_i/M_{i-1} is a highest weight module for $1 \leqslant i \leqslant s$.

(c) For all $\mu \in \mathfrak{h}_0^*$, the weight space M^μ is finite dimensional.

Lemma 2.2.3 ([Mus12, Cor. 8.2.12]). *If M is an object in category $\mathcal{O}$, then M has finite length.*

Lemma 2.2.4 ([Mus12, Lem. 8.2.14]). *The Verma module $M(\lambda)$ has a finite composition series as a $\mathbb{Z}_2$-graded module with composition factors of the form $L(\mu)$ where $\chi_\lambda = \chi_\mu$ and $\mu \leqslant \lambda$. Furthermore $L(\lambda)$ is a composition factor of $M(\lambda)$ with multiplicity 1.*

In the Lie algebra case, $\chi_\lambda = \chi_\mu$ if and only if $\mu = w.\lambda$ for some $w \in \mathcal{W}$ (where $\mathcal{W}$ denotes the Weyl group). (This result is due to Harish-Chandra [HC51]; for a good exposition see for instance [Hum08, Sec. 1.7–1.10].) In the Lie superalgebra case, the relationship is somewhat more complex, as we shall see in Theorem 2.5.2.

2.3 PARTITIONS AND CHARACTERS

Let $Q^+ = \sum_{\eta \in \Pi} \mathbb{N}\eta$. For $\lambda \in \mathfrak{h}^*$, set $D(\lambda) = \lambda - Q^+$ and let $\mathcal{E}$ be the set of functions on $\mathfrak{h}^*$ which are supported on a finite union of sets of the form $D(\lambda)$. Elements of $\mathcal{E}$ can be written as formal linear combinations $\sum_{\lambda \in \mathfrak{h}^*} c_\lambda \epsilon^\lambda$ where $\epsilon^\lambda(\mu) = \delta_{\lambda,\mu}$. We can make $\mathcal{E}$ an algebra via the convolution product

$$(fg)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu).$$

To describe some elements of $\mathcal{E}$, we use partitions. If $\eta \in Q^+$, a *partition* of η is a map $\pi : \Delta^+ \to \mathbb{N}$ such that $\pi(\xi) = 0$ or 1 for all $\xi \in \Delta_1^+$ and

$$\sum_{\xi \in \Delta^+} \pi(\xi)\xi = \eta.$$

For π a partition, set $|\pi| = \sum_{\xi \in \Delta^+} \pi(\xi)$. We denote by $\mathbf{P}(\eta)$ the set of partitions of η, and for $\xi \in \Delta_1^+$ we define

$$\mathbf{P}_\xi(\eta) = \{\pi \in \mathbf{P}(\eta) : \pi(\xi) = 0\}.$$

Set $\mathbf{p}(\eta) = |\mathbf{P}(\eta)|$ and $\mathbf{p}_\xi(\eta) = |\mathbf{P}_\xi(\eta)|$. The *partition function* p is defined by $p = \sum \mathbf{p}(\eta)\epsilon^{-\eta}$. Thus $p(\mu) = \sum \mathbf{p}(\eta)\epsilon^{-\eta}(\mu) = \mathbf{p}(-\mu)$. It is easy to see that

$$p = \frac{\prod_{\zeta \in \Delta_1^+}(1 + \epsilon^{-\zeta})}{\prod_{\zeta \in \Delta_0^+}(1 - \epsilon^{-\zeta})},$$

since the coefficient of $\epsilon^{-\zeta}$ in this expression is precisely the number of ways of writing ζ as a sum of elements of Δ^+ where elements of Δ_1^+ have coefficient in $\{0, 1\}$. Similarly for $\xi \in \Delta_1^+$ we define $p_\xi = \sum \mathbf{p}_\xi(\eta)\epsilon^{-\eta}$. Then we have

$$p_\xi = \frac{p}{(1 + \epsilon^{-\xi})}.$$

We can readily compute also $\mathbf{p}(\eta) = \mathbf{p}_\xi(\eta) + \mathbf{p}_\xi(\eta - \alpha)$.

Partitions are useful in indexing a basis for $\mathfrak{U}(\mathfrak{n}^\pm)$, as in the following lemma. Take a basis for $\mathfrak{g}$ of elements $e_\xi \in \mathfrak{g}^\xi$, $e_{-\xi} \in \mathfrak{g}^{-\xi}$ for each $\xi \in \Delta^+$ such that

$$[e_\xi, e_{-\xi}] = h_\xi,$$

where $h_\xi \in \mathfrak{h}$ such that $\zeta(h_\xi) = (\zeta, \xi)$ for all $\zeta \in \Delta$, and fix an ordering on Δ^+. (Note that the basis given in Table 1.1 is such a basis for $\mathfrak{osp}(3, 2)$.) Then for a partition π set

$$e_{-\pi} = \prod_{\xi \in \Delta^+} e_{-\xi}^{\pi(\xi)},$$

where the product is taken with respect to the order. In addition set

$$e_\pi =^t e_{-\pi} = \prod_{\xi \in \Delta^+} e_\xi^{\pi(\xi)},$$

where the product is taken in the opposite order. The following lemma follows directly from the PBW theorem:

Lemma 2.3.1 ([Mus12, Lem. 8.4.1]). *The elements* $e_{\pm\pi}$ *with* $\pi \in \mathbf{P}(\eta)$ *form a basis of* $\mathfrak{U}(\mathfrak{n}^\pm)^{\pm\eta}$. *Thus* $\dim(\mathfrak{U}(\mathfrak{n}^\pm)^{\pm\eta}) = \mathbf{p}(\eta)$.

If M is an object of $\mathcal{O}$, the *character* $\operatorname{ch} M$ of M is defined by

$$\operatorname{ch} M = \sum \dim(M^\eta)\epsilon^\eta.$$

Remark 2.3.2. *Since* $M(\lambda)^{\lambda-\mu}$ *has a basis consisting of all* $e_{-\pi}v_\lambda$ *with* $\pi \in \mathbf{P}(\mu)$, *it follows that*

$$\operatorname{ch} M(\lambda) = \epsilon^\lambda p.$$

Remark 2.3.3. *Note that* $\operatorname{ch} M(\lambda) \in \mathcal{E}$, *so* $\mathcal{E}$ *is useful in calculations involving characters. Also if* $M \in \mathcal{O}$ *and* E *is a finite dimensional simple module, we have*

$$\operatorname{ch}(M \otimes E) = \operatorname{ch} M \operatorname{ch} E \in \mathcal{E}.$$

Remark 2.3.4. *If* $M, M', M'' \in \mathcal{O}$ *such that the sequence*

$$0 \to M' \to M \to M'' \to 0$$

is exact,

$$\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M'' \in \mathcal{E}.$$

The *Grothendieck group* of the category $\mathcal{O}$, denoted $K(\mathcal{O})$ is defined as follows. For $M \in \mathcal{O}$, write $[M]$ for the isomorphism class of M. Then $K(\mathcal{O})$ is the free Abelian group generated by the symbols $[M]$ with relations $[M] = [M'] + [M'']$ whenever

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence in $\mathcal{O}$. It follows from Lemmas 2.2.1 and 2.2.3 that $\operatorname{ch} M \in \mathcal{E}$ for any module $M \in \mathcal{O}$. The same results imply that $K(\mathcal{O})$ is free Abelian on the $[L(\lambda)]$ with $\lambda \in \mathfrak{h}^*$. Let $C(\mathcal{O})$ be the additive subgroup of $\mathcal{E}$ generated by the characters $chL(\lambda)$ for $\lambda \in \mathfrak{h}^*$. Then it is easy to show the following (see [Jan79, Satz 1.11]).

Theorem 2.3.5 ([Mus12, Thm. 8.4.6])**.** *There is an isomorphism from the group $K(\mathcal{O})$ to $C(\mathcal{O})$ sending $[M]$ to $\operatorname{ch} M$ for all modules $M \in \mathcal{O}$.*

2.3.1 Characters of Induced Modules

We can generalize this notation to describe the modules induced from a parabolic subalgebra. Suppose $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{r}$, where $\mathfrak{p}$ is a parabolic subalgebra, and let

$$\Delta^{\mathfrak{r}} = \{\zeta \in \Delta : g^{\zeta} \subset \mathfrak{r}\}.$$

For a weight η, a $\mathfrak{p}$-*partition* of η is a map $\mathfrak{r} \to \mathbb{N}$ such that $\pi(\zeta) = 0$ or 1 for all $\zeta \in \Delta_1^{\mathfrak{r}}$ and

$$\sum_{\zeta \in \Delta^{\mathfrak{r}}} \pi(\zeta) = -\eta.$$

Denote by $\mathbf{P}_{\mathfrak{p}}(\eta)$ the set of all $\mathfrak{p}$-partitions of η, and let $\mathbf{p}_{\mathfrak{p}}(\eta) = |\mathbf{P}_{\mathfrak{p}}(\eta)|$. Finally, define the $\mathfrak{p}$-*partition function* $p_{\mathfrak{p}}$ by

$$p_{\mathfrak{p}} = \sum \mathbf{p}_{\mathfrak{p}}(\eta)\epsilon^{-\zeta} = \frac{\Pi_{-\zeta \in \Delta_1^{\mathfrak{r}}}(1 + \epsilon^{-\zeta})}{\Pi_{-\zeta \in \Delta_0^{\mathfrak{r}}}(1 - \epsilon^{-\zeta})}$$

The same argument used in Remark 2.3.2 tells us that if L is a $\mathfrak{p}$-module,

$$\operatorname{ch} \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L = \operatorname{ch} L \cdot p_{\mathfrak{p}}.$$

2.4 CHANGING THE BOREL SUBALGEBRA

We consider the behavior of highest weight modules when the Borel subalgebra is changed. Take $\mathfrak{b}, \mathfrak{b}'$ adjacent Borel subalgebras and β an isotropic root such that

$$\mathfrak{g}^{\beta} \subset \mathfrak{b} \qquad \mathfrak{g}^{-\beta} \subset \mathfrak{b}'.$$

Note that $\rho(\mathfrak{b}) = \rho(\mathfrak{b}') + \beta$. We see the following results.

Lemma 2.4.1 ([Mus12, Lem. 8.6.1]). *Suppose $V = \mathfrak{U}(\mathfrak{g})v_\lambda$, where v_λ is a highest weight vector for $\mathfrak{b}$ with weight λ. Let $w = e_{-\beta}v_\lambda$. Then either $w = 0$ or w is a highest weight vector of weight $\lambda - \beta$ for $\mathfrak{b}'$. Moreover, one of the following holds.*

(a) $(\lambda, \beta) \neq 0$ and $\mathfrak{U}(\mathfrak{g})w = V$, or

(b) $(\lambda, \beta) = 0$ and w generates a proper $\mathfrak{U}(\mathfrak{g})$-submodule of V.

The following corollary follows immediately.

Corollary 2.4.2 ([Mus12, Cor. 8.6.2]). *Assume V is as in Lemma 2.4.1 and V is simple. Then one of the following holds.*

(a) $(\lambda, \beta) \neq 0$ and V has highest weight $\lambda - \beta$ with respect to $\mathfrak{b}'$, or

(b) $(\lambda, \beta) = 0$ and V has highest weight λ with respect to $\mathfrak{b}'$.

2.5 VERMA MODULE EMBEDDINGS

Here, we describe a sufficient condition for an embedding of Verma modules $M(\mu) \hookrightarrow M(\lambda)$. In describing Verma module embeddings, we will frequently call upon the following hypothesis. Here we assume we have a basis Π of simple roots for $\mathfrak{g}$ containing at most one isotropic odd simple root, and we denote by $\mathcal{W}'$ the subgroup of $\mathcal{W}$ generated by the reflections σ_ξ, $\xi \in \Pi_0$.

Hypothesis 2.5.1. *The positive root ζ is in the $\mathcal{W}'$-orbit of a simple root.*

It is also useful to define the following sets of weights. For $\lambda \in \mathfrak{h}^*$ denote

$$A(\lambda)_0 = \{\xi \in \overline{\Delta}_0^+ : (\lambda + \rho, \xi^\vee) \in \mathbb{N}^+\}$$

$$A(\lambda)_1 = \{\xi \in \Delta_1^+ \backslash \overline{\Delta}_1^+ : (\lambda + \rho, \xi^\vee) \in \mathbb{N}^{\text{odd}}\}$$

$$A(\lambda) = A(\lambda)_0 \cup A(\lambda)_1$$

$$B(\lambda) = \{\xi \in \overline{\Delta}_1^+ : (\lambda + \rho, \xi) = 0\}.$$

We are now able to supply a sufficient condition for a Verma module embedding.

Theorem 2.5.2 ([Mus12, Thm. 9.3.3]). *Assume Hypothesis 2.5.1, with $\zeta \in A(\lambda)$. Then $M(\sigma_\zeta.\lambda) \hookrightarrow M(\lambda)$.*

2.6 ŠAPAVOLOV ELEMENTS

The following discussion will allow us to describe the highest weight vectors of a Verma module of given highest weight. These results are analogous to results for Lie algebras first studied in [Šap72]; for a good exposition of the theory in the Lie algebra case, see [Hum08, Sec. 4.12].

We determine the conditions on a weight ζ under which a *Šapavolov element* $\theta \in \mathfrak{U}(\mathfrak{b}^-)$ exists such that θv_λ is a highest weight vector in $M(\lambda)$ of weight $\lambda - \zeta$.

For simple roots ζ, the element θ_ζ is easy to describe.

Lemma 2.6.1 ([Mus12, Lem. 9.2.1]).

(a) If $\zeta \in \Pi_{\mathfrak{b}} \cap \Delta_0^+$ and $(\lambda + \rho, \zeta^\vee) = m \in \mathbb{N}^+$, then $e_{-\zeta}^m v_\lambda$ generates a submodule of $M(\lambda)$ isomorphic to $M(\sigma_\zeta.\lambda)$.

(b) If $\zeta \in \Pi_{\mathfrak{b}} \cap \Delta_1^+$, $(\zeta, \zeta) \neq 0$, and $(\lambda + \rho, \zeta^\vee) = 2m + 1 \in \mathbb{N}^{\text{odd}}$, then $e_{-\zeta}^{2m+1} v_\lambda$ generates a submodule of $M(\lambda)$ isomorphic to $M(\sigma_\zeta.\lambda)$.

(c) If $\zeta \in \Pi_{\mathfrak{b}} \cap \Delta_1^+$ is an isotropic root and $(\lambda + \rho, \zeta) = 0$, then $e_{-\zeta} v_\lambda$ generates a proper submodule of $M(\lambda)$.

For ease of exposition, we will assume our basis of simple roots has the following property. This holds, in particular, for both Borels for $\mathfrak{osp}(3, 2)$.

Hypothesis 2.6.2. Π *is a basis of simple roots for $\mathfrak{g}$ containing at most one isotropic odd simple root.*

When this hypothesis holds, let β denote the unique simple isotropic root.

Lemma 2.6.3 ([Mus12, Lem. 9.2.3]). *If Hypothesis 2.6.2 holds, then:*

(a) For all positive roots ζ and $w \in \mathcal{W}'$, we have $r_\beta(\gamma) = r_\beta(w\gamma)$.

(b) If ζ is a positive isotropic root and $\zeta \neq \beta$, then $(\zeta, \alpha^\vee) > 0$ for some $\alpha \in \Pi_0$.

(c) We have $\overline{\Delta}_1^+ = \mathcal{W}'\beta$.

2.6.1 Šapavolov Elements for Nonisotropic Roots

Assume Hypothesis 2.5.1 with ζ a positive nonisotropic root. If $\zeta \in \Delta_0^+$, we assume that $\zeta \in \overline{\Delta}_0^+$ since otherwise we can consider instead the root $\zeta/2$. Suppose one of the following holds.

(a) $\gamma \in \overline{\Delta}_0^+$ and $m \in \mathbb{N}^+$.

(b) $\gamma \in \Delta_1^+ \backslash \overline{\Delta}_1^+$ and $m \in \mathbb{N}^{\text{odd}}$.

Let $\pi^0 \in \overline{\mathbf{P}}(m\zeta)$ be the unique partition of $m\zeta$ such that $\pi^0(\xi) = 0$ for $\xi \in \Delta^+ \backslash \Pi$. The result below is an analog of a result of Šapavolov, [Šap72, Lem. 1], for semisimple Lie algebras; see also [Hum08, Sec. 4.1.3].

Theorem 2.6.4 ([Mus12, Thm. 9.2.6]). *There exists an element $\theta_{\zeta,m} \in \mathfrak{U}(\mathfrak{b}^-)^{-m\zeta}$ such that the following hold.*

(a) For all $\xi \in \Delta^+$,

$$e_\xi \theta_{\zeta,m} \in \mathfrak{U}(\mathfrak{g})(h_\zeta + \rho(h_\zeta) - m(\zeta,\zeta)/2) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+.$$

(b) $\theta_{\gamma,m} = \sum_{\pi \in \overline{\mathbf{P}}(m\gamma)} e_{-\pi} H_\pi$ for some $H_\pi \in \mathfrak{U}(\mathfrak{h})$, with $H_{\pi^0} = 1$.

Corollary 2.6.5 ([Mus12, Cor. 9.2.7]). *For each $\lambda \in \mathfrak{h}^*$ such that $(\lambda + \rho, \zeta) = m(\zeta,\zeta)/2$, there is a nonzero map of Verma modules $\omega_\lambda : M(\lambda - m\zeta) \to M(\lambda)$ sending $xv_{\lambda - m\zeta}$ to $x\theta_{\gamma,m}(\lambda)v_\lambda$ for $x \in \mathfrak{U}(\mathfrak{n}^-)$.*

While there are general formulas which may be used to compute the element $\theta_{\gamma,m}$, in the case of $\mathfrak{osp}(3,2)$, they may be readily computed via change of Borels, as we shall do in chapters 3 to 5.

2.7 THE JANTZEN FILTRATION AND THE THE JANTZEN SUM FORMULA

Given a Verma module $M(\lambda)$, we consider a specific filtration, the *Jantzen Filtration*

$$M(\lambda) \supseteq M_1(\lambda) \supseteq M_2(\lambda) \supseteq \cdots$$

which has the following properties.

Lemma 2.7.1 ([Mus12, Lem. 10.2.2]). *For all $\lambda \in \mathfrak{h}^*$ and $\eta \in \mathbb{Q}^+$:*

(a) $M_1(\lambda)$ is the unique maximal proper submodule of $M(\lambda)$.

(b) Each $M_n(\lambda)$ is a $\mathfrak{U}(\mathfrak{g})$-submodule of $M(\lambda)$.

(c) $M_n(\lambda) = 0$ for n sufficiently large.

The construction of this filtration and many additional properties are described in [Mus12, Sec. 10.2]. This is an analog of a result shown for the Lie algebra case in [Jan79]. This filtration is related to the Verma submodules via the following equation concerning characters.

Theorem 2.7.2 (The Jantzen Sum Formula, [Mus12, Thm. 10.3.1]). *For all $\lambda \in \mathfrak{h}^*$*

$$\sum_{i>0} \operatorname{ch} M_i(\lambda) = \sum_{\eta \in A(\lambda)} \operatorname{ch} M(\sigma_\eta.\lambda) + \sum_{\eta \in B(\lambda)} \epsilon^{\lambda-\eta} p_\eta.$$

Remark 2.7.3. *In the case where all terms are characters of modules, in particular, when $\epsilon^{\lambda-\eta}p_\eta$ is the character of a module for each $\eta \in B(\lambda)$, we can regard the Jantzen sum formula instead as a sum in the Grothendieck group, by Theorem 2.3.5. We shall see in chapter 5 that this holds for $\mathfrak{osp}(3,2)$.*

2.8 THE CASIMIR ELEMENT

As in the Lie algebra case, we construct a central element Ω of $\mathfrak{U}(\mathfrak{g})$ known as the *Casimir element*. Assume $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are bases of $\mathfrak{g}$ such that $(x_i, y_j) = \delta_{i,j}$ and x_i, y_i are homogeneous elements of $\mathfrak{g}$ of the same degree d_i. Fix $x \in \mathfrak{g}_c$ ($c \in \{0, 1\}$) and write

$$[x, x_i] = \sum_j a_{ij} x_j, \qquad [x, y_i] = \sum_j b_{ij} y_j.$$

Then computation shows that

$$a_{ik} = -(-1)^{cd_i} b_{ki}.$$

Now set $\Omega = \sum (-1)^{d_i} x_i y_i \in \mathfrak{U}(\mathfrak{g})$.

The above computation allows us to readily show the following lemma.

Lemma 2.8.1 ([Mus12, Lem. 8.5.1])**.** *The Casimir element Ω is central in $\mathfrak{U}(\mathfrak{g})$.*

The following lemmas will be helpful in determining whether a module is simple.

Lemma 2.8.2 ([Mus12, Lem. 8.5.3])**.** *Let Ω denote the Casimir element of $\mathfrak{g}$. Then Ω acts on any $\mathfrak{g}$-module M as scalar multiplication by $\chi_\lambda(\Omega) = (\lambda + 2\rho, \lambda)$.*

Lemma 2.8.3 ([Mus12, Lem. 8.5.4])**.** *Suppose μ is a highest weight of M, a $\mathfrak{g}$-module of generated by a highest weight vector of weight λ. Then*

$$(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu).$$

From Lemma 2.8.3 and easy computations, we can obtain the following.

Lemma 2.8.4. *Suppose μ is a highest weight vector in M, a $\mathfrak{g}$-module of highest weight λ. Write $\mu = \lambda - \zeta$; then $2(\lambda + \rho, \zeta) = (\zeta, \zeta)$.*

Proof. We compute

$$\begin{aligned}
(\lambda + 2\rho, \lambda) &= (\lambda - \zeta + 2\rho, \lambda - \zeta) \\
(\lambda, \lambda) + 2(\rho, \lambda) &= (\lambda, \lambda) - (\lambda, \zeta) - (\lambda, \zeta) + (\zeta, \zeta) + 2(\rho, \lambda) - 2(\rho, \zeta) \\
0 &= -(\lambda, \zeta) - (\lambda, \zeta) + (\zeta, \zeta) - 2(\rho, \zeta) \\
2(\lambda + \rho, \zeta) &= (\zeta, \zeta)
\end{aligned}.$$

as desired. □

CHAPTER 3
VERMA MODULES OF TYPICAL WEIGHT

3.1 TERMINOLOGY AND NOTATION

Here, and for the remainder of the thesis, we make the assumption that $\mathfrak{g} = \mathfrak{osp}(3,2)$. We begin by studying the structure of Verma modules of typical weight with respect to all isotropic roots. We say that the module V generated by a highest weight vector is $\mathfrak{b}$-typical if it is generated by a vector v_λ of highest weight λ with respect to $\mathfrak{b}$, where $(\lambda + \rho, \xi) \neq 0$ for any isotropic root ξ. We also say that the weight λ is $\mathfrak{b}$-typical. The following lemma will be useful.

Lemma 3.1.1. *A module V is $\mathfrak{b}^{(1)}$-typical if and only if it is $\mathfrak{b}^{(2)}$-typical.*

Proof. Let $\{i, j\} = \{1, 2\}$. Suppose V is $\mathfrak{b}^{(i)}$-typical, say V is generated by $v_{\lambda^{(i)}}$. Then by Lemma 2.4.1, V is generated by $e_{-\beta^{(i)}} v_{\lambda^{(i)}}$, of highest weight $\lambda^{(i)} - \beta^{(i)} = \lambda^{(j)}$ with respect to $\mathfrak{b}^{(j)}$. A simple computation or Lemma 2.4.1 tells us that $\lambda^{(j)} + \rho^{(j)} = \lambda^{(i)} + \rho^{(i)}$, and so $(\lambda^{(j)} + \rho^{(j)}, \xi) \neq 0$ for any isotropic root ξ. Thus by definition, V is $\mathfrak{b}^{(j)}$-typical. $\square$

Lemma 3.1.1 tells us that we need not specify the Borel when considering whether an $\mathfrak{osp}(3,2)$-module is typical, and therefore, we may unambiguously state that a module M is typical without reference to the specific Borel. The following lemma relates typical weights to their reflections.

Lemma 3.1.2. *Fix a Borel $\mathfrak{b}$. Then if λ is $\mathfrak{b}$-typical, then $\sigma_\zeta.\lambda$ is as well for any non-isotropic root ζ.*

Proof. By computation, $\sigma_\zeta.\lambda + \rho = \sigma_\zeta(\lambda + \rho)$, so

$$(\sigma_\zeta.\lambda + \rho, \xi) = (\sigma_\zeta(\lambda + \rho), \xi)$$

$$= (\lambda + \rho, \sigma_\zeta(\xi)) \qquad \text{because } \sigma_\zeta \text{ preserves the inner product}$$

$$\neq 0 \qquad \text{because } \sigma_\zeta(\xi) \text{ is again an isotropic root.} \qquad \square$$

3.2 VERMA SUBMODULES

Let $M = M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)})$ be a typical Verma module; then by Lemma 2.4.1, $\lambda^{(1)} + \rho^{(1)} = \lambda^{(2)} + \rho^{(2)}$. Let

$$m = (\lambda^{(i)} + \rho^{(i)}, \varepsilon^\vee)$$

$$n = (\lambda^{(i)} + \rho^{(i)}, \delta^\vee).$$

(Note that $m \pm n \neq 0$, since otherwise $(\lambda^{(i)} + \rho^{(i)}, \varepsilon \pm \delta) = 0$ and M is atypical. These cases are treated in chapters 4 and 5.)

Lemma 3.2.1 follows immediately from Theorem 2.5.2.

Lemma 3.2.1. *V has the following Verma submodules.*

(a) If $m \in \mathbb{N}^+$, $M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)})$ is a submodule of V.

(b) If $n \in \mathbb{N}^{\text{odd}}$, $M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$ is a submodule of V.

The following lemma follows after a little calculation.

Lemma 3.2.2.

(a) $\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)} + \rho^{(1)} = \sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)} + \rho^{(2)}$.

(b) $\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)} + \rho^{(2)} = \sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} + \rho^{(1)}$.

Since these are typical, it follows that

$$M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) = M^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)})$$

$$M^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) = M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)}).$$

3.3 THE STRUCTURE OF $M(\lambda)$

We are now ready to describe the structure of the Verma module

$$M = M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)})$$

where $\lambda^{(1)} + \rho^{(1)} = \lambda^{(2)} + \rho^{(2)}$. Again we write

$$(\lambda^{(i)} + \rho^{(i)}, \varepsilon^\vee) = m \qquad (\lambda^{(i)} + \rho^{(i)}, \delta^\vee) = n.$$

First, we establish a sufficient (and also, we shall see, necessary) condition for M to be simple.

Lemma 3.3.1. *If $m \notin \mathbb{N}^+$ and $n \notin \mathbb{N}^{\text{odd}}$, then M is simple.*

Proof. This follows from Theorem 2.7.2. In this case, with either Borel,

$$A(\lambda^{(i)}) = B(\lambda^{(i)}) = \varnothing,$$

and so M_1, the unique maximal submodule, is the zero module. □

Next, we find two cases with composition series of length two. Let L denote $L^{(1)}(\lambda^{(1)}) = L^{(2)}(\lambda^{(2)})$ the unique simple submodules of these weights (which exist by Lemma 2.2.1, and are equal by Corollary 2.4.2).

Lemma 3.3.2. *If $m \in \mathbb{N}^+$ and $n \notin \mathbb{N}^{\text{odd}}$, let $N^1 = M^{(1)}(\sigma_{\varepsilon_{(1)}} \cdot \lambda^{(1)}) = M^{(2)}(\sigma_{\varepsilon_{(2)}} \cdot \lambda^{(2)})$. Then we have short exact sequence*

$$0 \to N^1 \to M \to L \to 0.$$

with N^1, L simple.

Proof. First, N^1 is simple by Lemma 3.3.1, so to prove the theorem, all that must be shown is that M/N^1 is simple. To demonstrate this, we turn again to Theorem 2.7.2. In this case $A(\lambda) = \{\varepsilon\}$ and $B(\lambda) = \varnothing$. So the right hand side in Theorem 2.7.2 has unique term N^1, and thus $N^1 = M_1$ is the unique maximal submodule of M, that is, N^1 is maximal, and so M/N^1 is simple. □

Lemma 3.3.3. *If $n \in \mathbb{N}^{\text{odd}}$ and $m \notin \mathbb{N}^+$, let $N^2 = M^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) = M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$. Then we have short exact sequence*

$$0 \to N^2 \to M \to L \to 0.$$

with N^2, L simple.

Proof. This follows a similar argument to Lemma 3.3.2. We know N^2 is simple by Lemma 3.3.1, and the right hand side in Theorem 2.7.2 has unique term N^2, since in this case $A(\lambda) = \{\delta\}$ and $B(\lambda) = \varnothing$. Thus $N^2 = M_1$ the unique maximal submodule of M, and so M/N^2 is simple. □

Finally, the most complicated case has composition series of length four.

Lemma 3.3.4. *If $m \in \mathbb{N}^+$ and $n \in \mathbb{N}^{\text{odd}}$, M has the structure seen in fig. 3.1, where a node labeled by a weight indicates a submodule of that highest weight.*

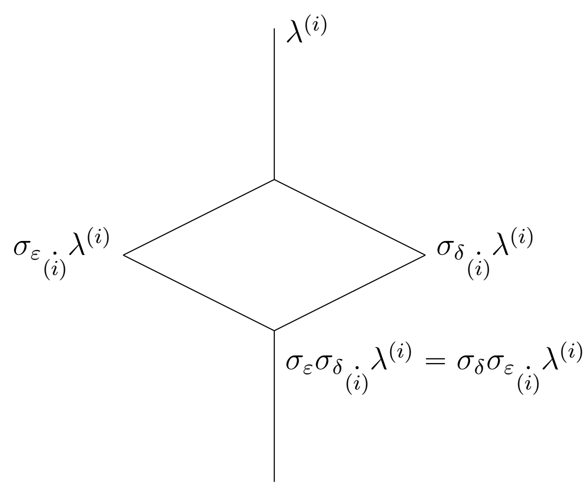


Figure 3.1: The structure of $M = M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)})$ when $(\lambda^{(i)} + \rho^{(i)}, \varepsilon^\vee) \in \mathbb{N}^+$ and $(\lambda^{(i)}\rho^{(i)}, \delta^\vee) \in \mathbb{N}^{\text{odd}}$.

Proof. In this case, the right-hand side in Theorem 2.7.2 has two terms, as $A(\lambda) = \{\varepsilon, \delta\}$ and $B(\lambda) = \varnothing$. For the following argument, we treat the Jantzen Sum Formula as an equality in the Grothendieck group (not problematic, as $B(\lambda) = \varnothing$, so each term on the right hand side is the character of a Verma module), and let $\boxplus$ denote the addition in the Grothendieck group. Let

$$R^1 = M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) = M^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)})$$

$$R^2 = M^{(2)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) = M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)}).$$

Then the right hand side of the Jantzen Sum Formula is $R^1 \boxplus R^2$. But by Lemma 3.3.2, $R^1 = N^1 \boxplus L$, and by Lemma 3.3.3. $R^2 = N^2 \boxplus L$. So the right hand side becomes $N^1 \boxplus N^2 \boxplus 2L$. Thus either $M_1 = N^1 \boxplus N^2 \boxplus L$ and $M_2 = L$ or $M_1 = N^1 \boxplus N^2 \boxplus 2L$ and $M_2 = 0$.

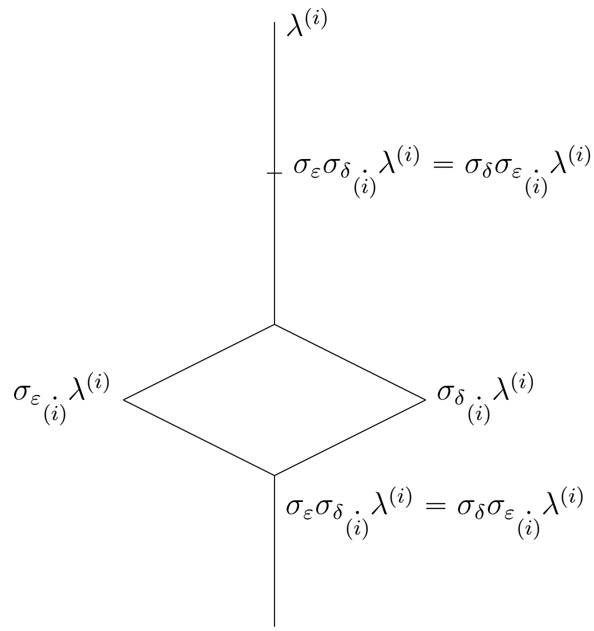


Figure 3.2: To show that the structure of M is that shown in fig. 3.1, we must eliminate this possibility.

To eliminate the second possibility, seen in fig. 3.2, we consider the weight

$$\mu^{(i)} = \sigma_\varepsilon \sigma_\delta \underset{(i)}{\cdot} \lambda^{(i)} = \sigma_\delta \sigma_\varepsilon \underset{(i)}{\cdot} \lambda^{(i)}.$$

If the structure in fig. 3.2 is correct, then L is a composition factor of the quotient M/R^1. This requires the existance of a singular vector u of weight $\mu^{(1)}$ in $M\backslash R^1$ such that u is highest weight mod R^1, that is, such that $\mathfrak{U}(\mathfrak{b})u \subseteq R^1$.

It is easy to compute that $\mu^{(i)} = \lambda^{(i)} - m\varepsilon - n\delta$. If $n > 1$, $\mu^{(1)}$ has multiplicity 4 in M, with $M^{\mu^{(1)}}$ having basis

$$\left\{ \begin{array}{cc} e_{-\delta}^{n} e_{-\varepsilon}^{m} v_\lambda, & e_{-\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1} v_\lambda, \\ e_{\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m+1} v_\lambda, & e_{-\varepsilon-\delta} e_{\varepsilon-\delta} e_{-\delta}^{n-2} e_{-\varepsilon}^{m} v_\lambda \end{array} \right\}$$

where v_λ is a vector of highest weight $\lambda^{(1)}$ with respect to $\mathfrak{b}^{(1)}$. If $n = 1$, the weight has multiplicity 3 in M, with $M^{\mu^{(1)}}$ having basis

$$\left\{ \begin{array}{cc} e_{-\delta}^{n} e_{-\varepsilon}^{m} v_\lambda, & e_{-\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1} v_\lambda, \\ e_{\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m+1} v_\lambda & \end{array} \right\}.$$

In either case, by eliminating basis elements which are clearly contained in R^1 (generated by $e_{-\varepsilon}^{m} v_\lambda$), we see that if the required singular vector exists we may take

$$u = e_{-\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1} v_\lambda.$$

To prove that no such singular vector exists, it suffices to show that $e_\varepsilon u \notin R^1$.

Noting that $n \in \mathbb{N}^{\text{odd}}$, we note first that

$$\left[e_\varepsilon, e_{-\delta}^{n-1}\right] = \left[e_\varepsilon, e_{-2\delta}^{(n-1)/2}\right] = 0.$$

Then, computing, we obtain

$$\begin{aligned} [e_\varepsilon, e_{-\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1}] v_\lambda &= \left([e_\varepsilon, e_{-\varepsilon-\delta}] e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1} + e_{-\varepsilon-\delta}[e_\varepsilon, e_{-\delta}^{n-1} e_{-\varepsilon}^{m-1}]\right) v_\lambda \\ &= \left(e_{-\delta}^{n} e_{-\varepsilon}^{m-1} + e_{-\varepsilon-\delta}\left([e_\varepsilon, e_{-\delta}^{n-1}] e_{-\varepsilon}^{m-1} + e_{-\delta}^{n-1}[e_\varepsilon, e_{-\varepsilon}^{m-1}]\right)\right) \\ &= \left(e_{-\delta}^{n} e_{-\varepsilon}^{m-1} + e_{-\varepsilon-\delta}\left(0 + e_{-\delta}^{n-1}[e_\varepsilon, e_{-\varepsilon}^{m-1}]\right)\right) v_\lambda \\ &= \left(e_{-\delta}^{n} e_{-\varepsilon}^{m-1} + 2\tbinom{m}{2} e_{-\varepsilon-\delta} e_{-\delta}^{n-1} e_{-\varepsilon}^{m-2}\right) v_\lambda \end{aligned}$$

which is clearly not an element of $\mathfrak{U}((n^-)^{(1)}) e_{-\varepsilon}^{m} v_\lambda = R^1$. Thus the required singular vector does not exist, and L is not a composition factor of M/R^1. This rules out the structure in fig. 3.2, leaving the structure in fig. 3.1 the only possibility. □

CHAPTER 4

VERMA MODULES OF ATYPICAL WEIGHT: TYPE 1

Next we consider weights λ orthogonal to a simple isotropic root $\beta^{(i)}$. Note that since $\beta^{(1)} = -\beta^{(2)}$, λ is orthogonal to $\beta^{(1)}$ if and only if it is orthogonal to $\beta^{(2)}$.

Again letting $(\lambda + \rho^{(i)}, \varepsilon^\vee) = m^{(i)}$ and $(\lambda + \rho^{(i)}, \delta^\vee) = n^{(i)}$ $(i = 1, 2)$, we see that in this case $m^{(i)} = -n^{(i)}$. In particular, this means that if $m^{(i)} \in \mathbb{N}^+$ then $n^{(i)} \notin \mathbb{N}^{\text{odd}}$, and conversely, if $n^{(i)} \in \mathbb{N}^{\text{odd}}$ then $m^{(i)} \notin \mathbb{N}^+$.

Note that we do not introduce the notation $\lambda^{(i)}$ in this case. Since λ is orthogonal to $\beta^{(i)}$, Lemma 2.4.1 tells us that if $M^{(i)}(\lambda)$ is a Verma module generated by highest weight vector v_λ, then $e_{-\beta}v_\lambda$ generates a proper submodule of $M^{(i)}(\lambda)$. We examine this submodule in the first section.

It is convenient to note that if $(\lambda + \rho^{(i)}, \beta^{(i)}) = 0$, then λ is a scalar multiple of $\beta^{(i)}$, since $\rho^{(i)} = -\frac{1}{2}\beta^{(i)}$. In particular, we can write $\lambda = C(\varepsilon - \delta)$ for a constant C.

4.1 A PARABOLIC SUBALGEBRA

Partition Δ into

$$\Delta^{\mathfrak{r}} = (-\varepsilon, -2\delta, -\delta, -\varepsilon - \delta), \qquad \Delta^{\mathfrak{p}} = (\varepsilon, 2\delta, \varepsilon - \delta, \varepsilon + \delta, \delta, -\varepsilon + \delta),$$

and define

$$\mathfrak{r} = \bigoplus_{\alpha \in \Delta^{\mathfrak{r}}} \mathfrak{g}^\alpha, \qquad \mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^{\mathfrak{p}}} \mathfrak{g}^\alpha$$

(so $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{p}$). Note that $\mathfrak{r} \subseteq \mathfrak{n}^-$ for either choice of Borel, and that $\mathfrak{r}_0 = \mathfrak{n}_0^-$. Then the following lemmas hold.

Lemma 4.1.1. *For $\lambda = C(\varepsilon - \delta)$, let $\overline{M}(\lambda) = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V_\lambda$, where V_λ is as in Lemma 2.1.1. Then we have a short exact sequence*

$$0 \to \overline{M}(\lambda - \beta) \xrightarrow{\iota} M(\lambda) \xrightarrow{\pi} \overline{M}(\lambda) \to 0$$

where V_ζ *is as in Lemma 2.1.1.*

Proof. We first establish the maps ι and π. Let v_λ be a highest weight vector in $M(\lambda)$, $w_{\lambda-\beta}$, w_λ highest weight vectors in $\overline{M}(\lambda-\beta)$, $\overline{M}(\lambda)$, respectively. By the universality of Verma modules, there exists a unique surjection $\pi : v_\lambda \mapsto w_\lambda$. Further, since β is atypical, by Lemma 2.4.1 $\iota : w_{\lambda-\beta} \to e_{-\beta}v_\lambda$ maps $\overline{M}(\lambda-\beta)$ into the proper submodule of $M(\lambda)$ generated by $e_{-\beta}v_\lambda$. Finally, noting that

$$\pi(e_{-\beta}v_\lambda) = e_{-\beta}\pi(v_\lambda) = e_{-\beta}w_\lambda = 0,$$

where the last equality holds because $-\beta \in \Delta^{\mathfrak{p}}$, we see that $\ker\pi \subseteq \operatorname{im}\iota$.

Now the sequence is exact by comparing characters. Note from Remark 2.3.2 that

$$\operatorname{ch} M(\lambda) = \frac{\epsilon^\lambda(1+\epsilon^{-\beta})(1+\epsilon^{-\varepsilon-\delta})}{(1-\epsilon^{-\varepsilon})(1-\epsilon^{-\delta})}$$

and from the discussion in section 2.3.1 that

$$\begin{aligned} \operatorname{ch}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}V_\lambda &= \frac{\epsilon^\lambda(1+\epsilon^{-\varepsilon-\delta})}{(1-\epsilon^{-\varepsilon})(1-\epsilon^{-\delta})} \\ \operatorname{ch}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}V_{\lambda-\beta} &= \frac{\epsilon^{\lambda-\beta}(1+\epsilon^{-\varepsilon-\delta})}{(1-\epsilon^{-\varepsilon})(1-\epsilon^{-\delta})}, \end{aligned}$$

and so it is easy to see that $\operatorname{ch} M(\lambda) = \operatorname{ch}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}V_{\lambda-\beta} + \operatorname{ch}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}V_\lambda$. □

Note: If λ is orthogonal to β, then $\lambda-\beta$ is as well. So by studying $\overline{M}(\lambda) = \operatorname{Ind}_p^{\mathfrak{g}}V_\lambda$, where $\lambda = \lambda^{(1)}(\varepsilon+\delta)$, we also discover the structure of $\overline{M}(\lambda-\beta)$. To study the action of $\mathfrak{U}(\mathfrak{g})$ (and thus of $\mathfrak{g}$) on $\overline{M}(\lambda)$, we construct a map

$$\phi : \mathfrak{U}(\mathfrak{g}) \to M_4(A_2)$$

and describe an action of $M_4(A_2)$ on $\overline{M}(\lambda)$. To show that such a function ϕ exists, we first consider $\overline{M}(\lambda)$ as a free module.

Lemma 4.1.2. $\overline{M}(\lambda) \cong \mathfrak{U}(\mathfrak{r})v_\lambda$ *is a free* $\mathfrak{U}(\mathfrak{r}_0)$*-module.*

Proof. By the PBW theorem. Take a basis for $\mathfrak{r}$ by letting $s = e_{-2\delta}$, $t = e_{-\varepsilon}$, $a = e_{-\delta}$,

$b = e_{-\varepsilon-\delta}$. The PBW theorem implies that we can write

$$\mathfrak{U}(\mathfrak{r}) = \mathfrak{U}(\mathfrak{r}_0) \oplus ab\mathfrak{U}(\mathfrak{r}_0) \oplus a\mathfrak{U}(\mathfrak{r}_0) \oplus b\mathfrak{U}(\mathfrak{r}_0),$$

where $\mathfrak{U}(\mathfrak{r}_0) = \mathfrak{U}(\mathfrak{n}_0^-)$ has basis $\{s^i t^j | i, j \in \mathbb{N}\}$. □

Note that, since $s = e_{-2\delta}$ and $t = e_{-\varepsilon}$ commute, as vector spaces $\mathfrak{U}(\mathfrak{r}_0) \cong \mathbb{C}[s,t]$ a polynomial ring. So

$$\overline{M}(\lambda) \cong \bigoplus_{i=1}^{4} x_i A$$

where $A = \mathbb{C}[s,t]$ a polynomial ring, and $\{x_i\} = \{1, ab, a, b\}$. Recalling that $\overline{M}(\lambda)$ is a left $\mathfrak{U}(\mathfrak{g})$-module, we define a map

$$\phi : \mathfrak{U}(\mathfrak{g}) \to M_4(\operatorname{End}_{\mathbb{C}} A)$$

such that, for $u \in \mathfrak{U}(\mathfrak{g})$, $p \in A$,

$$u x_i p v_\lambda = \sum_{i=1}^{4} x_j \left(\phi(u)_{ji}(a)\right).$$

Calculations show that the image of ϕ is contained in $M_4(D(A)) = M_4(A_2)$, where $D(A)$ denotes the differential operators on A and A_2 denotes the second Weyl algebra $\mathbb{C}[s, t, \partial_s, \partial_t]$.

Lemma 4.1.2 shows that the action of $\mathfrak{g}$ on $\overline{M}(\lambda)$ provides a mapping of $\mathfrak{g}$ into $\operatorname{End}_{k[s,t]}\mathfrak{U}(\mathfrak{r})$; computation shows that the image of this map is asubring of the matrix ring $M_4(A_2)$, where A_2 denotes the second Weyl algebra $k[s, t, \partial_s, \partial_t]$.

To compute the map, we fix a basis for $\mathfrak{g}$ and compute the action of each basis element. This induces a map $\mathfrak{g} \to M_4(A_2)$, and by the universality of $\mathfrak{U}(\mathfrak{g})$, a map $\mathfrak{U}(\mathfrak{g}) \to M_4(A_2)$.

4.1.1 Explicit Computation of the Map ϕ

Take a basis for $\mathfrak{g}$ with $e_\alpha \in \mathfrak{g}^\alpha$ for each $\alpha \in \Delta$. Using the basis on page 11, basic elements have images as follows.

First, images of elements of $\mathfrak{r}_0$ are easy to compute.

$$s = e_{-2\delta} \mapsto \left[\begin{array}{cc|cc} s & & & \\ & s & & \\ \hline & & s & \\ & & & s \end{array}\right] \qquad t = e_{-\varepsilon} \mapsto \left[\begin{array}{cc|cc} t & & & \\ & t & & \\ \hline & & t & \\ & & & t \end{array}\right]$$

Images of elements of $\mathfrak{r}_1$ are computed by induction on commutators.

$$a = e_{-\delta} \mapsto \left[\begin{array}{cc|cc} & & s & 0 \\ & & \partial_t & 1 \\ \hline 1 & 0 & & \\ -\partial_t & s & & \end{array}\right] \qquad b = e_{-\varepsilon-\delta} \mapsto \left[\begin{array}{cc|cc} & & & \\ & & -1 & \\ \hline & & & \\ 1 & & & \end{array}\right]$$

Next, an induction result proved in [Hum78, Sec 21.1] allows us to compute the action of $\mathfrak{b}^0$ on $\mathfrak{U}(\mathfrak{r}_0)v_\lambda$. Extending to $\overline{M}(\lambda)$ we obtain the following.

$$h_\varepsilon \mapsto \left[\begin{array}{cc|cc} C - t\partial_t & & & \\ & C - t\partial_t - 1 & & \\ \hline & & C - t\partial_t & \\ & & & C - t\partial_t - 1 \end{array}\right]$$

$$e_\varepsilon \mapsto \left[\begin{array}{cc|cc} (C - \frac{1}{2}t\partial_t)\partial_t & -s & & \\ & (C - 1 - \frac{1}{2}t\partial_t)\partial_t & & \\ \hline & & (C - \frac{1}{2}t\partial_t)\partial_t & 1 \\ & & & (C - 1 - \frac{1}{2}t\partial_t)\partial_t \end{array}\right]$$

$$h_\delta \mapsto \left[\begin{array}{cc|cc} -C - 2s\partial_s & & & \\ & -C - 2s\partial_s - 2 & & \\ \hline & & -C - 2s\partial_s - 1 & \\ & & & -C - 2s\partial_s - 1 \end{array}\right]$$

$$e_{2\delta} \mapsto \left[\begin{array}{cc|cc} (-C - s\partial_s)\partial_s & t & & \\ & (-C - 2 - s\partial_s)\partial_s & & \\ \hline & & (-C - 1 - s\partial_s)\partial_s & \\ & & & (-C - 1 - s\partial_s)\partial_s \end{array}\right]$$

The remaining basis elements are computed as commutators of these.

$$e_\delta = [e_{2\delta}, e_{-\delta}] \mapsto \left[\begin{array}{cc|cc} & & -C - s\partial_s + t\partial_t & t \\ & & -\partial_s\partial_t & -\partial_s \\ \hline -\partial_s & -t & & \\ \partial_s\partial_t & -C + t\partial_t - s\partial_s & & \end{array}\right]$$

$$e_{-\varepsilon+\delta} = [e_{-\varepsilon}, e_\delta] \mapsto \left[\begin{array}{cc|cc} & & -t & 0 \\ & & \partial_s & 0 \\ \hline 0 & 0 & & \\ -\partial_s & -t & & \end{array}\right]$$

$$e_{\varepsilon-\delta} = [e_\varepsilon, e_{-\delta}] \mapsto \left[\begin{array}{cc|cc} & & -s\partial_t & -2s \\ & & -\frac{1}{2}\partial_t{}^2 & -\partial_t \\ \hline -\partial_t & 2s & & \\ \frac{1}{2}\partial_t{}^2 & -s\partial_t & & \end{array}\right]$$

$$e_{\varepsilon+\delta} = [e_\varepsilon, e_\delta] \mapsto \left[\begin{array}{cc|cc} & & (C + s\partial_s - \frac{1}{2}t\partial_t)\partial_t & 2C + t\partial_t - 2s\partial_s \\ & & \frac{1}{2}\partial_s\partial_t{}^2 & \partial_s\partial_t \\ \hline \partial_s\partial_t & 2C - 1 + t\partial_t - 2s\partial_s & & \\ -\frac{1}{2}\partial_s\partial_t{}^2 & (C + s\partial_s - \frac{1}{2}t\partial_t)\partial_t & & \end{array}\right]$$

4.2 HIGHEST WEIGHT VECTORS FOR $\overline{M}(\lambda)$

4.2.1 Predicted Highest Weights

4.2.1.1 *Nonisotropic Simple roots*

When

$$(\lambda + \rho^{(1)}, \varepsilon^\vee) = m^{(1)} = 2C + 1 \in \mathbb{N}^+$$

(that is, when $C \in \frac{1}{2}\mathbb{N}$), by Theorem 2.5.2 we have a Verma module embedding

$$M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda) \hookrightarrow M^{(1)}(\lambda)$$

where, by Lemma 2.6.1,

$$v_{\sigma_\varepsilon \underset{(1)}{\cdot} \lambda} \mapsto e_{-\varepsilon}^{2C+1}\lambda. \tag{4.2.1}$$

Similarly, when

$$(\lambda + \rho^{(2)}, \delta^\vee) = n^{(2)} = -2C + 1 \in \mathbb{N}^{\text{odd}}$$

(that is, when $C \in -\mathbb{N}$), we have a Verma module embedding

$$M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda) \hookrightarrow M^{(2)}(\lambda)$$

with

$$v_{\sigma_\delta \underset{(2)}{\cdot} \lambda} \mapsto e_{-\delta}^{-2C+1} v_\lambda = e_{-2\delta}^{-C} e_{-\delta} v_\lambda. \tag{4.2.2}$$

However, we find that these highest weight vectors lie in $\overline{M}(\lambda)$, rather than $\overline{M}(\lambda - \beta^{(i)})$, since $\overline{M}(\lambda - \beta^{(i)})$ is of highest weight $\lambda - \beta^{(i)}$, and

$$\sigma_\varepsilon \underset{(1)}{\cdot} \lambda \nleqslant \lambda - \beta^{(1)} \qquad \sigma_\delta \underset{(2)}{\cdot} \lambda \nleqslant \lambda - \beta^{(2)}.$$

4.2.1.2 *Nonisotropic Non-Simple Roots*

We expect to see a submodule of highest weight $\sigma_\varepsilon \underset{(2)}{\cdot} \lambda$ when

$$(\lambda + \rho^{(2)}, \varepsilon^\vee) = m^{(2)} = 2C - 1 \in \mathbb{N}^+,$$

and similarly a submodule of highest weight $\sigma_\delta \underset{(2)}{\cdot} \lambda$ when

$$(\lambda + \rho^{(1)}, \delta^\vee) = n^{(2)} = -2C - 1 \in \mathbb{N}^{\text{odd}}.$$

(Since Hypothesis 2.5.1 is not satisfied, these need not be Verma submodules.) However,

$$\sigma_\varepsilon \underset{(2)}{\cdot} \lambda \leqslant \lambda - \beta^{(2)} \qquad \sigma_\delta \underset{(1)}{\cdot} \lambda \leqslant \lambda - \beta^{(1)}.$$

In fact, we will see that the vectors of these highest weights lie in $\overline{M}(\lambda - \beta^{(i)})$. However, in these cases, we can find another highest weight of $\overline{M}(\lambda)$.

To see this, we first note that in $\overline{M}(\lambda)$, v_λ is highest weight for both Borels (since $\mathfrak{b}^{(1)} \cup \mathfrak{b}^{(2)} \subseteq \mathfrak{p}$), and we change Borels, per Lemma 2.4.1.

Beginning with the highest weight vector obtained above for $\mathfrak{b}^{(2)}$ and changing Borels, we obtain as a highest weight vector for $\mathfrak{b}^{(1)}$, of weight $\sigma_{\delta} \underset{(2)}{\cdot} (\lambda) + \beta^{(1)}$:

$$\begin{aligned} e_{-\beta^{(2)}} e_{-2\delta}^{-C} e_{-\delta} v_\lambda = e_{\beta^{(1)}} e_{-2\delta}^{-C} e_{-\delta} v_\lambda \\ &\propto e_{-2\delta}^{-C-1}(e_{-\varepsilon} e_{-2\delta} + C e_{-\delta} e_{-\varepsilon-\delta}) v_\lambda \\ &= e_{-2\delta}^{-C-1}(e_{-\varepsilon} e_{-2\delta} - e_{-\delta} e_{-\varepsilon-\delta} h_\delta) v_\lambda \end{aligned}$$

when $C \leqslant -1$, with the last equality holding because $h_\delta v_\lambda = -C v_\lambda$. When $C = 0$, $(\lambda + \rho^{(1)}, \varepsilon^\vee) = 1 \in \mathbb{N}^+$, and the highest weight vector obtained in this computation is $e_{-\varepsilon+\delta} e_{-\delta} v_\lambda \propto e_{-\varepsilon} v_\lambda$, the highest weight vector obtained directly in eq. (4.2.1).

Similarly, beginning with the highest weight vector for $\mathfrak{b}^{(1)}$ and changing Borels, we obtain as a highest weight vector for $\mathfrak{b}^{(2)}$, of weight $\sigma_{\varepsilon} \underset{(1)}{\cdot} (\lambda) + \beta^{(2)}$:

$$\begin{aligned} e_{-\beta^{(1)}} e_{-\varepsilon}^{2C+1} v_\lambda = e_{\beta^{(2)}} e_{-\varepsilon}^{2C+1} v_\lambda \\ &\propto e_{-\varepsilon}^{2C-1}(e_{-\varepsilon}^2 + (2C+1) e_{-\varepsilon} e_{-\delta} - C(2C+1) e_{-\varepsilon-\delta}) v_\lambda \\ &= e_{-\varepsilon}^{2C-1}(e_{-\varepsilon}^2 + e_{-\varepsilon} e_{-\delta} + 2 e_{-\varepsilon} e_{-\delta} h_\varepsilon - e_{-\varepsilon-\delta} h_\varepsilon - 2 e_{-\varepsilon-\delta} h_\varepsilon^2) v_\lambda \end{aligned}$$

when $C \geqslant \frac{1}{2}$. When $C = 0$, $(\lambda + \rho^{(2)}, \delta^\vee) = 1 \in \mathbb{N}^{\text{odd}}$, and the highest weight vector obtained in this computation is $e_{\varepsilon-\delta} e_{-\varepsilon} v_\lambda \propto e_{-\delta} v_\lambda$, again, the same as obtained directly in eq. (4.2.2).

Finally, we rewrite the weights in terms of the dot action for the appropriate Borel, noting that $\rho^{(i)} = \rho^{(j)} + \beta^{(j)}$ ($\{i, j\} = \{1, 2\}$), and so for $\zeta \in \{\varepsilon, \delta\}$,

$$\begin{aligned} \sigma_\zeta \underset{(i)}{\cdot} (\lambda) + \beta^{(j)} &= \sigma_\zeta(\lambda + \rho^{(i)}) - \rho^{(i)} + \beta^{(j)} \\ &= \sigma_\zeta(\lambda + \rho^{(j)} + \beta^{(j)}) - \rho^{(j)} \\ &= \sigma_\zeta \underset{(j)}{\cdot} (\lambda + \beta^{(j)}). \end{aligned}$$

Thus, the weights above are $\sigma_\delta \underset{(1)}{\cdot} (\lambda + \beta^{(1)})$ and $\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda + \beta^{(2)})$.

4.2.2 Explicit Computation via Matrices

To confirm that those found above are truly highest weights in $\overline{M}(\lambda)$, and there are no further highest weights, we compute directly from the matrices found in section 4.1.1.

First, note that any highest weight vector must be contained in a single weight space, and therefore (in terms of the basis given earlier) must be of the following forms:

$$(s^i t^j + ks^{i-1}t^{j-1}ab)v_\lambda \in \overline{M}(\lambda)^0, \text{ weight } \lambda - i\varepsilon - 2j\delta \tag{4.2.3}$$

$$(s^i t^j a + ks^i t^{j-1}b)v_\lambda \in \overline{M}(\lambda)^1, \text{ weight } \lambda - i\varepsilon - (2j+1)\delta. \tag{4.2.4}$$

Representing these vectors as column vectors in the free module $\overline{M}(\lambda)$, with basis ordered as in Lemma 4.1.2, we use matrix multiplication to identify all highest weight vectors. Further, we will introduce the following notation:

$$\left[\begin{smallmatrix} * \\ * \end{smallmatrix}\right]_0 := \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \qquad \left[\begin{smallmatrix} * \\ * \end{smallmatrix}\right]_1 := \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix}$$

The matrices in section 4.1.1 are block-diagonal or block-antidiagonal, and act on $\overline{M}(\lambda)$ in a straight forward manner. Elements of $\mathfrak{g}_0$ map $\overline{M}(\lambda)_0$ to itself as multiplication by the upper left block, $\overline{M}(\lambda)_1$ to itself as multiplication by the lower right block. Similarly, elements of $\mathfrak{g}_1$ map $\overline{M}(\lambda)_0$ to $\overline{M}(\lambda)_1$ as multiplication by the lower left block, $\overline{M}(\lambda)_1$ to $\overline{M}(\lambda)_0$ as multiplication by the upper right block. In order to simplify the computations to come, we introduce the following notation: for an element x of $\mathfrak{g}_0 \cup \mathfrak{g}_1$, write x^i for the block of the matrix image of x that acts on $\overline{M}(\lambda)_i$ ($i = 0, 1$). Thus, if $x \in \mathfrak{g}_0$,

$$x \mapsto \left[\begin{array}{c|c} x^0 & \\ \hline & x^1 \end{array}\right],$$

and if $x \in \mathfrak{g}_1$,

$$x \mapsto \left[\begin{array}{c|c} & x^1 \\ \hline x^0 & \end{array}\right].$$

Consider first the action of $\mathfrak{g}_0$, with positive roots $\{\varepsilon, 2\delta\}$. The elements $e_\varepsilon, e_{2\delta}$ will kill any highest weight vector for either Borel. For elements of $\overline{M}(\lambda)_0$, as expressed in eq. (4.2.3), multiplication by these two elements gives us

$$e_\varepsilon^0 = (C - \tfrac{1}{2}t\partial_t)\partial_t I + \begin{bmatrix} 0 & -s \\ 0 & -\partial_t \end{bmatrix}$$

$$e_\varepsilon^0 \begin{bmatrix} s^i t^j \\ n s^{i-1} t^{j-1} \end{bmatrix}_0 = \begin{bmatrix} (Cj - \frac{1}{2}j(j-1) - n)s^i t^{j-1} \\ (nC(j-1) - \frac{1}{2}n(j-1)(j-2) + n(j-1))s^{i-1}t^{j-2} \end{bmatrix}_0$$

$$e_{2\delta}^0 = (-C - s\partial_s)\partial_s I + \begin{bmatrix} 0 & t \\ 0 & -2\partial_s \end{bmatrix}$$

$$e_{2\delta}^0 \begin{bmatrix} s^i t^j \\ n s^{i-1} t^{j-1} \end{bmatrix}_0 = \begin{bmatrix} (-Ci - i(i-1) + n)s^{i-1}t^j \\ (-Cn(i-1) - n(i-1)(i-2) - 2n(i-1))s^{i-2}t^{j-1} \end{bmatrix}_0 .$$

Setting these equal to zero yields the following system of equations

$$\begin{aligned} 0 =& Cj - \tfrac{1}{2}j(j-1) - n \\ 0 =& n(j-1)(C - \tfrac{1}{2}(j-2) + 1) \\ 0 =& -Ci - i(i-1) + n \\ 0 =& n(i-1)(-C - (i-2) - 2) \end{aligned}$$

which has solutions as shown in table 4.1. Since e_δ lies in both Borels, and has relatively simple matrix image, it is a convenient element of $\mathfrak{g}_1$ to check first. (Note that it suffices to check elements of simple root spaces, so e_δ need only be checked for $\mathfrak{b}^{(2)}$, but since $e_\delta \in \mathfrak{b}^{(1)}$ also, any vector that fails to be killed by e_δ is not highest weight for either Borel.)

$$e_\delta^0 = \begin{bmatrix} -\partial_s & -t \\ \partial_s\partial_t & -C + t\partial_t - s\partial_s \end{bmatrix}$$

$$e_\delta^0 \begin{bmatrix} s^i t^j \\ n s^{i-1} t^{j-1} \end{bmatrix}_0 = \begin{bmatrix} (-i-n)s^{i-1}t^j \\ (ij - Cn + n(j-1) - n(i-1))s^{i-1}t^{j-1} \end{bmatrix}_1 .$$

	n	i	j	$i,j \in \mathbb{N}$	Notes
(1)	0	0	0	all C	$= v_\lambda$
(2)	0	$-C+1$	0	$C \in -\mathbb{N} \cup \{1\}$	Not killed by e_δ
(3)	0	0	$2C+1$	$C \in \frac{1}{2}\mathbb{N}$	
(4)	0	$-C+1$	$2C+1$	$C = 1$	same as (3)
(5)	C	1	1	all C	Not killed by e_δ
(6)	C	$-C$	1	$C \in -\mathbb{N}$	
(7)	C	1	$2C$	all C	Not killed by e_δ
(8)	C	$-C$	$2C$	$C = 0$	same as (1)

Table 4.1: Parameters for Highest Weight Vectors in $\overline{M}(\lambda)^0$

Setting this vector equal to zero gives the equations

$$0 = -i - n$$

$$0 = ij - Cn + n(j-1) - n(i-1)$$

which are satisfied only by (3) and (6). To decide for which Borels, if any, the vectors in (3) and (6) are highest weight, having already found that both are killed by e_ε and e_δ, we must check whether they are killed by $e_{\varepsilon-\delta}$ or $e_{-\varepsilon+\delta}$. These correspond to the highest weight vectors

$$e_{-\varepsilon}^{2C+1} v_\lambda = \begin{bmatrix} t^{2C+1} \\ 0 \end{bmatrix}_0 \tag{4.2.5}$$

$$(e_{-2\delta}^{-C} e_{-\varepsilon} + C e_{-\varepsilon}^{-C-1} e_{-\delta} e_{-\varepsilon-\delta}) v_\lambda = \begin{bmatrix} s^{-C} t \\ C s^{-C-1} \end{bmatrix}_0 . \tag{4.2.6}$$

These are both highest weight with Borel $\mathfrak{b}^{(1)}$:

$$e_{-\varepsilon+\delta}^0 = \begin{bmatrix} 0 & 0 \\ -\partial_s & -t \end{bmatrix}$$

$$e_{-\varepsilon+\delta}^0 \begin{bmatrix} t^{2C+1} \\ 0 \end{bmatrix}_0 = 0$$

$$e_{-\varepsilon+\delta}^0 \begin{bmatrix} s^{-C} t \\ C s^{-C-1} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ C s^{-C-1} t - C s^{-C-1} t \end{bmatrix}_1 = 0$$

	n	i	j	$i,j \in \mathbb{N}$	Notes
(1)	0	0	0	all C	$= v_\lambda$
(2)	0	$-C$	0	$C \in -\mathbb{N}$	
(3)	0	0	$2C+1$	$C \in \frac{1}{2}\mathbb{N}$	Not killed by e_δ
(4)	0	$-C$	$2C+1$	$C = 0$	same as (5)
(5)	$-C$	0	1	all C	Not killed by e_δ
(6)	$-C$	$-C$	1	$C \in -\mathbb{N}$	Not killed by e_δ
(7)	$-C$	0	$2C$	all C	
(8)	$-C$	$-C$	$2C$	$C = 0$	same as (1)

Table 4.2: Parameters for Highest Weight Vectors in $\overline{M}(\lambda)^1$

but neither is highest weight with Borel $\mathfrak{b}^{(2)}$:

$$e^0_{\varepsilon-\delta} = \begin{bmatrix} -\partial_t & 2s \\ \frac{1}{2}{\partial_t}^2 & -s\partial_s \end{bmatrix}$$

$$e^0_{\varepsilon-\delta}\begin{bmatrix} t^{2C+1} \\ 0 \end{bmatrix}_0 = \begin{bmatrix} -(2C+1)t^{2C} \\ \frac{1}{2}(2C+1)(2C)t^{2C-1} \end{bmatrix}_1 \neq 0$$

$$e^0_{\varepsilon-\delta}\begin{bmatrix} s^{-C}t \\ Cs^{-C-1} \end{bmatrix}_0 = \begin{bmatrix} -s^{-C}+2Cs^{-C} \\ 0 \end{bmatrix}_1 \neq 0.$$

We next perform the same calculations on elements of $\overline{M}(\lambda)_1$ as expressed in eq. (4.2.4). Checking $\mathfrak{g}_0$ first,

$$e^1_\varepsilon = (C - \tfrac{1}{2}\partial_t)\partial_t I + \begin{bmatrix} 0 & 1 \\ 0 & -\partial_t \end{bmatrix}$$

$$e^1_\varepsilon \begin{bmatrix} s^i t^j \\ ns^i t^{j-1} \end{bmatrix}_1 = \begin{bmatrix} C(j - \frac{1}{2}j(j-1) + n)s^i t^{j-1} \\ n(C(j-1) - \frac{1}{2}(j-1)(j-2) - (j-1))s^i t^{j-2} \end{bmatrix}_1$$

$$e^1_{2\delta} = (-C - 1 - s\partial_s)\partial_s I$$

$$e^1_{2\delta} \begin{bmatrix} s^i t^j \\ ns^i t^{j-1} \end{bmatrix}_1 = \begin{bmatrix} (-Ci - i - i(i-1))s^{i-1}t^j \\ n(-Ci - i - i(i-1))s^{i-1}t^{j-1} \end{bmatrix}_1 .$$

This gives the system of equations

$$0 = i(-C - 1 - (i-1))$$

$$0 = Cj - \tfrac{1}{2}j(j-1) + n$$
$$0 = n(j-1)(C - \tfrac{1}{2}(j-2) - 1)$$

with solutions as shown in table 4.2.

Again, we check e_δ next.

$$e_\delta^1 = \begin{bmatrix} -C - s\partial_s + t\partial_t & t \\ -\partial_s\partial_t & -\partial_t \end{bmatrix}$$
$$e_\delta^1 \begin{bmatrix} s^i t^j \\ n s^i t^{j-1} \end{bmatrix}_1 = \begin{bmatrix} (-C - i + j + n)s^i t^j \\ (-ij - ni)s^{i-1}t^{j-1} \end{bmatrix}_0$$

which fails to kill all but (2) and (7). So we have two possible highest weight vectors,

$$e_{-2\delta}^C e_{-\delta} v_\lambda = \begin{bmatrix} s^{-C} \\ 0 \end{bmatrix}_1 \tag{4.2.7}$$
$$(e_{-\varepsilon}^{2C} e_{-\delta} - C e_{-\varepsilon}^{2C-1} e_{-\varepsilon-\delta}) v_\lambda = \begin{bmatrix} t^{2C} \\ -Ct^{2C-1} \end{bmatrix}_1. \tag{4.2.8}$$

Checking with remaining primitive root vectors, we see that neither of these is highest weight with $\mathfrak{b}^{(1)}$:

$$e_{-\varepsilon+\delta}^1 = \begin{bmatrix} -t & 0 \\ \partial_s & 0 \end{bmatrix}$$
$$e_{-\varepsilon+\delta}^1 \begin{bmatrix} s^{-C} \\ 0 \end{bmatrix}_1 = \begin{bmatrix} -s^{-C}t \\ -Cs^{-C-1} \end{bmatrix}_0 \neq 0$$
$$e_{-\varepsilon+\delta}^1 \begin{bmatrix} s^{-C} \\ 0 \end{bmatrix}_1 = \begin{bmatrix} -t^{2C+1} \\ 0 \end{bmatrix}_0 \neq 0$$

but both are highest weight with $\mathfrak{b}^{(2)}$:

$$e_{\varepsilon-\delta}^1 = \begin{bmatrix} -s\partial_t & -2s \\ -\frac{1}{2}\partial_t{}^2 & -\partial_t \end{bmatrix}$$
$$e_{\varepsilon-\delta}^1 \begin{bmatrix} s^{-C} \\ 0 \end{bmatrix}_1 = 0$$
$$e_{\varepsilon-\delta}^1 \begin{bmatrix} t^{2C} \\ -Ct^{2C-1} \end{bmatrix}_1 = \begin{bmatrix} -2Cst^{2C-1} + 2Cst^{2C-1} \\ -\frac{1}{2}(2C)(2C-1)t^{2C-2} + C(2C-1)t^{2C-2} \end{bmatrix}_0 = 0.$$

The argument above shows the following:

Lemma 4.2.1. *Highest weight vectors of $\overline{M}(\lambda)$ are as follows.*

(a) With $\mathfrak{b}^{(1)}$, if $C \in \frac{1}{2}\mathbb{N}$, then $\varepsilon \in A(\lambda)$ and ε is simple. In this case, $\overline{M}(\lambda)$ contains a unique highest weight vector

$$e_{-\varepsilon}^{2C+1} v_\lambda$$

of weight $\sigma_\varepsilon \underset{(1)}{\cdot} \lambda$.

(b) With $\mathfrak{b}^{(1)}$, if $C \in -\mathbb{N}, C \neq 0$, then $\delta \in A(\lambda)$ and δ is non-simple. In this case, $\overline{M}(\lambda)$ contains a unique highest weight vector

$$(e_{-2\delta}^{-C} e_{-\varepsilon} + C e_{-2\delta}^{-C-1} e_{-\delta} e_{-\varepsilon-\delta}) v_\lambda$$

of weight $\sigma_\delta \underset{(1)}{\cdot} (\lambda + \beta^{(1)})$.

(c) With $\mathfrak{b}^{(2)}$, if $C \in -\mathbb{N}$, then $\delta \in A(\lambda)$ and δ is simple. In this case, $\overline{M}(\lambda)$ contains a unique highest weight vector

$$e_{-2\delta}^{-C} e_{-\delta} v_\lambda$$

of weight $\sigma_\delta \underset{(2)}{\cdot} \lambda$.

(d) With $\mathfrak{b}^{(2)}$, if $C \in \frac{1}{2}\mathbb{N}, C \neq 0$, then $\varepsilon \in A(\lambda)$ and ε is non-simple. In this case, $\overline{M}(\lambda)$ contains a unique highest weight vector

$$(e_{-\varepsilon}^{2C} e_{-\delta} - C e_{-\varepsilon}^{2C-1} e_{-\varepsilon-\delta}) v_\lambda$$

of weight $\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda + \beta^{(2)})$.

(e) In all other cases, $\overline{M}(\lambda)$ contains no highest weight vector except v_λ.

Proof. It remains only to determine the weights of the highest weight vectors found above. This may be done directly by examining the weights of the given vectors, or by noting the computations in section 4.2.1, above. □

4.3 DECOMPOSITION OF $\overline{M}(\lambda)$

We are now in the position to fully describe the structure of $\overline{M}(\lambda)$ in all cases.

Theorem 4.3.1. *If $C \in \mathbb{Z} \cup \frac{1}{2}\mathbb{N}$, $\overline{M}(\lambda)$ has a unique simple submodule with highest weight as given in Lemma 4.2.1 and simple quotient. In all other cases, $\overline{M}(\lambda)$ is simple.*

Proof. In light of Lemma 4.2.1, we need only show that, when $\overline{M}(\lambda)$ has a highest weight submodule, the quotient is simple, as the remaining results follow from the fact that every submodule must contain a highest weight vector. Call the highest weight μ, $N_\mu = \mathfrak{U}(\mathfrak{g})v_\mu$, and write $L = \overline{M}(\lambda)/N_\mu$. We call on Lemma 2.8.4 to show that L contains no highest weight vector, and is thus simple.

Potential highest weights of L are of the form $\lambda - \zeta$, where $\zeta = A\varepsilon + B\delta$, $A, B \in \mathbb{N}$ (since $\Delta_{\mathfrak{r}} = \{-\varepsilon, -\varepsilon-\delta, -\delta, -2\delta\}$). Recall that in the current case $\lambda = C(\varepsilon - \delta)$, and depending on choice of Borel, $\rho = \pm\frac{1}{2}(\varepsilon - \delta)$; so $2(\lambda - \rho) = (2C \pm 1)(\varepsilon - \delta)$. Thus, if $\lambda - \zeta$ is a highest weight in L, we must have

$$\begin{aligned} 2(\lambda + \rho, \zeta) &= (\zeta, \zeta) \\ (2C \pm 1)(\varepsilon - \delta, A\varepsilon + B\delta) &= (A\varepsilon + B\delta, A\varepsilon + B\delta) \\ (2C \pm 1)(A + B) &= A^2 - B^2. \end{aligned}$$

Note that $\zeta = 0$ is uninteresting, and A, B are nonnegative, so $A + B \neq 0$, and thus ζ must satisfy

$$2C \pm 1 = A - B. \tag{4.3.1}$$

We now pass to cases, and show that in each case from Lemma 4.2.1, all possible solutions are contained in N_μ.

(a) $\mathfrak{b} = \mathfrak{b}^{(1)}$, $C \in \frac{1}{2}\mathbb{N}$, $\mu = \lambda - (2C + 1)\varepsilon$. In this case and the next, $\rho = \frac{1}{2}(\varepsilon - \delta)$, so eq. (4.3.1) becomes

$$2C + 1 = A - B.$$

Note that with $C \in \frac{1}{2}\mathbb{N}$, $2C+1$ is a positive integer. Nonnegative integer solutions to this equation are of the form

$$A = B + 2C + 1$$

so potential highest weights are of the form

$$\lambda - (2C + 1 + B)\varepsilon - B\delta = \mu - B(\varepsilon + \delta).$$

But these are of lower weight than μ and so contained in N_μ.

(b) $\mathfrak{b} = \mathfrak{b}^{(1)}$, $C \in -\mathbb{N}$, $C \leqslant -1$, $\mu = \lambda - \varepsilon + 2C\delta$. In this case, $2C + 1$ is a negative integer, so nonnegative integer solutions are of the form

$$B = A - 2C - 1$$

and so potential highest weights are of the form

$$\lambda - A\varepsilon - (A - 2C - 1)\delta = \mu - (A - 1)(\varepsilon + \delta).$$

If $A > 0$, these are lower weights than μ and so contained in N_μ.

If $A = 0$, we have a potential highest weight $\mu + \varepsilon + \delta$. The weight space is one-dimensional, with basis vector $v = e_{-\varepsilon}^{-C-1} e_{-\varepsilon+\delta} v_\lambda$. But multiplying this vector by $e_{-\varepsilon+\delta}$ gives a vector of weight $\mu + 2\delta$, which is a higher weight than μ and thus not in N_μ. Thus the image of v in L is not killed by $e_{-\varepsilon+\delta}$, so $\mu + \varepsilon + \delta$ is not a highest weight either.

(c) $\mathfrak{b} = \mathfrak{b}^{(2)}$, $C \in -\mathbb{N}$, $C \leqslant -1$, $\mu = \lambda - (-2C + 1)\delta$. In this case and the next, $\rho = -\frac{1}{2}(\varepsilon - \delta)$, so eq. (4.3.1) becomes

$$2C - 1 = A - B.$$

In this case, $2C - 1$ is a negative integer, so nonnegative integer solutions are of the form

$$B = A - 2C + 1$$

and so potential highest weights are of the form

$$\lambda - A\varepsilon - (A - 2C + 1)\delta = \mu - A(\varepsilon + \delta).$$

These are lower weights than μ and so are contained in N_μ.

(d) $\mathfrak{b} = \mathfrak{b}^{(2)}$, $C \in \frac{1}{2}\mathbb{N}$, $C > 0$, $\mu = \lambda - 2C\varepsilon - \delta$. In this case $2{-}C - 1$ is a nonnegative integer, so nonnegative integer solutions are of the form

$$A = B + 2C - 1$$

and so potential highest weights are of the form

$$\lambda - (B + 2C - 1)\varepsilon - B\delta = \mu - (B - 1)(\varepsilon + \delta).$$

When $B > 0$, these are lower weights than μ and thus contained in N_μ.

When $B = 0$, we have potential highest weight $\mu + \varepsilon + \delta$. Multiplying a vector of this weight by $e_{\varepsilon-\delta}$ gives a vector of weight $\mu + 2\varepsilon$, which is a higher weight than μ, and so not in N_μ. Thus the image of this vector is not highest weight in L.

So in all cases where $\overline{M}(\lambda)$ is not simple, the quotient L has no highest weight other than λ and is thus simple. □

Thus (recalling, Lemma 2.2.1, that a simple highest weight module is uniquely determined by its highest weight), $\overline{M}(\lambda)$ decomposes as follows.

If $\mathfrak{b} = \mathfrak{b}^{(1)}, (\lambda + \rho^{(1)}, \varepsilon^\vee) \in \mathbb{N}^+$ or $\mathfrak{b} = \mathfrak{b}^{(2)}, (\lambda + \rho^{(2)}, \delta^\vee) \in \mathbb{N}^{\text{odd}}$,

$$0 \to L(\sigma_\alpha.\lambda) \to \overline{M}(\lambda) \to L(\lambda) \to 0.$$

If $\mathfrak{b} = \mathfrak{b}^{(1)}, (\lambda + \rho^{(1)}, \delta^\vee) \in \mathbb{N}^{\text{odd}}$ or $\mathfrak{b} = \mathfrak{b}^{(2)}, (\lambda + \rho^{(2)}, \varepsilon^\vee) \in \mathbb{N}^+$,

$$0 \to L(\sigma_{\alpha+\beta}.(\lambda + \beta)) \to \overline{M}(\lambda) \to L(\lambda) \to 0.$$

4.4 HIGHEST WEIGHT VECTORS FROM $\overline{M}(\lambda)$

We now return from $\overline{M}(\lambda)$ to consider the structure of $M(\lambda)$. Recall (Lemma 4.1.1) that $M(\lambda)$ has short exact sequence

$$0 \to \overline{M}(\lambda - \beta) \to M(\lambda) \to \overline{M}(\lambda) \to 0.$$

Thus $\overline{M}(\lambda - \beta)$ embeds in $M(\lambda)$, so highest weight vectors in $\overline{M}(\lambda - \beta)$ continue to be highest weight in $M(\lambda)$. On the other hand, $\overline{M}(\lambda)$ is a quotient; so highest weight vectors in $\overline{M}(\lambda)$ need only be highest weight in $M(\lambda)$ modulo $\overline{M}(\lambda-\beta)$; in particular, they need only be highest weight modulo $e_{-\beta}\mathfrak{U}(\mathfrak{g})$. In the discussion to follow, special attention is paid to these vectors. The associated submodules will not in general be highest weight modules.

4.5 CASES WHERE $M(\lambda)$ HAS LENGTH 3

When $\lambda = -\rho$, $M(\lambda)$ is uniserial of length 3.

When $\lambda = -\rho^{(1)} = -\frac{1}{2}(\varepsilon - \delta)$, $b = \mathfrak{b}^{(1)}$, note that $\lambda - \beta^{(1)} = \frac{1}{2}(\varepsilon - \delta)$. Thus, by Theorem 4.3.1, $\overline{M}(\lambda)$ is simple, but $\overline{M}(\lambda - \beta^{(1)})$ has length 2, with a unique simple submodule of highest weight

$$\sigma_{\overset{\cdot}{\varepsilon}_{(1)}}(\lambda - \beta^{(1)}) = \lambda - \beta^{(1)} - 2\varepsilon$$

generated by the highest weight vector $e_{-\varepsilon}^2 e_{-\beta^{(1)}} v_\lambda$. Thus $M^{(1)}(-\rho^{(1)})$ has a unique composition series

$$0 \subseteq L(\sigma_{\overset{\cdot}{\varepsilon}_{(1)}}\lambda) \subseteq \overline{M}(\lambda - \beta^{(1)}) \subseteq M^{(1)}(\lambda).$$

This is unique because each term in the series is its successor's unique maximal submodule.

When $\lambda = -\rho^{(2)} = \frac{1}{2}(\varepsilon - \delta)$, $b = \mathfrak{b}^{(2)}$, note that $\lambda - \beta^{(2)} = -\frac{1}{2}(\varepsilon - \delta)$. Thus, by Theorem 4.3.1, $\overline{M}(-\rho^{(2)} - \beta^{(2)})$ is simple, but $\overline{M}(\lambda)$ has length 2, with unique simple submodule of highest weight

$$\sigma_{\varepsilon_{\dot{(2)}}}(\lambda + \beta^{(2)}) = \lambda + \beta^{(2)} - 2\varepsilon$$

generated by the highest weight vector $w = (e_{-\varepsilon}e_{-\delta} - \frac{1}{2}e_{-\varepsilon-\delta})v_\lambda$. (Note that w is a singular vector in $M^{(2)}(\lambda)$.) Let N be the submodule of $M^{(2)}(\lambda)$ generated by w. Thus $M^{(2)}(-\rho^{(2)})$ has unique composition series

$$0 \subseteq \overline{M}(\lambda - \beta^{(2)}) \subseteq N \subseteq M^{(2)}(\lambda).$$

This series is unique because N contains no highest weight vector other than $\lambda - \beta^{(2)}$, and thus unique submodule is $\overline{M}(\lambda - \beta^{(2)})$; N is in turn the unique maximal submodule of $M^{(2)}(\lambda)$.

4.6 CASES WHERE $M(\lambda)$ HAS LENGTH 4

When $C \in \frac{1}{2}\mathbb{N} \cup -\mathbb{N}$ and $\lambda + \rho \neq 0$, $C \pm 1 \in \frac{1}{2}\mathbb{N} \cup -\mathbb{N}$, so $\overline{M}(\lambda - \beta)$ has length 2 precisely when $\overline{M}(\lambda)$ does. So in these cases, $M(\lambda)$ has length 4.

4.6.1 Simple Root Case

These are the cases appearing in parts (a) and (c) of Lemma 4.2.1, where $\mathfrak{b} = \mathfrak{b}^{(1)}$ and $(\lambda + \rho, \varepsilon^\vee) \in \mathbb{N}^+$ or $\mathfrak{b} = \mathfrak{b}^{(2)}$ and $(\lambda + \rho, \delta^\vee) \in \mathbb{N}^{\text{odd}}$.

In these cases, we see the submodule lattice shown in fig. 4.1, where a node labeled by a weight represents a composition factor of that highest weight.

When $\mathfrak{b} = \mathfrak{b}^{(1)}$ (and $C \in \frac{1}{2}\mathbb{N}$) the highest weight vectors are

$$\begin{aligned} &v_\lambda \\ v_{\lambda-\beta} &= e_{\varepsilon-\delta}v_\lambda \\ v_{\sigma_\varepsilon.\lambda} &= e_{-\varepsilon}^{2C+1}v_\lambda \\ v_{\sigma_\varepsilon.(\lambda+\varepsilon-\delta)} &= e_{-\varepsilon}^{2C+3}e_{\varepsilon-\delta}v_\lambda. \end{aligned}$$

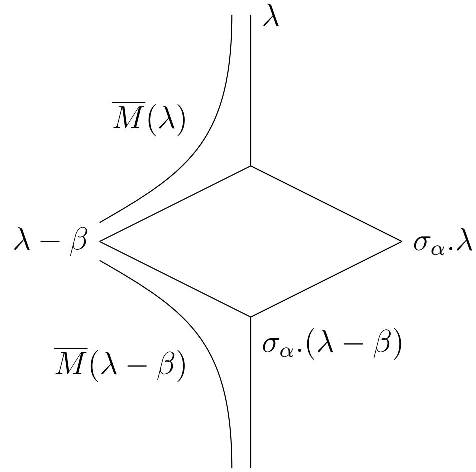


Figure 4.1: Submodule lattice when $(\lambda+\rho,\beta)=0$ and $(\lambda+\rho,\alpha^\vee)$ is integral.

When $\mathfrak{b}=\mathfrak{b}^{(2)}$ (and $C\in-\mathbb{N}$) the highest weight vectors are

$$\begin{aligned} &v_\lambda \\ v_{\lambda-\beta} &= e_{-\varepsilon+\delta}v_\lambda \\ v_{\sigma_\delta.\lambda} &= e_{-2\delta}^{-C}e_{-\delta}v_\lambda \\ v_{\sigma_\delta.(\lambda-\varepsilon+\delta)} &= e_{-2\delta}^{-C+1}e_{-\delta}e_{-\varepsilon+\delta}v_\lambda. \end{aligned}$$

4.6.2 Non-Simple Root Case

These are the cases (b) and (d) of Lemma 4.2.1, where $\mathfrak{b}=\mathfrak{b}^{(1)}$ and $(\lambda+\rho^{(1)},\delta^\vee)\in\mathbb{N}^{\text{odd}}$ or $\mathfrak{b}=\mathfrak{b}^{(1)}$ and $(\lambda+\rho^{(2)},\varepsilon^\vee)\in\mathbb{N}^+$. In these cases, we see the submodule lattice shown in fig. 4.2, where a node labeled by a weight represents a composition factor of that highest weight.

The module on the right, labeled N in the diagram, is not a highest weight module. It is the preimage under the projection $\pi: M(\lambda)\to\overline{M}(\lambda)$ described in Lemma 4.1.1 of the submodule of $\overline{M}(\lambda)$ described in Lemma 4.2.1 part (b) or (d).

When $\mathfrak{b}=\mathfrak{b}^{(1)}$ (and $C\in-\mathbb{N}, C\leqslant 1$) the highest weight vectors are

$$v_\lambda$$

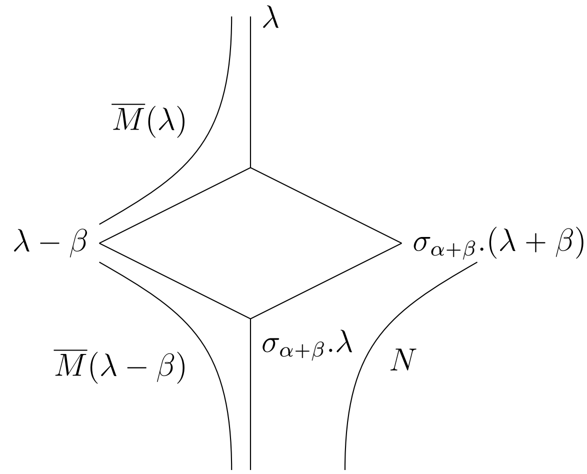


Figure 4.2: Submodule lattice when $(\lambda+\rho,\beta)=0$ and $(\lambda+\rho,(\alpha+\beta)^\vee)$ is integral.

$$v_{\lambda-\beta} = e_{\varepsilon-\delta}v_\lambda$$

$$v_{\sigma_\delta.\lambda} = (e_{-2\delta}^{-C-1}e_{-\varepsilon} + (C+1)e_{-2\delta}^{-C-2}e_{-\delta}e_{-\varepsilon-\delta})e_{\varepsilon-\delta}v_\lambda$$

and the singular vector which generates the module N is

$$v_{\sigma_\delta.(\lambda+\beta)} = (e_{-2\delta}^{-C}e_{-\varepsilon} + Ce_{-2\delta}^{-C-1}e_{-\delta}e_{-\varepsilon-\delta})v_\lambda.$$

When $\mathfrak{b}=\mathfrak{b}^{(2)}$ (and $C\in\frac{1}{2}\mathbb{N}, C\geqslant 1$) the highest weight vectors are

$$v_\lambda$$

$$v_{\lambda-\beta} = e_{-\varepsilon+\delta}v_\lambda$$

$$v_{\sigma_\varepsilon.\lambda} = (e_{-\varepsilon}^{2C-2}e_{-\delta} - (C-1)e_{-\varepsilon}^{2C-3}e_{-\varepsilon-\delta})e_{-\varepsilon+\delta}v_\lambda$$

and the singular vector which generates the module N is

$$v_{\sigma_\varepsilon.(\lambda+\beta)} = (e_{-\varepsilon}^{2C}e_{-\delta} - Ce_{-\varepsilon}^{2C-1}e_{-\varepsilon-\delta})v_\lambda.$$

4.7 CASES WHERE $M(\lambda)$ HAS LENGTH 2

For all $\lambda=C(\varepsilon-\delta)$, $M(\lambda)$ has length at least 2, per Lemma 4.1.1. When both $\overline{M}(\lambda)$ and $\overline{M}(\lambda-\beta)$ are simple, this is a complete decomposition, and $M(\lambda)$ has length 2.

This occurs in all cases except those discussed above.

4.8 COMPOSITION FACTORS OF $M(\lambda)$

This section summarizes the structure seen above. From section 4.5, we see Corollaries 4.8.1 and 4.8.2.

Corollary 4.8.1. *If $\lambda + \rho^{(1)} = 0$, $M^{(1)}(\lambda)$ has composition factors of highest weights*

(a) λ,

(b) $\lambda - \beta^{(1)}$, *and*

(c) $\sigma_{\varepsilon_{(1)}} \cdot (\lambda - \beta^{(1)})$.

Corollary 4.8.2. *If $\lambda + \rho^{(2)} = 0$, $M^{(2)}(\lambda)$ has composition factors of highest weights*

(a) λ,

(b) $\lambda - \beta^{(2)}$, *and*

(c) $\sigma_{\varepsilon_{(2)}} \cdot (\lambda + \beta^{(2)}) = \sigma_{\varepsilon_{(2)}} \cdot \lambda - (\varepsilon + \delta)$.

All composition factors are of multiplicity 1.

From section 4.6.1 we see Corollaries 4.8.3 and 4.8.4, and from section 4.6.2 we see Corollaries 4.8.5 and 4.8.6.

Corollary 4.8.3. *If $\lambda = C(\varepsilon - \delta)$, $C \in \frac{1}{2}\mathbb{N}$, $M^{(1)}(\lambda)$ has composition factors of highest weight*

(a) λ,

(b) $\lambda - \beta^{(1)}$,

(c) $\sigma_{\varepsilon_{(1)}} \cdot \lambda$, *and*

(d) $\sigma_{\varepsilon_{(1)}} \cdot (\lambda - \beta^{(1)})$.

Corollary 4.8.4. *If $\lambda = C(\varepsilon - \delta)$, $C \in -\mathbb{N}$, $M^{(2)}(\lambda)$ has composition factors of highest weight*

(a) λ,

(b) $\lambda - \beta^{(2)}$,

(c) $\sigma_{\delta_{(2)}} \cdot \lambda$, *and*

(d) $\sigma_{\delta_{(2)}} \cdot (\lambda - \beta^{(2)})$.

Corollary 4.8.5. *If* $\lambda = C(\varepsilon-\delta)$, $C \in -\mathbb{N}$, $C \leqslant -1$, $M^{(1)}(\lambda)$ *has composition factors of highest weight*

(a) λ,

(b) $\lambda - \beta^{(1)}$,

(c) $\sigma_{\delta_{(1)}} \cdot (\lambda + \beta^{(1)}) = \sigma_{\delta_{(1)}} \cdot \lambda - (\varepsilon+\delta)$, *and*

(d) $\sigma_{\delta_{(1)}} \cdot \lambda$.

Corollary 4.8.6. *If* $\lambda = C(\varepsilon - \delta)$, $C \in \frac{1}{2}\mathbb{N}$, $C \geqslant 1$, $M^{(2)}(\lambda)$ *has composition factors of highest weight*

(a) λ,

(b) $\lambda - \beta^{(2)}$,

(c) $\sigma_{\varepsilon_{(2)}} \cdot (\lambda + \beta^{(2)}) = \sigma_{\varepsilon_{(2)}} \cdot \lambda - (\varepsilon+\delta)$, *and*

(d) $\sigma_{\varepsilon_{(2)}} \cdot \lambda$.

Finally, section 4.7 shows Corollary 4.8.7.

Corollary 4.8.7. *If* $\lambda = C(\varepsilon - \delta)$, $C \notin \frac{1}{2}\mathbb{N} \cup -\mathbb{N}$, $M^{(i)}(\lambda)$ *has composition factors of highest weight*

(a) λ, *and*

(b) $\lambda - \beta^{(i)}$.

CHAPTER 5

VERMA MODULES OF ATYPICAL WEIGHT: TYPE 2

In this chapter, we consider weights orthogonal to $\gamma = \varepsilon + \delta$, so $\lambda + \rho^{(i)} = \frac{m}{2}(\varepsilon + \delta)$. This notation is chosen for the coefficient because we will frequently use

$$(\lambda + \rho^{(i)}, \varepsilon^\vee) = (\lambda + \rho^{(i)}, \delta^\vee) = m.$$

Note that in this case λ is $\beta^{(i)}$-typical, and so if we write

$$\lambda^{(1)} + \rho^{(1)} = \lambda^{(2)} + \rho^{(2)} = \tfrac{m}{2}(\varepsilon + \delta),$$

then by Lemma 2.4.1, $M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)})$. When this is the case, write

$$M = M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)}).$$

In the remainder of the section, we will denote by $L^{(i)}(\zeta)$ the simple module of highest weight ζ with respect to $\mathfrak{b}^{(i)}$.

5.1 HIGHEST WEIGHT $\lambda - \gamma$

Throughout this section, we suppose $m \in \mathbb{N}$ and $m \geqslant 3$. In this case, there are elements $\theta_\gamma^{(i)}$ of the weight space $\mathfrak{U}(\mathfrak{n}^{-(i)})^\gamma$ such that, if $v_{\lambda^{(1)}}, w_{\lambda^{(2)}}$ are highest weight vectors of M with respect to $\mathfrak{b}^{(1)}, \mathfrak{b}^{(2)}$ respectively, $\theta_\gamma^{(1)} v_{\lambda^{(1)}}, \theta_\gamma^{(2)} w_{\lambda^{(2)}}$ are highest weight vectors of M with respect to $\mathfrak{b}^{(1)}, \mathfrak{b}^{(2)}$ of weight $\lambda^{(1)} - \gamma, \lambda^{(2)} - \gamma$, respectively. We show that these elements exist by computing them, specifically,

$$\begin{aligned} \theta_\gamma^{(1)} &= \tbinom{m}{2} e_{-\varepsilon-\delta} - \tbinom{m}{1} e_{-\varepsilon} e_{-\delta} + e_{-\varepsilon}^2 e_{\varepsilon-\delta} \\ \theta_\gamma^{(2)} &= \tbinom{m}{2} e_{-\varepsilon-\delta} - \tbinom{m}{1} e_{-\delta} e_{-\varepsilon} - \tbinom{m}{1} e_{-\delta}^2 e_{-\varepsilon+\delta}. \end{aligned}$$

These elements are known as Šapovalov elements.

This easily shows the following lemma.

Lemma 5.1.1. *If $(\lambda^{(i)} + \rho^{(i)}, \gamma) = 0$, then $\lambda^{(i)} - \gamma$ is a highest weight of $M^{(i)}(\lambda^{(i)})$.*

We next consider the submodule of highest weight $\lambda^{(i)} - \gamma$ generated by $\theta_\gamma^{(1)} v_{\lambda^{(1)}}$, $\theta_\gamma^{(2)} w_{\lambda^{(2)}}$.

Lemma 5.1.2. *Let $v_{\lambda^{(1)}}$ be an element of M of highest weight $\lambda^{(1)}$ with respect to $\mathfrak{b}^{(1)}$. Let $w_{\lambda^{(2)}}$ be an element of M of highest weight $\lambda^{(2)}$ with respect to $\mathfrak{b}^{(2)}$.*

(a) $\mathfrak{U}(\mathfrak{g})\theta_\gamma^{(1)} v_{\lambda^{(1)}} = \mathfrak{U}(\mathfrak{g})\theta_\gamma^{(2)} w_{\lambda^{(2)}}$ as submodules of M.

(b) This submodule (hereafter $M_{\lambda-\gamma}$) is a proper submodule of M.

Proof. First, note that, since λ is typical with respect to $\beta^{(1)}, \beta^{(2)}$, by Lemma 2.4.1, $w_{\lambda^{(2)}} \propto e_{-\beta^{(1)}} v_{\lambda^{(1)}}$ and $v_{\lambda^{(1)}} \propto e_{-\beta^{(2)}} w_{\lambda^{(1)}}$.

Now, since $\lambda - \gamma$ is a typical weight, showing part (a) amounts to showing that

$$\theta_\gamma^{(1)} v_{\lambda^{(1)}} \in \mathfrak{U}(\mathfrak{g})\theta_\gamma^{(2)} w_{\lambda^{(2)}} \tag{5.1.1}$$

and

$$\theta_\gamma^{(2)} w_{\lambda^{(2)}} \in \mathfrak{U}(\mathfrak{g})\theta_\gamma^{(1)} v_{\lambda^{(1)}}. \tag{5.1.2}$$

First, to show eq. (5.1.1), suppose without loss of generality that $w_{\lambda^{(2)}} = e_{-\beta^{(1)}} v_{\lambda^{(1)}}$. Then computation shows that

$$\theta_\gamma^{(1)} v_{\lambda^{(1)}} \propto e_{-\varepsilon+\delta} \theta_\gamma^{(2)} e_{\varepsilon-\delta} v_{\lambda^{(1)}} = e_{-\varepsilon+\delta} \theta_\gamma^{(2)} w_{\lambda^{(2)}}$$

which shows the desired inclusion.

Next, to show eq. (5.1.2), suppose, again without loss of generality, that $v_{\lambda^{(1)}} = e_{-\beta^{(2)}} v_{\lambda^{(1)}}$. Another computation shows that

$$\theta_\gamma^{(2)} w_{\lambda^{(2)}} \propto e_{\varepsilon-\delta} \theta_\gamma^{(1)} e_{-\varepsilon+\delta} w_{\lambda^{(2)}} = e_{\varepsilon-\delta} \theta_\gamma^{(1)} v_{\lambda^{(1)}}$$

which shows the other inclusion.

Part (b) follows easily from the fact that $M_{\lambda-\gamma}$ is generated by a highest weight vector of weight less than $\lambda^{(i)}$ for each Borel. □

5.2 CASES WHERE M HAS LENGTH 8

We see the most complex structure, length 8, when $m \in \mathbb{N}^{\text{odd}}$, $m > 1$. In this section, we determine the composition factors of M by considering the quotients of M by its Verma submodules, and then comparing the two quotients; see Theorems 5.2.6 and 5.2.11, which combine to show Theorem 5.2.12.

5.2.1 Verma Submodules

Lemma 5.2.1. *M has the following Verma submodules:*

(a) $N^1 := M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)})$, *generated by* $e_{-\varepsilon}^m v_\lambda$, *and*

(b) $N^2 := M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$, *generated by* $e_{-\delta}^m v_\lambda$.

Proof. Noting that with $\mathfrak{b}^{(1)}$, the root ε is simple and Hypothesis 2.5.1 holds, (a) follows by Theorem 2.5.2. The same argument with $\mathfrak{b}^{(2)}$ and δ shows part (b). □

Computation shows that

$$\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)}) = \tfrac{m}{2}(-\varepsilon + \delta) - \rho^{(1)} = -\tfrac{m-1}{2}(\varepsilon - \delta)$$
$$\sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)}) = \tfrac{m}{2}(\varepsilon - \delta) - \rho^{(2)} = \tfrac{m-1}{2}(\varepsilon - \delta),$$

so the composition factors of N^1 and N^2 are given by Corollary 4.8.5 and Corollary 4.8.6, respectively.

5.2.2 The Quotient M/N^1

Lemma 5.2.2. $M = M^{(1)}(\lambda^{(1)})$ *has highest weight vectors of the following weights:*

(a) $\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}$,

(b) $\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)} - \beta^{(1)}$,

(c) $\sigma_\delta \sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}$,

(d) $\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} - \beta^{(1)} = \sigma_\delta \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)$,

(e) $\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}$, *and*

(f) $\lambda^{(1)} - \gamma$.

Proof. (a) – (c) follow from Lemma 5.2.1 and the discussion in section 4.6.2. To see (d) and (e), note from the discussion in section 4.6.2 that the $\beta^{(1)}$-atypical weights $\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)}$ and $\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)} - \beta^{(2)}$ are highest weights of $M^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$, which is a submodule of M by Lemma 5.2.1. By Lemma 2.4.1, these are also highest weights of M. Computation shows that

$$\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)} - \beta^{(2)} = \sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} \qquad \sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)} = \sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} - \beta^{(1)} = \sigma_\delta \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma),$$

which are the weights given in (d) and (e). Finally, (f) follows from Lemma 5.1.1. □

Next, we turn our attention to the quotient

$$Q^1 = M/N^1.$$

To describe the structure of Q^1 we turn again to a parabolic subalgebra. Partition Δ as

$$\Delta^{\mathfrak{q}} = \{-\varepsilon, -\varepsilon+\delta, \delta, 2\delta, \varepsilon+\delta, \varepsilon\} \qquad \Delta^{\mathfrak{m}} = \{\varepsilon-\delta, -\delta, -2\delta, -\varepsilon-\delta\},$$

and define

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\eta \in \Delta^{\mathfrak{q}}} \mathfrak{g}^\eta \qquad \mathfrak{m} = \bigoplus_{\eta \in \Delta^{\mathfrak{m}}} \mathfrak{g}^\eta \qquad \mathfrak{m}^- = \bigoplus_{\eta \in \Delta^{\mathfrak{m}}} \mathfrak{g}^{-\eta}$$

(so $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$). Let $\mathfrak{l} \subset \mathfrak{q}$ be the copy of $\mathfrak{so}(3)$ generated by $\{e_{\pm\varepsilon}, h_\varepsilon\}$, and let L be the finite-dimensional simple $\mathfrak{l}$-module of dimension m. (Note that while $\mathfrak{so}(3) \cong \mathfrak{sl}(2)$, the basis for $\mathfrak{l}$ resembles a canonical basis for $\mathfrak{so}(3)$, rather than a canonical basis for $\mathfrak{sl}(2)$. In particular, $[h_\varepsilon, e_\varepsilon] = e_\varepsilon$, not $2e_\varepsilon$ as one would expect with a canonical $\mathfrak{sl}(2)$ basis. This affects the character of L, computed later in this section.) Extend the action on L to $\mathfrak{q}$ by letting h_δ act on L as multiplication by $\lambda^{(1)}(h_\delta) = \frac{m+1}{2}$, and allowing $\mathfrak{m}^-$ to act trivially.

Lemma 5.2.3. $Q^1 \cong \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$

Proof. We prove this by comparing characters. By Remark 2.3.2,

$$\operatorname{ch} M = \frac{\epsilon^{\lambda^{(1)}}(1+\epsilon^{-(\varepsilon+\delta)})(1+\epsilon^{-(-\varepsilon+\delta)})}{(1-\epsilon^{-\varepsilon})(1-\epsilon^{-\delta})}$$
$$\operatorname{ch} N^1 = \frac{\epsilon^{(\lambda^{(1)}-m\varepsilon)}(1+\epsilon^{-(\varepsilon+\delta)})(1+\epsilon^{-(-\varepsilon+\delta)})}{(1-\epsilon^{-\varepsilon})(1-\epsilon^{-\delta})}$$

and so, Remark 2.3.4,

$$\operatorname{ch} Q^1 = \operatorname{ch} M - \operatorname{ch} N^1 = \frac{\epsilon^{\lambda^{(1)}}\left(\sum_{k=0}^{m-1}\epsilon^{-k\varepsilon}\right)(1+\epsilon^{-(\varepsilon+\delta)})(1+\epsilon^{-(-\varepsilon+\delta)})}{(1-\epsilon^{-\delta})}.$$

On the other hand, $\mathfrak{so}(3)$ theory, together with the action of h_δ on L, tells us that

$$\begin{aligned}\operatorname{ch} L &= \epsilon^{((m+1)/2)\delta}\left(\sum_{k=-(m-1)/2}^{(m-1)/2}\epsilon^{-k\varepsilon}\right)\\ &= \epsilon^{\lambda^{(1)}}\left(\sum_{k=0}^{m-1}\epsilon^{-k\varepsilon}\right)\end{aligned}$$

and so, following the discussion in section 2.3.1,

$$\operatorname{ch} \operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L = \frac{\epsilon^{\lambda^{(1)}}\left(\sum_{k=0}^{m-1}\epsilon^{-k\varepsilon}\right)(1+\epsilon^{-(\varepsilon+\delta)})(1+\epsilon^{-(-\varepsilon+\delta)})}{(1-\epsilon^{-\delta})},$$

Thus, $\operatorname{ch} \operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L = \operatorname{ch} Q^1$.

Next, we establish a surjection $Q^1 \twoheadrightarrow \operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$. Noting the construction of Q^1 gives the short exact sequence

$$0 \to N^1 \to M \to Q^1 \to 0.$$

Thus it suffices to construct a surjection $\psi : M \twoheadrightarrow \operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$ such that

$$N^1 = \mathfrak{U}(\mathfrak{g}) e_{-\varepsilon}^m v_\lambda \subseteq \ker \psi.$$

But $\operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$ is a highest weight module generated by a vector w_λ of highest weight λ; let ψ be the surjection

$$\psi : M \to \operatorname{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$$
$$v_\lambda \mapsto w_\lambda$$

which exists uniquely by the universality of M. To show $N^1 \subseteq \ker\psi$, we need only show that $\psi(e_{-\varepsilon}^m v_\lambda) = 0$. ψ is a $\mathfrak{U}(\mathfrak{g})$-module homomorphism, so

$$\psi(e_{-\varepsilon}^m v_\lambda) = e_{-\varepsilon}^m \psi(v_\lambda) = e_{-\varepsilon}^m w_\lambda.$$

Since $e_{-\varepsilon} \in \mathfrak{l}$, this is merely the $\mathfrak{U}(\mathfrak{l})$-action on L. L is finite dimensional of dimension m, with basis

$$\{w_\lambda, e_{-\varepsilon} w_\lambda, \ldots, e_{-\varepsilon}^{m-1} w_\lambda\},$$

and so $e_{-\varepsilon}^m w_\lambda = 0$, as desired.

Thus by isomorphism theorems, Q^1 surjects onto $\mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$, and so by comparing characters, $Q^1 = \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{g}} L$. □

Now to determine the composition factors of $M^{(1)}(\lambda^{(1)})$, we need only determine the composition factors of Q^1, since the composition factors of N^1 are known.

Lemma 5.2.4. *If $\mu = \lambda^{(1)} - (A\varepsilon + B\delta)$ is a highest weight of Q^1, then either $\mu = \lambda^{(1)}$, or $A \in \mathbb{N} \cup \{-1\}$, $B \in \mathbb{N}^+$ and $A = B$ or $A + B = m$ (where $\lambda^{(1)} + \rho^{(1)} = \frac{m}{2}(\varepsilon + \delta)$).*

Proof. By Lemma 2.8.4, if $\lambda^{(1)} - (A\varepsilon + B\delta)$ is a highest weight of Q^1, then

$$2(\lambda^{(1)} + \rho^{(1)}, A\varepsilon + B\delta) = (A\varepsilon + B\delta, A\varepsilon + B\delta).$$

Computing, obtain the equation

$$m(A - B) = A^2 - B^2$$

Solving this, we find that either $A = B$ or $A + B = m$.

Further, we must have $A\varepsilon + B\delta$ sums of positive roots with respect to $\mathfrak{b}^{(1)}$, with isotropic roots having coefficient at most one. Recalling that

$$\Delta^+_{\mathfrak{b}^{(1)}} = \{\varepsilon,\ \varepsilon + \delta = \gamma,\ \delta,\ 2\delta,\ -\varepsilon + \delta\}$$

we see that $A \in \mathbb{N} \cup \{-1\}$, $B \in \mathbb{N}$.

When $B = 0$, $A = m$,

$$\lambda^{(1)} - (A\varepsilon + B\delta) = \lambda^{(1)} - m\varepsilon = \sigma_{\varepsilon} \underset{(1)}{\cdot} \lambda^{(1)},$$

which is not a weight of Q^1 by its construction as a quotient. The remaining cases are the ones described in the lemma. □

Lemma 5.2.5. *Q^1 has highest weights*

(a) $\lambda^{(1)}$,

(b) $\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}$,

(c) $\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} - \beta^{(1)}$, *and*

(d) $\lambda^{(1)} - \gamma$.

Proof. (a) is clear. (b), (c) and (d) are highest weights of M by Lemma 5.2.2. However, by the discussion in section 4.6.2, they are not highest weights of N^1, so they must be highest weights of Q^1. □

We are now in a position to describe some constraints on the possible composition factors of $M = M^{(1)}(\lambda^{(1)})$.

As vector spaces (and indeed as $\mathfrak{l}$-modules),

$$Q^1 \cong \bigoplus_{k \in \mathbb{N}} Q^1_k$$

where

$$\begin{aligned} Q^1_0 &= L \\ Q^1_1 &= e_{-\delta} L \oplus e_{\varepsilon-\delta} L \oplus e_{-\varepsilon-\delta} L \\ Q^1_k &= 2e^k_{-\delta} L \oplus e^{k-1}_{-\delta} e_{\varepsilon-\delta} L \oplus e^{k-1}_{-\delta} e_{-\varepsilon-\delta} L \quad \text{for } k \geqslant 2. \end{aligned}$$

Each Q^1_k is a finite-dimensional $\mathfrak{l}$-module; for $k \geqslant 2$, the $\mathfrak{l}$-weight spaces of Q^1_k have the dimensions shown in table 5.1. These dimensions can be computed by noting that L has weight spaces of weight

$$\lambda_\varepsilon, \lambda_\varepsilon - \varepsilon, \ldots, \lambda_\varepsilon - (m-1)\varepsilon,$$

Weight	Dimesion
$\lambda^{(1)} + \varepsilon - k\delta$	1
$\lambda^{(1)} - k\delta$	3
$\lambda^{(1)} - \varepsilon - k\delta$	4
$\vdots$	4
$\lambda^{(1)} - (m-2)\varepsilon - k\delta$	4
$\lambda^{(1)} - (m-1)\varepsilon - k\delta$	3
$\lambda^{(1)} - m\varepsilon - k\delta$	1

Table 5.1: Dimensions of weight spaces in Q^1_k

each of dimension one, where λ_ε denotes $\lambda^{(1)}(h_\varepsilon)\varepsilon = \frac{m-1}{2}\varepsilon$. ($Q^1_0$ is simple and Q^1_1 replaces the dimensions 3 and 4 by 2 and 3, respectively.)

Each Q^1_k is finite-dimensional, and thus semi-simple, so can be written as direct sum of simple finite-dimensional $\mathfrak{l}$-modules L_{j_i}, each of dimension j_i. By counting dimensions, we observe that

$$Q^1_0 = L_m \tag{5.2.1}$$

$$Q^1_1 = L_{m+2} \oplus L_m \oplus L_{m-2} \tag{5.2.2}$$

$$Q^1_k = L_{m+2} \oplus 2L_m \oplus L_{m-2} \qquad k \geqslant 2. \tag{5.2.3}$$

With respect to the action of $\mathfrak{l} \cong \mathfrak{so}(3)$, the highest weights of L_{m+2}, L_m, and L_{m-2} are $\lambda_\varepsilon + \varepsilon$, λ_ε, and $\lambda_\varepsilon - \varepsilon$, respectively.

This means that if $\mu = \lambda - A\varepsilon - B\delta$ is a highest weight of Q^1, then $A \in \{-1, 0, 1\}$. So the possible highest weights are:

$$\mu_a = \lambda^{(1)} \in Q^1_0$$

$$\mu_b = \lambda^{(1)} - \varepsilon - \delta = \lambda - \gamma \in Q^1_1$$

$$\mu_c = \lambda^{(1)} + \varepsilon - (m+1)\delta = \sigma_{\delta}\underset{(1)}{\cdot}(\lambda^{(1)} + \gamma) \in Q^1_{m+1}$$

$$\mu_d = \lambda^{(1)} - m\delta = \sigma_{\delta}\underset{(1)}{\cdot}\lambda^{(1)} \in Q^1_m$$

$$\mu_e = \lambda^{(1)} - \varepsilon - (m-1)\delta = \sigma_{\delta}\underset{(1)}{\cdot}(\lambda^{(1)} - \gamma) \in Q^1_{m-1}.$$

Recalling that $m \geqslant 3$, these are all distinct. All but μ_e are noted as highest weights in Lemma 5.2.5, but we must also consider their multiplicity.

(a) $(Q_0^1)^{\mu_a}$ is one-dimensional, and so the highest weight μ_a is multiplicity free.

(b) Observing eq. (5.2.2), we see that $(Q_1^1)^{\mu_b}$ contains a unique $\mathfrak{l}$-highest weight vectors. So again, the highest weight μ_b is multiplicity free.

(c) Again, $(Q_{m+1}^1)^{\mu_c}$ is one-dimensional, and so the highest weight μ_c is multiplicity free.

(d) Observing eq. (5.2.3), we see that $(Q_m^1)^{\mu_d}$ contains two $\mathfrak{l}$-highest weight vectors.

(e) Finally, observing eq. (5.2.3) again, we see that $(Q_{m-1}^1)^{\mu_e}$ contains a unique $\mathfrak{l}$-highest weight vector.

We can now give a partial description of the composition factors of $M = M^{(1)}(\lambda^{(1)})$. We remind the reader that ours strategy will be to compare the result below to Theorem 5.2.11 to completely determine the composition factors and their multiplicities.

Theorem 5.2.6. *The composition factors of $M = M^{(1)}(\lambda^{(1)})$ satisfy the following.*

(a) Each of the following composition factors has multiplicity exactly one.

$$
\begin{aligned}
L_a &= L^{(1)}(\lambda^{(1)}) && = L^{(2)}(\lambda^{(2)}) \\
L_b &= L^{(1)}(\lambda^{(1)} - \gamma) && = L^{(2)}(\lambda^{(2)} - \gamma) \\
L_c &= L^{(1)}(\sigma_\delta \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) && = L^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)}) \\
L_e &= L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) && = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma)) \\
L_f &= L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} - \gamma)) && = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)}) \\
L_g &= L^{(1)}(\sigma_\delta \sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) && = L^{(2)}(\sigma_\varepsilon \sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma)) \\
L_h &= L^{(1)}(\sigma_\delta \sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) && = L^{(2)}(\sigma_\varepsilon \sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})
\end{aligned}
$$

(b) The following composition factor has multiplicity one or two.

$$L_d = L^{(1)}(\sigma_\delta \cdot_{(1)} \lambda^{(1)}) \qquad\qquad = L^{(2)}(\sigma_\delta \cdot_{(2)} (\lambda^{(2)} - \gamma))$$

(c) The following simple module is a possible composition factor, occurring with multiplicity at most one.

$$L_* = L^{(1)}(\sigma_\delta \cdot_{(1)} (\lambda^{(1)} - \gamma)) \qquad\qquad = L^{(2)}(\sigma_\delta \cdot_{(2)} (\lambda^{(2)} - 2\gamma))$$

(d) $M = M^{(1)}(\lambda^{(1)})$ *has no other composition factors.*

Proof. In part (a), the composition factors L_e, L_f, L_g and L_h are the composition factors of N^1, described in Corollary 4.8.5. The remainder of (a) and parts (b), (c) and (d) follow from Lemma 5.2.5 together with the discussion above. □

5.2.3 The Quotient M/N^2

We now determine the composition factors of $M = M^{(2)}(\lambda^{(2)})$. The argument is similar to the one in the previous section, with some key differences.

Lemma 5.2.7. $M^{(2)}$ *has highest weight vectors of the following weights:*

(a) $\sigma_\delta \cdot_{(2)} \lambda^{(2)}$,

(b) $\sigma_\delta \cdot_{(2)} \lambda^{(2)} - \beta^{(2)}$,

(c) $\sigma_\varepsilon \sigma_\delta \cdot_{(2)} \lambda^{(2)}$,

(d) $\sigma_\varepsilon \cdot_{(2)} \lambda^{(2)} - \beta^{(2)} = \sigma_\varepsilon \cdot_{(2)} (\lambda^{(2)} + \gamma)$,

(e) $\sigma_\varepsilon \cdot_{(2)} \lambda^{(2)}$, *and*

(f) $\lambda^{(2)} - \gamma$.

Proof. Follows the same argument as Lemma 5.2.2. □

Now we turn again to the quotient $Q^2 = M/N^2$. To determine the structure of this quotient, we introduce a parabolic subalgebra different than the one in the previous section. Partition Δ as

$$\Delta^{\mathfrak{q}'} = \{2\delta, \delta, \varepsilon + \delta, \varepsilon, \varepsilon - \delta, -\delta, -2\delta\} \qquad \Delta^{\mathfrak{m}'} = \{-\varepsilon + \delta, -\varepsilon, -\varepsilon - \delta\},$$

and define

$$\mathfrak{q}' = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^{\mathfrak{q}'}} \mathfrak{g}^\alpha \qquad \mathfrak{m}' = \bigoplus_{\alpha\in\Delta^{\mathfrak{m}'}} \mathfrak{g}^\alpha$$

(so $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$). Let $\mathfrak{l}' \subset \mathfrak{q}$ be the copy of $\mathfrak{osp}(1,2)$ generated by $\{e_{\pm\delta}, h_\delta\}$, and let L' be the finite-dimensional simple $\mathfrak{l}'$-module of dimension m.

The structure of L' requires some discussion, as the representation theory of $\mathfrak{osp}(1,2)$ is less well known than that of $\mathfrak{sl}(2)$. A good discussion can be found in [Mus12, Ex. A.4.4]. Unlike the case for $\mathfrak{sl}(2)$, $\mathfrak{osp}(1,2)$ has a unique finite dimensional submodules only of dimension any odd dimension m, with one-dimensional weight spaces of weights

$$-\tfrac{m-1}{2}\delta, -\tfrac{m-3}{2}\delta, \ldots, \tfrac{m-1}{2}\delta$$

and thus character

$$\operatorname{ch} L = \left(\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \epsilon^{-k\delta} \right).$$

Now, extend the action on L' to $\mathfrak{q}'$ by letting h_ε act on L' as multiplication by $\lambda^{(2)}(h_\varepsilon) = \frac{m+1}{2}$ and remaining elements of $\mathfrak{q}$ act trivially.

Lemma 5.2.8. $Q^2 = \operatorname{Ind}_{\mathfrak{p}'}^{\mathfrak{g}} L'$

Proof. We begin by noting that

$$\begin{aligned}
\operatorname{ch} Q^2 &= \operatorname{ch} M - \operatorname{ch} N^2 \\
&= \frac{\epsilon^{\lambda^{(2)}}(1+\epsilon^{-(\varepsilon-\delta)})(1+\epsilon^{-(\varepsilon+\delta)})}{(1-\epsilon^{-\delta})(1-\epsilon^{-\varepsilon})} - \frac{\epsilon^{\lambda^{(2)}-m\delta}(1+\epsilon^{-(\varepsilon-\delta)})(1+\epsilon^{-(\varepsilon+\delta)})}{(1-\epsilon^{-\delta})(1-\epsilon^{-\varepsilon})} \\
&= \frac{\epsilon^{\lambda^{(2)}}\left(\sum_{k=0}^{m-1} \epsilon^{k\delta}\right)(1+\epsilon^{-(\varepsilon-\delta)})(1+\epsilon^{-(\varepsilon+\delta)})}{(1-\epsilon^{-\varepsilon})}
\end{aligned}$$

and that

$$\operatorname{ch} L' = \epsilon^{\left(\frac{m+1}{2}\right)\varepsilon} \left(\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \epsilon^{-k\delta} \right)$$

$$= \epsilon^{\lambda^{(2)}} \left(\sum_{k=0}^{m-1} \epsilon^{-k\delta} \right).$$

The remainder of the proof follows the same argument as that for Lemma 5.2.3. □

Lemma 5.2.9. *If $\mu = \lambda^{(2)} - (A\varepsilon + B\delta)$ is a highest weight of Q^1, then either $\mu = \lambda$ and $A \in \mathbb{N}^+$, $B \in \mathbb{N} \cup \{-1\}$, and $A = B$ or $A + B = m$ (where $\lambda^{(2)} + \rho^{(2)} = \frac{m}{2}(\varepsilon + \delta)$).*

Proof. Follows the same argument as Lemma 5.2.4. □

Lemma 5.2.10. *Q^2 has highest weights*

(a) $\lambda^{(2)}$,

(b) $\sigma_{\underset{(2)}{\varepsilon}} \cdot \lambda^{(2)}$,

(c) $\sigma_{\underset{(2)}{\varepsilon}} \cdot \lambda^{(2)} - \beta^{(2)}$, *and*

(d) $\lambda^{(2)} - \gamma$.

Proof. Follows the same argument as Lemma 5.2.5. □

We now employ an argument similar to that preceding Theorem 5.2.6 to describe the possible composition factors of $M = M^{(2)}(\lambda^{(2)})$.Again. we decompose Q^2 (this time as $\mathfrak{l}'$-modules) as

$$Q^2 = \bigoplus_{k \in \mathbb{N}} Q^2_k$$

and we see that

$$Q^2_0 = L'$$

$$Q^2_k = e_{-\varepsilon} L' \oplus e_{-\varepsilon+\delta} L' \oplus e_{-\varepsilon-\delta} L'$$

$$Q^2_k = 2e^k_{-\varepsilon} L' \oplus e^{k-1}_{-\varepsilon} e_{-\varepsilon+\delta} L' \oplus e^{k-1}_{-\varepsilon} e_{-\varepsilon-\delta} L' \quad \text{for } k \geqslant 2.$$

This allows us to compute the dimensions of the weight spaces of Q^2_k, as seen in table 5.2. We note that there exist unique $\mathfrak{osp}(1,2)$-modules of dimensions $m + 2, m, m-2$, which we will denote L'_{m+2}, L'_m, L'_{m-2}, having highest weights

$$\lambda_\delta + \delta, \lambda_\delta, \lambda_\delta - \delta,$$

Weight	Dimesion
$\lambda^{(2)} + \delta - k\varepsilon$	1
$\lambda^{(2)} - k\varepsilon$	3
$\lambda^{(2)} - \delta - k\varepsilon$	4
$\vdots$	4
$\lambda^{(2)} - (m-2)\delta - k\varepsilon$	4
$\lambda^{(2)} - (m-1)\delta - k\varepsilon$	3
$\lambda^{(2)} - m\delta - k\varepsilon$	1

Table 5.2: Dimensions of weight spaces in Q^2_k

respectively, where $\lambda_\delta = (\lambda^{(2)}(\delta))\delta$. We also note that each Q^2_k is finite-dimensional and thus completely decomposable. Then we see by counting dimensions that

$$\begin{aligned} Q^2_0 &= L'_m \\ Q^2_k &= L'_{m+2} \oplus L'_m \oplus L'_{m-2} \\ Q^2_k &= L'_{m+2} \oplus 2L'_m \oplus L'_{m-2} \quad \text{for } k \geqslant 2. \end{aligned}$$

Now a dimension-counting argument similar to that preceding Theorem 5.2.6 allows us to show the following; recall again that our strategy will be to compare this theorem to Theorem 5.2.6.

Theorem 5.2.11. *The composition factors of $M = M^{(2)}(\lambda^{(2)})$ satisfy the following.*

(a) Each of the following composition factors has multiplicity exactly one.

$$\begin{aligned}
L_a &= L^{(1)}(\lambda^{(1)}) && = L^{(2)}(\lambda^{(2)}) \\
L_b &= L^{(1)}(\lambda^{(1)} - \gamma) && = L^{(2)}(\lambda^{(2)} - \gamma) \\
L_c &= L^{(1)}(\sigma_\delta \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) && = L^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)}) \\
L_d &= L^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) && = L^{(2)}(\sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)} - \gamma)) \\
L_e &= L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) && = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma)) \\
L_g &= L^{(1)}(\sigma_\delta \sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) && = L^{(2)}(\sigma_\varepsilon \sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma))
\end{aligned}$$

$$L_h = L^{(1)}(\sigma_\delta \sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) \qquad = L^{(2)}(\sigma_\varepsilon \sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$$

(b) The following composition factor has multiplicity one or two.

$$L_f = L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} - \gamma)) \qquad = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)})$$

(c) The following simple module is a possible composition factor, occurring with multiplicity at most one.

$$L_{**} = L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} - 2\gamma)) \qquad = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda^{(2)} - \gamma))$$

(d) $M = M^{(2)}(\lambda^{(2)})$ *has no other composition factors.*

Proof. In part (a), the composition factors L_c, L_d, L_g and L_h are the composition factors of N^1, described in Corollary 4.8.6. The remainder of (a) and parts (b), (c) and (d) follow from Lemma 5.2.10 together with the discussion above. □

5.2.4 The Structure of M

Comparing the lists of composition factors in Theorem 5.2.6 and in Theorem 5.2.11, we obtain the following.

Theorem 5.2.12. $M = M^{(1)}(\lambda^{(1)}) = M^{(2)}(\lambda^{(2)})$ *has the following composition factors, each with multiplicity exactly one.*

$$L_a = L^{(1)}(\lambda^{(1)}) \qquad = L^{(2)}(\lambda^{(2)})$$

$$L_b = L^{(1)}(\lambda^{(1)} - \gamma) \qquad = L^{(2)}(\lambda^{(2)} - \gamma)$$

$$L_c = L^{(1)}(\sigma_\delta \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) \qquad = L^{(2)}(\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$$

$$L_d = L^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) \qquad = L^{(2)}(\sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)} - \gamma))$$

$$L_e = L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) \qquad = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma))$$

$$L_f = L^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} - \gamma)) \qquad = L^{(2)}(\sigma_\varepsilon \underset{(2)}{\cdot} \lambda^{(2)})$$

$$L_g = L^{(1)}(\sigma_\delta\sigma_\varepsilon \underset{(1)}{\cdot} (\lambda^{(1)} + \gamma)) \qquad = L^{(2)}(\sigma_\varepsilon\sigma_\delta \underset{(2)}{\cdot} (\lambda^{(2)} + \gamma))$$

$$L_h = L^{(1)}(\sigma_\delta\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) \qquad = L^{(2)}(\sigma_\varepsilon\sigma_\delta \underset{(2)}{\cdot} \lambda^{(2)})$$

Proof. Note that the possible composition factor L_* in Theorem 5.2.6 does not appear as a possible composition factor in Theorem 5.2.11, and similarly, L_{**} in Theorem 5.2.11 does not appear in Theorem 5.2.6. Thus, L_*, L_{**} do not appear as composition factors of M. Note also that Theorem 5.2.6 shows that L_d has multiplicity exactly one, and that Theorem 5.2.11 shows that L_f has multiplicity exactly one. The remainder of the theorem is clear from either of Theorems 5.2.6 and 5.2.11. □

We next wish to determine the Jantzen filtration. For the following computations, we fix $\mathfrak{b} = \mathfrak{b}^{(1)}$; if we had fixed $\mathfrak{b} = \mathfrak{b}^{(2)}$, the computations would be similar. Having fixed our Borel, we omit the notations indicating the choice of Borel in the following computations.

To determine the Jantzen filtration, we return to the Jantzen sum formula, Theorem 2.7.2,

$$\sum \operatorname{ch} M_i = \sum_{\eta \in A(\lambda)} \operatorname{ch} M(\sigma_\eta.\lambda) + \sum_{\eta \in B(\lambda)} \epsilon^{\lambda-\eta} p_\eta.$$

It is easy to see that $A(\lambda) = \{\varepsilon, \delta\}$ and $B(\lambda) = \{\gamma\}$. Our next step is to show that, in this case, $\epsilon^{\lambda-\gamma} p_\gamma$ is the character of a module.

Lemma 5.2.13. *Suppose* $m = (\lambda + \rho, \varepsilon^\vee) = (\lambda + \rho, \delta^\vee) \in \mathbb{N}^{\text{odd}}$, $m \geqslant 3$. *Let* $M_\lambda = M/M_{\lambda-\gamma}$ *(where* $M_{\lambda-\gamma}$ *is the submodule described in Lemma 5.1.2). Then* $\operatorname{ch} M_\lambda = \epsilon^\lambda p_\gamma$.

Proof. Showing the lemma amounts to showing that $\dim(M_\lambda^{\lambda-\eta}) = \mathbf{p}_\gamma(\eta)$. This in turn amounts to showing that

$$S_\gamma = \{e_{-\pi} v_\lambda : \pi \in \mathbf{P}_\gamma(\eta)\}$$

forms a basis for $M_\lambda^{\lambda-\eta}$. It is clear that $M_\lambda^{\lambda-\eta}$ is spanned by

$$S = \{e_{-\pi}v_\lambda : \pi \in \mathbf{P}(\eta)\}.$$

But writing $\eta = A\varepsilon + B\delta$, we see that with a properly ordered basis for $\mathfrak{n}^-$, when $A \geqslant 1, B \geqslant 2$,

$$S = = \left\{ \begin{array}{cc} e_{-\varepsilon}^A e_{-\delta}^B v_\lambda, & e_{-\varepsilon}^{A+1} e_{-\delta}^{B-1} e_{\varepsilon-\delta} v_\lambda, \\ e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon-\delta} v_\lambda, & e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon-\delta} v_\lambda \end{array} \right\}$$

$$S_\gamma = \left\{ e_{-\varepsilon}^A e_{-\delta}^B v_\lambda, \qquad e_{-\varepsilon}^{A+1} e_{-\delta}^{B-1} e_{\varepsilon-\delta} v_\lambda \right\}$$

so we need only show that $e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon-\delta} v_\lambda, e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon-\delta} v_\lambda$ are in the span of S_γ. (The cases for smaller A, B simply eliminate some elements of these vectors; the argument below also applies in these cases.)

By the construction of M_λ, in this module $\theta_\gamma^{(1)} v_\lambda = 0$, so

$$\binom{m}{2} e_{-\varepsilon-\delta} v_\lambda = m e_{-\varepsilon} e_{-\delta} v_\lambda - e_{-\varepsilon}^2 e_{\varepsilon-\delta} v_\lambda.$$

This identity allows us to show the desired result. We begin by noting that

$$w_1 = \binom{m}{2} e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon-\delta} v_\lambda = \binom{m}{1} e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon} e_{-\delta} v_\lambda - e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon}^2 e_{\varepsilon-\delta} v_\lambda \quad (5.2.4)$$

$$w_2 = \binom{m}{2} e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon-\delta} v_\lambda = \binom{m}{1} e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon} e_{-\delta} v_\lambda - e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon}^2 e_{\varepsilon-\delta} v_\lambda. \quad (5.2.5)$$

By computing

$$e_{\varepsilon-\delta} e_{-\varepsilon} e_{-\delta} = -e_{-\varepsilon} e_{-\delta} e_{\varepsilon-\delta} - e_{-\delta}^2$$

$$e_{\varepsilon-\delta} e_{-\varepsilon}^2 e_{\varepsilon-\delta} = -2 e_{-\varepsilon} e_{-\delta} e_{\varepsilon-\delta} - 2 e_{-\delta}^2 - e_{\varepsilon-\delta} e_{-\varepsilon-\delta},$$

we can rewrite eq. (5.2.5) as

$$e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{\varepsilon-\delta} e_{-\varepsilon-\delta} v_\lambda \propto e_{-\varepsilon}^A e_{-\delta}^B v_\lambda + e_{-\varepsilon}^A e_{-\delta}^{B-2} e_{-\varepsilon} e_{-\delta} e_{\varepsilon-\delta} v_\lambda. \quad (5.2.6)$$

First, we show that $w_1 \in \operatorname{span} S_\gamma$ when $B = 2k + 1$ is odd. To show this, we note that $e_{-\delta}^{B-1} = e_{-2\delta}^{k}$ commutes with $e_{-\varepsilon}$, and so

$$e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon} e_{-\delta} v_\lambda = e_{-\varepsilon}^{A} e_{-\delta}^{B} v_\lambda \in S_\gamma$$

$$e_{-\varepsilon}^{A-1} e_{-\delta}^{B-1} e_{-\varepsilon}^{2} e_{\varepsilon-\delta} v_\lambda = e_{-\varepsilon}^{A+1} e_{-\delta}^{B-1} e_{\varepsilon-\delta} \in S_\gamma$$

which gives the desired result.

Next, we show that $w_2 \in \operatorname{span} S_\gamma$ when $B = 2k+2$ is even. Noting that again $e_{-\delta}^{B-2} = e_{-2\delta}^{k}$ commutes with $e_{-\varepsilon}$,

$$e_{-\varepsilon}^{A} e_{-\delta}^{B} \in S_\gamma$$

$$e_{-\varepsilon}^{A} e_{-\delta}^{B-2} e_{-\varepsilon} e_{-\delta} e_{\varepsilon-\delta} v_\lambda = e_{-\varepsilon}^{A+1} e_{-\delta}^{B-1} e_{\varepsilon-\delta} v_\lambda \in S_\gamma$$

which gives the desired result.

When B is even, the computation for w_1 is complicated by the fact that w_2 appears as a commutator. However, we have already shown that in this case $w_2 \in \operatorname{span} S_\gamma$, and thus $w_1 \in \operatorname{span} S_\gamma$ also. Similarly, when B is odd, the computation for w_2 is complicated because w_1 appears as a commutator, but we have already seen that $w_1 \in \operatorname{span} S_\gamma$ in this case.

Thus for each fixed η, S_γ forms a basis for $(M_\lambda)^{\lambda-\eta}$, which proves the desired result. □

Corollary 5.2.14. $\operatorname{ch} M_{\lambda-\gamma} = \epsilon^{\lambda-\gamma}\mathbf{p}_\gamma$

Proof. By construction we have a short exact sequence

$$0 \to M_{\lambda-\gamma} \to M(\lambda) \to M_\lambda \to 0.$$

So, noting that $p = (1+\epsilon^{-\gamma})p_\gamma$,

$$\begin{aligned} \operatorname{ch} M_{\lambda-\gamma} &= \operatorname{ch} M(\lambda) - \operatorname{ch} M_\lambda \\ &= \epsilon^\lambda p - \epsilon^\lambda p_\gamma \\ &= \epsilon^{\lambda-\gamma} p_\gamma \end{aligned}$$

as desired. □

Corollary 5.2.15. *The notation M_λ introduced in Lemma 5.2.13 is unambiguous, that is, M_λ is the submodule of $M(\lambda+\gamma)$ generated by $\theta_\gamma^{(1)} v_{\lambda+\gamma}$.*

Proof. This is easy to see by comparing characters and the universality of $M(\lambda)$. □

Corollary 5.2.16. *M_λ has composition factors L_a, L_c, L_e, L_g, and $M_{\lambda-\gamma}$ has composition factors L_b, L_d, L_f, L_h.*

Proof. Comparing the composition factors of $M(\gamma)$ and $M(\lambda+\gamma)$ in Theorems 5.2.6 and 5.2.11, we see that L_a, L_c, L_e, L_g are precisely their common factors, and thus must be the composition factors of the common M_λ. The remaining factors must be composition factors of $M_{\lambda-\gamma}$. □

Theorem 5.2.17. *M has Jantzen filtration*

$$\begin{aligned} M_1 &= L_b \boxplus L_c \boxplus L_e \boxplus L_d \boxplus L_f \boxplus L_g \boxplus L_h \\ M_2 &= L_d \boxplus L_f \boxplus L_g \boxplus L_h \\ M_3 &= L_h \\ M_i &= 0 \text{ for } i \geqslant 4. \end{aligned}$$

Proof. Observing that $\epsilon^{\lambda-\gamma} p_\gamma = \operatorname{ch} M_{\lambda-\gamma}$, we see that the right-hand side of the Jantzen Sum Formula (Theorem 2.7.2) contains only characters of modules, and can thus be understood as a sum in the Grothendieck group. Letting $\boxplus$ denote addition in the Grothendieck group, we see the following equation from the Jantzen Sum Formula.

$$\boxplus_i M_i = M^{(1)}(\sigma_\varepsilon \underset{(1)}{\cdot} \lambda^{(1)}) \boxplus M^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)}) \boxplus M_{\lambda-\gamma} \tag{5.2.7}$$

Note, to avoid confusion, that $M^{(1)}(\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)})$ is not a submodule of M. However, computing

$$\sigma_\delta \underset{(1)}{\cdot} \lambda^{(1)} = \tfrac{m-1}{2}(\varepsilon - \delta),$$

we see by Corollary 4.8.3 that

$$M^{(1)}(\sigma_{\delta_{(1)}} \cdot \lambda^{(1)}) = L_c \boxplus L_d \boxplus L_g \boxplus L_h$$

As noted before, by Corollary 4.8.5,

$$M^{(1)}(\sigma_{\varepsilon_{(1)}} \cdot \lambda^{(1)}) = L_e \boxplus L_f \boxplus L_g \boxplus L_h,$$

and finally, Corollary 5.2.16 gives us

$$M_{\lambda-\gamma} = L_b \boxplus L_d \boxplus L_f \boxplus L_h.$$

This, together with eq. (5.2.7), gives

$$\boxplus_i M_i = L_b \boxplus L_c \boxplus L_e \boxplus 2L_d \boxplus 2L_f \boxplus 2L_g \boxplus 3L_h. \tag{5.2.8}$$

Finally, eq. (5.2.8) combined with the observation that each composition factor is multiplicity-free gives the desired result. □

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APPENDIX A
MULTIPLICATION TABLE FOR $\mathfrak{osp}(3,2)$

Multiplication table for $\mathfrak{osp}(3,2)$. Entries are brackets $[x,y]$, $\cdot$ indicates a bracket of 0.

$x \backslash y$	h_ε	h_δ	e_ε	$e_{-\varepsilon}$	$e_{2\delta}$	$e_{-2\delta}$	e_δ	$e_{-\delta}$	$e_{\varepsilon-\delta}$	$e_{\varepsilon+\delta}$	$e_{-\varepsilon-\delta}$	$e_{-\varepsilon+\delta}$
h_ε	$\cdot$	$\cdot$	e_ε	$-e_{-\varepsilon}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$e_{\varepsilon-\delta}$	$e_{\varepsilon+\delta}$	$-e_{-\varepsilon-\delta}$	$-e_{-\varepsilon+\delta}$
h_δ	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$2e_{2\delta}$	$-2e_{-2\delta}$	e_δ	$-e_{-\delta}$	$-e_{\varepsilon-\delta}$	$e_{\varepsilon+\delta}$	$-e_{-\varepsilon-\delta}$	$e_{-\varepsilon+\delta}$
e_ε	$-e_\varepsilon$	$\cdot$	$\cdot$	h_ε	$\cdot$	$\cdot$	$e_{\varepsilon+\delta}$	$e_{\varepsilon-\delta}$	$\cdot$	$\cdot$	$e_{-\delta}$	e_δ
$e_{-\varepsilon}$	$e_{-\varepsilon}$	$\cdot$	$-h_\varepsilon$	$\cdot$	$\cdot$	$\cdot$	$e_{\varepsilon-\delta}$	$e_{-\varepsilon-\delta}$	$e_{-\delta}$	e_δ	$\cdot$	$\cdot$
$e_{2\delta}$	$\cdot$	$-2e_{2\delta}$	$\cdot$	$\cdot$	$\cdot$	h_δ	$\cdot$	e_δ	$e_{\varepsilon+\delta}$	$\cdot$	$e_{-\varepsilon+\delta}$	$\cdot$
$e_{-2\delta}$	$\cdot$	$2e_{-2\delta}$	$\cdot$	$\cdot$	$-h_\delta$	$\cdot$	$e_{-\delta}$	$\cdot$	$\cdot$	$e_{\varepsilon-\delta}$	$\cdot$	$e_{-\varepsilon-\delta}$
e_δ	$\cdot$	$-e_\delta$	$-e_{\varepsilon+\delta}$	$-e_{\varepsilon-\delta}$	$\cdot$	$-e_{-\delta}$	$-2e_{2\delta}$	h_δ	e_ε	$\cdot$	$e_{-\varepsilon}$	$\cdot$
$e_{-\delta}$	$\cdot$	$e_{-\delta}$	$-e_{\varepsilon-\delta}$	$-e_{-\varepsilon-\delta}$	$-e_\delta$	$\cdot$	h_δ	$2e_{-2\delta}$	$\cdot$	$-e_{-\varepsilon}$	$\cdot$	$-e_{-\varepsilon}$
$e_{\varepsilon-\delta}$	$-e_{\varepsilon-\delta}$	$e_{\varepsilon-\delta}$	$\cdot$	$-e_{-\delta}$	$-e_{\varepsilon+\delta}$	$\cdot$	e_ε	$\cdot$	$\cdot$	$\cdot$	$-2e_{-2\delta}$	$-h_\varepsilon - h_\delta$
$e_{\varepsilon+\delta}$	$-e_{\varepsilon+\delta}$	$-e_{\varepsilon+\delta}$	$\cdot$	$-e_\delta$	$\cdot$	$-e_{\varepsilon-\delta}$	$\cdot$	$-e_\varepsilon$	$\cdot$	$\cdot$	$h_\varepsilon - h_\delta$	$2e_{2\delta}$
$e_{-\varepsilon-\delta}$	$e_{-\varepsilon-\delta}$	$e_{-\varepsilon-\delta}$	$-e_{-\delta}$	$\cdot$	$-e_{-\varepsilon+\delta}$	$\cdot$	$e_{-\varepsilon}$	$\cdot$	$-2e_{-2\delta}$	$h_\varepsilon - h_\delta$	$\cdot$	$\cdot$
$e_{-\varepsilon+\delta}$	$e_{-\varepsilon+\delta}$	$-e_{-\varepsilon+\delta}$	$-e_\delta$	$\cdot$	$\cdot$	$-e_{-\varepsilon-\delta}$	$\cdot$	$-e_{-\varepsilon}$	$-h_\varepsilon - h_\delta$	$2e_{2\delta}$	$\cdot$	$\cdot$

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Education

Ph.D., University of Wisconsin – Milwaukee, May 2013
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–Graduate Teaching Assistant, 2009–present
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–Online Tutor (Physics, Mathematics, Chemistry, General Science), 2002–2006
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Teaching Experience

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Computer adaptive developmental math courses
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–Preparation for College Mathematics
–Essentials of Algebra
–Intermediate Algebra

These combined courses are part of UWM's program to help marginally prepared students meet university competency requirements for mathematics in a timely manner. This sequence of courses cover high school material (all content is un-starred in the Common Core State Standards). Using the ALEKS (`www.alex.com`) computer adaptive testing system, we provide individualized instruction to students from a variety of backgrounds, enabling them to complete up to three semesters of developmental coursework in a single semester. Courses are intensive, meeting twice a day, five days a week, with one-on-one instruction by a team of instructors, teaching assistants, and undergraduate classroom assistants. The primary goal of instruction is to build real understanding of mathematics as a way of creating knowledge, rather than rote memorization of procedures. I have taught these courses both as teaching assistant and (Spring 2012, Spring 2013) as primary instructor.

University of Wisconsin – Milwaukee
Calculus and Analytic Geometry
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First in a three-semester course in calculus, including basic differential and integral calculus as well as the study of limits. Taught as primary instructor.

University of Wisconsin – Milwaukee
Intermediate Algebra
Fall 2008

A traditional classroom course satisfying the university-wide mathematical competency requirement. Taught as primary instructor in coordination with other graduate students; combined course final.

University of Wisconsin – Milwaukee
Survey in Calculus and Analytic Geometry
Fall 2007

Single-semester course in calculus, offered primarily for business and health science students. Taught as teaching assistant, responsible for conduction discussion sections and grading.

University of Wisconsin – Milwaukee
Discrete Probability and Statistics for Elementary Education Majors
Fall 2006

This course covers basic concepts in probability and statistics, with a focus on hands-on activities and the understanding of mathematics as fundamentally comprehensible study. Taught as a classroom assistant in cooperation with a professor and a visiting teacher from Milwaukee Public Schools. Responsible for grading, assistance with classroom activities, and occasional lecture.

The Princeton Review
LSAT (Law School Admissions Test) Preparation

Summer 2010

This is an intensive privately-offered test preparation course for the Law School Admissions Test, a high-stakes test required for admission to most US law school programs. The course is taught to a detailed class plan, to maintain uniform presentation among all the company's courses. Skills taught include logical reasoning, reading comprehension, and time management.

Honors/Awards

GAANN Fellowship, 2006–2009

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Publications

Image: Representation of the Gaussian primes in the complex plane
Cover image, *Elementary Number Theory: An Algebraic Approach* by Ethan D. Bolker, Dover Publications, 2007

Relevant Computer/Technical Experience

D2L Course Management Software: 7 years

PeopleSoft (academic applications, as instructor): 5 years

LaTeX Typesetting Language: 5 years

ALEKS Computer Adaptive Testing Tool: 4 years