On the Nonlinear Tribological Jerk Dynamics at Sliding Interfaces

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ON THE NONLINEAR TRIBOLOGICAL JERK DYNAMICS AT SLIDING INTERFACES

By

Divyeshkumar Patel

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Engineering at The University of Wisconsin-Milwaukee December 2014
As the world desires the next industrial revolution, the potential threats that will undermine energy efficient innovations include detrimental frictional effects that exacerbate wear, hasten equipment breakdowns, and worsen heat dissipation. Capturing the inherently nonlinear manifestations of friction fundamentally has been difficult. A fundamental modeling scheme elucidating friction will bolster novel technologies synthesizing wear resistant materials and lubricants needed for sustainable energy efficiency.

Frictional dissipation at dynamical sliding interfaces has been studied for generations. Interfacial sliding frictional effects are prevalent in natural and artificial phenomena such as earthquake, hip and knee joints, and the moving parts of energy-producing and energy-consuming equipment. Hitherto, despite significant research efforts, no consensus fundamental modeling technique exists that deterministically ties friction with system degradation. Yet, elucidating the basic physics of nonlinear friction-resisted motion will clarify how heat generation, efficiency, lubrication, wear, and material lifetime evolve in sliding contacts.

In this study, we unify Newtonian mechanics with classical thermodynamics to elicit nonlinear tribological jerk dynamics at a sliding interface. Jerk, the rate of
change of acceleration has been elusive in classical mechanics. By showing jerk originating in a friction-resisted motion a new fundamental scientific modeling tool emerges. For example, although Coulomb’s law of friction precludes significant friction-velocity coupling our reassessment using jerk dynamics results shows otherwise. We find Coulomb’s law may seemingly be an oversimplification by reproducing the Strubeck effect known to capture friction-velocity coupling. Furthermore, negative frictional jerk opposes relative motion while positive lubricating jerk supports relative motion. A frictionless unconstrained motion recaptures constant acceleration Newtonian-Galilean mechanics. Using the kinematic and dynamic results as inputs, we quantified wear and wear rates, subsurface temperature and mechanical sliding efficiency. Our modeling results quantitatively match experimental results from tribometer and thermal compliance tests very well.

We constructed an analytical algebraic partitioning technique to solve the jerk balance equations which are third order and nonlinear ordinary differential equations. The algebraic technique works well and may facilitate engineering and scientific modeling efforts.

By placing jerk in basic physics context, we proffer a fundamental tool that likely will transform how relative motions in artificial and natural phenomena are modeled.
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## NOMENCLATURE AND LIST OF ABBREVIATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_0$</td>
<td>Initial sliding distance [m]</td>
</tr>
<tr>
<td>$s$</td>
<td>Sliding distance [m]</td>
</tr>
<tr>
<td>$u_0$</td>
<td>Initial velocity [m/s]</td>
</tr>
<tr>
<td>$u, v$</td>
<td>Velocity of the block [m/s]</td>
</tr>
<tr>
<td>$a_0$</td>
<td>Initial acceleration [m/s$^2$]</td>
</tr>
<tr>
<td>$a$</td>
<td>Acceleration [m/s$^2$]</td>
</tr>
<tr>
<td>$j$</td>
<td>Rate of change of acceleration, jerk [m/s$^3$]</td>
</tr>
<tr>
<td>$t$</td>
<td>Time [seconds]</td>
</tr>
<tr>
<td>K</td>
<td>Temperature [Kelvin]</td>
</tr>
<tr>
<td>°C</td>
<td>Temperature [Celsius]</td>
</tr>
<tr>
<td>T</td>
<td>Temperature [°C or K]</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass [kg]</td>
</tr>
<tr>
<td>$m_0$</td>
<td>Initial mass [kg]</td>
</tr>
<tr>
<td>$m^*$</td>
<td>Heat penetrated portion of the mass [kg]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of the material [kg/m$^3$]</td>
</tr>
<tr>
<td>$\forall$</td>
<td>Volume of the material [m$^3$]</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational acceleration [9.81 m/s$^2$]</td>
</tr>
</tbody>
</table>
\(\mu\)  Coefficient of friction

\(\mu_d\)  Dry coefficient of friction

\(\gamma\)  Lubricity of the powder lubricant [s/m]

\(\kappa\)  Wear coefficient

\(\kappa_{eff}\)  Modified dimensionless wear coefficient

H  Hardness of the material [N/m\(^2\)]

\(\lambda_i\)  Partitioning parameters for APT

\(v_e\)  Velocity of the escaping wear mass

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>COF</td>
<td>Coefficient of Friction</td>
</tr>
<tr>
<td>RFFL</td>
<td>Rate Form of First Law</td>
</tr>
<tr>
<td>RRM</td>
<td>Resisted Relative motion</td>
</tr>
<tr>
<td>PETRE</td>
<td>Power to Energy Transfer Rate Equation</td>
</tr>
<tr>
<td>JBE</td>
<td>Jerk Balance Equation</td>
</tr>
<tr>
<td>HIETRE</td>
<td>Heat to Internal Energy Transfer Rate Equation</td>
</tr>
<tr>
<td>IFOV</td>
<td>Influence of Friction on Velocity</td>
</tr>
<tr>
<td>PGS</td>
<td>Planetary Gear Systems</td>
</tr>
<tr>
<td>REB</td>
<td>Rolling Element Bearings</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>TCE</td>
<td>Thermal Compliance Experiment</td>
</tr>
<tr>
<td>Hp</td>
<td>Horsepower</td>
</tr>
<tr>
<td>kN</td>
<td>Kilo-Newton</td>
</tr>
<tr>
<td>N</td>
<td>Newton</td>
</tr>
<tr>
<td>W</td>
<td>Watts</td>
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<tr>
<td>IVP</td>
<td>Initial Value Problem</td>
</tr>
<tr>
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<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
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<td>APT</td>
<td>Algebraic Partitioning Technique</td>
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<td>APP</td>
<td>Algebraic Partitioning Parameter</td>
</tr>
<tr>
<td>APC</td>
<td>Algebraic Partitioning Condition</td>
</tr>
<tr>
<td>RDE</td>
<td>Reduced Differential Equations</td>
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<tr>
<td>DET</td>
<td>Differential Equation Term</td>
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Chapter 1 : Introduction

As the world contemplates lasting solutions to the ever-dwindling supply of non-renewable energy sources, attempts at improving energy-efficient innovations have significantly increased. World-renowned scientists suggest that a major component of the next industrial revolution must find ways of reducing friction to sustain global energy efficiency [1].

Energy-efficient innovations that bolster burgeoning clean energy technologies such as fuel cell, wind turbines, advanced manufacturing and turbine engines will be vital to the next industrial revolution. A major roadblock that hinders revolutionary sustainable energy and manufacturing innovations and threatens their significant progress is frictional dissipation. Frictional dissipation in energy and manufacturing systems has always been a major scientific problem [2, 3]. If left unchecked, friction deteriorates equipment performance, shortens lifespan, and increases maintenance costs [4, 5]. Friction affects not just energy-consuming and energy-producing equipment but also the relative sliding motion in natural phenomena such as earthquake, glacier melts, blood flow, hip and knee joints, planetary tidal motions, and interstellar evolutions. Therefore, any technique that advances the understanding of nonlinear friction during interfacial sliding likely will transform a variety of fields as well.

During relative motion accompanied by normal and tangential forces in sliding contact, frictional energy dissipation in the form of heat usually occurs. Under high load and speed operating condition, an increase in friction amplifies the generated heat. Sliding friction influences the general behavior of the dynamic systems with interfacial sliding contact [6-10]. Frictional energy loss occurs at different scales [11] and is measured as retardation in torque. The heat generated at the interface between
two surfaces is conducted away through both surfaces. The temperature reached will be determined by the coefficient of friction, the relative sliding velocity of the surfaces, their conductivity and their heat capacity [12, 13]. The fundamental property of two surfaces in relative motion is their velocity which becomes difficult to quantify if dynamical interfacial friction were admitted into the basic physics. The modeling difficulties worsen further when interfacial wear and deformation become significant.

It is well known that friction force opposes relative motion and all real surfaces are rough. For example, under an atomic force microscope (AFM), what to the naked eye appears as a flat surface shows up as asperities which are protuberances (see Figure 1). The sliding action creates heat because the surfaces tend to resist motion due to interlocking of asperities [8, 10, 14].

Furthermore, the asperities in a sliding interface must interact with different forms of lubricant (solids, liquids, and/or gases or solid-liquid, or solid-gas mixtures). This creates multi-scale difficulties for a problem that is already multi-physics. What further complicates issues is that friction is itself already inherently nonlinear [15].

In tribology experimentation techniques, the coefficient of friction and velocity are considered to be constant. The heat transfer is quantified primarily at steady state.

Figure 1: Schematic of a multi-scale, tribologically dynamic interface illustrating the multi-scale, interactions between lubricants and asperities, (A) two nominally flat surfaces in sliding contact with a sandwiched lubricant, (B) interfacial asperities and crevices observed with proximal probes such as an atomic force microscope.
and asperities on the surfaces are considered as statistical entities [16-19]. This undermines deterministic and predictive modeling.

The lubricant between the interacting surfaces not only reduces metal to metal contacts but also takes away the heat. If the heat generation rate exceeds the heat removal rate, unfavorable high thermal stresses within the components causes lubricant degradation and ultimate machine failure [20]. Technological advances have resulted in larger, faster, and more powerful machines. Modern energy producing and energy consuming devices such as advanced engines, turbines, expanders, and compressors are required to operate under extreme speed and high load conditions to meet the specific applications and operating conditions.

In this study, a deterministic model is constructed that captures all the complexities of dynamic frictional effects in a sliding contact. The modeling approach unifies Newtonian mechanics with classical thermodynamics. Together with basic laws of friction and wear, a mechanistic framework results that captures nonlinear tribological jerk behavior during sliding. The resulting kinematic and dynamics results lay the basis for tying friction to heat generation, material wear, energy efficiency, and for predicting equipment lifetimes.
Chapter 2 : Historical Review

The dissipation of energy in a sliding contact and/or machines is an antiquated problem. Humankind has always confronted the good and bad side in friction [21]. The study of friction, lubrication, and wear, is the science of tribology. Unlike other established fields in science, the formal inception of tribology occurred only about sixty years ago [21, 22]. The opportunities for studying the complexities of friction, lubrication, and wear, and their individual or joint impact on technologies that directly or indirectly use energy are limitless. Friction occurs at different length and time scales [23, 24], and affects many natural and man-made phenomena such as: earthquakes [25, 26], drag heating [27, 28], artificial hip, knee, and joint replacements [29-33], and computer read-write head technology [34]. Any fundamental theory on frictional heat generation will potentially transform all these affiliated fields.

There are three known laws of friction: (1) Friction is directly proportional to applied load attributed to Amontons, (2) Friction is independent of sliding velocity (attributed to Coulombs), and (3) Friction is independent of area of contact (attributed to Amontons) [6, 8-10]. Leonardo da Vinci (1452–1519) was the first to analyze the mechanism of friction and Guillaume Amontons (1663–1705) developed the formula for coefficient of friction. Coulomb investigated the influence of load, materials, surface area and the time of contact on friction [21]. Many years after notable contributions from scientists like Hertz and Reynolds [35], Bowden and Tabor [22] established the basis for the formal modern of friction, lubrication, and wear as known to us today. All the fundamental laws of friction predate this formal study of tribology since they were enacted between 1699 and 1789. Since its formal inception, it has been problematic to unite the nonlinearity of friction with Newtonian and non-Newtonian mechanics.
Friction has many deteriorating effects such as wear, heat generation, interfacial temperature rise, thermo-elastic instability, stick-slip, melting of metals, and lubricant degradation. A historical review on some of these topics is presented in the following sections.

2.1: Interfacial Wear in Tribology

In tribology, wear is defined as the mass loss due to interaction between contacting surfaces. Wear leads to inferior performance, replacement of the parts and reduction in efficiency of a system. In an unlubricated sliding motion the wear rate depends on the normal load, the sliding speed, initial temperature and the thermal, mechanical, and chemical properties of the materials in contact [14]. There are different types of wear processes associated with sliding or rolling motion. The major types include: 1) Adhesive wear, 2) Abrasive wear, 3) Erosive wear, 4) Fretting and corrosive wear, 5) Rolling contact surface fatigue, and 6) Wear by abrasive contaminants [36, 37]. Each type of wear has different effects on system performance. A single wear mode can be acting alone. Otherwise, the combined effect of multiple wear modes can occur at the same time. During the running-in period i.e. during the startup of any mechanical equipment, multiple wear modes act together and cause a large amount of wear. Once the running-in is over the system will run almost in a steady condition and the amount of wear is negligible. Usually, the running-in period in any sliding or rolling event is where asperities interlock and exchange momenta, energy, and cause excessive frictional heat generation and wear. Paradoxically, tribologists rarely study the running-in domain and most friction and wear data are taken after the running-in period [2]. But a seamless inclusion of the running-in is necessary to capture the entire range of frictional effects during sliding.
Figure 2 shows various contacting conditions and associated types of wear what are the effects of such events.

![Diagram of wear types and mechanisms](image)

Figure 2: The different facets of wear and accompanying interactions [38]

To deterministically model wear and predict its rate at lubricated and unlubricated interfaces is difficult. Most wear models are based on the empirical results. Hatchett (1803) and Rennie (1829) pioneered the experimental studies of wear [39]. There are several mechanisms by which wear can take place but the phenomenon is perhaps the least understood in the field of tribology and would require more extensive study. Some tribologists contemplate sliding wear as a result of adhesion only but it is only one of the many possible factors that complicates the wear [40].

Modeling wear during sliding motion is very challenging, but various researchers have laid a solid foundation. For example, Archard formulated wear by assuming that wear is proportional to the area of contact, applied load, and by comparisons with experimental evidence [41]. Archard’s wear law is a basis for many studies on wear
Yin and Komvopoulos studied adhesive wear that takes into account the effect of interfacial adhesion on the total load for three-dimensional fractal surfaces in normal contact. The fraction of fully-plastic asperity contacts, wear rate, and wear coefficient are expressed in terms of the total normal load (global interference), fractal (topography) parameters, elastic–plastic material properties, surface energy, material compatibility, and interfacial adhesion characteristics controlled by the environment of the interacting surfaces [45]. A low cycle fatigue wear mechanism is used to predict the wear coefficient for metallic sliding friction [46]. Some researchers extended Archard’s wear law using multiple stochastic models to determine the wear coefficient for different wear mechanism to predict the life of the high-power dry clutch system [47]. Also, finite element analysis is used to determine the material wear by considering the structural analysis analogy and load distribution [48]. Other computational approaches such as boundary elements, fracture mechanics, molecular dynamics are used to investigate the wear mechanism [49].

2.2: Heat Generation in Planetary Gear Systems and Rolling Element Bearings

Most energy-consuming and energy-producing equipment feature critical components that require bearings and gears. Heating by friction in sliding contacts in a planetary gear system (PGS) and rolling contacts in rolling element bearings (REB) degrades the performance of lubricants which depends directly on temperature. PGS are used in high speed and high torque applications for geared turbofans, wind turbine gearboxes, mixing equipment, and mining machinery while rolling bearings are used in turbine engine technologies, fuel cell-powered automobiles, stationary and mobile power plants, and environmentally-friendly advanced manufacturing systems. Any change in operating conditions that changes the temperature at the interface disrupts
the systems [50]. In fact, while gears are highly efficient at transmitting torque at high efficiencies reliability can be compromised when lubrication becomes adversely affected by churning, splashing, and windage [51]. Modes of failure in gear sets and bearings occur from lubricant starvation or excessive heat generation [52, 53]. Most PGS reside in gearboxes where sumps supply needed lubricant. Sump lubricant supply can generate excessive heat because of the high speed kinematics of a PGS which adversely affect the equilibrium temperature in the gearbox [54]. Figure 3 below shows the failure mode triggered by excessive wear and heat generation [55].

Figure 3: Failure mode in PGS due to (A) wear due to Adhesion (B) Plastic deformation due to heat generation [55]

Rolling element bearings (REBs) are considered antifriction since the rolling elements and supporting races have point contact instead of surface contact [20]. Bearings operating at high speed cause shearing of the lubricant, which results in an increase in metal-to-metal contact. Friction causes energy loss and this loss is measured as retardation in torque [56]. The thermal behavior of REBs is controlled by the balance between various power losses and the heat evacuated mainly by the lubricant [57]. Unfavorable high thermal stresses within the bearing elements cause lubricant degradation and ultimate bearing failure.
Lubricant degradation can also trigger other bearing failure modes; for example, flaking, cracks and fractures, cavities and indentation, smearing, wear, corrosion, and cage failure [58]. A recurrent theme from studies on bearing failure attributes the main cause of bearing failure to lubricant deterioration initiated by uncontrolled temperature rise. Lubricant churning also generates heat due to fluid friction [59]. Figure 4 below shows failure mode in REBs due to lubrication degradation and heat generation.

Figure 4: Failure modes triggered by lubrication degradation in REBs [53], (A) Flaking: due to temperature difference (B) wear

2.3: Surface Temperature and Distribution of Heat During Sliding

Various researchers have investigated temperature distribution and surface temperature at the sliding interface [60-66]. Sometimes the interfacial temperature can be very high and results in modification of mechanical and metallurgical properties and drastically changes system behavior. The rise in temperature at the interface can cause oxidation, phase transformation, and reduced wear resistance of the materials [41, 67-70].

There are many theoretical and experimental studies to determine the contact surface temperature. In 1937, a key insight into the impact of heat generation was given by Blok [71] who developed a first deterministic model to calculate the
temperature rise at the sliding surfaces under boundary lubricated conditions. In this study he considered the circular, square and partially contacting heat sources sliding over a half-space. The model considered low and high Peclet number effects. Blok partitioned the heat at the interface in order to attain equal surface temperature on both contacting surfaces. This partitioning of heat is known as Blok’s conjecture in the field of tribology. In 1942 Jaeger [72] extended the work of Blok and developed the solutions for various shapes of moving heat sources at the surface of a semi-infinite medium. In his work average temperature on both rubbing surfaces was used to obtain the expression for a heat partitioning function. Later in 1960, Archard [73] derived a simple theoretical formulation for “flash temperature” which is the highest temperature at the contacting surface using graphical methods. It was shown that the elasto-hydrodynamic film in a contact zone has the highest temperature which can be very high even when compared to the flash temperature of the surface. The high temperatures produced at rubbing contacts, the so-called flash temperatures, are of short duration (say $10^{-3}$ s or less) and occur only over small dimensions (say $10^{-4}$ m or less) [41]. Blok [74] provided a review of studies on the concept of flash temperature for the highly loaded gear teeth, cams and tappets. Francis [75] developed an analytic expression for the temperature field in a Hertzian contact using the ellipsoidal heat source. Greenwood [76] proposed a rule to estimate the surface temperature rise in the body employing the moving heat source with slower and faster velocities. Tian et.al [77] developed a mathematical model with circular, square and parabolic moving heat sources for the wide range of Peclet numbers using the Green’s function method and obtained the analytical and approximate solution for maximum and average surface temperatures. A fractal theory was provided to approximate the statistical temperature distribution at fast sliding surfaces in dry and boundary lubricated interfaces [78]. In
the historical models to estimate the surface temperature thermal conductivity of the
sliding solid was assumed to be constant. Abdel-Aal [79, 80] accounts for temperature
dependent thermal conductivity using the Kirchoff transformation. The influence of
varying thermal conductivity affects the magnitude of the flash temperature. Most of
the surfaces modeling interfacial temperature do not quantify the nonlinear effects
introduced by friction to the substrates that constitute the heat sources or sinks.

It is very difficult to obtain the surface temperature during sliding using
experiments [81]. Furey [82] measured the surface temperature for a fixed ball riding
on a rotating cylinder with imbedded thermocouples. Advancement in technology and
temperature measuring equipment made it possible to monitor the temperature rise
during sliding motion. Pin-on-ring experiments using a thermal video imaging system
showed real-time temperature distribution [83], with the surface temperature
increasing rapidly at the beginning of the sliding motion. Tian et al. [84] developed a
procedure to form thin film thermocouples on the surfaces to measure contact
temperature. Experimental studies to measure the contact temperature for dry and
boundary lubricated sliding systems with multiple heat source over the same area
were also compared with the theoretical model for the same test configuration [85].

The capability of infrared technology is also used to determine the temperature
field at a near surface contact between a flat and an aluminum alloy cylinder [86]. The
Fourier law of heat conduction is used to model transient heat conduction contact
problems for example in breaks [87]. The governing equations for heat generation are
based on classical Fourier and non-Fourier heat conduction [10, 88]. Laplace
transform and numerical techniques are used to solve the resulting partial differential
equations [89]. Fourier's law implies that any thermal disturbance on a body is
instantaneously felt throughout the body, which requires the propagation speed of
thermal disturbances is infinite. In fact, thermal waves travel with a finite speed in any material. For extremely short periods of time, extreme temperature gradients, and temperatures near absolute zero Fourier’s law is inadequate. Therefore, non-Fourier model can be used to overcome the shortcoming in the analysis of the temperature field in the laser applications [90].

To complement Fourier and non-Fourier heat diffusion models, a deterministic modeling approach that quantifies substrate transient nonlinear velocity is critical for elucidating the complex nonlinear heat generation and transfer caused by friction. This opportunity is explored in this study.

2.4: Thermo-Elastic Instabilities in Tribology

When two surfaces are in contact over a large area, the irregularities in the surfaces will cause a non-uniform pressure distribution between them. The generation of heat due to friction at the interface will also be non-uniform and the thermal expansion of asperities will cause distortion of the solids. The highest parts of the surface will carry the greatest pressure, reach the highest temperature, and consequently expand more than the surroundings. Thus the thermal distortion tends to exaggerate the initial irregularity of the surface. The wear at the interface has the opposite effect, but under suitable conditions the process can be unstable [91].

It is well known that in brakes, clutches and seals, frictional heat generation depends on applied load and for a given coefficient of friction there is a critical sliding speed above which the system becomes unstable. It generates small regions with local high temperature known as hot spots [62]. Barber developed a solution method of setting up thermoelastic contact problems using the surface displacement of an equivalent traction-free body and then solved an isothermal contact problem for this distorted body using green’s functions [92]. Willner investigated numerically
generated fractal rough surface in the elasto-plastic normal contact regime. The contact simulation uses an iterative elastic half space solution based on a variational principle [93]. If the velocity in a thermo-elastic problem is considered as constant then the problem will become linear and it is easy to obtain the solution using the existing methods. Olesiak [94] investigated the friction heat and wear in thermoelastic solids with non-linear boundary conditions using Volterra-Hammerstein non-linear integral equations and numerical techniques.

Considering velocity as a constant paints an unrealistic picture that overlooks the nonlinearities of frictional dissipation. In this study, the inherent nonlinear velocity from a tribological contact is determined.

2.5: Stick-Slip Phenomenon and Friction Induced Vibrations

Stick-slip, in which two sliding surfaces cycle between relative rest and motion, is a widely observed phenomenon. The frictional interactions of the two surfaces in relative motion cause stick-slip vibrations. Stick-slip vibrations are self-sustained oscillations induced by dry friction [95]. Dry friction appears in two different phenomena in nature: (i) as a resistance against the beginning of a motion starting from equilibrium (stick mode); (ii) as a resistance against an existing motion (slip mode). Whenever the coefficient of kinetic friction is less than the static coefficient of friction the motion becomes intermittent. The two contacting surfaces will stick until the pull force of one surface equals the static friction. The surface will slide over one another after that by decreasing values of kinetic coefficient of friction and again they stick together. This motion becomes jerky due to the sudden increase in the velocity of the moving surface. When machines move slowly, they exhibit stick-slip or a periodic cycle of alternating motion and arrest stick slip determines the lower
performance bound of a machine: lowest sustainable speed and the shortest
governable motion [96].

Stick-slip oscillations are normally analyzed in terms of the kinetic friction-
velocity and the static friction-time of stick characteristics of the rubbing surfaces. 
Rabinowicz [97] discussed the intrinsic frictional characteristics of the surfaces to
determine the stick-slip oscillation and its magnitude. The apparatus used for this
study is shown in Figure 5 which is capable of producing the stick-slip oscillations.
They postulated that a sliding velocity reached instantaneously during slip gives rise
to a friction coefficient characteristic, not of the velocity itself, but of the average
velocity over some previous slid distance. Using the critical distance concept, they
deduced a simple relationship between the static and kinetic coefficients of friction.
This critical distance is of the order of $10^{-4}$ cm.

![Figure 5: Schematic friction apparatus capable of producing stick-slip oscillations [97]](image)

The friction-velocity coupling captured in Rabinowicz’ studies as well as the
critical distance phenomenon prove to be key concepts that facilitate fundamental
deterministic friction modeling. In fact, the critical distance provides an important
initial sliding distance for relative motion. This critical distance is used as the initial
sliding distance throughout the modeling in this study.

Stick-slip vibrations can be observed in mechanical systems as well as in day to
day life. Some common examples are vibration of the string, squeaking of door,
squealing in brakes, musical instruments, a hinged bar with periodic impulsive loading, a vibrating bean, rattling gear drives, etc. Stick-slip vibrations are obviously undesirable because of their negative effects on the functionality and performance of many mechanical systems (machine tool slides, automatic transmissions, hydraulic driving systems, etc.) as they can cause serious wear of the components, fatigue and positioning errors [98].

2.6: Experiments to Simulate the Sliding Motion

In tribology experiments, friction, lubrication and wear tests are carried out using tribometers. Most tribometers operate individual modules for specific tests such as ball-on-disk (BOD), pin-on-disk (POD), four ball, and block-on-ring (BOR), just to mention a few [6, 8, 14, 99]. Typically, block-on-ring (BOR) tests are used to simulate sliding motion. The BOR experiment is good at quantifying line contact friction, but pure sliding may be too severe to simulate the true contact in some cases [100]. The American Society for Testing and Materials (ASTM) International standard for BOR experiments is ASTM G 77 [100]. BOR experiments have been used for evaluation of piston rings, engine blocks, liners, valve guides, as well as surface coatings [101].

During typical tribometer tests, data associated with the incipient stages of sliding, known as the running-in are ignored. The preferred data is taken at steady state when all the transients have died out. Moreover, tribometers typically feature a velocity feedback essentially masking any direct friction-velocity coupling effects. This state of affairs seems to align with Coulomb’s law proffering friction as independent of sliding velocity. Tribometer tests that featured no feedback [102, 103] capture essential friction-velocity coupling that will be pivotal in eliciting nonlinear system
behavior. Since deterministic friction models are scarce, matching experimental results with theory is often troubling.

In this study, the transient friction, lubrication, and wear models provide a basis for interrogating experimental tribological results.
Chapter 3: Research Motivation and Methodology

3.1: Motivation and Objectives

The main research problem here is that despite the existence of the three key laws of friction (Amontons 1st and 2nd laws and Coulomb’s law of friction), understanding friction in a constrained sliding interface is not sufficient. Tribologists study surfaces in relative motion with emphasis on friction, lubrication, and wear, and the design of bearings. Friction, lubrication, and wear are multi-scale, multi-physics phenomena that are intertwined in transient and steady state occurrences. Moreover, since friction itself is inherently nonlinear, it exacerbates difficulties associated with predictive and deterministic modeling.

Recent studies show that solid lubricants and liquid-solid lubricant mixtures are most viable for operating the critical mating components (bearing, gears and seals) of energy efficient equipment, devices, and manufacturing technologies [102]. Also, from recent thermal compliance studies, it was noticed frictional heat generation increases with interfacial speed [104, 105]. This excessive heat generation can lead to lubrication degradation, wear or failure during a long run. By constructing a fundamental deterministic model that captures kinematic properties of distance, velocity, and acceleration, transient friction, lubrication, and wear can be quantified which are needed for predicting frictional impact on heat generation, mechanical efficiency, and equipment lifetime.
3.1.1 Objectives of Research

The main aims of this study are as follows:

1. Create a fundamental jerk dynamics model to quantify the nonlinear friction, lubrication, and wear in sliding contacts.
2. Apply the tribological jerk dynamics results to predict frictional heat generation and mechanical efficiency.
3. Perform block-on-ring, pin-on-disk tribometer experiments to validate the theoretical models.
4. Carry out parametric studies to refine or modify the theoretical model and experiments to identify a universal theme or themes.

The stated objectives will be carried out through experiments and modeling.

3.2 Experimentation

The purpose of the experimental procedure is to construct tests that identify and monitor friction, lubrication, wear, and the incidence of frictional heat generation in a lubricated sliding interface. To measure heat generation, surface and near surface temperatures of blocks and rings in sliding contact are measured using embedded thermocouples and infrared temperature guns. The onset of thermal deformation or stresses is captured using a thermal imaging camera. Two sets of test rigs are used: a) a special in-house built thermal compliance experimental (TCE) test rig which allows for the testing of actual bearings, gears, and seals, and b) a universal multi-specimen tribometer that allows typical bench top friction, wear, and lubrication tests to be conducted. Additionally, the multi-specimen tribometer is retrofitted to capture temperature measurements for determining the incidence of frictional heat generation.
3.2.1 Thermal Compliance Experiment

A thermal compliance experiment (TCE) is designed to test an *in situ* cooling mechanism aimed at mitigating heat losses and performance degradation during frictional heat generation (FHG). The schematic of the TCE test rig is shown in Figure 6. In this test, planetary gear sets (PGS) and rolling element bearings (REBs) are tested. The PGS and REB are selected as test units, and the *in situ* heat exchanger is constructed and subjected to various tests. Parameters such as speed and load are important considerations in designing test experiments for mechanical systems.

![Figure 6: Schematic setup of the thermal compliance experiment (TCE) test rig](image)

The heat generation experiment is used to determine the amount of heat generated at different speeds. Figure 7 (A and B) show the heat generation in the REB outer race and PGS ring gear respectively. By using a variable frequency drive (VFD) a single speed (i.e., 1800 rpm), three phase AC motor was used to run the bearing at different rotational speeds. Thermocouples were used to continuously monitor the temperature variation at various critical locations.
The heat generation in the REB outer race and PGS ring gear increases with speed as exhibited in Figure 7 (A and B). The largest temperature rise of 2.5 Kelvin corresponds to the highest speed. The temperature increases rapidly from 0 to 600 seconds and starts to level off.
to a linear increase in temperature. The temperature rise in the PGS follows trends similar to the ones noticed in the REBs. It is important to note the strong effect of the environment on the temperature measurement. Outside disturbances can have a significant effect on the temperature of the system.

### 3.2.2: Block-on-Ring (BOR) Tribometer Testing:

A block-on-ring (BOR) tribometer test is used to study sliding friction in this preliminary study. The BOR tribometer tests simulate contact studies that can help to quantify the amount of frictional heating taking place. The tests were conducted using a universal multi-specimen tribometer (UMT). This tribometer has a capability to run pin-on-disc, block-on-ring and reciprocating motion tests with environmental controls with cooling or heating chambers. Additionally, the UMT has an inbuilt data acquisition system which enables monitoring the real-time data for force, speed, coefficient of friction during the test. Additional thermocouples were used with a separate data acquisition system in order to measure the temperature of the lubricant in the reservoir.

Figure 8 shows the tribometer with the BOR module in place, and Figure 9 is a schematic drawing of the BOR module. For the BOR module the motor provides the rotary motion to the ring and a 50 N load is applied through a force transducer which is mounted on the top carriage. The force transducer contains a load cell to measure the normal load applied.
A lateral force transducer measures the frictional force as indicated in Figure 9. The lubricant reservoir is located below the ring to facilitate dip lubrication.

The standard dimensions specified by ASTM [100] for the block-on-ring test specimen are shown in Figure 10. Other properties of the material, roughness, and hardness of the specimens are listed in Table 1.
Figure 10: Schematics of the (A) block and (B) ring specimens used in the tribometer tests.

Table 1: Properties of the tribometer specimens

<table>
<thead>
<tr>
<th></th>
<th>Block</th>
<th>Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>SAE 01 Tool Steel</td>
<td>SAE 4620 Tool Steel</td>
</tr>
<tr>
<td>Surface Roughness (μ in)</td>
<td>4 - 8</td>
<td>6 - 12</td>
</tr>
<tr>
<td>Hardness (HRC)</td>
<td>27 - 33</td>
<td>58 - 63</td>
</tr>
</tbody>
</table>

Heat Generation in the Block and Reservoir

Temperature is measured in the block and reservoir for the block-on-ring experiments. Type-K thermocouples are used with a range of 200 °C to 1000 °C with an accuracy of ± 3° C. Typical block specimens do not have sites to insert thermocouples. Therefore, a hole with diameter 5/64 inch is bored in the block to facilitate the thermocouple tip. The temperature of the reservoir is also measured. A type-K thermocouple is placed in the lubricant sump and secured. The lubricant reservoir is cleaned after each trial and refilled with gear oil having the same properties as in the TCE tests.

Results of Block-on-Ring (BOR) Testing

The BOR tests used to simulate the frictional contact encountered in REB and PGS are used to examine the amount of heat generation due to frictional heating. Figure 11(A and B) show the temperature rise of the block and reservoir, respectively.
The temperature rise is observed to increase with speed in a similar fashion as in the TCE experiments. The highest temperature rise of 11 Kelvin in the block is observed for the highest speed. The highest temperature rise in the sump is 5 Kelvin.

Figure 11: Temperature rise in the (A) Block, and (B) Lubricant with increasing speed

The coefficient of friction (COF) is measured simultaneously with the temperature. Figure 12 shows the COF for each test at different speeds.
Figure 12: The coefficient of friction exhibits different trends with changing speed

Lower speeds correlate to a lower coefficient of friction and higher speeds correlate to higher coefficient of friction. Two regimes for COF are observed. In the first regime, at lower speeds of 500 and 750 rpm COF decreases with time. Initially it starts with a very higher values and decrease with time. For the speed of 1250 rpm the behavior is interesting. The COF initially decreases for 400 seconds and then begins to increase. In the regime-2 at the higher speeds of 1500 and 1750 rpm COF increases throughout the test duration. Also, the COF is higher at the beginning of the test indicating the speed is having a large influence on the COF. Moreover, the COF appears to have a strong dependence on temperature which confirms different studies in tribology.
3.2.3: Temperature Measurement Using a Thermal Imaging Camera

The universal multi-specimen tribometer can was used to run the pin-on-disk test using the rotary module. Figure 13 shows the tribometer with a pin-on-disk module and Table 2 shows the properties and dimensions of the test specimen used for these tests.

![Figure 13: The universal multi-specimen tribometer with a pin-on-disk module](image)

Table 2: Properties and dimension of pin-on disk test specimens

<table>
<thead>
<tr>
<th></th>
<th>Pin</th>
<th>Disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>2024 Aluminum</td>
<td>2024 Aluminum</td>
</tr>
<tr>
<td>Dimensions</td>
<td>Diameter: 12.7 mm</td>
<td>Diameter: 70 mm</td>
</tr>
<tr>
<td></td>
<td>Length: 50 mm</td>
<td>Thickness: 6 mm</td>
</tr>
</tbody>
</table>

The tests were performed at same speed under the two different normal forces, 10 N and 15 N at a speed of 1 m/s. The results for this test are shown in Figure 14. It is clear that as the normal load increases the temperature of the pin under the same operating conditions also increases.
Figure 14: The effect of load on frictional heat generation

During the tests temperature rise inside the pin is monitored using the FLIR™ infrared thermal imaging camera. The images of the pin temperature at different times during the tests are shown in Figure 15 below.

Figure 15: The thermal images of an aluminum pin specimen at 15 N load at 1 m/s speed
Experimental studies using thermal compliance experiments, block-on-ring tests and pin-on-disk tests clearly show the temperature at the contacting surfaces increases with speed as well as load. Therefore, a predictive mathematical model that captures these trends fundamentally is a key to examining these experimental and other essential tribo-thermal experimental results.

During modeling, a fundamental basis for quantifying the kinematic properties of sliding distance, velocity, and acceleration is constructed. These kinematic properties serve as input to determine the dynamic effects of force, work, energy, and power. The thermal and mechanical system behaviors can be then be obtained.
3.3: Theoretical Modeling

The essence of the theoretical modeling is to construct a unified deterministic tribological jerk dynamics model for quantifying friction, lubrication, and wear at sliding interfaces. The modeling framework may be generalized to capture the essential sliding details including surface roughness, lubricant adhesion, interfacial asperity locking, and ploughing. It was found that even for nominally flat surfaces, the resulting jerk dynamics is highly nonlinear and extremely challenging to solve. This study however focuses only on nominally flat interfacial sliding and prepares the groundwork for future modifications that will incorporate the additional complexities where needed.

Even though tribologists study surfaces in relative motion, the application of Newtonian mechanics with the inclusion of friction remains difficult. Besides, Coulomb’s law of friction suggests that friction is independent of sliding velocity [8-10, 14] and it is this supposedly controversial law that has hindered a deterministic assessment of dynamical interactions involving friction.

In this study, a tribological sliding event was analyzed by including the running-in. The interfacial sliding friction problem was approached differently by synthesizing insights from classical mechanics [106], heat transfer [7, 88], thermodynamics [107, 108], fluid mechanics, friction, lubrication and wear theories to construct a power to energy transfer rate equation (PETRE) and a heat to internal energy transfer rate equation (HIETRE). The novelty here is that we start our tribo-thermal analysis from an energy point of view using the rate form of the first law (RFFL) of thermodynamics. In tribology the surfaces in sliding contact are called first body and the lubricant in between is called the third body. Each third body and first bodies are treated individually and collectively as control volume undergoing tribo-thermal
sliding events. We do not presume any form of the applied force, work, or energy transfer but rather allow the rate forms to capture the essential transients in the sliding events. For example, consider a simply lubricated block that is undergoing frictional wear and heat generation during sliding. Even in the absence of asperities it is a complex tribo-thermal dynamical problem.

### 3.3.1: Newton’s Second Law and The Concept of Jerk

Newton’s second law of motion is the basis for describing different types of motion mathematically or in differential form. The equations of motion are derived by applying a force balance on the moving object. Generally, acceleration is considered as a constant or assumed to be continuous and the significance of jerk is glossed over. This formulation is helpful when either the form of the applied force or the variation of distance as a function of time is known. During relative resisted motion both the force and the variation of distance are unknown due to effects of the resisted force. In that case, even if the form of the resisted force is known, Newton’s second law of motion gives a differential equation with two unknowns, i.e. applied force and distance as a function of time. Jerk analysis removes this difficulty by producing an equation of motion which is a third order differential equation in a single variable, distance.

Jerk is an important factor in some engineering applications such as design of cam, valve train, and Geneva mechanism [109, 110]. In the literature, various studies are available on the concept of jerk. An experiment illustrating the notion of jerk [111] in the spring-mass system and hanging mass-thread system is a classic example. Schot [112] studied several geometric properties of the jerk vectors in planar motion. Sandin [113] developed kinematic equations for a constant jerk system using newton’s second law. Nzotungicimpaye [114] established group theoretical methods
for jerk using Galilei-Newton laws of motion. Tan et al. [115] introduced the concept of angular jerk in angular motion which represents the Keplerian motion. Newtonian jerky motions were developed by differentiating Newton’s second law of motion with respect to time [116, 117] and some criterion were specified for being Newtonian jerky or hyper jerky [118, 119] systems. Furthermore, jerk has been applied in modern computer numerically controlled production [120], transporting Bose-Einstein condensates [121], and noticed in plastic strains within superconductors and superconducting composites [122].

Despite its significantly reported usage jerk has hardly been studied in basic physics [123], mechanics, and tribology [124, 125] because it rarely enters dynamical formulations involving force, momentum, or energy [112, 113, 117]. Paradoxically, jerk occurs not just in artificial but also natural phenomena. In fact, the Earth’s internal torsional oscillations periodically triggers geomagnetic jerk [126, 127], while planetary tidal forcing influences the Sun’s recurrent, cyclical, solar jerk motion, and sunspots [128-130]. Recent evidence of cosmic jerk in supernovae data illustrates how evolving dark energy affects the expansion and fate of our universe [131].

In this study, the jerk phenomenon is placed within classical mechanics context. We show that even though Newton’s second law is a fundamental basis for defining jerk to express it completely requires unifying Newtonian mechanics with classical thermodynamics. The jerk balance equation resulting from what we have termed mechanistic unification helps in reducing the differential equation of motion with only one unknown i.e. distance, in case of the constant mass motion. Further modification of the case of variable mass produces a jerk balance equation with two unknowns: mass and distance. It is relatively easy to solve one equation with one unknown using the prescribed initial conditions. For the case with variable mass, a
suitable conservation of mass law suffices to reconvert into a single variable jerk balance equation.

Apart from elucidating complex nonlinear friction and its effects, jerk dynamics in general scientific terms may transform how a diversity of natural and artificial phenomena is modeled.

3.3.2: The Rate Form of the First Law of Thermodynamics

The first law of thermodynamics has formed the basics of numerous groundbreaking fundamental laws in various facets of scientific explorations, from fluid flow and heat dissipation, nuclear reactions, and plasmonics [107, 108]. The universality and flexibility of this law allow for accommodating different forms of energy and work transfer modes under one platform. The first law in rate form is given by:

$$
\dot{Q} - \dot{W}_a = \frac{dE_T}{dt} = \dot{E}_T = \dot{K}_e + \dot{G}_{pe} + \dot{U}
$$

(3.1)

where, $\dot{Q}$[W] is the rate of heat input into a control volume and $\dot{W}_a$[W] is the work transfer rate or power and $\dot{E}_T$[W] is the total energy transfer rate. The total energy transfer rate includes the sum of the rates of kinetic $\dot{K}_e$, gravitational potential $\dot{G}_{pe}$, and internal $\dot{U}$, energies respectively. At steady state, the total energy transfer rate vanishes (i.e. $\dot{E}_T = 0$). At once we observe by simple integration that the total energy in a steady state process is constant (i.e. $E_T = 0$). It is this steady state format that is currently prevalent in most dynamical modeling. Contrarily, during an unsteady dynamical process, the transient energy transfer rate is neither vanishing nor constant (i.e., $\dot{E}_T \neq 0$) but is captured as the interaction between heat and work.
transfer rates as given by the rate form of the first law (RFFL) in Eq. (3.1) by taking care of each term appropriately [108].

This essence of the rate form of the first law (RFFL) of thermodynamics is rarely used in dynamical modeling. We explored this tool to elucidate the onset of nonlinear tribological jerk during interfacial sliding.

To implement the RFFL, we first decouple it into the power to energy transfer rate equation (PETRE):

\[ -\dot{W} = \dot{K}_e + \dot{G}_{pe} \]  

and the heat to internal energy transfer rate equation (HIETRE):

\[ \dot{Q} = \dot{Q}_{in} - \dot{Q}_{out} = \dot{U} \]  

By decoupling the RFFL the heat transfer and mechanical energy transfer rates can be modeled separately. At the same time, assuming negligible heat transfer and internal energy transfer rates reduces the RFFL to a power to energy transfer rate equation (PETRE) alone. Equations (3.2) and (3.3) represent different aspects of our problem that combines heat transfer, tribology, and classical mechanics theories to yield the generalized nonlinear frictional jerk and transient temperature models.

The modeling flow diagram for uniting Newtonian mechanics and classical thermodynamics is shown in Figure 16. Figure 16 schematically outlines how the laws of thermodynamics, heat transfer, and tribology can be used with Newton’s second law of motion to obtain the dynamic behavior of different systems under transient resistance such as friction. This mechanistic unification produces a third order differential equation which is a jerk balance equation (JBE) by which kinematic properties (distance, velocity, acceleration, and jerk) and dynamic causes and/or effects (force, work, energy and power) can be obtained. This information can be used
to quantify the effect of friction on heat generation and mechanical efficiency of the sliding motion.

Figure 16: Modeling flowchart showing a mechanistic unification of classical laws

In the modeling approach, Eq. (3.2) is applied to study the kinematics and dynamics while Eq. (3.3) is applied to study the associated heat transfer during interfacial sliding. Prototypical examples during interfacial sliding are used as illustrative examples in the sections that follow.
Chapter 4: Nonlinear Jerk Dynamics in Boundary Lubricated Sliding

In the previous chapter, the power to energy transfer rate equation (PTETRE) was derived using Newtonian mechanics and classical thermodynamics as:

\[-\dot{W}_a = \dot{K}_e + \dot{E}_{pe}\]  \hspace{1cm} (4.1)

To apply the PETRE to study a sliding block, different scenarios in the sliding process were considered:

1) Sliding without lubrication and wear, evaluates the impact of a non-lubricated sliding event without any wear. This modeling captures the best case scenario in a premium unlubricated tribological process especially for self-lubricating solids with negligible or no wear during sliding.

2) Sliding with lubrication and no wear, quantifies the ideal lubrication process in which friction is not eliminated completely but wear is negligible.

3) Friction, wear, and lubrication all occurring simultaneously as is typically found in all actual tribological events occur in bearings, gears, and seals. This third case is the real life scenario where the detrimental effects of friction and wear simultaneously occur even in the presence of lubrication.

The prototypical sliding block is the simplified version of different sliding interfaces found in mechanical systems such as the reciprocation motion of a piston-cylinder engine, brake pad of vehicles, earthquake gouge lubrication, sliding bearings, gears, seals, frictional melting of rocks, artificial hip and knee joints. Because of the boundary lubrication nature of the sliding problem, it applies similarly to liquid, solid, and liquid-solid mixture lubricants.
In the power to energy transfer rate balance analysis we consider only work and mechanical energy transfer rates. The mechanical energy part considers the kinetic energy and potential energy of the block. Since the block is sliding on a horizontal surface the change in potential energy with time is very small and is neglected in this analysis.

### 4.1 Sliding Without Lubrication or Wear

Consider a slider block (See Figure 17) of mass $m$ [kg] displaced through a distance $s$ [m] under the action of an external force $F_a$ [N], opposed by a frictional resistance $F_f$ [N]. According to Amonton’s law of friction $F_f = \mu F_n$, where, $\mu$ is the coefficient of friction, the normal load $F_n$ [N] is taken as the block’s self-weight $mg$, and $g$ is the gravitational acceleration. From Coulomb’s law of friction, friction is assumed to be independent of sliding velocity. This law allows us to keep $\mu$ as a constant in this portion of our analysis.

At the beginning of the motion, the block starts with an initial velocity $u_0$ [m/s$^2$] and a very small sliding displacement $s_0$ [m] which was reported by Rabinowicz as pre-sliding distance. Using Newton’s second law of motion we obtain the undetermined applied force needed to move the block through a sliding distance $s$ as:
Because there is no wear, it is assumed that the block’s mass remains constant. Thermodynamically, the work done by a system is defined as the multiplication of the force applied by the system and the distance travelled. Therefore, the work done on the block by the externally applied force \( F_a \) is:

\[
-W_a = F_a s = ms \frac{d^2 s}{dt^2} + \mu F_n s
\]

The negative algebraic sign aligns with thermodynamic convention for work done by the external force on the sliding block. The work transfer rate or power becomes:

\[
-W_a = \mu F_n \frac{ds}{dt} + ms \frac{d^3 s}{dt^3} + m \frac{ds}{dt} \frac{d^2 s}{dt^2}
\]

Since gravitational potential energy (G.P.E.) remains negligible because of the choice of the G.P.E datum our total energy is simply the system’s kinetic energy given by \( K_s = m(\frac{ds}{dt})^2 / 2 \). The total energy transfer rate therefore becomes the block’s kinetic energy transfer rate given by:

\[
\dot{E}_r = m \frac{ds}{dt} \frac{d^2 s}{dt^2}
\]

By combining Eqs. (4.3) and (4.4) the result of our power to energy transfer rate equation (PETRE) given by \(-\dot{W}_a = \dot{E}_r\) from Eq. (4.1) becomes:

\[
\mu mg \frac{ds}{dt} + ms \frac{d^3 s}{dt^3} + m \frac{ds}{dt} \frac{d^2 s}{dt^2} = m \frac{ds}{dt} \frac{d^2 s}{dt^2}
\]

\[
\frac{d^3 s}{dt^3} = -\frac{\mu g}{s} \frac{ds}{dt}
\]

Equation (4.5) is very interesting because it reveals that the third derivative of distance with time \( \frac{d^3 s}{dt^3} = \dddot{s} \), which is also the second derivative of velocity with time \( \frac{d^2 v}{dt^2} = \dddot{v} \), or the first derivative of acceleration with time \( \frac{da}{dt} = \dddot{a} \), is non-
vanishing. In classical mechanics, the rate of acceleration with time is called \textit{jerk}, $j$ \text{[m/s}^3\text{]}, which is not usually used. Equation (4.5) shows a non-vanishing inertial jerk $j_i = d^3s/dt^3$ created by a nonlinear frictional jerk $j_f = -\mu g(ds/dt)/s = -\mu gv/s$.

This frictional jerk $j_f$ is proportional to sliding velocity $v = ds/dt$, and inversely proportional to sliding distance $s$, with $\mu g$, as proportionality constant. The negative sign showing opposing frictional and inertial jerks conforms to a tangential jerk opposing sliding velocity [112]. Also, it confirms the basic attributes of non-conservative force such as friction which reduces the mechanical energy of a system.

If there is no friction then the work done by the block equals the total mechanical energy of the system. From Eq. (4.5), increasing friction and velocity amplify inertial jerk, and completes an initial value problem when initial conditions for distance $s(t = 0) = s_0$, velocity, $v(t = 0) = u_0$, and acceleration $a(t = 0) = a_0$ are applied.

Furthermore, during frictionless sliding ($\mu = 0$), Equation (4.5) becomes a zero inertial jerk \textit{(i.e.,} $d^3s/dt^3 = 0$), a conclusion obtained independently using Eq. (4.1) directly for a frictionless sliding block [132].

Directly integrating the zero jerk equation \textit{(d}^3s/dt^3 = 0\text{)} produces: (1) constant initial acceleration $d^2s/dt^2 = a_0$, (2) velocity $v = ds/dt = u_0 + a_0 t$, (3) distance, $s = u_0 t + a_0 t^2/2 + s_0$, and (4) the velocity-distance relation, $v^2 = u_0^2 + 2a_0(s - s_0)$. These kinematic results resemble well-known precursors from basic physics [133], the difference being the requirement of nonzero initial acceleration $a_0$ and sliding distance $s_0$ necessitated by zero jerk. Indeed, a $a_0 > 0$ necessarily alters the block’s rest or uniform motion \textit{(s}_0 \geq 0, u_0 \geq 0\text{)} state in conformity with Newton’s first law. Therefore, zero jerk apparently links Newton’s first and
second laws of motion, conforming to constant acceleration Galilei-Newtonian mechanics [114]

The non-vanishing frictional jerk $j_f$ being inversely proportional to distance $s$ apparently verifies tribological observations during running-in [134, 135] where changing accelerating effects vanish as $s$ increases. Also, $j_f$ opposes the system’s velocity with a damping frequency $\omega_{d}\left[s^{-1}\right] \sim \sqrt{(\mu g) / s}$ analogizing the frequency of a friction-induced vibration. Thus, $\omega_{d}$ increases with high friction coefficient $\mu$ and decreasing $s$ but diminishes at low $\mu$ and increasing $s$ potentially explaining the high squealing noises associated with sliding at high $\mu$ [136, 137].

Solving the nonlinear third order ODE in Equation (4.5) for exact distance-time is nontrivial. To apply an order reduction a velocity-space transformation was used to enable the integration of Eq. (4.5). Setting $v = ds / dt$ gives $d^2s / dt^2 = vdv / ds$ and $d^3s / dt^3 = vd / ds\left(vdv / ds\right)$. This $v-s$ transformation reduces Eq. (4.5) to:

$$s \frac{d}{ds}\left(v \frac{dv}{ds}\right) + \mu g = 0 \tag{4.6}$$

Integrating Eqn. (4.6) once with respect to $t$ and applying the initial conditions, $d^2s / dt^2\left(t = 0\right) = a_0$; $s\left(t = 0\right) = s_0$, an exact solution for acceleration can be obtain as:

$$a = \frac{d^2s}{dt^2} = -\mu g \ln s + a_0 + \mu g \ln s_0$$

$$a = \frac{d^2s}{dt^2} = a_0 + \mu g \ln \left[s_0 \over s\right] = a_0 - \mu g \ln \left[s \over s_0\right] \tag{4.7}$$

A similar transformation converts Eq. (4.7) into a reduced first order ODE to obtain the integrable form in Eq. (4.8).
After integrating Eq. (4.8) and applying applicable initial conditions the exact solution for velocity in Eq. (4.9) was obtained.

\[
v^2 = \left( \frac{ds}{dt} \right)^2 = u_0^2 + 2(a_0 + \mu g)(s - s_0) + 2\mu gs \ln \frac{s_0}{s}
\]

After integrating Eq. (4.8) and applying applicable initial conditions the exact solution for velocity in Eq. (4.9) was obtained.

\[
v = \frac{ds}{dt} = \sqrt{u_0^2 + 2(a_0 + \mu g)(s - s_0) + 2\mu gs \ln \frac{s_0}{s}}
\]

By combining Eqs. (4.5) and (4.9) the exact nonlinear dependence of jerk on sliding distance is given as

\[
\frac{d^3s}{dt^3} = -\frac{\mu g}{s} \left( u_0^2 + 2(a_0 + \mu g)(s - s_0) + 2\mu gs \ln \frac{s_0}{s} \right)^{1/2}
\]

Equation (4.9) is an important, exact, though nonlinear solution that illustrates the dependence of sliding distance on sliding velocity. Separating variables in the nonlinear first order ODE in Eq. (4.9) and integrating once, gives the elliptic integral in Eq.(4.11) with \(c_i\) being a constant of integration.

\[
\int dt = t + c_i = \int_{s_0}^{s} \frac{dx}{\sqrt{(u_0^2 + 2(a_0 + \mu g)(s - s_0) + 2\mu gs \ln \frac{s_0}{s})}}
\]

To solve Eq. (4.11) for the dependence of distance on time is problematic using the existing elliptical integral techniques. So far, we have been unable to evaluate the exact elliptic integral in Eq. (4.11) for an explicit time-distance relationship. Analytical elliptic integral techniques for solving Eq. (4.11) seem rare. Equation (4.11) appears to be neither Jacobi, Weierstrass, nor generalized elliptic integral format [132, 138, 139]. To cursorily evaluate the effects of frictional sliding at the interface, we approximated the integral in Eq. (4.11) using truncated series expansion.
\[
\ln x = 2 \left( \frac{x-1}{x+1} + \frac{1}{3} \frac{(x-1)^3}{x+1} + \frac{1}{5} \frac{(x-1)^5}{x+1} + \ldots \right); \quad x > 0; \quad (4.12)
\]

In Equation (4.11), the series expansion for \(\ln x\) is truncated by taking only the first term in Equation (4.12) (Taken from reference [139]) to get \(\ln x = 2(x-1)/(x+1)\).

The approximated form of Eq. (4.11) is shown as Equation (4.13):

\[
\int dt = t + c_1
\]

\[
= \int_{s_0}^{s} \frac{\sqrt{(x+1)} dx}{\sqrt{x^2 \left[ 2(a_0 + \mu g) + 2\mu g \ln(s_o) - 4\mu g \right] + x\left[ u_0^2 + 2(a_0 + \mu g)(1-s_0) + 2\mu g \ln(s_o) + 4\mu g \right] + u_0^2 - 2s_0(a_0 + \mu g)}}
\]

OR:

\[
\int dt = t + c_1 = \int_{s_0}^{s} \frac{(x+1)}{\sqrt{n_1 x^2 + n_2 x + n_3}} dx
\]

\[
n_1 = \left[ 2(a_0 + \mu g) + 2\mu g \ln(s_o) - 4\mu g \right];
\]

\[
n_2 = \left[ u_0^2 + 2(a_0 + \mu g)(1-s_0) + 2\mu g \ln(s_o) + 4\mu g \right]
\]

\[
n_3 = u_0^2 - 2s_0(a_0 + \mu g)
\]

The reduced version of the elliptic integral in Equation (4.13) is now solvable using elementary functions. To do so, requires one more conversion into the known format in Eq. (4.14)
\[ n_1^{1/2} \int dt + c_1 = \int_s^t \sqrt{\frac{x+1}{n_1 x^2 + n_2 x + n_3}} \, dx = \int_s^t \sqrt{\frac{x+1}{x^2 + b_1 x + b_2}} \, dx \]

where, \( b_1 = n_2 / n_1; b_2 = n_1 / n_1 \)

\[ n_1^{1/2} \int dt + c_1 = \int_s^t \sqrt{\frac{x+1}{x^2 + b_1 x + b_2}} \, dx = \int_s^t \sqrt{\frac{x+1}{(x + b_1 / 2)^2 - (b_1^2 / 4 - b_2^2)}} \, dx \]

\[ = \int_s^t \sqrt{\left( x + b_1 / 2 - \left[ b_1^2 / 4 - b_2^2 \right] \right) \left( x + b_1 / 2 + \left[ b_1^2 / 4 - b_2^2 \right] \right)} \, dx \]  \hspace{1cm} (4.14)

\[ n_1^{1/2} \int dt + c_1 = \int_s^t \sqrt{\frac{x-b}{(x-a)(x-c)}} \, dx; \]

where,

\[ a = \left[ b_1^2 / 4 - b_2 \right] - b_1 / 2; b = -1; c = \left[ b_1^2 / 4 - b_2 \right] - b_1; \]

Clearly, Eq. (4.11) is the principal elliptic integral of concern. This integral is given by Eq. (4.15) using elementary functions from [132].

\[ n_1^{1/2} \int dt + c_1 = \int_s^t \sqrt{\frac{x-b}{(x-a)(x-c)}} \, dx; \]

\[ a = \left[ b_1^2 / 4 - b_2 \right] - b_1 / 2; b = -1; c = \left[ b_1^2 / 4 - b_2 \right] - b_1; \]

\[ n_1^{1/2} \int dt + c_1 = \frac{2(a-b)}{\sqrt{a-c}} F(\phi, q) - 2 \sqrt{a-c} E(\phi, q) + \sqrt{(s-a)(s-c)} \left( s-b \right); \]

\[ \phi = \sin^{-1} \left[ \frac{s-a}{s-b} \right]; q = \frac{b-c}{a-b}; s > a > b > c; \]

To completely evaluate Eq. (4.15), we need to find \( F(\phi, q) \) and \( E(\phi, q) \) which are elliptic integrals of the first and second kinds respectively [132]. These are well-known functions which are either tabulated or can be generated as infinite series. Mathematica\textsuperscript{TM} software has functions for generating both \( F(\phi, q) \) and \( E(\phi, q) \) to the accuracy requirement of the user. For preliminary analyses of frictional effects during sliding, we impose the initial condition on Equation (4.15) appropriately. An
The important upshot of Equation (4.15) is that it gives the implicit distance-time relation that is reminiscent of classical mechanics problem exhibiting nonlinearity [138]. By applying initial conditions \( t = 0, s = s_0 \) in Eq. (4.15) the integration constant \( c_1 \) can be obtained as:

\[
c_1 = \frac{2(a-b)}{\sqrt{a-c}} F(\{ \phi = \sin^{-1} \left[ \frac{s_0 - a}{s_0 - b} \right], q \}) - 2\sqrt{a-c} E(\{ \phi = \sin^{-1} \left[ \frac{s_0 - a}{s_0 - b} \right], q \}) + \sqrt{\frac{(s_0 - a)(s_0 - c)}{(s_0 - b)}},
\]

Substituting \( c_1 \) into the general solution in Eq. (4.15) and simplifying, the approximate time-distance relationship becomes:

\[
t = \frac{1}{n_1^{1/2}} \left[ \frac{2(a-b)}{\sqrt{a-c}} \left( F(\phi, q) - F(\phi_0, q) \right) - 2\sqrt{a-c} \left[ E(\phi, q) - E(\phi_0, q) \right] + \sqrt{\frac{(s-a)(s-c)}{(s-b)}} - \sqrt{\frac{(s_0 - a)(s_0 - c)}{(s_0 - b)}} \right]
\]

It should be noted that Eq. (4.16) is an analytical approximation because the original rare elliptic integral in (4.11) had to be simplified before applying the elementary being used. In the future, numerical and analytical integration techniques may be sought to determine a more precise answer to the rare elliptic integral in Eq. (4.11).

Using Mathematica™ software, the elliptic integral of the first kind can be approximated in the form of a series expansion:

\[
F(\phi, q) \approx \sin^{-1} \left[ \frac{-a + s}{-b + s} \right] + \frac{1}{6} q \sin^{-1} \left[ \frac{-a + s}{-b + s} \right]^3 + \frac{1}{120} (-4q + 9q^2) \sin^{-1} \left[ \frac{-a + s}{-b + s} \right]^5 + O\left[ \sin^{-1} \left[ \frac{-a + s}{-b + s} \right]^6 \right]
\]

Similarly, the elliptic integral of the second kind can also be approximated in the form of a series expansion as:
Having established an explicit though approximate time-distance relationship in Eq. (4.16), the time dependencies associated with velocity, acceleration, and jerk can be explored. The results from the kinematic properties also help to evaluate the interacting forces on the block: friction force, inertial force, and applied force.

A few illustrative parametric studies were carried out to check the behavior of the sliding block under the action of applied force and constant friction force. Using the established kinematic relationships for acceleration (Eq. (4.7)), velocity (Eq. (4.9)), jerk (Eq. (4.10)) and distance (Eq. (4.16)) we determine the interacting forces causing the kinematic effects. For example, by combining Eqs. (4.5) and (4.7), we obtain the applied force as:

\[
F_a = ma + \mu mg = m \frac{d^2s}{dt^2} + \mu mg;
\]

\[
F_a = m(a_o + \mu g) - \mu mg \ln[s/s_0] \tag{4.17}
\]

The resultant or inertial force becomes:

\[
F_{resultant} = ma = m \frac{d^2s}{dt^2} = F_a = ma_o - \mu mg \ln[s/s_0] \tag{4.18}
\]

The frictional force \( F_f = \mu mg \) was already considered a constant. Using the distance-time relationship in Eq. (4.16) together with appropriate simplification using initial conditions, the kinematic and dynamic properties are evaluated for time dependencies. All the distance and time-based mechanical (i.e., kinematic and dynamic) properties are illustrated graphically in the figures that follow. Figure 18 shows the change in applied force as the block travels though certain distance.
Figure 18: Interaction of forces on the block, (A) variation of applied force with distance, (B) variation of applied force with time

Figure 18 shows the applied force decreases with distance travelled because of the detrimental effect of friction. Also, because friction force was kept constant it does not vary with distance. The resultant inertial force on the block also decreases with distance. This is obvious because the force required to initiate the motion is always greater than the force required to keep it moving once it is set in motion. Similar trends were observed for the variation of forces over time. The input parameters used to obtain these plots for Figure 18 are shown in Table 3.

<table>
<thead>
<tr>
<th>Table 3: Input parameters</th>
</tr>
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<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>$g$</td>
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<tr>
<td>$a_0$</td>
</tr>
<tr>
<td>$u_0$</td>
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<tr>
<td>$s$</td>
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<tr>
<td>$\mu$</td>
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</table>

In the following sections, the effects of friction on kinematic properties are illustrated graphically. First, the effect of friction is analyzed by increasing the coefficient of friction (COF) for different quantities like total distance travelled, time to travel a specified distance, variation of velocity, acceleration and jerk with distance
and time. The rationale for exploring frictional effects on distance, velocity, acceleration, and jerk is that once we determined a fundamental dynamical model, we can interrogate Coulomb’s law quantitatively to ascertain its generality or otherwise. So, if friction is independent of sliding velocity, then regardless of the chosen COF, all the velocity, acceleration, distance, and jerk graphs must collapse onto each other. This was clearly not so, even as our results agree with real life experience.

In the second section the effect of varying the initial acceleration on different parameters is discussed. In most machines, for example electric motors that drive energy consuming and energy producing equipment, there is always a high initial torque requirement, which deteriorates equipment performance overtime. This initial torque emanates from the required initial acceleration which is overlooked in current modeling approaches. By comparing different initial accelerations, we illustrate an essential kinematic insight from jerk dynamics: initial acceleration is a critical parameter for analyzing sliding motions.
4.1.1 Effect of Friction on Kinematic Properties

Using the time-distance relation from elliptical integral the variation of distance travelled (Eq. (4.16)) under different coefficient of friction was studied parametrically with an illustration shown in Figure 19. Clearly, Figure 19 indicates the nonlinear behavior of distance with time for different COF values.

![Figure 19: The influence of friction on sliding distance](image)

Friction seemingly influences sliding distance (Figure 19). As the COF increases the time required to travel the same distance increases, it is the commonly understood physical phenomenon from real life captured here with our basic jerk model. Contrarily, a lower COF requires a shorter time to travel the same distance.

![Figure 20: The influence of friction on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time](image)
Velocity is the rate of change of distance with time. So, since friction seems to affect sliding distance (Figure 19) it should correspondingly affect sliding velocity. To find this out, Figure 20 is plotted to evaluate the effect of friction on the block’s sliding velocity. The maximum velocity of the block is observed for the lowest COF case, (Figure 20). Also, as the COF increases it reduces the velocity at which the block travels, in total agreement of what is expected in real life. Figure 20 clearly indicates that Coulomb’s law of friction may have oversimplified the basics of sliding friction. In our illustrations, the block is started from rest, so its initial velocity is assumed to be zero.

Since friction affects sliding velocity, it follows that properties that depend on velocity such as acceleration and jerk must also be affected similarly by friction. We tested this hypothesis with illustrations that follow.

Figure 21 evaluates the influence of friction as acceleration varies with distance (Figure 21A) and time (Figure 21B). In Figure 21B, the time for different friction coefficient is different because we had an implicit relation between distance and time and we solved time implicitly as a function of distance. In the illustrative plots distance is the input used to obtain the time because of the implicit distance-time relationship involved. Since the plots are obtained for the same values of distance, the time axis expectedly shows a longer time for the highest COF. Figure 20 (A and B) clearly captures effect of increasing the COF with the inherent nonlinearities of friction displayed.

The acceleration decreasing significantly with increasing COF Figure 21 (A and B) clearly indicates why more force is required when the interfacial sliding surfaces are rough. Previously, in our analysis it was shown that the initial acceleration cannot be zero because a nonzero force is needed nonzero to initiate the motion.
For consistency, the change in different parameters will be shown using both distance and time throughout the thesis.

Figure 21: The influence of friction on sliding acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time

Figure 22 shows the effect of increasing friction coefficient on the block’s jerk. Jerk is a rate of change of acceleration. If we recall, the differential equation for frictional jerk was obtained using the power to energy transfer equation (PETRE) in our modeling. While providing other useful kinematic insights, the frictional jerk is worth examining on its own merit. In Figure 22 the influence of friction on jerk is illustrated. It is observed that the frictional jerk decreases with distance which
confirms the relation obtained in Eq. (4.5). In that relation frictional jerk is inversely proportional to the sliding distance. Additionally, the change in jerk is very sharp initially before tapering off gradually with both distance and time. Clearly, the frictional jerk represents some phenomena observed in everyday life. For example, when pressing the accelerator paddle during driving we are changing the power required to initiate the car’s motion but we do not require the same power after the car is in motion. Additionally, the area under the jerk-distance curve (Figure 22A) indicates the power lost due to friction per unit mass. Jerk has units of $m/s^3$ and distance has a units of $m$. Checking the units:

\[
\frac{m}{s^3} \times m = \frac{kg \cdot m}{s^2} \times \frac{m}{kg \cdot s} = \frac{N \cdot m}{s \cdot kg} = \frac{W}{kg}
\]

The friction jerk has negative values because it opposes the motion and causes the dissipation of energy. Similarly, the area under the jerk-time curve Figure 22B represents the force needed to overcome the friction per unit mass (N/kg) which is clear from checking the units:

\[
\frac{m}{s^3} \times s = \frac{kg \cdot m}{s^2} \times \frac{1}{kg} = \frac{N}{kg}
\]

Different input parameters used for the parametric studies (for Figure 19 through Figure 22) are shown in Table 4.

<table>
<thead>
<tr>
<th>Table 4: Input parameters</th>
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<tbody>
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<td>m</td>
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<td>$s$</td>
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<tr>
<td>$\mu$</td>
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</table>
4.1.2 Effect of Increasing Initial Acceleration

It is required to have a nonzero acceleration to initiate any type of motion. In most modeling in classical mechanics, without incorporating nonzero jerk, this requirement of initial acceleration is overlooked. In this section parametric studies of the effect of changing initial acceleration are shown.

Figure 23: The influence of initial acceleration on sliding distance

Clearly, Figure 23 indicates that the time required traveling the same distance decreases as a larger initial acceleration is used. This confirms the fact that if more force is applied then an object moves faster.

Figure 24 shows the effect of increasing the initial acceleration on the velocity of the block. Increasing the initial acceleration increases the block’s velocity overall.

Figure 24: The influence of initial acceleration on velocity, (A) variation of velocity with distance (B) variation of velocity with time
In this illustrative example, the block is started from rest with a zero initial velocity. Apart from distance and velocity, the block’s instantaneous acceleration is also influenced by the initial acceleration that initiates the motion. As Figure 25 shows even as frictional effects decrease the acceleration, the largest acceleration is retained when the initial acceleration is largest.

Figure 25: The influence of initial acceleration on block’s instantaneous acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time

Jerk being the rate of change of acceleration, it is expected that any influences that increase acceleration must also increase the jerk. This scenario is manifested in Figure 26 where the steepest jerk descent corresponds with the largest initial acceleration.

Figure 26: The influence of initial acceleration on jerk, (A) variation of jerk with distance, (B) variation of jerk with time
Table 5 gives different input parameters used for the illustration in Figure 23 through Figure 26.

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<th>Table 5: Input parameters</th>
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### 4.2 Boundary Lubricated Sliding with Interfacial Friction- Velocity Coupling

During boundary lubricated sliding, the effect of lubrication is to minimize friction thereby reducing the needed externally applied force $F_a$ which translates into less power expenditure and therefore more energy transfer. The schematic sliding block from Figure 17 was reconsidered to account for a velocity-dependent friction in the modified version shown in Figure 27. The rationale for this modeling approach is to create a fundamental modeling framework for lubricated sliding in which the effect of friction under dynamical conditions can be evaluated deterministically. This way, a fundamental basis for testing Coulomb’s law for friction-velocity dependence during interfacial sliding is obtained. Besides, it is imperative for technologists, scientists, and engineers, in their energy efficiency innovations to quantify the detrimental effects of friction by understanding how velocity evolves during sliding.

![Schematic of a block sliding against a stationary surface lubricated with a boundary lubricant L](image)

Figure 27: Schematic of a block sliding against a stationary surface lubricated with a boundary lubricant L

The power to energy transfer rate equation (PETRE) was applied to the boundary lubricated sliding conditions (see schematic in Figure 27) using Eq. (4.1) with inputs from a linear friction-velocity coupling. In fact a number of studies and semi-empirical evidence already suggest that the dependence of friction on velocity is significant [26, 102, 103]. The $\mu$-$v$ coupling used in this study is taken from [102] and given by Eq. (4.19) as:
\[ \mu = \mu_d - \gamma v \]  

(4.19)

The chosen first order friction-velocity coupling shows that interfacial friction varies partially as the initial dry friction \( \mu_d \) and is directly proportional to sliding velocity \( V \) with the negative slope \( \gamma \text{[sm}^{-1}\text{]} \) or proportionality constant termed lubricant lubricity. The incidence of negative slope from friction-velocity is coupling documented tribology literature [6, 8, 97, 140] and often called the Stribeck effect [141]. Figure 28 shows different lubrication regimes. The Stribeck effect is observed in the non-hydrodynamic (i.e. boundary lubrication) regime where the lubricity or slope is negative. For this example, the maximum values of COF in this regime are less than 0.2. The Stribeck effect also extends to the hydrodynamic regime where the lubricity or slope is positive.

![Stribeck curve showing different lubrication regimes](image)

Figure 28: Stribeck curve showing different lubrication regimes [99].

The consequence of this Stribeck-like friction-velocity coupling is explored using jerk dynamics. To obtain the applied force \( F_a \) the constant \( \mu \) in Amonton’s law of friction \( F_J = \mu F_n \) was replaced with \( \mu = \mu_d - \gamma v \). Therefore, friction force will be:
\[
F_f = \mu F_n = (\mu_a - \gamma v) F_n = \left(\mu_a - \gamma \frac{ds}{dt}\right) mg
\]

Using Newton’s second law of motion the applied force is obtained as:

\[
F_a = \mu_a mg - \gamma mg \frac{ds}{dt} + m \frac{d^2s}{dt^2}
\]

where \( \mu_a mg \) is the dry friction force, \( \gamma mg (ds / dt) \) is the lubricating force and \( m(d^2s / dt^2) \) is the inertial force. The negative lubricating force shows that it aids the sliding motion. Consequently, a good lubrication reduces the amount of force needed to maintain the sliding motion. Predicting the force or power reduction fundamentally will facilitate innovations that target novel lubrication schemes. The work done by the applied force is given as:

\[
-W_a = F_a s = \mu_a mgs - \gamma mgs \frac{ds}{dt} + ms \frac{d^2s}{dt^2}
\]

Thus the power transfer rate becomes:

\[
-W_a = \mu_a mg \frac{ds}{dt} - \gamma mgs \frac{d^2s}{dt^2} - \gamma mg \left(\frac{ds}{dt}\right)^2 + m \frac{ds}{dt} \frac{d^2s}{dt^2} + ms \frac{d^3s}{dt^3}
\]

The block’s total mechanical energy is its kinetic energy \( K_e = m(ds / dt)^2 / 2 \) with a transfer rate:

\[
\dot{K}_e = m \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

Combining Eqs. (4.21) and (4.22) the power to energy transfer rate equation (PETRE) leads to the jerk balance equation (JBE) in Eq. (4.23):

\[
\frac{d^3s}{dt^3} = \gamma g \frac{d^2s}{dt^2} + \gamma g \left(\frac{ds}{dt}\right)^2 - \mu_a g \frac{ds}{s} \quad (4.23)
\]

Equation (4.23) is a third order nonlinear ODE which is difficult to solve completely for distance-time correlation. Using the following differential transformation:
\[
\frac{d^3s}{dt^3} = \frac{1}{2} \frac{d}{dt} \left( \frac{d^2s}{dt^2} \right)^2 - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} \quad \text{and} \quad \frac{ds}{dt} \frac{d^2s}{dt^2} = \frac{1}{2} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2
\]  
(4.24)

the jerk balance equation in Eqn. (4.23) becomes:

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{d^2s}{dt^2} \right)^2 - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} = \gamma g \frac{d}{dt} \left( s \frac{ds}{dt} \right) - \mu g \frac{ds}{dt}
\]  
(4.25)

This ODE is then integrated once for the interfacial acceleration:

\[
\frac{d^2s}{dt^2} = \frac{1}{2s} \left( \frac{ds}{dt} \right)^2 + \gamma g \frac{ds}{dt} - \mu g + \frac{a_s v_s^2 - (1/2)u_0^2 - \gamma g u_0 s_0 + \mu g s_0}{s}
\]  
(4.26)

Furthermore, in trying to solve the acceleration equation so obtained for velocity we used reduction of order by substituting \( ds / dt = v \). This procedure leads to:

\[
\frac{dv}{ds} = \frac{v^2 - u_0^2 + 2 \gamma g (v s - u_0 s_0) - 2 \mu g (s - s_0) + 2 a_s s_0}{2 v s} = \frac{Q(s, v)}{R(s, v)}
\]  
(4.27)

Equation (4.27) is a classic first order nonlinear ODE which is of the form of Hilbert’s sixteenth problem (here with \( n = 2 \)) [142, 143]. Hilbert’s sixteen problem [143] has indicated that there is no known technique for solving a first order nonlinear ODE of the form: \( dy / dx = Y(x, y) / X(x, y) \) where \( Y(x, y) \) and \( X(x, y) \) are nth order multivariate polynomials. As such, while Eq. (4.27) illustrates the expected nonlinearities associated with friction, it nevertheless has no known analytical solution technique. For an instructive study of friction and its effects, a quantitative solution is sought. To overcome existing analytical solution limitations we explored other solution techniques.

Equation (4.27) is a classic first order nonlinear ODE which is of the form of Hilbert’s sixteenth problem (here with \( n = 2 \)) [142, 143]. Hilbert’s sixteen problem [143] has indicated that there is no known technique for solving a first order nonlinear ODE of the form: \( dy / dx = Y(x, y) / X(x, y) \) where \( Y(x, y) \) and \( X(x, y) \) are nth order multivariate polynomials. As such, while Eq. (4.27) illustrates the expected nonlinearities associated with friction, it nevertheless has no known analytical solution technique. For an instructive study of friction and its effects, a quantitative solution is sought. To overcome existing analytical solution limitations we explored other solution techniques.
Chapter 5 : Solution Techniques

5.1: Non-Dimensionalization Technique

Equation (4.23) is a third order nonlinear ODE which is problematic to solve using current techniques. Moreover, the order reduction from applying a velocity-distance transformation approach does not readily resolve any analytical solution difficulties. A non-dimensionalization approach was considered to determine how simplified versions of the nonlinear ODE may be obtained. The characteristic variables of running-in length \( L_r \) and time \( t_r \) were selected to non-dimensionalize Eq. (4.23). Using a characteristic distance, \( (s^* = L_r) \), the aim of the non-dimensionalization approach is to extract an appropriate characteristic time \( t^* \). The non-dimensional distance is \( \bar{s} = s / s^* \Rightarrow s = \bar{s}s^* \). That is, \( s = \bar{s}s^* = \bar{s}L_r \), and the unknown non-dimensional time is \( \bar{t} = t / t^* \Rightarrow t = t^*\bar{t} \). Using these non-dimensional variables, we reduce Eq. (4.23) to:

\[
\frac{L_r}{t^*^3} \frac{d^3\bar{s}}{dt^3} = \gamma g \frac{L_r}{t^*^2} \frac{d^2\bar{s}}{dt^2} + \frac{\gamma g}{L} \frac{L_r^2}{t^*^2} \frac{1}{\bar{s}} \left( \frac{d\bar{s}}{dt} \right)^2 - \frac{\mu_d g}{L} \frac{1}{t^*} \frac{d\bar{s}}{dt} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.1)
\]

From Eq. (5.1) we have two possible choices for a characteristic time, \( t^* \). If we choose our first characteristic time to be \( t^* = 1 / (\gamma g) \), then Eqn. (5.1) becomes:

\[
\left( \frac{d^3\bar{s}}{dt^3} \right) = \left( \frac{d^2\bar{s}}{dt^2} \right) + \frac{1}{\bar{s}} \left( \frac{d\bar{s}}{dt} \right)^2 - \frac{\mu_d}{gL_r \gamma^2} \left( \frac{1}{\bar{s}} \frac{d\bar{s}}{dt} \right) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.2)
\]

Equation (5.2) yields a dimensionless group \( \pi_1 = \mu_d / (gL_r \gamma^2) \) that dictates the impact of lubrication ability \( \gamma \), sliding distance \( L_r \), and maximum friction coefficient \( \mu_d \) on
the jerk balance in a lubricated sliding contact. Furthermore, $\pi_1$ gives important physical insights into the dynamical sliding process. For example, when $\gamma$ is very large, $\pi_1$ decreases thereby reducing Eq. (5.2) to:

$$ \left( \frac{d^3 \bar{s}}{dt^3} \right) = \left( \frac{d^2 \bar{s}}{dt^2} \right) + \frac{1}{\bar{s}} \left( \frac{d \bar{s}}{dt} \right)^2 \tag{5.3} $$

Choosing the second characteristic time as $t^{*2} = L_r / (\mu_d g) \Rightarrow t = \sqrt{L_r / (\mu_d g)}$, then Eq. (5.1) becomes:

$$ \left( \frac{d^3 \bar{s}}{dt^3} \right) = \gamma \sqrt{\frac{gL_r}{\mu_d}} \left( \frac{d^2 \bar{s}}{dt^2} \right) + \gamma \sqrt{\frac{gL_r}{\mu_d}} \frac{1}{\bar{s}} \left( \frac{d \bar{s}}{dt} \right)^2 - \left( \frac{1}{\bar{s}} \frac{d \bar{s}}{dt} \right) $$

$$ \left( \frac{d^3 \bar{s}}{dt^3} \right) = \pi_2 \left[ \left( \frac{d^2 \bar{s}}{dt^2} \right) + \frac{1}{\bar{s}} \left( \frac{d \bar{s}}{dt} \right)^2 \right] - \left( \frac{1}{\bar{s}} \frac{d \bar{s}}{dt} \right) \tag{5.4} $$

Equation (5.4) yields a second dimensionless group $\pi_2 = \gamma \sqrt{gL_r / \mu_d}$ that dictates the impact of lubrication ability $\gamma$, sliding distance $L_r$, and maximum friction coefficient $\mu_d$ on the jerk balance in a lubricated sliding contact. Clearly, $\pi_2$ also gives important physical insights into the dynamical sliding process. For example, when $\gamma$ is vary small, $\pi_2$ increases thereby reducing Eq. (5.4) to the form:

$$ \left( \frac{d^3 \bar{s}}{dt^3} \right) = - \left( \frac{1}{\bar{s}} \frac{d \bar{s}}{dt} \right) \tag{5.5} $$

Equation (5.5) is similar to the jerk balance equation capturing sliding without lubrication that was discussed previously. Furthermore, when $\gamma$ is very large, $\pi_2$ increases thereby reducing Eq. (5.4) to the form:

$$ \left( \frac{d^2 \bar{s}}{dt^2} \right) \left( \frac{d \bar{s}}{dt} \right)^2 = 0 \tag{5.6} $$
Dropping the bars the transformed Eq. (5.6) can be solved using the initial conditions $t = 0; s = s_0; ds/dt = u_0$ as follows:

$$s \frac{d^2s}{dt^2} + \left( \frac{ds}{dt} \right)^2 = 0 \Rightarrow \frac{d}{dt} \left( s \frac{ds}{dt} \right) = 0 \Rightarrow s \frac{ds}{dt} = c_1; c_1 = s_0u_0;$$

$$s \frac{ds}{dt} = s_0u_0 \Rightarrow sds = s_0u_0dt; \quad s^2 / 2 = s_0u_0t + c_2; c_2 = s_0^2 / 2$$

$$\Rightarrow s^2 = 2s_0u_0t + s_0^2 \quad (5.7)$$

We see from Eq. (5.7) that pre-sliding or initial distance $s_0$ can never be zero in a frictional sliding. From Eq. (5.7), the distance traveled becomes:

$$s = \pm \sqrt{2s_0u_0t + s_0^2} \quad (5.8)$$

In this case Eq. (5.8) indicates the necessity of a nonzero initial velocity. But that is an unrealistic physical restriction on the sliding motion notwithstanding the simplifications given by Eq. (5.8). Using Eq. (5.8) the sliding velocity is obtained in Eq. (5.9):

$$\frac{ds}{dt} = \pm \frac{d}{dt} \left( \sqrt{2s_0u_0t + s_0^2} \right) = \pm \frac{s_0u_0}{\sqrt{2s_0u_0t + s_0^2}} \quad (5.9)$$

Differentiate Eq. (5.9) with respect to time the acceleration becomes:

$$a = \frac{d^2s}{dt^2} = \pm \frac{d}{dt} \left( \frac{s_0u_0}{\sqrt{2s_0u_0t + s_0^2}} \right) = \mp \frac{(s_0u_0)^2}{(2s_0u_0t + s_0^2)^{3/2}} \quad (5.10)$$

Finally, differentiating Eq. (5.10) the jerk becomes:

$$j = \frac{d^3s}{dt^3} = \mp \frac{d}{dt} \left( \frac{(s_0u_0)^2}{(2s_0u_0t + s_0^2)^{3/2}} \right) = \pm \frac{3(s_0u_0)^3}{(2s_0u_0t + s_0^2)^{5/2}} \quad (5.11)$$

Depending on the choice of $S_0$ and $u_0$ jerk can be very small or very large. The non-dimensionalization technique therefore reduces the complexities introduced by the nonlinear ODE. Nevertheless, important physical details are sacrificed in this
approach which also introduces a new requirement for nonzero initial velocity which seems at odds with real life experience.

Our modeling attempts elicited the inherent nonlinearities associated with friction fundamentally, at least in a nominally flat surface not populated with asperities. What this shows is that in a real sliding event with millions and billions of interfacial asperities, the consideration of the each asperity will compound the computational needs. At the moment, notwithstanding advances in computational algorithms and computing infrastructure, we are still a long way off at crafting exact or even approximate analytical solutions for complex nonlinear ODEs and PDEs [138, 144-146]. Over the past two decades, various researchers have provided solutions to the nonlinear equations involving jerk. Unfortunately, there is no existing method capable of providing the exact solution to the nonlinear equations; most of the existing methods are merely approximations. Solution methods includes harmonic balance approach to periodic solution [147], homotopy analysis to periodic solution [148], analytical and approximate solutions with an artificial parameter-Linstedt–Poincaré procedure [145], Asymptotic methods[149]. Therefore, it is required to obtain a new method to solve this type of nonlinear equations. Since tribologically dynamic sliding is inherently nonlinear, it is imperative to create alternative exact or approximate analytical techniques. We created a technique using algebra which we have nicknamed an “Algebraic Partitioning Technique (APT)” [150] to solve the nonlinear differential equations encountered in this study.
5.2: The Algebraic Partitioning Technique (APT)

Ordinary and Partial Differential Equations (ODEs and PDEs) are ubiquitous scientific tools for modeling artificial and natural phenomena [151-154]. Fluid flows [155], chemical reactions [156], tribology [157], plasma physics [158], and prostate cancer growth [159], are but a few of the innumerable scientific discourses dominated by ODEs and PDEs. The power of a scientific model depends on its ability to organize, predict, and/or manipulate valuable data that corroborate experiments and simulations, becoming invaluable for design optimization or a key decision-making resource. Historically, an analytical model, even if approximate, offers the surest bet in problem-solving [160]. Unfortunately, not all differential equations (DEs) can be solved analytically using existing methods. For seemingly intractable DEs successful attempts adopt simplifying analytical assumptions such as linearization, transform methods such as Fourier and Laplace transforms [151], or numerical methods [138, 161]. In fact, many linear, most nonlinear ODEs, certain linear and most nonlinear PDEs currently are only solved using numerical methods [138, 159, 161-163]. In some isolated instances, numerical methods have proven inadequate [163].

A growing sophistication in numerical techniques [164-167], algorithmic manipulations [168-170], and computational infrastructure [171] continue to significantly transform the speed, accuracy, stability, and reliability of numerical solutions. Nevertheless, numerical techniques are inherently approximate, often specific, usually requiring expensive computing infrastructure for implementation. Consequently, making timely, universal, and dependable predictions with numerical solutions may be difficult. Moreover, simplified analytical techniques often mask essential intrinsic characteristics associated with the original DE [138, 172, 173]. Here, we proffer an algebraic partitioning technique (APT) for analytically solving
linear and nonlinear ODEs and PDEs regardless of order or homogeneity. So far, our APT-based solutions capture important characteristics of DEs and match known analytical and numerical solutions well [150].

The underlying proposition of the algebraic partitioning technique (APT) is that any linear or nonlinear differential equation generalizable by individual terms \( DET_i \) such that:

\[
DET_1 + DET_2 + \cdots + DET_N = \sum_{i=1}^{N} DET_i = 0
\]  

(5.12)
can be partitioned into a set of reduced differential equations (RDEs) using an algebraic partitioning parameter (APP), \( \lambda_i \), to represent each component differential equation term (DET):

\[
DET_1 = \lambda_1; \quad DET_2 = \lambda_2; \quad \cdots; \quad DET_N = \lambda_N
\]  

(5.13)

The initial and/or boundary conditions that combine with Eq. (5.12) to form initial, boundary, and/or an initial-boundary value problem (IVP, BVP, and/or IBVP), apply similarly to the RDEs in Eq. (5.13). The equivalence of Eqs. (5.12) and (5.13) is established through an algebraic partitioning condition (APC):

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_N = \sum_{i=1}^{N} \lambda_i = 0
\]  

(5.14)

The solution to each individual RDE in Eqn. (5.13) becomes partial solutions to the original DE in Eq. (5.12) obtained by regarding each APP \( \lambda_i \) as a constant. To obtain the final analytical solution to the original DE in Eq. (5.12), all the partial solutions from Eq. (5.13) are combined and the APPs are eliminated using the APC in Eq. (5.14) to yield an algebraic equation with terms \( AET_i \):

\[
AET_1 + AET_2 + \cdots + AET_N = \sum_{i=1}^{N} AET_i = 0
\]  

(5.15)
Simplifying Eq. (5.15) produces the desired explicit or implicit, spatial, temporal, or spatio-temporal solution. The implementation of the APT follows the application of Eqs. (5.12) through (5.15).

5.2.1: Solving a Nonlinear Ordinary Differential Equation Using the APT

The following example is a taken from “A Treatise on Ordinary and Partial Differential equations” by William Woolsey Johnson [174].

\[
x \left( \frac{dy}{dx} \right)^2 - y \frac{dy}{dx} + a = 0
\]  
(5.16)

with initial condition: \( y(0) = 0 \). Applying the partitioning:

\[
x \left( \frac{dy}{dx} \right)^2 = \lambda_1 \quad (5.17) \quad -y \frac{dy}{dx} = \lambda_2 \quad (5.18)
\]

\[a = \lambda_3 \quad (5.19) \quad \lambda_1 + \lambda_2 + \lambda_3 = 0 \quad (5.20)
\]

Solve individual partition to get algebraic partitioning parameters. From Eq. (5.17):

\[
\left( \frac{dy}{dx} \right)^2 = \frac{\lambda_1}{x} \implies \frac{dy}{dx} = \sqrt{\lambda_1} x^{-1/2}
\]

Integrate once:

\[
y = \sqrt{\lambda_1} x^{-1/2 + 1} + c_1 = 2\sqrt{\lambda_1} x^{1/2} + c_1
\]

Apply the initial condition \( y(0) = 0 \) \( \Rightarrow c_1 = 0 \), therefore:

\[
\lambda_1 = \frac{y^2}{4x}
\]  
(5.21)

From partitioned Eq. (5.18):

\[
-\int ydy = \lambda_2 \int dx \implies -\frac{y^2}{2} = \lambda_2 x + c_2
\]
Applying the initial condition $y(0) = 0 \Rightarrow c_2 = 0$, therefore:

$$\lambda_2 = -\frac{y^2}{2x} \quad (5.22)$$

Invoking the algebraic partitioning condition of Eq. (5.20) and simplifying leads to the exact solution shown in Eq. (5.23):

$$\frac{y^2}{4x} - \frac{y^2}{2x} + a = 0$$

$$y^2 = 4ax \quad (5.23)$$

Equation (5.23) satisfies the given non-linear ODE (5.16) and the given initial condition exactly.

The promising impact of the APT is applied to solve the tribological jerk (i.e., tribojerk) equations. It must be noted that for the algebraic portioning process to yield the exact solution of a linear, nonlinear ODE or PDE of any order, the chosen boundary and/or initial conditions must be congruent or the problems must be well-posed [150].
5.3: APT Solution for the Nonlinear Frictional Jerk Balance Equation

In Chapter 4, the equation of jerk for sliding motion with friction was developed and it was approximately solved using an elliptic integral technique. The solution using the elliptic integral technique is compared to the APT solution for the same equation in this section. Recalling from Chapter 4, the frictional jerk balance equation:

\[
\frac{d^3 s}{dt^3} = -\frac{\mu_d g}{s} \frac{ds}{dt} \quad \Rightarrow \frac{d^3 s}{dt^3} + \frac{\mu_d g}{s} \frac{ds}{dt} = 0 \tag{4.5}
\]

Subject to three initial conditions: \(s(t = 0) = s_0\), \(ds/dt(t = 0) = u_0\), and \(d^2s/dt^2(t = 0) = a_0\). Applying the partitioning procedure:

\[
\frac{d^3 s}{dt^3} = \lambda_1 \tag{5.24} \quad \frac{\mu_d g}{s} \frac{ds}{dt} = \lambda_2 \tag{5.25}
\]

\[
\lambda_1 + \lambda_2 = 0 \tag{5.26}
\]

The detailed procedure of APT solution for nonlinear frictional jerk balance equation is shown in Appendix A with the essential details summarized here. From Eq. (5.24):

\[
\lambda_1 = \frac{-3a_0 t^2 - 6u_0 t + 6(s - s_0)}{t^3} \tag{5.27}
\]

From Eq. (5.25):

\[
\lambda_2 = \frac{\mu_d g}{t} \ln \left[ \frac{s}{s_0} \right] \tag{5.28}
\]

Combining Eqs. (5.27) and (5.28) to invoke the APC and grouping terms:

\[
\left(3a_0 - \mu_d g \ln \left[ \frac{s}{s_0} \right] \right) t^2 + 6u_0 t - 6(s - s_0) = 0 \tag{5.29}
\]

Equation (5.29) an implicit quadratic equation for time with distance as a coefficient functions.
\[ i^2 A(s) + tB + C(s) = 0 \]
\[ A(s) = [3a_0 - \mu_d g \ln(s / s_0)]; \quad B = 6u_0; \quad C(s) = -6(s - s_0); \quad (5.30) \]

Using the quadratic formula, the time-distance relationship was obtained:

\[ t = -\frac{6u_0 \pm \left\{36u_0^2 + 24(s - s_0)[3a_0 - \mu_d g \ln(s / s_0)]\right\}^{1/2}}{2[3a_0 - \mu_d g \ln(s / s_0)]} \]

Only one of the two roots satisfies the initial condition: \( s(t = 0) = s_0 \). This gives the final solution for time as:

\[ t = -\frac{3u_0 + \left\{9u_0^2 + 6(s - s_0)[3a_0 - \mu_d g \ln(s / s_0)]\right\}^{1/2}}{[3a_0 - \mu_d g \ln(s / s_0)]} = \frac{M(s)}{N(s)} \quad (5.31) \]

The form of Eq. (5.31) is promising because it gives an explicit time-distance relationship that captures the general format of the approximate elliptic integral solution in Eq. (4.16). Subsequent implicit differentiations of Eq. (5.31) lead to velocity and acceleration at the lubricated interface. For example, differentiating Eq. (5.31) with respect to \( t \) gives the velocity in Eq.(5.32).

\[ v = \frac{ds}{dt} = \frac{\left[N(s)\right]^2}{N(s)M'(s) - M(s)N'(s)}; \]

\[ v = \frac{ds}{dt} = \left[\begin{array}{c}
\frac{3[3a_0 - \mu_d g \ln(s / s_0)]}{[3a_0 - \mu_d g \ln(s / s_0)]^{1/2}} \\
\frac{-3(s - s_0)[\mu_d g / s]}{\left\{9u_0^2 + 6(s - s_0)[3a_0 - \mu_d g \ln(s / s_0)]\right\}^{1/2}} \\
\frac{\left\{9u_0^2 + 6(s - s_0)[3a_0 - \mu_d g \ln(s / s_0)]\right\}^{1/2}}{(\mu_d g / s)}
\end{array}\right] \quad (5.32) \]

Equation (5.32) satisfies the initial condition for velocity \( v(t = 0) = u_0 \). Similarly, the second derivative of Eq. (5.31) with respect to time gives the acceleration in Eq.(5.33)
\[
\ddot{a} = \frac{d^2 s}{dt^2} = -v^3 \left( \frac{N \left( N M'' - M N'' \right) - 2N' \left( NM' - MN' \right)}{N^3} \right)
\]

Equation (5.33) also satisfies the initial condition for acceleration, \( \dot{a}(t = 0) = a_0 \).

Previously, we obtained the approximate solution of the same jerk balance equation in Chapter 4, using the elliptic integrals technique. To see the effectiveness of the APT solution we considered the same values when comparing results with the approximate elliptic integral solution. This comparison is shown in the next section.

5.3.1: The Influence of Friction on Kinematic Properties

The effectiveness of the APT solution was assessed by comparing the influence of friction on the various kinematic properties (distance, velocity, acceleration, and jerk) that was illustrated using the solutions from the elliptic integral solution. This comparison is aided by juxtaposing the corresponding plots from the alternative solution techniques.

To assess the influence of friction on sliding distance, the time-distance relation from the APT solution was used.
Figure 29: The influence of friction on sliding distance

Clearly, Figure 29 recaptures the nonlinear behavior of distance with time for different COF values. As the COF increases the time required to travel the same distance increases. This is same trend as we obtained using the elliptic integral solution method for frictional jerk balance equation (see Figure 19) when there was no lubrication.

The impact of friction on sliding velocity translates similarly to the velocity which is the change of distance with time. The influence of friction on velocity is illustrated in Figure 30 showing the maximum velocity of the block is occurring at the lowest COF. As the COF increases it reduces the velocity at which the block travels. In this case the block is assumed to start from rest with initial velocity zero. The illustrations from Figure 30 shows the APT solution similarly recaptures the elliptical integral solution trends (see Figure 20)
Figure 30: The influence of friction on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time

Figure 31 shows the effect friction on the acceleration of the block. These plots mimic the trends found in the elliptic integral solution for acceleration shown in Figure 21

Figure 31: The influence of friction on sliding acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time
Figure 32 shows the effect of friction coefficient on the block’s jerk. Again, the curves for jerk follow the same trends and very close values to the elliptic integral solution shown in Figure 22.

The magnitude of the frictional jerk increases with the increasing coefficient of friction with high magnitudes initially that reduce with the sliding distance. Moreover, the solution obtained using APT matches well with the elliptic integral solutions obtained using the series approximation for nonlinear logarithmic term. This shows the dependability of the solution obtained by APT. This gives a confidence that APT can be used to obtain the solution for the boundary lubricated sliding.
5.4: The APT Solution for the Nonlinear Lubrication Jerk Balance Equation

In chapter 4, the power to energy transfer rate equation produced a jerk balance equation was obtained for the friction-velocity coupling under powder lubricated conditions. Even the non-dimensionalization and reduction in order techniques were not able to produce the full solution to the lubrication jerk balance equation. We obtained the solution using the algebraic partitioning technique. The JBE for lubricated sliding without wear was:

\[
\gamma g \frac{d^2 s}{dt^2} + \gamma g \left( \frac{ds}{dt} \right)^2 - \frac{\mu_d g}{s} \frac{ds}{dt} - \frac{d^3 s}{dt^3} = 0
\]  (4.23)

Apply partitioning of Eq. (4.23) as:

- \[
\frac{-d^3 s}{dt^3} = \lambda_1
\]  (5.34)

- \[
\frac{\gamma g}{s} \left( \frac{ds}{dt} \right)^2 = \lambda_3
\]  (5.36)

- \[
\frac{\mu_d g}{s} \frac{ds}{dt} = \lambda_5
\]  (5.37)

\[
\sum_{i=1}^{4} \lambda_i = 0
\]  (5.38)

The detailed APT procedure is shown in Appendix B with the essential results summarized here.

From partitioned Eqn. (5.34):

\[
\lambda_i = -3a_d t^2 + 6u_0 t - 6(s - s_0) \quad (5.39)
\]

From partitioned Eq. (5.35):

\[
\lambda_2 = \frac{2\gamma g(s - u_0 t - s_0)}{t^2} \quad (5.40)
\]

From partitioned Eq. (5.36):

\[
\lambda_3 = \frac{4\gamma g(s^{1/2} - s_0^{1/2})^2}{t^2} \quad (5.41)
\]
From partitioned Eq. (5.37):

\[
\lambda_i = \frac{\mu_d g}{t} \ln \left[ \frac{s_0}{s} \right] \tag{5.42}
\]

Invoking the algebraic partitioning condition (APC) of Eq. (5.38):

\[
\frac{3a_d t^2 + 6a_d - 6(s - s_o)}{t^3} + 2\gamma g (s - u_d,t - s_o) + \frac{4\gamma g (s^{1/2} - s_o^{1/2})^2}{t^2} + \frac{\mu_d g}{t} \ln \left[ \frac{s_0}{s} \right] = 0 \tag{5.43}
\]

An implicit solution result which can be rearranged by multiplying through by \(t^3\):

\[
3a_d t^2 + 6a_d - 6(s - s_o) + 2\gamma g (s - u_d,t - s_o) + 4\gamma g (s^{1/2} - s_o^{1/2})^2 + \mu_d g t^2 \ln \left[ \frac{s_0}{s} \right] = 0 \tag{5.44}
\]

Clearly, the implicit solution in Eq. (5.44) satisfies the I.C. at \(t = 0\), where \(s = s_0\). The solution that results from the nonlinear ODE is a nonlinear but implicit equation as captured in Eq. (5.44). Although Eq. (5.44) is implicit in distance-time it seems explicit in time. This outlook provides an outlet to determine at least an approximate (or probably exact) though nonlinear relation between distance and time. From Eq. (5.44) we factorize in terms of time:

\[
t^2[3a_o - 2\gamma g u_o + \mu_d g \ln(\frac{s_o}{s})]
+ t[6u_o + 2\gamma g (s - s_o) + 4\gamma g (s^{1/2} - s_o^{1/2})^2] - 6(s - s_o) = 0 \tag{5.45}
\]

Equation (5.45) illustrates that the analytical solution obtained from the APT may be factorized in terms of the independent variable \(t\). In fact, we see that Eq. (5.45) is a quadratic equation in time with distance as the coefficient functions as shown in Eq.(5.46).

\[
t^2 A(s) + tB(s) + C(s) = 0
\quad A(s) = [3a_o - 2\gamma g u_o + \mu_d g \ln(\frac{s_o}{s})];
\quad B(s) = [6u_o + 2\gamma g (s - s_o) + 4\gamma g (s^{1/2} - s_o^{1/2})^2];
\quad C(s) = -6(s - s_o); \tag{5.46}
\]

Using the quadratic formula, we obtain time-distance relationship as:
\[-[6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2] \]
\[\pm \left[ \frac{[6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2]^2}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]} \right]^{1/2} \]
\[t = \frac{24(s - s_0)[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]} \tag{5.47} \]

To check whether our initial condition is satisfied explicitly, we observe that setting \(s = s_0\) Eq. (5.47) becomes:

\[t = -\frac{[6u_0]}{2[3a_0 - 2\gamma gu_0]} \pm \frac{\sqrt{[6u_0]^2}}{2[3a_0 - 2\gamma gu_0]} \tag{5.48} \]

Obviously, \(u_0 = 0\), and \(\gamma = 0\), are all tolerable but it seems that is impractical from the physical perspective of the problem for a zero initial acceleration \((a_0 = 0)\). In fact, it seems that for a constraint to be imposed on a dynamical system to alter its state of rest or uniform motion in a specific direction, initial acceleration must be nonzero. Furthermore, Eq. (5.48) yields zero initial time only when the positive sign is maintained in the last term on the RHS of Eq. (5.48). Thus, the choice of time that satisfies our initial condition is:

\[-[6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2] \]
\[\pm \left[ \frac{[6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2]^2}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]} \right]^{1/2} \]
\[t = \frac{24(s - s_0)[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]} \tag{5.49} \]

Equation (5.49) is our desired solution. Before doing any further differentiations to capture transient kinematic and dynamic (i.e. mechanical) behavior of the tribological object in resisted relative motion, we simplify Eq. (5.49) further. This facilitates obtaining the velocity and acceleration. The procedure is as follows: The interfacial velocity from implicit differentiation is given in Eq. (5.50):
\[ t = Q(s) + R(s), \Rightarrow 1 = [Q'(s) + R'(s)] \dot{s} = [Q'(s) + R'(s)] v \]

\[ v = \frac{1}{[Q'(s) + R'(s)]} \]

The final form of velocity can be given as:

\[ v = \frac{[3a_0 - 2\gamma g u_0 + \mu_g g \ln(s_0 / s)]^2}{-3a_0 - 2\gamma g u_0 + \mu_g g \ln(s_0 / s) [\gamma g + 2\gamma g (s^{1/2} - s_0^{1/2}) s^{-1/2}] - (\mu_g g / s) [3a_0 + \gamma g (s - s_0) + 2\gamma g (s^{1/2} - s_0^{1/2}) \dot{s}^2]}
\]

\[ + [3a_0 - 2\gamma g u_0 + \mu_g g \ln(s_0 / s)] \left\{ \frac{[3a_0 + \gamma g (s - s_0) + 2\gamma g (s^{1/2} - s_0^{1/2}) \dot{s}^2]^{1/2}}{6(s - s_0) [3a_0 - 2\gamma g u_0 + \mu_g g \ln(s_0 / s)]} + \left( \mu_g g / s \right) \left\{ [3a_0 + \gamma g (s - s_0) + 2\gamma g (s^{1/2} - s_0^{1/2}) \dot{s}^2]^{1/2} \right\} + 6(s - s_0) [3a_0 - 2\gamma g u_0 + \mu_g g \ln(s_0 / s)] \right\} \]  

Equation (5.51) satisfies the initial condition: \( s = s_0, v = u_0 \).

Subsequent differentiations of Eq. (5.50) gives the acceleration in Eq.(5.52)

\[ \ddot{s} = \frac{d^2 s}{dt^2} = -s^2 \left[ Q''(s) + R''(s) \right] \frac{[Q'(s) + R'(s)]}{[Q'(s) + R'(s)]} = -v^3 [Q''(s) + R''(s)] \]

The various terms in Eq. (5.52) are as follows:

\[ Q''(s) = \frac{4[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)]^2 \left( -2\gamma g (s^{1/2} - s_0^{1/2}) s^{-3/2} + 2\gamma g / s \right)]}{-2 \left( 2\mu_d g / s^2 \right) 2[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)] \left[ 6u_0 + 2\gamma g (s - s_0) + 4\gamma g (s^{1/2} - s_0^{1/2})^2 \right]}
\]

\[ R''(s) = -2 \left( 2\mu_d g / s \right) \left[ -2 \left[ \frac{3a_0 - 2\gamma g u_0}{+ \mu_d g \ln(s_0 / s)} \right] (2\gamma g + 4\gamma g (s^{1/2} - s_0^{1/2}) s^{-1/2}) \right] + 6u_0 + 2\gamma g (s - s_0) + 4\gamma g (s^{1/2} - s_0^{1/2})^2 \left( 2\mu_d g / s \right) \]

and \( R''(s) \):
\[
R''(s) = \frac{4[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]^2 H''}{8[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]^3}
\]

which includes:

\[
H' = \frac{\left\{6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2 \left(2\gamma g + 4\gamma g(s^{1/2} - s_0^{1/2}) s^{-1/2}\right)\right\}}{+12(s - s_0)(\mu_d g / s) + 12[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]}
\]

and

\[
H'' = \frac{\left\{6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2 \left(2\gamma g + 4\gamma g(s^{1/2} - s_0^{1/2}) s^{-1/2}\right)\right\}}{+12(s - s_0)(\mu_d g / s) + 12[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_0 / s)]}
\]

and the velocity shown in Eq. (5.51). Also, Eq. (5.52) satisfies the initial condition for acceleration: \(s = s_0; a = a_0\).
A key basis for the analysis of lubricated sliding is the friction-velocity coupling \( \mu = \mu_d - \gamma v \). Having obtained the time-distance correlation, the transient velocity becomes an essential input to extract the transient friction allowing primarily the Striebeck effect to be reevaluated in the solutions obtained. As a bonus, effect of the jerk in the lubricated interface may also be evaluated. Illustrative examples are given based on the results obtained.

5.4.1: The Effect of Dry Friction Coefficient of Friction on Lubricated Sliding

Using the time-distance relation from APT solution the influence of dry friction coefficient on distance, velocity, acceleration, jerk and kinetic or instantaneous friction was studied parametrically. This parametric study is similar to the case when there is no lubrication. Because insufficient lubrication has a negligible effect on reducing the coefficient of friction during sliding the lubricity here is set at a very low value (\( \gamma = 0.0005 \text{ s/m} \)) in Table 6. The other values used for the parametric studies (for Figure 33 through Figure 37) are shown in Table 6.

<table>
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<th>Table 6: Input parameters</th>
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<tr>
<td>( m )</td>
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<td>( S )</td>
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<td>( \mu_d )</td>
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<td>( \gamma )</td>
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</table>
Figure 33: The influence of dry friction on sliding distance during poor lubrication

Clearly, Figure 33 indicates the nonlinear behavior of distance with time for different dry COF values. When there is not enough lubrication, as the dry COF increases the time required to travel the same distance increases. Contrarily, a lower COF requires shorter time than the higher COF motion.

Figure 34: The influence of dry kinetic friction during poor lubrication, (A) variation of COF with distance, (B) variation of COF with time

Figure 34 exhibits the minor change in kinetic friction over the total sliding distance and time because of the poor lubrication. The effect of good lubrication on different kinematic and dynamic properties are shown later in this chapter. First, the effect of dry friction on velocity, acceleration, and jerk is shown in the following figures. This illustration shows that one type of lubricant is not suitable for all
materials. A higher value of dry COF is an indication of higher surface roughness. This result is critical for assessing how different materials will perform at the onset of sliding where dry friction predominates.

Figure 35 shows the effect of dry friction the block’s sliding velocity. The maximum block velocity corresponds to the lowest COF.

![Figure 35](image)

Figure 35: The influence of dry friction on velocity during poor lubrication (A) variation of velocity with distance, (B) variation of velocity with time

To assess how dry friction affects acceleration, illustrations using Figure 36 show acceleration decreases with increasing COF, a clear indication of why more force is required when the sliding surfaces are rough. That also indicates high initial torque requirements in rotating machinery.

![Figure 36](image)

Figure 36: The influence of dry friction on acceleration during poor lubrication, (A) variation of acceleration with distance (B) variation of acceleration with time
The influence of dry friction on jerk is captured in Figure 37 showing the effect of increasing dry COF on the jerk or rate of change of acceleration of the block. It is observed that the frictional jerk decreases with increasing distance which confirms the relation obtained in Eq.(4.23). In that relation frictional jerk is inversely proportional to the sliding distance.

Figure 37: The influence of dry friction on jerk during poor lubrication, (A) variation of jerk with distance, (B) variation of Jerk with time

Since there is poor lubrication there is a minor effect of lubricity on the distance, velocity and acceleration of the block. Consequently, the results shown in this section are much similar to the case with no lubrication shown previously.

Figure 38: Verification of friction velocity coupling
Interestingly, even though a minor decrease in kinetic friction is evident, friction velocity curves are still almost linear with negative slopes (Figure 38). This verifiably recaptures the friction-velocity coupling used for modeling the boundary lubricated sliding motion. By recapturing the friction-velocity coupling even in the poorly lubricated case, this evidence strengthens the viability of the mechanistic unification that produces the jerk effects and the promise of the APT solutions obtained.

The effect of dry friction on applied force is captured in Figure 39 which follows the trend exhibited by the acceleration in Figure 36: Sliding with higher dry COF reduction in the applied force more significantly compared to the motion with lower COF.

![Figure 39](image.png)

Figure 39: The influence of dry friction on applied force during poor lubrication, (A) variation of applied force with distance, (B) variation of applied force with time

### 5.4.2: Effect of Lubricity on Sliding

In this section, the effects of good lubrication on different kinematic sliding properties of the block are considered. The instantaneous kinematic friction is also examined. The desirable effects of lubrication in aiding sliding motion is illustrated through choice of lubricity $\gamma$. To mimic good lubrication, higher values of lubricity, $\gamma$ will be used. In the following results the value of dry coefficient of friction $\mu_d$ is kept...
constant while the lubricity ($\gamma$) is varied. The lubricity ($\gamma$), is a property of the lubricating particulate solid powder or boundary lubricant. It is different for different solid lubricants or liquid-solid particulate lubricant mixtures.

Three different values of $\gamma$ are chosen to illustrate the effects of lubricity. Care must be taken while choosing the values of $\gamma$, such that the resulting values of coefficient of friction during the sliding remains positive. In actual practice, the suitable range of lubricity values must be obtained using the carefully conducted experimentation using with specific boundary lubricants.

The input values used for the parametric studies (illustrated by Figure 40 through Figure 45) are shown in Table 7.

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<th>Table 7: Input parameters</th>
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<tr>
<td>$\mu_d$</td>
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<td>$\gamma$</td>
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To examine the effect of lubrication on sliding distance, the illustration with Figure 40 is used. Clearly, the nonlinear behavior of distance with time for is captured. Additionally, larger values of lubricity $\gamma$ correspond with shorter times to travel the same distance indicating the attributes of proper lubrication in overcoming the effects of friction.
Figure 40: The influence of lubricity on sliding distance

The benign effect of lubrication is the reduction of kinematic friction as exhibited in Figure 41. Figure 41 shows that there is a significant change in coefficient of friction over the total sliding distance because of the increase in lubrication effects.

Figure 41: The influence of lubricity kinetic friction, (A) variation of COF with distance, (B) variation of COF with time

Apart from the instantaneous friction, the beneficial effect of lubrication on velocity is shown in Figure 42 where a high lubricity corresponds to high velocity. This increment in velocity from good lubrication is expected from real life experience.
Figure 42: The influence of lubricity on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time

Just like velocity, good lubrication also supports acceleration as Figure 43 shows: acceleration increases with increasing lubricity $\gamma$. This clearly indicates why less force is required when there is a good lubrication.

Figure 43: The influence of lubricity on sliding acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time

Because jerk is the rate of change of acceleration with time, it is expected that any influence that increases acceleration must have a similar on jerk. This is exactly the case as exhibited in Figure 44 showing how an increasing lubricity decreases the magnitude of block’s jerk.
It is observed that the frictional jerk decreases with distance agreeing with the relation obtained in Eq. (4.23) where frictional jerk is shown to be inversely proportional to sliding distance. In this case good lubrication provides lower values of jerk compared contrary to the case where the larger values of dry friction produced higher jerk. The competition between lubrication and friction jerk obviously dictate the interfacial sliding.

From the current parametric studies the effects of lubricity on the distance, velocity, and acceleration of the block are shown. Since we used the friction velocity coupling for this lubrication jerk section, the plots for coefficient of friction versus velocity is also shown to verify the results obtained using APT. Figure 45 shows the change of COF with increasing lubricity. The velocity values obtained using the time-distance relationship resulted from the APT solution.
Interestingly, we obtain the decreasing values of COF with the increase in velocity. Additionally, the effect of high lubricity is the produce an overall low friction that increases the sliding velocity. This shows that good lubrication reduces the COF to some extent. Also, by recapturing the linear friction-velocity coupling used in the modeling, the soundness of the mechanistic unification protocol eliciting the jerk balance equations is proven. Even though a simple linear friction-velocity coupling is used here, the method may be extended to and friction that depend on velocity nonlinearly, and/or with other parameters such as temperature etc.

![Figure 46: The influence of lubricity on applied force, (A) variation of applied force with distance, (B) variation of applied force with time](image)

Ultimately, the aim of energy efficient innovations is to reduce the effect of friction thereby guaranteeing lower force expenditure for transporting a given load over the same distance. To gauge the effect of lubrication on the interaction forces on the block, Figure 46 is constructed. The applied force decreases more for low lubricity signifying the scenario of sliding with poor lubrication. Contrarily, the applied force decreases less for the high lubricity, confirming that the motion with good lubrication requires less force.

So far, the modeling elicits not just kinematic properties of velocity, distance, acceleration and jerk, dynamic properties of applied and inertial forces, but also captures transient interfacial friction. The consequence of this modeling insight will
be transformative for the science of tribology. For the first time, a fundamental modeling approach that elicits transient dynamical friction is shown.

Apart from friction, the jerk dynamical framework is also tested to quantify deterministic surface wear.
Chapter 6: Nonlinear Jerk Dynamics during Variable Mass Sliding

6.1: Variable Mass Systems in Mechanics

In mechanics, a variable mass system is a system whose mass does not remain constant during motion. Mass loss can have a significant effect on the dynamics of various mechanical systems. Newton’s second law cannot be applied directly because it works best for a constant mass system [175, 176]. A term can be added to account for the momentum carried by the leaving or entering mass into Newton’s second law. This modification can be used to calculate the time dependence of mass. The change in mass is taken into consideration using the impulse momentum theorem or by the conservation of linear momentum where external force is considered as the difference of the initial and final momenta [177-179].

Variable mass systems can be divided into two classes: (1) A continuous mass variation e.g. a rocket and, (2) A discrete mass variation e.g. robots picking up or releasing objects, or a moving vehicle dropping off some of its payload in discrete chunks [180]. The equation of motion for this type of systems depends on whether the mass is entering or leaving the body. The processes are called mass accretion and mass ejection respectively.

In systems where there is an external force like applied force and resistive forces for example, friction force or air drag, the momentum is not conserved because from Newton’s second law of motion the net external force causes the change in momentum of the system. Also, when the system is losing its mass at a certain rate and some velocity then the escaping mass exerts a force on the system. In the case of rocket propulsion, if we only consider the rocket as our system without the leaving gases the momentum is not conserved because the escaping gases exert a force on the
rocket by Newton’s third law. However, if we consider both the rocket and gases as the system and there is no external force such as gravity or air resistance and, we can conserve momentum by equating the initial and final momenta of the system. The final equation of motion for a variable mass system is given as [181]:

\[
F_{net} + v_{rel} \frac{dm}{dt} = m \frac{dv}{dt} \tag{6.1}
\]

where \( v_{rel} = v_e - v \), \( v \) is the relative velocity of the escaping mass to the body and \( v_e \) is the velocity of the entering or leaving mass. \( F_{net} = F_a - F_f \) is the net external or inertial force acting on the body and is the difference between the applied and frictional force. From Eq. (6.1) we can see that not only the external force but also the changing mass can cause a change in acceleration. For example, in case of rocket propulsion, the gases are continuously leaving while the rocket’s mass decreases with time which can increase its acceleration because the escaping gases provide thrust on the rocket. This formulation is widely used in modeling the motion of rockets [182], two dimensional flatcar machine gun systems [183], particles moving at relativistic velocities [184], motion of particles through clouds of stationary dust, falling rain drop, an oscillator with variable mass [185-187], the time dependent rise of a liquid in a capillary tube [188], the propulsion and filling of a wagon by an incoming water [189].

We adopted the fundamental insights of rocket propulsion to construct a deterministic wear model for sliding contacts.
6.2: Variable Mass Sliding Contact in Tribology

In tribology, wear is the measure of a system’s change in mass. Wear is defined as the unwanted loss of solid material from sliding surfaces due to mechanical interaction [190, 191]. Wear is an inevitable companion of friction and it has many detrimental effects on the mechanical system.

In previous wear modeling efforts, the common assumptions were that the deformation of the hard surface is negligible compared to the wear of the softer material and a single asperity of the hard material moving across the softer material surface. These models are phenomenological models and cannot be applied to different types of applications. The wear rate of a sliding or rolling contact is defined as the volume lost from the wearing surface per unit sliding distance. In 1953 Archard developed a wear law based on the experimental evidence [41]. In this law, wear is proportional to the area of contact and the applied load on the asperity. Using Archard’s adhesive wear law, the mass lost during sliding contact or wear mass is given by:

\[
m_w = \frac{\kappa \rho F_s s}{H}
\]

where, \(m_w [\text{kg}]\) is the mass of wear, \(\kappa\) is the dimensionless wear coefficient, and \(H [\text{N/m}^2]\) is the hardness of the softer sliding object. There are several ranges for \(\kappa\) which gives the idea of the severity of wear. When \(\kappa\) is in the range \(10^{-6} – 10^{-8}\), wear is considered mild; for \(\kappa\) larger than \(10^{-4}\) it is severe. Rabinowicz suggested an alternative scheme for the wear, with severe wear for \(\kappa \approx 10^{-2} – 10^{-4}\), moderate wear for \(K \approx 10^{-4} – 10^{-6}\), and burnishing wear for \(\kappa \approx 10^{-6} – 10^{-8}\) [37]. The empirical nature of the wear coefficient in Archard’s wear law is also investigated analytically [192]. It is impossible to eliminate wear completely, but it can be reduced using the
lubrication, formation, of sufficiently smooth surfaces, modification of near-surface materials of rubbing components, correct assembling of fitted component parts [39].

In this chapter, it was assumed that a single moving block of mass \( m \) made up of a softer material slides against a harder stationary surface and loses mass during its motion. Newton’s second law of motion, the rate form of first law (RFFL) of thermodynamics and Archard’s adhesive wear equation are used for the modeling purposes. The power to energy transfer rate equation (PETRE) helps in investigating the mass loss during the running-in where more than one type of wear processes may operate simultaneously.

In our modeling, we made the following assumptions:

1. The block is continuously losing mass during its motion.
2. The escaping mass has a constant velocity.
3. The remaining instantaneous mass is considered as a control volume.
4. To use the rate form of first law of thermodynamics we considered the system of block and the wear mass as an open system where energy as well as the mass can cross the system boundary. Here, the exiting mass is the wear debris that exits the open system.
5. We considered two scenarios: i) friction coefficient is constant, and ii) friction is coupled with velocity, to create deterministic wear models for un lubricated and lubricated sliding respectively.
6.3: Mass Loss During an Unlubricated Sliding Motion

Consider a block of initial mass \( m_0 \) sliding against a stationary surface. The initial mass changes decreases with time under the action of two external forces an applied force \( F_a \) and a friction force at the interface \( F_f \).

![Diagram of block sliding against a stationary unlubricated surface](image)

Figure 47: Block sliding against a stationary unlubricated surface

The general equation of variable-mass motion is written as:

\[
F_a - F_f = m \frac{dv}{dt} - (v_e - v) \frac{dm}{dt}
\]  

(6.3)

where \( F_a \) = Applied force (N), \( F_f \) = Friction force (N), \( m \) = Instantaneous block mass (kg), \( \frac{dm}{dt} \) = Rate of change of mass with time (kg/s), \( v \) = Block velocity (m/s), \( v_e \) = velocity of the exiting mass (m/s).

For this system the friction force, \( F_f = \mu F_N = \mu mg \) depends on the mass of the block and it will also change over time. Using this form of friction force, Eq. (6.3) gives the applied force as:

\[
F_a = \mu mg + m \frac{dv}{dt} - (v_e - v) \frac{dm}{dt}
\]  

(6.4)

In the case of rocket propulsion, the velocity of the escaping gases is provided by the motor which controls the flow of the gases and pushes them out of the rocket.
Here, in case of block wearing due to its motion in the forward direction only and there is no force inside the block that induces the mass to escape. Therefore, the velocity at which mass is escaping from the block can be zero or a small value compared to the velocity of the block. In a typical tribometer experiment, the wear mass fly out of contact by centripetal action of a rotating disk.

The instantaneous block mass is the difference between the initial and escaping wear mass: \( m = m_0 - m_e \). As the block travels a distance \( s \) under the influence of the applied force then the work done by the applied force is:

\[
-W_a = F_a s = \mu mgs + ms \frac{d^2 s}{dt^2} - s \left( v_e - \frac{ds}{dt} \right) \frac{dm}{dt}
\]

Thus, the work transfer rate or power becomes:

\[
-W_a = \mu mg \frac{ds}{dt} + \mu gs \frac{dm}{dt} + ms \frac{d^3 s}{dt^3} + m \frac{ds}{dt} \frac{d^2 s}{dt^2} + 2s \frac{dm}{dt} \frac{d^2 s}{dt^2}
\]

For a body with constant mass, the momentum is linearly proportional to its velocity. This relation applies even in the case where the force is changing, as long as the mass is not constant [184]. In classical mechanics, kinetic energy can be obtained by integrating the force over the distance travelled which leads to the expression of kinetic energy \( Ke = (1/2)mv^2 \). The kinetic energy for a variable mass case is slightly more complicated. In this case, the force and mass of the system are not constant so integration is not straightforward. By definition,

\[
Ke = \frac{1}{2}mv^2 = \frac{1}{2} m \left( \frac{ds}{dt} \right)^2
\]
In this equation, \( m = m_0 - m_e \), where, \( m_0 \) and \( m_e \) are the original mass of the block and escaping mass respectively. The rate of change of mass is \( \frac{dm}{dt} = -\frac{dm_e}{dt} \).

Therefore, the kinetic energy transfer rate becomes:

\[
\dot{Ke} = m \frac{ds}{dt} \frac{d^2 s}{dt^2} + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt} + \frac{1}{2} \left( \frac{ds}{dt} \right) \frac{d^2 m}{dt^2}
\]

(6.6)

Applying the rate form of the first law (RFFL) of thermodynamics for the open system (neglecting potential energy changes) the power to energy transfer rate equation (PETRE) becomes [107]:

\[
\frac{dE_{CV}}{dt} = \dot{Ke} - \frac{1}{2} \frac{v_e^2}{2} \frac{dm_e}{dt}
\]

(6.7)

Combining equations (6.5), (6.6) and (6.7):

\[
m \frac{ds}{dt} \frac{d^2 s}{dt^2} + \frac{1}{2} \frac{dm}{dt} \left( \frac{ds}{dt} \right)^2 = \mu mg \frac{ds}{dt} + \mu gs \frac{dm}{dt} + ms \frac{d^3 s}{dt^3} + m \frac{ds}{dt} \frac{d^2 s}{dt^2} + 2s \frac{dm}{dt} \frac{d^2 s}{dt^2}
\]

\[
+ \left( \frac{ds}{dt} \right) \frac{d^2 m}{dt^2} - v_e \frac{ds}{dt} \frac{dm}{dt} - v_e \frac{d^2 m}{dt^2} + s \frac{d^2 m}{dt^2} \frac{d^2 s}{dt} + \frac{v_e^2}{2} \frac{dm}{dt}
\]

(6.8)

Rearranging the terms:

\[
\frac{d^3 s}{dt^3} + \frac{\mu g}{s} \frac{ds}{dt} + \frac{\mu g}{m} \frac{dm}{dt} + \frac{2}{m} \frac{d^2 s}{dt^2} \frac{dm}{dt} + \frac{1}{2ms} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt} + \frac{v_e^2}{2ms} \frac{dm}{dt} - \frac{v_e}{ms} \frac{ds}{dt} \frac{dm}{dt} - \frac{v_e}{m} \frac{d^2 m}{dt^2} + s \frac{d^2 m}{dt^2} \frac{d^2 s}{dt} + \frac{1}{m} \frac{d^2 s}{dt^2} \frac{d^2 m}{dt} = 0
\]

Equation (6.8) represents the jerk balance equation for the variable mass system under the externally applied and friction forces. Because Eq. (6.8) has two dependent variables (mass and distance) which change with time it is difficult to solve unless the
relationship between mass, velocity and/or distance is known. Furthermore, because Eq. (6.8) is obtained from fundamental laws, it may apply to a variety of variable mass sliding cases.

For simplicity, we considered two special cases. In the first case we assume that the block is sliding with a constant velocity which resembles the motion of most tribological systems after the running-in period. In the second case, we assume that the wear of the block obeys Archard’s adhesive wear law. The first case is an idealized case because the equation does not consider any material properties for example, hardness or density of the material. In the second case, we use Archard’s adhesive wear law to replace the change in mass in terms of sliding velocity. This case includes the density and hardness of the material and forms an important basis for interrogating experimental results involving wear. We may also use this approach to synthesize different materials for novel wear reduction techniques as needed for piston-cylinder engines, bearing seals, compressors, expanders, etc.
### 6.3.1: Block is Sliding with a Constant Velocity

In this case, it was assumed that the block slides with a constant velocity \( u_0 \).

Using the standard definition of velocity \( v = ds / dt = u_0 \), the distance (using the initial condition \( s(t = 0) = s_0 \)) becomes \( s = u_0 t + s_0 \). Using the constant velocity \( u_0 \) and the distance relationship obtained, Eq. (6.8) simplifies as:

\[
\frac{d^2 m}{dt^2} + \frac{(u_0 - v_e)^2 + 2 \mu g s}{2 (u_0 - v_e) s} \frac{dm}{dt} + \frac{\mu g u_0 m}{(u_0 - v_e)^2} = 0
\]

\[
\frac{d^2 m}{dt^2} + \frac{(u_0 - v_e)^2 + 2 \mu g (u_0 t + s_0)}{2 (u_0 - v_e)(u_0 t + s_0)} \frac{dm}{dt} + \frac{\mu g u_0 m}{(u_0 - v_e)(u_0 t + s_0)} = 0 \quad (6.9)
\]

Equation (6.9) is a second order ordinary differential equation with the mass \( m \) as the only unknown. We treat Eq. (6.9) as an initial value problem (IVP) By admitting two initial conditions: initial mass \( m(t=0) = m_0 \), and initial mass transfer rate \( \dot{m}(t=0) = -\dot{m}_0 \). Applying the algebraic partitioning technique:

\[
\frac{d^2 m}{dt^2} = \lambda_1 \quad (6.10)
\]

\[
\frac{(u_0 - v_e)^2 + 2 \mu g (u_0 t + s_0)}{2 (u_0 - v_e)(u_0 t + s_0)} \frac{dm}{dt} + \frac{\mu g u_0 m}{(u_0 - v_e)(u_0 t + s_0)} = \lambda_2 \quad (6.11)
\]

\[
\lambda_1 + \lambda_2 = 0 \quad (6.12)
\]

Integrate Eq.(6.10) once with respect to time and apply initial condition \( dm / dt(t=0) = -\dot{m}_0 \):

\[
\frac{dm}{dt} = \lambda_1 t + c_3; (-\dot{m}_0) = \lambda_1 (0) + c_1 \Rightarrow c_1 = -\dot{m}_0 \Rightarrow \frac{dm}{dt} = \lambda_1 t - \dot{m}_0
\]

Integrating again and applying initial condition: \( m(t=0) = m_0 \Rightarrow c_2 = m_0 \) gives:

\[
\Rightarrow m = \frac{\dot{m}_0 t^2}{2} - \dot{m}_0 t + m_0 \Rightarrow \lambda_1 = \frac{2(m - \dot{m}_0 t + \dot{m}_0)}{t^2} \quad (6.13)
\]
From Eq. (6.11):

\[
\frac{dm}{dt} + \frac{2\mu g u_0 m}{(u_0 - v_e)^2 + 2\mu g (u_0 t + s_0)} = \lambda_2 \frac{2(u_0 - v_e)(u_0 t + s_0)}{(u_0 - v_e)^2 + 2\mu g (u_0 t + s_0)}
\]

\[
\frac{dm}{dt} + \frac{m}{(u_0 - v_e)^2 / 2\mu g u_0 + s_0 / u_0 + t} = \lambda_2 \frac{(u_0 - v_e)(t + s_0 / u_0)}{\mu g (u_0 - v_e)^2 / 2\mu g u_0 + s_0 / u_0 + t}
\]

Using the integration factor:

\[
I.F. = e^{\int \frac{1}{(u_0 - v_e)^2 / 2\mu g u_0 + s_0 / u_0 + t} dt} = e^{\ln[(u_0 - v_e)^2 / 2\mu g u_0 + s_0 / u_0 + t]}
\]

we obtain:

\[
\frac{d}{dt}\left(m \left(\frac{(u_0 - v_e)^2}{2\mu g u_0} + \frac{s_0 + t}{u_0}\right)\right) = \frac{\lambda_2 (u_0 - v_e)(t + s_0)}{\mu g (u_0 - v_e)^2 / 2\mu g u_0 + s_0 / u_0 + t} + c_1
\]

Apply initial condition \(m(t = 0) = m_0\)

\[
\Rightarrow m_0 \left(\frac{(u_0 - v_e)^2}{2\mu g u_0} + \frac{s_0}{u_0}\right) = 0 + c_1 \Rightarrow c_1 = m_0 \left(\frac{(u_0 - v_e)^2}{2\mu g u_0} + \frac{s_0}{u_0}\right)
\]

Therefore,

\[
m \left((u_0 - v_e)^2 + 2\mu g (u_0 t + s_0)\right) = \lambda_2 (u_0 - v_e)(u_0 t + 2s_0) + m_0 \left((u_0 - v_e)^2 + 2\mu gs_0\right)
\]

Rearranging terms:

\[
\lambda_2 = \frac{m \left((u_0 - v_e)^2 + 2\mu g (u_0 t + s_0)\right) - m_0 \left((u_0 - v_e)^2 + 2\mu gs_0\right)}{(u_0 - v_e)(u_0 t + 2s_0) + t}
\] (6.14)

Insert Eqs. (6.13) and (6.14) into Eq. (6.12)
\[ m = \frac{2(m_0 - \dot{m}_0 t)(u_0 - v_c)(u_0 t + 2s_0) + m_0 \left( (u_0 - v_c)^2 + 2\mu gs_0 \right)}{2(u_0 - v_c)(u_0 t + 2s_0) + (u_0 - v_c)^2 t + 2\mu gt(u_0 t + s_0)} \]  

(6.15)

Differentiating Eq. (6.15) we obtain the rate of change of mass.

\[ \frac{dm}{dt} = \left[ \begin{array}{c}
\left\{ \frac{2(u_0 - v_c)(u_0 t + 2s_0)}{2(u_0 - v_c)(u_0 t + 2s_0) + (u_0 - v_c)^2 t + 2\mu gt(u_0 t + s_0)} \right.

\left. - \frac{2m_0(u_0 - v_c)(u_0 t + 2s_0)}{2m_0(u_0 - v_c) + (u_0 - v_c)^2 + 2\mu g(u_0 t + s_0) + 2\mu g\mu t} \right)

\left. + m_0 \left( (u_0 - v_c)^2 + 2\mu gs_0 \right) \right) 

\end{array} \right] 

\]  

(6.16)

Equation (6.15) is the APT solution of Eq. (6.9). Equations (6.15) and (6.16) satisfy the initial conditions. Using Eq. (6.15), the illustration in Figure 48, shows the change in mass when block is travelling with a constant velocity.

![Coefficient of friction, μ](image)

Figure 48: Change in block mass during constant velocity sliding

Figure 48 shows that the block’s mass decreases rapidly with time. This may not represent a specific situation during sliding motion unless more details about material properties are known. Interestingly, the trend in Figure 48 shows that the larger the
coefficient of friction (COF) the larger will be the block mass that is lost and therefore the smaller the instantaneous block mass. This trend agrees with real life expectations.

Using Eq. (6.15) Figure 49 was constructed to show the block’s mass wear rate. As Figure 49 shows, the wear rate is higher in the beginning of the sliding and then decreases with time which seems to capture the trends in experimental wear data. Also, the increasing and decreasing wear rates are severer for higher friction coefficient as expected. Without knowing the material properties, it is difficult to quantitatively ascertain whether the wear rate is realistic or not. Therefore, it is necessary to introduce the block’s material properties into Newtonian mechanics for variable mass sliding to get the actual amount of wear during the motion. As an example, we apply one case of wear, which is adhesive wear using Archard’s adhesive wear law, a phenomenological model that is widely accepted in tribology.
6.3.2: Adhesive Wear Mass during Unlubricated Sliding

Consider the same situation as in section 6.3.1 where the block is sliding with a variable velocity and losing its mass due to adhesive wear. The only modification is the normal force applied on the block is not only its self-weight. There is an external normal force applied by any mechanism during sliding motion. For example, in tribometer tests the applied load for block-on-ring or pin-on-disk tests is far greater than the mass of the block or the pin: \( F_n \gg mg \). A similar situation is considered for this case study.

Using Newton’s second law for variable mass system the applied force is given as:

\[
F_a = \mu F_n + m \frac{dv}{dt} - (v_e - v) \frac{dm}{dt}
\]

(6.17)

Using the force the power or work transfer rate will be:

\[
-W_a = \mu F_n \frac{ds}{dt} + ms \frac{d^3 s}{dt^3} + m s \frac{d^2 s}{dt^2} + 2s \frac{dm}{dt} \frac{d^2 s}{dt^2} + \frac{d}{dt} \left[ \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt} - v_e \frac{ds}{dt} \frac{dm}{dt} - v_e s \frac{d^2 m}{dt^2} + s \frac{d^2 m}{dt^2} \right] \]

(6.18)

The kinetic energy transfer rate of the block with changing mass will be:

\[
\dot{KE} = \frac{1}{2} \frac{dm}{dt} v_e^2 + mv \frac{dv}{dt} = m s \frac{d^2 s}{dt^2} + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt}
\]

(6.19)

As done previously, using the first law of thermodynamics for an open system [107] the power to energy transfer rate equation becomes:

\[
\frac{dE_{CV}}{dt} = \dot{KE} = -W_a - \frac{v_e^2}{2} \frac{dm}{dt}
\]

(6.7)

Combining Eqs. (6.18) and (6.19) the power to energy transfer rate equation (PETRE) becomes:
Equation (6.20) has two dependent variables: mass and distance which change with time. We introduce Archard’s adhesive wear law to represent the change of mass in terms of sliding distance. From Archard’s law, the mass lost during sliding contact or wear mass is given by:

\[ m_w = \frac{\kappa \rho F_n s}{H} \]  

(6.2)

where \( m_w \) [kg] is the mass of wear, \( \kappa \) is the dimensionless wear coefficient, and \( H \) [N/m^2] is the hardness of the softer sliding object. From the tribology literature [14, 191] it is clear that the values of \( \kappa \) are given for specific ranges of the COF. We assume that the values of \( \kappa \) includes the effect of COF. In our study, we want to show the effect of increasing COF on the wear of the material therefore we will use a modified Archard’s wear law which includes the effect of friction coefficient in the wear equation. Then the mass of wear can be given as:

\[ m_w = \frac{\mu \kappa_{\text{eff}} \rho F_n s}{H} \]  

(6.21)

where \( \kappa = \mu \kappa_{\text{eff}} \) is the modified wear coefficient that captures frictional effects. The modified Archard’s adhesive wear law in Eq. (6.21) incorporates the effect of friction directly as we notice that high COF leads to a high \( \kappa \) and induces a larger mass wear.

The instantaneous block mass is \( m(t) = m_0 - m_w(t) \), where \( m_0 \) [kg] is the original mass of the sliding block. Archard’s adhesive wear law gives us a facility to extract the instantaneous block mass as \( m = m_0 - (\mu \kappa_{\text{eff}} \rho F_n) s / H \). Setting \( (\mu \kappa_{\text{eff}} \rho F_n) / H = \beta \) then,
\[ m(t) = m_0 - \beta s \]  
(6.22)

Using this relation we can obtain the first and second order mass wear rates as:

\[ \dot{m} = \frac{dm}{dt} = -\beta \frac{ds}{dt} \]  
(6.23)

\[ \frac{d^2 m}{dt^2} = -\beta \frac{d^2 s}{dt^2} \]

Substituting Eqs. (6.22) and (6.23) into (6.20) the jerk balance equation for the variable mass sliding with adhesive wear law becomes:

\[
\frac{d^3 s}{dt^3} = \frac{\beta s d^2 s}{m_0 dt^2} + \frac{\mu F_s}{m_0 s dt} \frac{ds}{dt} \frac{3 \beta ds d^2 s}{m_0 dt^2} - \frac{\beta}{2m_0 s} \left( \frac{ds}{dt} \right)^3 + \frac{v_e \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 + \frac{v_e \beta d^2 s}{m_0 dt^2} - \frac{v_e^2 \beta ds}{2m_0 s dt} = 0
\]  
(6.24)

Equation (6.24) is a third order nonlinear ordinary differential equation. As done with the other nonlinear third order ODEs, we solve this jerk balance equation using the APT.

In Eq. (6.24) the terms: \( \frac{v_e \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 - \frac{v_e^2 \beta ds}{2m_0 s dt} \) represent the wear jerks, which are primarily inversely proportional to distance and directly proportional to the sliding velocity and escaping debris velocity \( v_e \). This relation is similar to the frictional jerk and shows that wear can also cause an energy loss during the sliding motion. One more jerk term with the escaping mass velocity is \( \frac{v_e \beta d^2 s}{m_0 dt^2} \) which indicates that because of the positive algebraic sign, wear aid motion since a decrease in mass may reduce the force required to maintain relative sliding motion.

Even applying the APT is not simple for the type of nonlinear equation found in Eq. (6.24). To simplify the partitioning procedure, some terms need to be transformed to integrable form. To apply the APT we used following transformation.
Using this transformation, Eq. (6.24) can be simplified to:

\[
- \frac{d^3 s}{dt^3} + \lambda_1 = 0
\]

\[
\frac{\beta}{2m_0} \frac{d^3 s^2}{dt} + \frac{1}{s} \frac{d s}{dt} + \frac{\beta}{2m_0} \left( \frac{ds}{dt} \right)^3
\]

\[
- \frac{v_e \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 - \frac{v_e \beta d^2 s}{m_0 \frac{dt^2}{dt}} = 0
\]

(6.25)

We treat Eq. (6.25) as an initial value problem (IVP) with three initial conditions: \( s(t = 0) = s_0 \); \( d s / dt (t = 0) = u_0 \) and \( d^2 s / dt^2 (t = 0) = a_0 \). The detailed APT procedure is shown in Appendix: C with only the key results shown here. We partition Eq. (6.25) as:

\[
- \frac{d^3 s}{dt} = \lambda_1
\]

\[
\frac{\beta}{2m_0} \frac{d^3 s^2}{dt} = \lambda_2
\]

(6.26)

(6.27)

\[
\frac{v_e \beta - 2 \mu F_n}{2m_0} \frac{1}{s} \frac{d s}{dt} = \lambda_3
\]

\[
\frac{\beta}{2m_0 s} \left( \frac{ds}{dt} \right)^3 = \lambda_4
\]

(6.28)

(6.29)

\[
- \frac{v_e \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 = \lambda_5
\]

\[
\frac{v_e \beta d^2 s}{m_0 \frac{dt^2}{dt}} = \lambda_6
\]

(6.30)

(6.31)

\[
\sum_{i=1}^{6} \lambda_i = 0
\]

(6.32)

Following the APT procedure, from partitioned Eq. (6.26):

\[
\lambda_1 = \frac{3 \alpha_0 t^2 + 6 \alpha_0 t - 6(s - s_0)}{t^3}
\]

(6.33)

From the partitioned Eq. (6.27):

\[
\lambda_2 = \frac{3 \beta \left( s^2 - s_0^2 \right) - \left( \alpha_0 s_0 + u_0 \right) t^2 - 2 \alpha_0 s_0 t}{m_0 t^3}
\]

(6.34)

From the partitioned Eq. (6.28):

\[
\lambda_3 = \frac{3 \alpha_0 t^2 + 6 \alpha_0 t - 6(s - s_0)}{t^3}
\]

\[
\lambda_4 = \frac{3 \beta \left( s^2 - s_0^2 \right) - \left( \alpha_0 s_0 + u_0 \right) t^2 - 2 \alpha_0 s_0 t}{m_0 t^3}
\]

\[
\lambda_5 = \frac{3 \beta \left( s^2 - s_0^2 \right) - \left( \alpha_0 s_0 + u_0 \right) t^2 - 2 \alpha_0 s_0 t}{m_0 t^3}
\]

\[
\lambda_6 = \frac{3 \beta \left( s^2 - s_0^2 \right) - \left( \alpha_0 s_0 + u_0 \right) t^2 - 2 \alpha_0 s_0 t}{m_0 t^3}
\]
\[ \lambda_3 = \frac{(v_c^2 \beta - 2\mu F_n)}{2m_0 t} \ln \left[ \frac{s}{s_0} \right] \]  

(6.35)

From the partitioned Eq. (6.29):

\[ \lambda_4 = \frac{27\beta \left( s^{2/3} - s_0^{2/3} \right)^3}{16 \ m_0 t^2} \]  

(6.36)

From the partitioned Eq. (6.30):

\[ \lambda_5 = \frac{-4v_c\beta(s^{1/2} - s_0^{1/2})^2}{m_0 t^2} \]  

(6.37)

From the partitioned Eq. (6.31):

\[ \lambda_6 = \frac{2v_c\beta(u_0 t - (s - s_0))}{m_0 t^2} \]  

(6.38)

Invoking the algebraic partitioning condition (APC) of Eq. (6.32):

\[
\frac{3a_0 t^2 + 6u_0 t - 6(s - s_0)}{t^3} + \frac{3\beta \left( (s^2 - s_0^2) - (a_0 s_0 + u_0^2) t^2 - 2u_0 s_0 t \right)}{m_0 t^3} \\
+ \frac{(v_c^2 - 2\mu F_n)}{2m_0 t} \ln \left[ \frac{s}{s_0} \right] + \frac{27\beta \left( s^{2/3} - s_0^{2/3} \right)^3}{16 \ m_0 t^3} - \frac{4v_c\beta(s^{1/2} - s_0^{1/2})^2}{m_0 t^2} \\
+ \frac{2v_c\beta(u_0 t - (s - s_0))}{m_0 t^2} = 0
\]  

(6.39)

Multiplying through by \(16m_0 t^3\) and simplifying gives an implicit time-distance relationship as:

\[
\left[ 48m_0a_0 - 48\beta \left( a_0 s_0 + u_0^2 \right) + 8\left( v_c^2 \beta - 2\mu F_n \right) \ln \left[ s / s_0 \right] + 32v_c\beta u_0 \right] t^2 \\
+ \left[ 96m_0u_0 - 96\beta u_0 s_0 - 64v_c\beta(s^{1/2} - s_0^{1/2})^2 - 32v_c\beta(s - s_0) \right] t \\
+ 27\beta(s^{2/3} - s_0^{2/3})^3 - 96m_0(s - s_0) + 48\beta(s^2 - s_0^2) = 0
\]  

(6.40)

Using the quadratic root formula:
\[ t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]
\[
A(s) = \left[ 48m_o a_o - 48\beta \left( a_o s_o + u_o^2 \right) + 8 \left( v^2 \beta - 2\mu F_o \right) \ln \left[ s / s_o \right] + 32v_e \beta u_o \right] 
\]
\[
B(s) = \left[ 96m_o u_o - 96\beta u_o s_o - 64v_e \beta (s^{1/2} - s_o^{1/2})^2 - 32v_e \beta (s - s_o) \right] 
\]
\[
C(s) = \left[ 27\beta (s^{2/3} - s_o^{2/3})^3 - 96m_o (s - s_o) + 48\beta (s^2 - s_o^2) \right] 
\]

To check for the initial condition: when \( s = s_o \), we should get \( t = 0 \). Thus, for
\[
t = \frac{-B + \sqrt{B^2 - 4AC}}{2A} 
\]
\[
A(s_o) = \left[ 48m_o a_o - 48\beta \left( a_o s_o + u_o^2 \right) + 32v_e \beta u_o \right] ; 
\]
\[
B(s_o) = 96u_o \left[ m_o - \beta s_o \right] ; C(s_o) = 0 
\]

only the positive root satisfies the initial condition. Consequently,
\[
t = \frac{-B + \sqrt{B^2 - 4AC}}{2A} 
\]

To obtain the velocity and acceleration we differentiate Eq. (6.42) with respect to time. The velocity resulting from the implicit differentiation is in Eq.(6.43).
\[
t = Q(s) + R(s) 
\]
\[
Q(s) = -\frac{B(s)}{2A(s)} ; R(s) = \frac{\sqrt{B(s)^2 - 4A(s)C(s)}}{2A(s)} 
\]
\[
t = Q(s) + R(s) \Rightarrow 1 = \left[ Q'(s) + R'(s) \right] \dot{s} = \left[ Q'(s) + R'(s) \right] v 
\]
\[
v = \frac{1}{\left[ Q'(s) + R'(s) \right]} 
\]

Similarly, the acceleration can be obtained by differentiating Eq. (6.42) twice with respect to time. The final form of the acceleration becomes:
\[
t = Q(s) + R(s) \Rightarrow 1 = \left[ Q'(s) + R'(s) \right] \ddot{s}; 
\]
\[
\Rightarrow 0 = \left[ Q'(s) + R'(s) \right] \ddot{s} + \dot{s}^2 \left[ Q'(s) + R'(s) \right] 
\]
\[
\ddot{s} = -\frac{v^2 \left[ Q'(s) + R'(s) \right]}{\left[ Q'(s) + R'(s) \right]} = -v^2 \left[ Q'(s) + R'(s) \right] 
\]
Here, we can see from Eq. (6.44) that acceleration is a function of velocity and time. The task of obtaining the velocity and acceleration can be made easy by assigning variables. The differentiation process using the assigned variables is shown below:

\[
Q(s) = \frac{-B}{2A} = \frac{F}{G}; \quad Q'(s) = \frac{GF' - FG'}{G^2};
\]

\[
R(s) = \frac{\sqrt{B^2 - 4AC}}{2A} = \frac{H}{G}; \quad R'(s) = \frac{GH' - HG'}{G^2};
\]

\[
Q''(s) = \frac{G[GF'' - G^3/F]}{G^3};
\]

\[
R''(s) = \frac{G[GH'' - G^3/H] - 2G'[GH' - HG']}{G^3};
\]

\[
A' = 8\left(\nu^2\beta - 2\mu F_n\right)/s; \quad A'' = -8\left(\nu^2\beta - 2\mu F_n\right)/s^2
\]

\[
B' = -32\nu\beta - 64\nu\beta(s^{1/2} - s_0^{1/2})s^{-1/2}; \quad B'' = 32\nu\beta(s^{1/2} - s_0^{1/2})s^{-3/2} - 32\nu\beta/s
\]

\[
C' = 54\beta(s^{2/3} - s_0^{2/3})^2 s^{-1/3} - 96m_0 + 96\beta s;
\]

\[
C'' = -18\beta(s^{2/3} - s_0^{2/3})^2 s^{1/3} + 72\beta(s^{2/3} - s_0^{2/3})s^{-1/3} + 96\beta
\]

\[
G' = 2A'; \quad G'' = 2A''; \quad F' = -B'; \quad F'' = -B'';
\]

\[
H' = \frac{(BB' - 2A'C - 2AC')}{H};
\]

\[
H'' = \frac{H\left(BB'' + [B']^2 - 4A'C' - 2A''C - 2AC''\right) - H'(BB' - 2A'C - 2AC')}{H^2}
\]

We applied the analytical results obtained for distance, velocity, acceleration, and jerk to test the impact of wear on interfacial sliding.
6.3.2.1: Effect of Friction on the Wear of the Material

Having obtained the solution of the jerk balance equation for mass wear during sliding motion without lubrication, parametric studies were performed to quantify the effect of increasing coefficient of friction on the wear and kinematic properties such as, distance, velocity, acceleration and jerk. The values of different parameters for Figure 50 through Figure 56 are shown in Table 8 below.

<table>
<thead>
<tr>
<th>Table 8: Input parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>$g$</td>
</tr>
<tr>
<td>$F_n$</td>
</tr>
<tr>
<td>$a_0$</td>
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<tr>
<td>$u_0$</td>
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<tr>
<td>$v_e$</td>
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<td>$s_0$</td>
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<tr>
<td>$S$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\kappa_{eff}$</td>
</tr>
<tr>
<td>$\rho_{steel}$</td>
</tr>
<tr>
<td>$H_{steel}$</td>
</tr>
</tbody>
</table>

Usually, the dimensionless wear coefficient $\kappa$ has different ranges based on the severity of wear. In this study, because there is no lubrication, we consider a severe wear condition and choose $K \approx 10^{-2} - 10^{-4}$. Since $\kappa = \kappa_{eff} \mu$, the values for $\kappa_{eff}$ is taken as $10^{-2}$. Also, the chosen values of $\mu$ lies in the range of the severe wear condition. At first the material selected is steel and its density and hardness are given in Table 8. Moreover, the normal load $F_n$ has a higher value compared to the mass of the block. Moreover, the normal load $F_n$ has a higher value compared to the mass of the block. This requires higher values of initial acceleration to move the block.
compared to the other cases where we considered that the normal load is the self-weight of the block. The distance travelled is considered to be 50 m to show the effect of wear clearly. For the small distances the wear volume might be very small.

Figure 50 shows the variation of block mass with the increasing coefficient of friction (COF). It confirms the fact that if the block travels on very rough surface then its mass would decrease rapidly.

![Graph showing the variation of block mass with distance and time](image)

Figure 50: The influence of friction on the block mass, (A) variation of block mass with distance, (B) variation of block mass with time

It can be seen that the mass of the block decreases more for the highest value of COF and less for the lowest value of the COF. The variation of block mass with distance shows the linear relation with distance conforming to Archard’s wear law where the wear mass is linearly proportional to the sliding distance. The mass of the block varies nonlinearly with time (Figure 50B) because the jerk dynamics gives the nonlinear distance-time relationship. The difference in time scale is the same reason because the time was obtained by supplying the distance travelled. At a particular time, if a vertical line is drawn which intersects all three curves then it can be seen that the block mass decreases more for the highest value of COF.
Figure 51: The influence of friction on block wear mass, (A) variation of wear mass with distance, (B) variation of wear mass with time

Figure 51 shows that the wear mass increases with distance as well as time. Sliding with higher COF gives more wear. The wear mass shows an almost linear variation with distance (Figure 51A), because it is linearly proportional to the sliding distance. As expected, the variation of wear mass with time (Figure 51B) follows a similar nonlinear trend can be observed for block mass (Figure 50B).

Figure 52: The influence of friction on wear rate, (A) variation of wear rate with distance, (B) variation of wear rate with time

Figure 52 shows the variation of wear rate with distance and time. The motion with higher values of COF gives the higher wear rate which represents the severe wear conditions.
Additionally, other kinematic properties; distance, velocity, acceleration and jerk are shown in the following figures.

Figure 53: Variation of distance with time for increasing COF and severe wear

Figure 53 shows a similar behavior found in the case of unlubricated sliding in chapter 4. Sliding with highest COF requires a longer time than the lower COF values.

Figure 54 shows the effect of increasing COF on the velocity of the block under the severe wear condition. The maximum velocity of the block is observed for the lowest COF case. This conforms to real life expectation.

Figure 54: The influence of friction on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time
Figure 55 shows the effect of increasing COF with adhesive wear on the acceleration of the block. Acceleration decreases with increasing COF. Figure 55 (A and B) shows, a large initial acceleration is required to initiate the sliding motion during a lubricated sliding with wear. The high initial acceleration may account for the large initial torque required in rotary equipment such as electric motors.

Figure 55: The influence of friction on sliding acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time

Figure 56 (A and B) shows the effect of increasing COF on the jerk of the block with adhesive wear. The severest jerk occurs at the highest COF.

Figure 56: The influence of friction on jerk, (A) variation of jerk with distance, (B) variation of jerk with time

It is observed that the frictional and wear jerk decrease with the increase in distance which confirms the relation obtained in Eq.(6.24). In that relation frictional jerk is inversely proportional to the sliding distance.
6.3.2.2: Effect of density and hardness on the wear of the material

To check the effect of mechanical properties such as hardness and density on the wear of the materials, we defined a ratio $\chi = \rho / H$ which is the ratio of material density to its hardness. Reconsidering the modified, Archard’s adhesive wear law:

$$m_w = \frac{\mu \kappa_{\text{eff}} \rho F_n s}{H}$$  \hspace{1cm} (6.21)

The density to hardness ratio $\chi$ is an integral part of the wear mass obtained using the Archard’s adhesive wear law. We will see the effect of these material properties for the same value of COF under the same severe wear conditions.

<table>
<thead>
<tr>
<th>Table 9: Input parameters for Figure 57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
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<tr>
<td>$g$</td>
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<tr>
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<tr>
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<tr>
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<tr>
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<tr>
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</tr>
<tr>
<td>$s$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\kappa_{\text{eff}}$</td>
</tr>
</tbody>
</table>

In this case study, different materials are selected. Table 10 shows the density and hardness of the materials used leading to the different values for $\chi$.

<table>
<thead>
<tr>
<th>Table 10: Mechanical properties of the different block materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>Steel</td>
</tr>
<tr>
<td>Aluminum</td>
</tr>
<tr>
<td>Copper</td>
</tr>
</tbody>
</table>

It is clear from Archard’s adhesive wear law that the density to hardness ratio is directly proportional to wear mass. Ordinarily, using hardness alone, it appears from
Eq. (6.21) that a material with the higher hardness should wear more. Yet, as Table 10 shows, the density of a material varies completely differently from the trends observed with its hardness. As such, the density to hardness ratio $\chi$ becomes a more realistic parameter that captures the expected wear and wear rates. Figure 57 illustrates the effect of $\chi$ on wear and wear rates.

![Figure 57](image.png)

Figure 57: Effect of density and hardness on the wear (A) variation of block mass with time (B) variation of wear mass with time (C) variation of wear rate with distance (D) variation of wear rate with time.

Figure 57A shows that the block made of steel wears more because it has a higher $\chi$. The instantaneous mass of the material with the highest $\chi$ is lowest and follows the trend: $\chi_{\text{steel}} > \chi_{\text{Aluminum}} > \chi_{\text{copper}} \Rightarrow m_{\text{steel}} < m_{\text{Aluminum}} < m_{\text{copper}}$. That is, for similar sliding conditions steel would wear more compared to aluminum and copper. The wear rate follows a similar trend as captured in Figure 57C and D. Steel has the highest wear rate while copper has the lowest wear rate.
6.4: Sliding with Simultaneous Friction, Lubrication, and Wear

In this section we consider the same block of mass $m$, whose mass changes under the influence of friction at the interface but the coefficient of friction is dependent on velocity due to the application of solid lubricant between two surfaces. This velocity and friction coefficient relation is also known as the ‘Strubeck effect’ in the field of tribology. We wanted to assess to what extent the nonlinear jerk dynamics reproduces the Strubeck effect.

Figure 58: Block sliding on a surface with boundary lubrication

Recall the general equation of variable-mass motion:

$$ F_a - F_f = m \frac{dv}{dt} - (v_e - v) \frac{dm}{dt} $$

(6.3)

The friction force $F_f = \mu F_n$ depends on the normal force on the block $F_n$ and the COF $\mu$ between the sliding surfaces. For the a boundary or powder lubricated interface the relation between the coefficient of friction and the sliding velocity is given by:

$\mu = \mu_d - \gamma v$. Using this form of friction force, the applied force is obtained from Eq. (6.3) as:

$$ F_a = (\mu_d - \gamma v)F_n + m \frac{dv}{dt} - (v_e - v) \frac{dm}{dt} $$

(6.46)
If the block travels distance \( s \), under the influence of the applied force than the work done by the applied force is:

\[
-W_a = F_a s = (\mu_d - \gamma v)F_n + ms \frac{d^2 s}{dt^2} - \left( v_e - \frac{ds}{dt} \right) \frac{dm}{dt}
\]

The work transfer rate or power becomes:

\[
-W_a = \mu_d F_n \frac{ds}{dt} - \gamma F_n s \frac{d^2 s}{dt^2} - \gamma F_n \frac{ds}{dt} \left( \frac{ds}{dt} \right)^2 + ms \frac{d^3 s}{dt^3} + m \frac{ds}{dt} \frac{d^2 s}{dt^2} + 2s \frac{dm}{dt} \frac{d^2 s}{dt^2} + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt}
\]

For the variable mass system the rate of kinetic energy can be obtained as:

\[
\dot{K}e = m \frac{ds}{dt} \frac{d^2 s}{dt^2} + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt}
\]

Recall the rate form of the first law of thermodynamics for an open system [107], the power to energy transfer rate equation (PETRE) becomes:

\[
\frac{dE_{cv}}{dt} = \dot{K}e = -\dot{W}_a - \frac{v_e^2}{2} \frac{dm_e}{dt}
\]

Combining Eqs. (6.47), (6.48) and (6.7):

\[
ms \frac{d^3 s}{dt^3} + \mu_d F_n \frac{ds}{dt} - \gamma F_n s \frac{d^2 s}{dt^2} - \gamma F_n \left( \frac{ds}{dt} \right)^2 + 2s \frac{dm}{dt} \frac{d^2 s}{dt^2} + \frac{1}{2} \left( \frac{ds}{dt} \right)^2 \frac{dm}{dt} = 0
\]

We introduce a modified Archard’s adhesive wear law to represent the change of mass in terms of sliding distance. Then the mass of wear can be given as:

\[
m_w = \mu \kappa_{eff} \rho F_n s
\]

where, \( \kappa = \mu \kappa_{eff} \), is the modified wear coefficient. If the Strubeck effect is introduced then the COF is a function of sliding velocity \( \mu = \mu_d - \gamma v \). The instantaneous mass becomes \( m(t) = m_o - m_w(t) \), where \( m_o [kg] \) is the original mass of
the sliding block. Archard’s adhesive wear law gives us a facility to extract the instantaneous block mass as \( m = m_0 - (\mu k_{eff} \rho F_n) s / H \). By setting \( (k_{eff} \rho F_n) / H = \beta_1 \) then we obtain the instantaneous block mass as:

\[
m(t) = m_0 - \mu_d \beta_1 s + \beta_1 \gamma s \frac{ds}{dt}
\]

Using this relation we can obtain the mass wear rate as:

\[
\dot{m} = \frac{dm}{dt} = -\mu_d \beta_1 \frac{ds}{dt} + \gamma \beta_1 \left( \frac{ds}{dt} \right)^2 + \gamma \beta_1 s \frac{d^2 s}{dt^2}
\]

Substituting Eqs. (6.51) and (6.52) into Eq.(6.49), and simplifying:

\[
\frac{d^3 s}{dt^3} = \frac{\mu_d \beta_1 s \frac{d^3 s}{dt^3}}{m_0} - \frac{v \gamma \beta_1 s \frac{d^3 s}{dt^3}}{m_0} + 2 \beta_1 \gamma s \frac{ds}{dt} \frac{d^2 s}{dt^2} - 4 \gamma \beta_1 \frac{ds}{dt} \frac{d^2 s}{dt^2} + \frac{\left( \gamma \beta_1 v_e^2 - 2 \gamma F_n + 2 \gamma \mu_d \beta_1 \right) \frac{d^2 s}{dt^2}}{2 m_0} - \frac{3 \mu_d \beta_1 \frac{ds}{dt} \frac{d^2 s}{dt^2}}{m_0} + \frac{2 \gamma \beta_1 s \left( \frac{d^2 s}{dt^2} \right)^2}{m_0} + \frac{\left( \mu_d \beta_1 + 2 \gamma \gamma \beta_1 \right) \left( \frac{ds}{dt} \right)^3}{2 m_0 s} + \frac{\gamma \beta_1 \left( \frac{ds}{dt} \right)^4}{2 m_0} + \frac{11 \gamma \beta_1 \left( \frac{ds}{dt} \right)^2 \frac{d^2 s}{dt^2}}{2 m_0} + \frac{\left( 2 \mu_d F_n - \mu_d \beta_1 v_e^2 \right) 1 \frac{ds}{dt}}{2 m_0 s} + \frac{\left( \gamma \beta_1 v_e^2 - 2 \gamma F_n + 2 \gamma \mu_d \beta_1 \right) 1 \left( \frac{ds}{dt} \right)^2}{2 m_0 s} = 0
\]

Equation (6.53) is the jerk balance equation when effects of friction, lubrication, and wear are occurring simultaneously. Because tribology is the study of friction, lubrication, and wear, we call this jerk balance equation with simultaneity of all three tribological components as the tribological jerk.

As we encountered in the friction and wear without lubrication jerk, partitioning Eq. (6.53) becomes problematic unless differential transformations are performed to make each partitioned term integrable. The detailed transformation procedure is shown in Appendix D with only the results shown here. Using the appropriate differential transformations and rearranging terms Eq. (6.53) becomes:
\[-\frac{d^3s}{dt^3} + \frac{\beta_i}{2m_0} \left( \mu_d + \gamma v_e \right) \frac{d^3s}{dt^3} + \frac{\mu_d \beta_i v_e^2 - 2 \mu_d F_n}{2m_0} \frac{ds}{dt} + \frac{\beta_i}{2m_0s} \left( \frac{ds}{dt} \right)^3 \]

\[-\frac{2m_0}{s} \left( \gamma v_e^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \frac{ds}{dt}^2 - \frac{2m_0}{s} \left( \gamma v_e^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \frac{ds}{dt} - \frac{\gamma^2 \beta_i}{2m_0s} \left( \frac{ds}{dt} \right)^4 \]

(6.54)

\[\frac{v \gamma \beta_i}{2m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 - \frac{7 \gamma \beta_i}{6m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right) - \frac{2 \beta \gamma}{m_0} \frac{d}{dt} \left( s \frac{ds}{dt} \right) + \frac{ds}{dt^2} = 0\]

The APT is now applied to solve Eq. (6.54) which is treated as an IVP with the initial conditions: \( s(t = 0) = s_0; \quad ds / dt(t = 0) = u_0 \) and \( d^2 s / dt^2(t = 0) = a_0 \). Equation (6.54) can be partitioned as:

\[\frac{d^3s}{dt^3} = \lambda_1\]

(6.55)

\[\frac{\beta_i}{2m_0} \left( \mu_d + \gamma v_e \right) \frac{d^3s}{dt^3} = \lambda_2\]

(6.56)

\[\frac{\mu_d \beta_i v_e^2 - 2 \mu_d F_n}{2m_0} \frac{ds}{dt} = \lambda_3\]

(6.57)

\[\frac{\beta_i}{2m_0s} \left( \frac{ds}{dt} \right)^3 = \lambda_4\]

(6.58)

\[-\frac{\left( \gamma v_e^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \frac{ds}{dt}^2}{2m_0} \frac{s}{dt} = \lambda_5\]

(6.59)

\[-\frac{\left( \gamma v_e^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \frac{ds}{dt}}{2m_0} \frac{ds}{dt} = \lambda_6\]

(6.60)

\[-\frac{\gamma \beta_i}{2m_0s} \left( \frac{ds}{dt} \right)^4 = \lambda_7\]

(6.61)

\[\frac{v \gamma \beta_i}{2m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 = \lambda_8\]

(6.62)

\[-\frac{7 \gamma \beta_i}{6m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^3 = \lambda_9\]

(6.63)
\[-\frac{2\beta^2 \gamma d}{m_0} \frac{ds}{dt} \left( s \frac{ds}{dt} \frac{d^2 s}{dt^2} \right) = \lambda_{10} \quad (6.64)\]

\[\sum_{i=1}^{10} \lambda_i = 0 \quad (6.65)\]

Following the APT procedure shown in detail in Appendix D, the implicit time-distance relationship is obtained in Eq. (6.66), which is a cubic polynomial in time with coefficients being the functions of distance.

\[A(s) t^3 + B(s) t^2 + C(s) t + D(s) = 0\]

\[A(s) = \begin{bmatrix} 3888m_0 a_0 + 2592\gamma \beta a_0 u_0 s_0 - 648\gamma \beta a_0^2 + 3888\beta_i (\mu_i + \gamma v_i) (a_i s_i + u_i^2) \\ +1296 m_0 \left( \gamma v_i^2 \beta_i + 2\gamma \beta_i u_i^2 - 2\gamma F_i \right) + 1512 \beta_i u_i^2 + 648 (\mu_i \beta_i v_i^2 - \gamma \beta_i u_i^2) \ln(s / s_i) \end{bmatrix}\]

\[B(s) = \begin{bmatrix} 7776 m_0 (s - s_0) - 7776 \beta_i (\mu_i + \gamma v_i) u_i s_i + 1296 \beta_i u_i^2 s_i - 2048 \gamma \beta_i (s^{3/4} - s_0^{3/4})^i \\ - \left( \gamma v_i^2 \beta_i + 2\gamma \beta_i u_i^2 - 2\gamma F_i \right) \left[ 2592 (s^{1/2} - s_0^{1/2})^2 + 1296 (s - s_0) \right] \end{bmatrix}\]

\[C(s) = \begin{bmatrix} -7776 m_0 (s - s_0) + (\mu_i + \gamma v_i) \left[ 3888 \beta_i (s^2 - s_0^2) + 2187 \beta_i (s^{2/3} - s_0^{2/3})^3 \right] \\ + 648 \gamma \beta_i (s - s_0)^2 \end{bmatrix}\]

\[D(s) = \begin{bmatrix} -216 \gamma \beta_i (s - s_0)^3 - 576 \gamma \beta_i (s^{3/2} - s_0^{3/2})^2 \end{bmatrix}\]

The roots of this cubic equation can be found using the following procedure from a standard mathematical handbook [139]:

\[t^3 + a_1 t^2 + a_2 t + a_3 = 0\]

\[Q = \frac{3a_2 - a_1^2}{9}, R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}\]

\[S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}\]

where \(ST = -Q\)

\[t_1 = S + T - \frac{1}{3} a_1\]

\[t_2 = -\frac{1}{2} (S + T) - \frac{1}{3} a_1 + \frac{1}{2} i \sqrt{3} (S - T)\]

\[t_3 = -\frac{1}{2} (S + T) - \frac{1}{3} a_1 - \frac{1}{2} i \sqrt{3} (S - T)\]

\[\frac{2\beta^2 \gamma d}{m_0} \frac{ds}{dt} \left( s \frac{ds}{dt} \frac{d^2 s}{dt^2} \right) = \lambda_{10} \quad (6.64)\]
Only one root satisfies the initial conditions. The root which satisfies the initial conditions is given as:

\[ t_i = S + T - \frac{1}{3} a_i \]  \hspace{1cm} (6.68)

Differentiating Eq. (6.68) with respect to time, the expression for velocity can be obtained as in Eq. (6.69):

\[ t = S + T - \frac{1}{3} a_i \]

\[ \Rightarrow 1 = \left[ S'(s) + T'(s) - \frac{a'_i}{3} \right] \dot{s} = \left[ S'(s) + T'(s) - \frac{a'_i}{3} \right] v \]

\[ v = \frac{1}{\left[ S'(s) + T'(s) - \frac{a'_i}{3} \right]} \]  \hspace{1cm} (6.69)

Subsequent differentiation of Eq. (6.69) gives the expression of acceleration as in Eq. (6.70):

\[ \Rightarrow 0 = \left[ S'(s) + T'(s) - \frac{a'_i}{3} \right] \ddot{s} + \dot{s}^2 \left[ S''(s) + T''(s) - \frac{a''_i}{3} \right] \]

\[ \dot{s} = \frac{d^2 s}{dt^2} = -\frac{v^2 \left[ S''(s) + T''(s) - \frac{a''_i}{3} \right]}{\left[ S'(s) + T'(s) - \frac{a'_i}{3} \right]} = -v^3 \left[ S''(s) + T''(s) - \frac{a''_i}{3} \right] \]  \hspace{1cm} (6.70)

As was done in the previous sections, assigning variables simplifies the differentiation process. The differential terms from the assigned variables can be obtained as:

\[ a'_i = \frac{AB' - BA'}{A^2}; a'_2 = \frac{AC' - CA'}{A^2}; a'_3 = \frac{AD' - DA'}{A^2} \]

\[ a''_i = \frac{A[AB'' - BA''] - 2A'[AB' - BA']}{A^3}; \]

\[ a''_2 = \frac{A[AC'' - CA''] - 2A'[AC' - CA']}{A^3}; \]

\[ a''_3 = \frac{A[AD'' - DA''] - 2A'[AD' - DA']}{A^3}; \]
\[ Q' = \frac{1}{9} \left[ 3a'_2 - 2a_1a'_1 \right]; \quad Q'' = \frac{1}{9} \left[ 3a''_2 - 2a_1a''_1 - 2(a'_1)^2 \right]; \]
\[ R' = \frac{1}{54} \left[ 9a'_2a'_2 + 9a_1a'_1 - 27a'_1 - 6a'_1^2 \right]; \]
\[ R'' = \frac{1}{54} \left[ 9a''_2a''_2 + 9a_1a''_1 + 18a'_1a'_2 - 27a'_1 - 6a'_1^2 a''_1 - 12a_1 (a'_1)^2 \right] \]

where various terms and their differentiations are as follows:

\[
A(s) = \left[ \begin{array}{c}
3888m_0 a_0 + 2592 \gamma \beta a_0 u_0 s_0 - 648 \nu \gamma \beta u_0^3 \\
-3888 \beta_1 \left( \mu_d + \gamma v_r \right) (a_0 s_0 + u_0^2) + 1296 u_0 \left( \gamma v_r^2 \beta_1 + 2 \nu \mu_d \beta_1 - 2 \gamma f_n \right) \\
+1512 \gamma \beta u_0^3 + 648 (\mu_d \beta_1 v_r^2 - 2 \mu_d f_n) \ln [s / s_0]
\end{array} \right]
\]
\[ A' = 648 \left( \mu_d \beta_1 v_r^2 - 2 \mu_d f_n \right) / s \]
\[ A'' = -648 \left( \mu_d \beta_1 v_r^2 - 2 \mu_d f_n \right) / s^2 \]

\[
B(s) = \left[ \begin{array}{c}
7776 m_0 u_0 - 7776 \beta_1 \left( \mu_d + \gamma v_r \right) u_0 s_0 + 1296 \gamma \beta u_0^3 s_0 - 2048 \gamma \beta_1 \left( s^{3/4} - s_0^{3/4} \right)^4 \\
- \left( \gamma v_r^2 \beta_1 + 2 \nu \mu_d \beta_1 - 2 \gamma f_n \right) \left[ 2592 (s^{1/2} - s_0^{1/2})^2 + 1296 (s - s_0) \right]
\end{array} \right]
\]
\[ B' = -6144 \gamma \beta_1 (s^{3/4} - s_0^{3/4})^3 s^{-1/4} \\
- \left( \gamma v_r^2 \beta_1 + 2 \nu \mu_d \beta_1 - 2 \gamma f_n \right) \left[ 2592 (s^{1/2} - s_0^{1/2}) s^{-1/2} + 1296 \right] \]
\[ B'' = 1536 \gamma \beta_1 (s^{3/4} - s_0^{3/4})^3 s^{-5/4} - 13824 \gamma \beta_1 (s^{3/4} - s_0^{3/4})^3 s^{-1/2} \\
- \left( \gamma v_r^2 \beta_1 + 2 \nu \mu_d \beta_1 - 2 \gamma f_n \right) \left[ -1296 (s^{1/2} - s_0^{1/2}) s^{-3/2} + 1296 s^{-1} \right] \]

\[
C(s) = \left[ \begin{array}{c}
-7776 m_0 (s - s_0) + (\mu_d + \gamma v_r) \left[ 3888 \beta_1 \left( s^2 - s_0^2 \right)^2 + 2187 \beta_1 \left( s^{2/3} - s_0^{2/3} \right)^3 \right] \\
+ 648 \nu \gamma \beta \left( s - s_0 \right)^2
\end{array} \right]
\]
\[ C' = -7776 m_0 + 1296 \nu \gamma \beta_1 (s - s_0) + \beta_1 (\mu_d + \gamma v_r) \left[ 7776 s + 4374 (s^{2/3} - s_0^{2/3})^2 s^{-1/3} \right] \]
\[ C'' = 1296 \nu \gamma \beta_1 + \beta_1 (\mu_d + \gamma v_r) \left[ 7776 - 1458 (s^{2/3} - s_0^{2/3})^2 s^{-4/3} + 5832 (s^{2/3} - s_0^{2/3}) s^{-2/3} \right] \]

\[
D(s) = \left[ \begin{array}{c}
-216 \gamma \beta (s - s_0)^3 - 576 \gamma \beta_1 \left( s^{3/2} - s_0^{3/2} \right)^2
\end{array} \right]
\]
\[ D' = -648 \gamma \beta_1 (s - s_0)^2 - 1728 \gamma \beta_1 \left( s^{3/2} - s_0^{3/2} \right) s^{1/2} \]
\[ D'' = -1296 \gamma \beta_1 (s - s_0) - 864 \gamma \beta_1 \left( s^{3/2} - s_0^{3/2} \right) s^{-1/2} - 2592 \gamma \beta_1 s \]

and:
\[
S = \sqrt{R + \sqrt{Q^3 + R^2}}; T = 3\sqrt{R - \sqrt{Q^3 + R^2}}
\]
\[
s = \left( R + \left( R^2 + Q^1 \right)^{1/2} \right)^{1/3} = \left( M + \left( M^2 + N \right)^{1/2} \right)^{1/3}
\]
\[
T = \left( R - \left( R^2 + Q^3 \right)^{1/2} \right)^{1/3} = \left( M - \left( M^2 + N \right)^{1/2} \right)^{1/3}
\]

where
\[
M = R; M' = R'; M'' = R''
\]
\[
N = Q^3; N' = 3Q^2Q'; N'' = 3Q^2Q'' + 6Q[Q']^2
\]
\[
S' = \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M' + \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right]
\]
\[
T' = \frac{1}{3} \left( M - \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M' - \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right]
\]
\[
S'' = -\left( \frac{2}{9} \right) \left[ M + \left( M^2 + N \right)^{1/2} \right]^{-2/3} \left[ M' + \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right]^2
\]
\[
+ \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M'' - \frac{1}{4} \left( M^2 + N \right)^{-3/2} \left( 2MM' + N' \right)^2 \right]
\]
\[
T'' = -\left( \frac{2}{9} \right) \left[ M - \left( M^2 + N \right)^{1/2} \right]^{-2/3} \left[ M' - \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right]^2
\]
\[
+ \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M'' + \frac{1}{4} \left( M^2 + N \right)^{-3/2} \left( 2MM' + N' \right)^2 \right]
\]

Using the implicit distance time relationship, the expression for velocity and acceleration, parametric studies were performed to show the effect of dry COF and lubricity γ on the wear of the block as well as on other kinematic properties.
6.4.1: Effect of Dry Friction on Adhesive Wear

Having obtained the solution of the jerk balance equation for the mass wearing during sliding motion with lubrication, parametric studies were performed to evaluate the effect of increasing dry coefficient of friction on the wear and kinematic properties such as, distance, velocity, and acceleration. The values of different input parameters for constructing Figure 59 through Figure 64 are shown in Table 11 below.

<table>
<thead>
<tr>
<th>Table 11: Input parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1.0 kg</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>$F_n$</td>
<td>15 N</td>
</tr>
<tr>
<td>$a_0$</td>
<td>3.0g m/s$^2$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>0.000000001 m/s</td>
</tr>
<tr>
<td>$v_e$</td>
<td>0.005 m/s</td>
</tr>
<tr>
<td>$s_0$</td>
<td>0.0001 m</td>
</tr>
<tr>
<td>$S$</td>
<td>20 m</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>[0.1, 0.15, 0.2]</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.0005 s/m</td>
</tr>
<tr>
<td>$\kappa_{\text{eff}}$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\rho_{\text{steel}}$</td>
<td>8050 kg/m$^3$</td>
</tr>
<tr>
<td>$H_{\text{steel}}$</td>
<td>$664 \times 10^6$ N/m$^2$</td>
</tr>
</tbody>
</table>

Figure 59 shows the variation of block mass with the increasing dry coefficient of friction (COF). It confirms the fact that if the block travels on very rough surface with a poor lubrication then its mass would decrease rapidly. It can be seen that the mass of the block decreases more for the highest value of COF and less for the lowest value of the COF. The variation of block mass with distance shows an almost linear relation with distance because in Archard’s wear law the wear mass is linearly proportional to the sliding distance. The mass of the block varies nonlinearly with time (Figure 59B) in conformity to the nonlinear distance-time relationship obtained from jerk dynamics (see Eq. (6.66) to (6.68)).
Figure 59: Effect of friction on the block mass, (A) variation of block mass with distance, (B) variation of block mass with time

The difference in time scale for each friction coefficient (Figure 59B) occurs because the time was obtained by supplying the distance travelled. At a particular time, if a vertical line is drawn which intersects all three curves then it can be seen that the block mass decreases more for the highest value of COF.

Figure 60: The influence of friction on wear mass, (A) variation of wear mass with distance, (B) variation of wear mass with time

Figure 60 shows that the wear mass increases with distance as well as time. Sliding with higher dry COF and poor lubrication gives more wear. The wear mass shows an almost linear behavior with respect to distance because it is linearly proportional to the sliding distance.
Figure 61: The influence of friction on wear rate, (A) variation of wear rate with distance, (B) variation of wear rate with time

Figure 61 shows the variation of wear rate with distance and time. The motion with higher values of dry COF and poor lubrication gives the higher wear rate which represents the severe wear conditions.

Additionally, other kinematic properties; distance, velocity, acceleration and friction coefficient are shown in the following figures.

Figure 62: Variation of distance with time for increasing dry COF and poor lubrication

Figure 62 shows a similar frictional sliding behavior found in the case of unlubricated sliding in section 6.3.2. Sliding with highest dry COF and poor lubrication requires longer time than the lower dry COF values.
Figure 63 shows the effect of dry COF on the velocity of the block under a poor lubrication condition. The maximum velocity of the block is observed for the lowest COF case.

Figure 63: The influence of friction on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time

Figure 64 shows the effect of dry COF with adhesive wear and very low lubrication on the acceleration of the block. Similar to the influence of friction on sliding velocity (Figure 63), a higher dry friction coefficient causes acceleration to decrease (Figure 64). High friction therefore adversely affects acceleration.

Figure 64: The influence of friction on sliding acceleration,(A) variation of acceleration with distance,(B) variation of acceleration with time

In this study, the friction-velocity coupling applied enables a deterministic quantification of instantaneous friction during simultaneous lubrication and wear. Figure 65 illustrates how friction coefficient begins at a high value (corresponding to
dry friction) and gradually decreases with distance (Figure 65A) or slightly linearly decreases with time (Figure 65B). To capture the frictional effects the lubricity was set to a minimal value ($\gamma = 0.0005$ s/m). Clearly, there is a very minor decrease in coefficient of friction when the lubrication effects are negligible and the dry coefficient of friction values are high.

![Figure 65: Instantaneous friction coefficient, (A) variation of COF with distance, (B) variation of COF with time](image)

To verify the friction-velocity coupling originally applied in the modeling, instantaneous friction is plotted against instantaneous velocity for three different input dry friction coefficients (Figure 66). Figure 66 shows that there is a minor change in coefficient of friction because of the lower lubricity value used for this parametric study which represents the poor lubrication condition. But the friction-velocity curve has a negative slope and reproduces the linearity of the friction velocity coupling used for the modeling. This validation is promising.
6.4.2: Effect of Lubricity on Adhesive Wear

In this section, the effects of good lubrication on wear, wear rate and different kinematic properties of the block during the sliding motion are considered. Higher values of lubricity $\gamma$, will be used. In the following results the value of dry coefficient of friction $\mu_d$ is kept constant while the lubricity ($\gamma$) is varied. The values used for parametric studies are shown in Table 12.

<table>
<thead>
<tr>
<th>Table 12: Input parameters</th>
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</thead>
<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>$g$</td>
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<tr>
<td>$F_n$</td>
</tr>
<tr>
<td>$a_0$</td>
</tr>
<tr>
<td>$u_0$</td>
</tr>
<tr>
<td>$v_e$</td>
</tr>
<tr>
<td>$s_0$</td>
</tr>
<tr>
<td>$S$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>$\mu_d$</td>
</tr>
<tr>
<td>$\kappa_{eff}$</td>
</tr>
<tr>
<td>$\rho_{steel}$</td>
</tr>
<tr>
<td>$H_{steel}$</td>
</tr>
</tbody>
</table>
Figure 67 shows the variation of block mass with the increasing lubricity.

![Figure 67](image)

Figure 67: Influence of lubricity on instantaneous block mass, (A) variation of block mass with distance, (B) variation of block mass with time

It confirms the fact that if good lubrication is provided during sliding then the block mass would decrease slowly. It can be seen that the mass of the block decreases more for the lowest value of lubricity.

![Figure 68](image)

Figure 68: Influence of lubricity on wear mass, (A) variation of wear mass with distance, (B) variation of wear mass with time

Figure 68 shows that the wear mass increases with distance as well as time. Furthermore, the sliding motion with the highest lubricity generates the lowest amount of wear mass. The instantaneous wear mass for different lubrication levels is evaluated through the use of lubricity. The instantaneous wear mass for different
lubrication levels is evaluated through the use of lubricity. As expected, good lubrication reduces the incidence of wear while poor lubrication does otherwise.

Figure 69: Influence of lubricity on wear rate, (A) variation of wear rate with distance, (B) variation of wear rate with time

Figure 69 shows the variation of wear rate with distance and time. The influence of lubricity on wear rate is similar to the trend observed for wear mass. Good lubrication corresponding to high lubricity produces low wear rates.

Additionally, the influence of lubrication (in the presence of wear and friction) on key kinematic properties like distance, velocity, and acceleration and the transient friction coefficient are evaluated. Figure 70 indicates the nonlinear behavior of distance with time for different values of lubricity.

Figure 70: Influence of lubricity on sliding distance
Higher values of lubricity corresponding to good lubrication show a lower time being required to travel a similar distance. This result agrees with real life experience and substantiates the relevance of good lubrication even in the presence of friction and wear.

Figure 71 shows the effect of increasing lubricity on the velocity of the block where the maximum block velocity corresponds to the highest lubricity or lubrication effect as expected from real life experience. As expected, the lubricating effect on

Figure 71: Influence of lubricity on sliding velocity, (A) variation of velocity with distance, (B) variation of velocity with time

Velocity is recaptured for acceleration as well (Figure 72) where good lubrication corresponds to increasing acceleration.

Figure 72: Influence of lubricity on sliding acceleration, (A) variation of acceleration with distance, (B) variation of acceleration with time
To quantify the effect of lubricity on friction coefficient directly, the instantaneous friction coefficient is plotted (Figure 73) for a given input initial or dry friction coefficient. Figure 73 shows that an increasing lubricity produces a significant decrement in the coefficient of friction over the total sliding distance and duration.

Figure 73: Influence of lubricity on the coefficient of friction, (A) variation of COF with distance, (B) variation of COF with time

Just as the friction-velocity coupling was tested for different dry or initial coefficients of friction, the influence of lubricity on the friction velocity coupling was assessed.

Figure 74: Verification of friction velocity coupling

Figure 74 shows that increasing lubricity representing good lubrication significantly increases the interfacial velocity attainable. Therefore, the impact of good lubrication is to increase velocity as the friction velocity-coupling model suggests. This
validation of the friction-velocity coupling or Stribeck effect for different lubricities is promising.

In this section, the results illustrating the effect of friction and lubrication on wear, wear rate and kinematic properties such as distance, velocity, and acceleration are promising. Despite the promising physical insight offered by the results a few difficulties encountered with the algebraic partitioning (Appendix D) of the highly nonlinear tribological jerk balance equation in Eq. (6.53) are noteworthy. The initial condition for distance in Eq. (6.68) checks out fine. But to capture the initial condition for velocity in Eq. (6.69), we set \( u_0 = 10^9 \text{ m/s} \) (Table 12) which is the nearest to zero initial velocity tolerable. Finally, the initial acceleration from Eq. (6.70) becomes unconnected or discontinued with the subsequent acceleration data points. To sidestep this issue, the initial acceleration (Table 12) was inserted into the time loop for plotting directly. It seems the difficulty of trapping all the initial conditions exactly may be due to the differential transformations carried out before integrable partitioning of Eq. (6.53) could be done. It may also be caused by the prevailing assumptions used in deriving Eq. (6.53) which sought to capture very nonlinear occurrences of friction, lubrication, and wear simultaneously. Perhaps, as we refine our algebraic partitioning technique, it may become clearer what partitioning strategies prevent such challenges.

Notwithstanding the problems with the APT for solving Eq. (6.53) because the parametric study results show expected trends from real life situations, it seems gratifying that at least the basic physics of the tribological (i.e., simultaneous friction, lubrication, and wear) sliding is well reproduced. As such, refined future APT partitions that yield exact solutions satisfying the initial conditions for distance velocity and acceleration will be completely satisfactory.
Chapter 7: Thermo-Mechanical Effects in Sliding Contact Using Jerk Dynamics

7.1: Heat to Internal Energy Transfer Rates

When two surfaces are rubbed together, most of the frictional work converts into heat which causes a temperature rise at the interface. The amount of heat generation depends mainly on two factors: the normal load and sliding velocity of the body. We use the velocity and applied force from our jerk balance equations (JBEs) as inputs of key nonlinear kinematic and dynamic variables respectively to quantify the thermal and mechanical effects resulting from the sliding. For example, putting these mechanical variables into the decoupled heat rate equation (HIETRE) in Eq. (3.3) the nonlinear interfacial temperature from frictional sliding can be obtained.

Consider a slider block (see Figure 75) of mass $m [\text{kg}]$ displaced through a distance $s [\text{m}]$ under the action of an external force $F_a [\text{N}]$, opposed by a frictional resistance $F_f [\text{N}]$. According to Amonton’s law of friction $F_f = \mu F_n$, where, $\mu$ is the coefficient of friction, the normal load $F_n [\text{N}]$ is taken as the block’s self-weight $mg$, and $g$ is the gravitational acceleration. The kinetic energy and its transfer rate are obtained as before in the jerk balance equations. Additionally, the work done by the external force on the moving block is similar to what was obtained for the jerk balance equations. The difference here is that we include the impact of the internal energy change.
There are three modes of heat transfer: (1) Conduction governed by Fourier’s law of heat conduction, (2) Convection, given by Newton’s law of cooling, and (3) Radiation is governed by the Stefan-Boltzmann law. In this study, we assume that conduction dominates the other modes of heat transfer because the surfaces are in contact during the motion. The generated heat is conducted into the block and increases the internal energy of the block. In this analysis, the following assumptions were made:

- The contact area is constant with no separation between the block and the stationary surface.
- There is no wear during the sliding process.
- The stationary block is nonconductive so that the heat flows only into the sliding block.

During a frictional sliding event, not all the heat is transmitted across the interface. This is because although very high temperatures are reached at the sliding surface, the small quantity of heat released at each actual contact area makes it impossible for the effect to penetrate far into the solid. Consequently, frictional heat generation (FHG) occurs through a finite penetration depth and volume and as such only affects a fraction of the overall sliding mass. A portion of the block mass ($m^*$)
acquires FHG through internal energy change. By designating this fraction as 

\(m^*/m = \epsilon_m \Rightarrow m^* = \epsilon_m m\), we obtain the internal energy change transfer rate as:

\[\dot{U}_{ip}^* = m \epsilon_m c_p \frac{dT}{dt}\] (7.1)

where, \(c_p\) is the block’s specific heat capacity (J/kg·K), and \(dT/dt\) is the temperature gradient [K/s]. By applying the heat to internal energy transfer rates equation (HIETRE) shown in Chapter 3:

\[\dot{Q} = \dot{Q}_{in} - \dot{Q}_{out} = \dot{U}\] (3.3)

In modifying Eq. (3.3) we capture the penetrated internal energy transfer rate as \(\dot{U}_{ip}^*\).

From the tribology viewpoint, frictional heat generation (FHG) can be represented as a product of the friction force and sliding velocity given in Eq. (7.2).

\[FHG = F_f \times v = F_f \frac{ds}{dt}\] (7.2)

Again, the friction force is given by Amonton’s law of friction: \(F_f = \mu F_n\). If we consider the block as a control volume the heat entering the control volume \(\dot{Q}_{in}\) is the heat generated at an interface during the sliding motion. Using Amonton’s law and Eq. (3.3) the heat to internal energy transfer rate equation (HIETRE) can be recast as:

\[\mu F_n \frac{ds}{dt} = m \epsilon_m c_p \frac{dT}{dt}\] (7.3)

Equation (7.3) can be integrated on both side with respect to time to obtain the relationship between the temperature rise and the sliding distance as:

\[\mu F_n (s(t) - s_0) = m \epsilon_m c_p (T - T_0)\] (7.4)

It is interesting to note that while Eq. (7.4) appears relatively simple, the exact form of \(s(t)\) is really what complicates the nonlinear or linearity of the evolving
temperature at the interface. Equation (7.4) plays an important boundary and/or initial condition role in solving the PDEs that govern the propagation of the friction generated heat into the substrates in sliding contact.

For this type of problem we need initial displacement \( s_0 \) and initial temperature which can be supplied based on the experiments and operating conditions. Here, we assumed that the initial temperature of the block \( (T_0) \) is the environmental temperature \( (T_{env}) \). Equation (7.4) can be rearranged to show the temperature distance relation as:

\[
T = \frac{\mu F}{m c_p} (s - s_0) + T_0 
\]  

\[ (7.5) \]

The novelty here is that we capture the nonlinear temperature through the nonlinear implicit distance-time from the jerk balance equations for the sliding with and without lubrication. Also, the effect of specific heat capacity on the temperature rise is readily captured in Eq. (7.5). From Eq. (7.5) we can see that the temperature varies linearly with the sliding distance but the distance-time relation obtained from jerk balance equation is nonlinear. Therefore, the temperature will change nonlinearly with time. This agrees with real life situations.

### 7.2: Temperature Rise During Sliding Without Lubrication.

In this case we consider Coulomb-like friction coefficient where friction coefficient is constant and it is independent of sliding velocity. Since the friction coefficient is constant the friction force must stay constant throughout the motion. The nonlinear distance – time relation from the power to energy transfer rate equation (Eq.(4.15) or Eq.(5.31)) was obtained for frictional sliding. The same nonlinear relationship is used to estimate the temperature rise of the block during its motion.
Therefore, the generated heat flux also changes with the changing velocity and distance.

### 7.2.1: Effect of Friction on Heat Generation

In Table 13, different input parameters used for obtaining Figure 76 are given:

<table>
<thead>
<tr>
<th>Table 13: Modeling input parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>$m^*$</td>
</tr>
<tr>
<td>$g$</td>
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<tr>
<td>$a_0$</td>
</tr>
<tr>
<td>$u_0$</td>
</tr>
<tr>
<td>$s_0$</td>
</tr>
<tr>
<td>$\delta$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$C_{p_{cu}}$</td>
</tr>
</tbody>
</table>

Figure 76 captures the nonlinear temperature at a finite depth below the sliding interface showing an interface is cooler at lower friction coefficient.

![Figure 76: Influence of friction on heat generation](image)

As expected, increasing the interfacial COF generates more heat by raising the temperature. Contrarily, an interface is cooler at lower friction coefficient, in agreement with real life.
7.2.2: Effect of Substrate Specific Heat Capacity on Frictional Heat Generation

The specific heat capacity of a material is defined as the amount of heat per unit mass required to increase its temperature by one degree [193] and is different for different materials. For this reason, blocks of different materials in similar sliding contact conditions will experience different near surface temperatures. We illustrate this effect and show how specific heat capacity affects the temperature rise of the sliding block. In Table 14 the different input parameters for Figure 76 are shown.

Table 14: Modeling input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>1.0 kg</td>
</tr>
<tr>
<td>( m^* )</td>
<td>0.005 kg</td>
</tr>
<tr>
<td>( g )</td>
<td>9.81 m/s²</td>
</tr>
<tr>
<td>( a_o )</td>
<td>1.0 m/s²</td>
</tr>
<tr>
<td>( u_0 )</td>
<td>0.0 m/s</td>
</tr>
<tr>
<td>( s_0 )</td>
<td>0.0001 m</td>
</tr>
<tr>
<td>( \delta )</td>
<td>20 m</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.25</td>
</tr>
</tbody>
</table>

For this parametric study we considered different material with different specific heat capacities shown in Table 15.

Table 15: List of specific heat capacities

<table>
<thead>
<tr>
<th>Material</th>
<th>Specific heat capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead</td>
<td>128 J/ kg-K</td>
</tr>
<tr>
<td>Silver</td>
<td>233 J/ kg-K</td>
</tr>
<tr>
<td>Copper</td>
<td>386 J/ kg-K</td>
</tr>
<tr>
<td>Carbon steel</td>
<td>490 J/ kg-K</td>
</tr>
<tr>
<td>Aluminum</td>
<td>910 J/ kg-K</td>
</tr>
</tbody>
</table>
Figure 77 shows that the material with lower specific heat capacity will heat up faster than the material with higher specific capacities. Consequently, as expected, the highest temperature rise is observed for the lead block (see Table 15). This insight may be valuable in thermally characterizing tribological materials in sliding contact.
7.3: Temperature Rise during Sliding with Lubrication

During boundary lubricated sliding, the effect of lubrication is to minimize friction. Here we introduce the linear friction-velocity coupling [26, 102, 103], given by Eq. (4.19) as:

$$\mu = \mu_d - \gamma v$$  \hspace{1cm} (4.19)

Amonton’s law of friction $F_f = \mu F_n$ was modified with $\mu = \mu_d - \gamma v$, giving the friction force as:

$$F_f = \left( \mu_d - \gamma \frac{ds}{dt} \right) F_n \hspace{1cm} (7.6)$$

From the tribology point of view, frictional heat generation at the interface is directly proportional to the frictional force and velocity:

$$FHG = F_f \times v = F_f \frac{ds}{dt} \hspace{1cm} (7.7)$$

By applying the heat to internal energy transfer rates equation (HIFETRE) shown in Chapter 3:

$$\dot{Q} = \dot{Q}_{in} - \dot{Q}_{out} = \dot{U} \hspace{1cm} (3.3)$$

Substituting Eq. (7.6) into (7.3) leads to:
\[ F_n \left( \mu_d - \gamma \frac{ds}{dt} \right) \frac{ds}{dt} = m \varepsilon_m c_p \frac{dT}{dt} \quad (7.8) \]

We carried out the parametric studies to predict the temperature variation of the block using Eqs. (5.49) and (7.9). Equation (7.8) can be solved using APT (see Appendix E) and the final implicit form of temperature as a function of time and distance is given by:

\[ T = T_0 + \frac{F_n}{m \varepsilon_m c_p \mu_d} \left[ (s - s_0) \mu_d t - \gamma (s - s_0)^2 \right] \quad (7.9) \]

Equation (7.9) shows the variation of temperature with distance and time. To show how the initial condition is satisfied, the L’Hopital’s rule is applied to Eq. (7.9) to obtain:

\[
\lim_{t \to 0} \left( T - T_0 \right) = \lim_{t \to 0} \frac{F_n}{d} \left[ \frac{d}{dt} \left( (s - s_0) \mu_d t - \gamma (s - s_0)^2 \right) \right] = 0
\]

showing that \( T = T_0 \) as expected. The temperature rise at the interface depends on the penetrated mass, specific heat of the material, applied normal load, dry coefficient of friction and lubricity of the lubricating medium.

**7.3.1: Effect of Dry Friction on Interfacial Temperature Rise**

First, the effect of increasing dry coefficient of friction on the temperature rise at the interface is considered. The lubricity value is kept constant. Other parameters for this case study are shown in Table 16.

The variation of temperature gradient with time can be obtained using Eq. (7.8). The evolution of temperature and temperature gradient are shown in Figure 79.
Table 16: Modeling input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1.0 kg</td>
</tr>
<tr>
<td>$m^s$</td>
<td>0.005 kg</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>$a_0$</td>
<td>2.0 g m/s$^2$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>0.00001 m/s</td>
</tr>
<tr>
<td>$s_0$</td>
<td>0.0001 m</td>
</tr>
<tr>
<td>$\delta$</td>
<td>20 m</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>[0.1 0.15 0.25]</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.0005 s/m</td>
</tr>
<tr>
<td>$C_{P_{cu}}$</td>
<td>386 J/kg-K (copper)</td>
</tr>
<tr>
<td>$T_0$</td>
<td>298 K</td>
</tr>
</tbody>
</table>

As Figure 79 (A and B) show, the motion with highest dry COF (at a fixed lower value of lubricity) gives the highest temperature rise and the highest temperature gradient.

Figure 79 shows similar trends as in case of heat generation in sliding without lubrication. For this case very small values of lubricity are considered which represents a very small lubrication effect. Therefore, when the lubrication is poor the temperature rise at the interface is significant and it causes undesirable temperature
gradient as well. The temperature gradient so estimated may be used to capture the diffusion of heat across the sliding interface assuming that the sliding block acts as the moving heat source.

7.3.2: Effect of Lubricity Interfacial Temperature

To show the effect of lubrication on the temperature rise near the interface, different values of lubricity can be used. Here, the dry friction value is kept constant and the three specified values of lubricity indicate different lubricating media. The results are shown in Figure 80 with the different input parameters used shown in Table 17.

| Table 17: Modeling input parameters |
|----------------------|------------------|
| \( m \)              | 1.0 kg           |
| \( m^8 \)            | 0.005 kg         |
| \( g \)              | 9.81 m/s\(^2\)   |
| \( a_0 \)            | 2.0g m/s\(^2\)   |
| \( u_0 \)            | 0.00001 m/s      |
| \( s_0 \)            | 0.0001 m         |
| \( S \)              | 20 m             |
| \( \mu_d \)          | 0.25             |
| \( \gamma \)         | [0.0005, 0.003, 0.005] s/m |
| \( C_{P_u} \)        | 386 J/kg-K (copper) |
| \( T_0 \)            | 298 K            |

Figure 80 shows the effect of lubricity on the temperature rise of the block. From Figure 80, the effect of lubricity can be clearly observed. Good lubrication (represented by high lubricity \( \gamma \)) keeps the temperature low near the sliding interface. The highest value of lubricity gives the lowest temperature and temperature gradient near the interface. Therefore lubricity may be used to thermally characterize different lubricants and lubrication schemes.
Figure 80: Influence of Lubricity on the heat generation (A) variation of temperature gradient with time (B) variation of temperature with time
### 7.4: Mechanical Efficiency of Sliding Motion

It is well known that friction hinders sliding motion and it requires extra energy to maintain motion. Since friction is a non-conservative force it reduces a system’s mechanical energy and dissipates energy in most machines and causes wear and tear. This dissipation occurs as a reduction in transmitted power or creates heat generation between contacting elements. The amount of dissipation depends on many factors such as surface roughness, load, and speed. In most automobile engines more than 50% of the energy loss occurs due to friction and other resistive parameters.

So far it has been difficult to predict friction-based efficiency. In most engine cycles major improvements have focused on minimizing irreversibilities within the cycles while maintaining isentropic compression and expansion segments [194-198]. But isentropic processes are reversible, adiabatic, and frictionless [194, 195]. Yet, actual engine cycles are rarely reversible with frictional dissipation exacerbating irreversibilities. Therefore, understanding fundamentally how friction controls the irreversible expansion and compression processes will be pivotal in reducing frictional dissipation mechanisms of wear and power losses [10, 199] thereby optimizing thermo-mechanical efficiency.

In this study, we construct a novel deterministic friction-based mechanical efficiency model using the kinematic and dynamic results from our jerk dynamics analyses. Using jerk dynamics we establish a deterministic model which shows the reduction in mechanical efficiency in the presence of friction. To construct friction-based efficiency, we compared the baseline frictionless motion with dry and lubricated sliding motions of the block.
7.4.1: Frictionless Sliding Motion

In the case of frictionless sliding, there is no jerk. This zero jerk also results primarily using the power to energy transfer rate equation (PETRE) with friction force set to zero. Solving this zero jerk motion, we recapture Newton’s second law of motion with constant acceleration: \( \frac{d^3s}{dt^3} = 0, \quad \therefore \frac{d^2s}{dt^2} = c_i = a_0 \), which results in the well-known kinematic relations for velocity \( \frac{ds}{dt} = v = u_0 + a_0t \), and distance \( s = u_0t + \frac{1}{2}a_0t^2 / 2 + s_0 \). Furthermore, the zero jerk equation with reduced order gives the velocity-distance relationship:

\[
v^2 = u_0^2 + 2a_0(s - s_0); \quad \text{or} \quad v = \left[u_0^2 + 2a_0(s - s_0)\right]^{1/2} \quad (7.11)
\]

The force required to maintain the motion with constant acceleration \( a_0 \) can be derived using Newton’s second law as \( F_a = ma_0 \). Also, the work done by the applied force during the sliding motion when it travels through the distance \( s \) is given by:

\[
-W_a = F_as = ma_0s
\]

(7.12)

The power required to produce the work can be calculated by differentiating the work done with respect to time:

\[
-\dot{W}_a = F_av = ma_0(u_0 + a_0t) = ma_0\left[u_0^2 + 2a_0(s - s_0)\right]^{1/2}
\]

(7.13)

The applied force, velocity, and power obtained here serve as the baseline for quantifying friction-based mechanical efficiency with or without lubrication.

7.4.2: Frictional Sliding and Mechanical Efficiency

In the case of frictional sliding without lubrication the applied force using Newton’s second law of motion is:
\[ F_a = m \frac{d^2s}{dt^2} + \mu F_n \]  

(4.2)
giving the work done by the applied force during the sliding motion when it travels through distance \( s \) is given by \( -W_a = F_a s = ms(d^2s/dt^2) + \mu mgs \). Thus, the power or work transfer rate becomes:

\[ -\dot{W}_a = \mu mg \frac{ds}{dt} + ms \frac{d^3s}{dt^3} + m \frac{ds}{dt} \frac{d^2s}{dt^2} \]  

(4.3)

As done previously, the power to energy transfer rate equation (PETRE) produced the frictional jerk as:

\[ \frac{d^3s}{dt^3} = -\frac{\mu g}{s} \frac{ds}{dt} \]  

(7.14)

Substituting Eq. (4.5) into (4.3) the required power becomes:

\[ -\dot{W}_a = m \frac{ds}{dt} \frac{d^2s}{dt^2} \]  

(7.15)

**Mechanical Efficiency**

If we define mechanical efficiency \( \eta_m \) as the ratio of power required for sliding with friction to the power required for sliding without friction, then:

\[ \eta_m = \frac{\text{Actual power with friction}}{\text{Frictionless power}} = \frac{\dot{W}_{\text{of}}}{\dot{W}_a} \]  

(7.16)

Substituting Eqs. (7.13) and (7.15) into Eq. (7.16) the mechanical efficiency becomes:

\[ \eta_m = \frac{ds}{dt} \frac{d^2s}{dt^2} \frac{1}{a_0 \left[ u_0^2 + 2a_0 (s - s_0) \right]^{1/2}} \]  

(7.17)
We can choose solutions for acceleration and velocity either from elliptical integration (Eq. (4.7) and Eq. (4.9)) or APT (Eq. (5.33) and Eq. (5.32)). Table 18 details the input parameters used to obtain illustrations with Figure 81.

| Table 18: Modeling parameters for Figure 81 |
|-----------------|-----|
| $m$             | 1.0 kg |
| $g$             | 9.81 m/s$^2$ |
| $a_0$           | 2.0g m/s$^2$ |
| $u_0$           | 0.05 m/s |
| $s_0$           | 0.001 m |
| $S$             | 50 m |
| $\mu$           | [0.01, 0.05, 0.1, 0.2] |

The plots showing variation of mechanical efficiency with distance and also with time for changing COF are shown in Figure 81. The illustrative plots show the continuous decrease in mechanical efficiency with the increasing COF. As expected, without lubrication or with poor lubrication (indicated by high COF), the performance of any mechanical system will suffer.

![Figure 81: Effect of friction on mechanical efficiency](image)

In tribology, the values of COF measured during standard sliding tests using the tribometer may be much lower than the values used in the illustrative plots in Figure 81. In the parametric studies higher values of COF are chosen to clearly show the sharp drops in the mechanical efficiency of the sliding block system. This represents
the fact that friction can deteriorate the system performance by triggering other mechanisms for example, wear, and heat generation. In this case we considered constant friction coefficient and no wear for simplicity. When lubrication is applied it will reduce the metal to metal contact between the mating components and in our chosen example it is the block and the stationary surface. In chapter 3, a boundary lubricated sliding concept was introduced to use the friction-velocity coupling for powder lubricated interfaces allowing the velocity dependent COF to be used. The mechanical efficiency for this friction-velocity coupling example is discussed in the next section.

7.4.3: Lubricated Sliding and Mechanical Efficiency

In case of lubricated sliding the applied force is

\[ F_{aft} = \mu_d F_n - \gamma F_n \frac{ds}{dt} + m \frac{d^2s}{dt^2} \] (4.20)

The work done by the applied force during the sliding motion when it travels through distance \( s \) is given by:

\[ -W_{aft} = F_{a} s = \mu_d F_n s - \gamma F_n s \frac{ds}{dt} + ms \frac{d^2s}{dt^2} \]

Thus, the power required or work transfer rate becomes:

\[ -\dot{W}_{aft} = \mu_d mg \frac{ds}{dt} - \gamma mgs \frac{d^2s}{dt^2} - \gamma mg \left( \frac{ds}{dt} \right)^2 + m \frac{ds}{dt} \frac{d^2s}{dt^2} + ms \frac{d^3s}{dt^3} \] (4.21)

From the power to energy transfer rate equation (PETRE) for the lubricated sliding motion the nonzero inertial jerk balanced by friction and lubrication jerks is:

\[ \frac{d^3s}{dt^3} = \gamma g \frac{d^2s}{dt^2} + \gamma g \frac{ds}{s} \frac{d^2s}{dt} - \mu_d g \frac{ds}{s} \] (4.23)

Simplifying Eq. (4.21) using Eq. (4.23):
\[-\dot{W}_{afl} = m \frac{ds}{dt} \frac{d^2s}{dt^2}\] (7.18)

The velocity from Eq. (5.50) and acceleration from Eq. (5.52) are substituted into Eq. (7.18) to obtain the required power for the lubricated sliding interface.

**Mechanical Efficiency**

Defining mechanical efficiency as the ratio of power required for lubricated sliding to the power required for sliding without friction, then:

\[\eta_m = \frac{\text{Actual power with friction and lubrication}}{\text{Frictionless power}} = \frac{\dot{W}_{afl}}{\dot{W}_a}\] (7.19)

Substituting Eq. (7.13) and (7.18) into Eq. (7.19) the mechanical efficiency becomes:

\[\eta_m = \frac{ds}{dt} \frac{d^2s}{dt^2} \frac{1}{a_0\left[u_0^2 + 2a_0(s - s_0)\right]^{1/2}}\] (7.20)

Equation (7.20) shows that the mechanical efficiency of sliding is the ratio of velocity × acceleration for lubricated sliding with friction velocity coupling to the velocity × acceleration for frictionless sliding. The illustrative mechanical efficiency plots from Eq. (7.20) are shown in Figure 82(A and B). It is clear that the use of a better powder lubrication for the sliding interface can at least maintain the mechanical efficiency of the system. There is a sudden drop in the mechanical efficiency due to jerk and other multiple wear effects which causes this reduction. Once there is sufficient lubricant there is no further reduction in the mechanical efficiency.

In the case of dry sliding the mechanical efficiency drops throughout the motion but in the case of lubricated sliding, the drop is not continuous. In other words, lubrication can aid the motion and reduces the effect of friction. The input parameters used for the illustrative plots in Figure 82 are shown in the Table 19.
Table 19: Modeling input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1.0 kg</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>$a_0$</td>
<td>2.0g m/s$^2$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>0.05 m/s</td>
</tr>
<tr>
<td>$s_0$</td>
<td>0.001 m</td>
</tr>
<tr>
<td>$S$</td>
<td>50 m</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>[0.0001, 0.0005, 0.001] s/m</td>
</tr>
</tbody>
</table>

Figure 82: Effect of lubricity on mechanical efficiency, (A) variation of mechanical efficiency with distance, and (B) variation of mechanical efficiency with time
Chapter 8: Conclusions and Suggestions for Future Research

8.1: Conclusions

To study the nonlinear effects of friction at a sliding interface, we unified Newton’s second law of motion with the universal first law of classical thermodynamics. The interesting result of this mechanistic unification is that it opens up a fundamental modeling tool that is seemingly absent in classical mechanics and tribology: jerk. Jerk, the rate of change of acceleration, turns out to be a resisted motion phenomenon that captures the kinematic aspects of motion in the presence of a varying inertial force.

Newton’s second law of motion offers a means for defining the interfacial applied force fundamentally. Apart from classical thermodynamics, key laws from tribology for friction, wear, and lubrication were invoked to elicit the required applied force under specific sliding conditions. Thus, the flexibility of Newton’s second law coupled with the universality of the first law of thermodynamics ensures that different work and energy transfer modes may be interrogated using jerk dynamics.

By considering the sliding block as a moving heat source, its nonlinear sliding is captured that clarifies temperature and temperature gradient evolve nonlinearly during sliding. Besides, we show how friction, wear, and lubrication occur as intertwined nonlinear phenomena. The following specific conclusions are drawn:

1. A basic fundamental model is constructed for sliding under tribological (friction, lubrication, and wear) conditions in which a key event such as running-in is not ignored but included from the onset of sliding.

2. Even though the jerk phenomenon is rarely used in classical mechanics by establishing its origin at least in the context of friction, lubrication and wear,
we show the unification of classical mechanics and thermodynamics as a way in which jerk may arise fundamentally. Thus, for other known cases where jerk is such to occur naturally for the Earth [126, 127], Sun [128-130], our universe [131] or artificially [109, 110], a basic mechanistic unification may elucidate how jerk arises in those instances. This mechanistic unification treatment may be extended to complex dynamical system in general, modeling resisted relative motion e.g. bearings, gears, spring mass system etc.

3. Equations such as the nonlinear third order ordinary differential equations (ODEs) generated in the jerk balance equations have whose place in mathematical modeling have hitherto been treated only for their mathematical values [145] without a unified solution technique. Primarily in this study, the analytical tool constructed in-house, the algebraic partitioning technique (APT) shows a way of solving these set of ODEs with promising results. They at least capture the essential physical attributes of friction, lubrication, and wear while exhibiting their nonlinear nature. Furthermore, the APT may serve as a fundamental solution technique for tackling other differential equations that govern natural and artificial phenomena encountered by engineers and scientists.

4. Three case studies were considered. The first case represents idealized dry sliding without any wear. It was shown that friction indeed affects kinematic properties nonlinearly. Distance, velocity, acceleration, and jerk all show nonlinear behavior. Interacting forces are also shown to be impacted by friction.

In the second case, idealized sliding with boundary lubrication was considered, representing good lubrication conditions with minimal wear. The
poor lubrication condition recaptures the results from the dry lubricated case. In addition the Strubeck effect was recaptured which shows the opposing nature of friction and velocity: good lubrication reduces the friction effects but high friction overshadows good lubrication.

In the third case friction, lubrication, and wear were considered simultaneously. This scenario captures real life conditions where the presence of lubrication may not completely overcome the effects of friction and wear. Archard’s adhesive wear law was used to elicit the effect of friction and lubrication on wear. Also, mechanical properties like density, hardness, density to hardness ratio were used to show their effect on wear theoretically.

5. Using the jerk dynamics results, frictional impacts on thermal and mechanical effects have been quantified fundamentally by assessing the interfacial temperature and mechanical efficiency. First, the interfacial or near surface temperature was obtained using heat to internal energy rate balances. The effect of thermal properties of the material for example, specific heat capacity was shown. Secondly, a friction based mechanical efficiency was constructed. Moreover, it was realized that zero jerk occurs from unconstrained motion tribologically when there is no friction, lubrication or wear. This captured constant acceleration, Galilean-Newtonian mechanics. It also serves as the baseline against which frictional sliding with and without lubrication can be compared to obtain the equation for mechanical efficiency of the sliding. This likely will facilitate quantification of bearings and gear lifetime and strategize equipment maintenance schedules, while developing innovative lubrication, bearing and energy efficient technologies.
6. According to Newton’s first law an object’s state of rest or uniform motion will be altered by an impressed force. In jerk dynamics, using the basic definition of an applied force from Newton’s second law and classical thermodynamics, the needed varying force that moves an object from rest against friction can be obtained. For an object to overcome the initial friction a huge initial force is required to initiate the sliding motion which is provided by an initial acceleration. This initial acceleration is pivotal but may be overlooked without considering jerk balance for a sliding block. It seems that jerk analysis reconnects Newton’s first and second laws. If the initial acceleration remains constant, a condition of no friction, no lubrication, and no wear results. Otherwise, the acceleration will change thereby causing the the onset of nonzero jerk.

7. In this work, some key dynamical aspects of tribology are clarified. Until now tribologists rarely study dynamics of frictional contact using basic physics. Data taken at steady state usually ignores running-in where microstructural and topographical rearrangements occur. By constructing a fundamental model that captures motion from the onset of sliding, this may be a game changer for studying complex tribological phenomena. It may now be possible to construct more sophisticated friction, lubrication and wear models that consider all the key parameters especially for running-in or the basic physics of tribology.

8. More tribology testing is needed for capturing lubricity which is the slope of the friction-velocity curve. To use lubricity as lubricant characterization tool requires constructing tribometers that do not primarily do a velocity feedback.
Wear coefficients based on friction may be determined through such experiments.

8.2: Suggestions for Future Research

The following suggestions are made for using the fundamental framework for interfacial sliding considered in this study:

1. So far the analyses have been restricted to nominally flat surfaces. Yet, because interfacial asperities affect the nonlinear dynamics of lubricated sliding. Future improvement to the jerk dynamic modeling scheme should incorporate surface roughness.

2. The mechanical deformation associated with sliding contact substrates must be considered.

3. The diffusion of heat across the sliding interfaces must quantified using Fourier and non-Fourier heat diffusion models with input from the jerk dynamics results.

4. The algebraic partitioning technique (APT) may be refined to get exact solutions for a variety of initial conditions.
References


Appendix A: APT Solution for Nonlinear Frictional Jerk

Balance Equation

In Chapter 4, the equation of jerk for sliding motion with friction was developed and it was approximately solved using an elliptic integral technique. The APT solution for the same equation is obtained as follows:

\[
\frac{d^3s}{dt^3} = -\frac{\mu g s}{s} \frac{ds}{dt} \Rightarrow \frac{d^3s}{dt^3} + \frac{\mu g s}{s} \frac{ds}{dt} = 0 \quad (4.5) \text{ or (A.1)}
\]

Equation (A.1) can be solved as an initial value problem. The applicable three initial conditions are: \(s(t = 0) = s_0\), \(ds/dt(t = 0) = u_0\), and \(d^2s/dt^2(t = 0) = a_0\).

Applying the algebraic partitioning:

\[
\frac{d^3s}{dt^3} = \lambda_1 \quad \text{(A.2)} \quad \frac{\mu g s}{s} \frac{ds}{dt} = \lambda_2 \quad \text{(A.3)}
\]

\[\lambda_1 + \lambda_2 = 0 \quad \text{(A.4)}\]

From partitioned Eq. (A.2):

\[
\frac{d^3s}{dt^3} = \lambda_1;
\]

Integrate once with respect to \(t\): \(\frac{d^2s}{dt^2} = \lambda_1 t + c_1\);

Apply initial condition: \(d^2s/dt^2(t = 0) = a_0\); \(c_1 = a_0\); \(\Rightarrow d^2s/dt^2 = \lambda_1 t + a_0\);

Integrate again: \(\frac{ds}{dt} = \frac{\lambda_1 t^2}{2} + a_0 t + c_2\);

Apply initial condition: \(ds/dt(t = 0) = u_0\); \(c_2 = u_0\); \(\Rightarrow \frac{ds}{dt} = \frac{\lambda_1 t^2}{2} + a_0 t + u_0\);

Integrate again: \(s = \frac{\lambda_1 t^3}{6} + \frac{a_0 t^2}{2} + u_0 t + c_3\);

Apply initial condition: \(s(t = 0) = s_0\); \(c_3 = s_0\); \(\Rightarrow s = \frac{\lambda_1 t^3}{6} + \frac{a_0 t^2}{2} + u_0 t + s_0\);

Rearrange terms: \(\lambda_1 = \frac{-3a_0 t^2 - 6u_0 t + 6(s - s_0)}{t^3}\);

From Partitioned Eq. (A.3):
\[
\frac{\mu_d g}{s} \frac{ds}{dt} = \lambda_2;
\]

Separate variables: \(\frac{ds}{s} = \frac{\lambda_2}{\mu_d g} dt;\)

Integrate on both sides: \(\ln s = \frac{\lambda_2}{\mu_d g} t + c_4;\)

Apply initial condition: \(s(t = 0) = s_0; c_4 = \ln s_0; \Rightarrow \frac{\lambda_2}{\mu_d g} t = \ln \left[ \frac{s}{s_0} \right] \)

Rearrange terms: \(\lambda_2 = \frac{\mu_d g}{t} \ln \left[ \frac{s}{s_0} \right] \)

Combining Eqs. (A.5) and (A.6) to get APC in Eq. (A.4):

\[
\frac{-3a_u t^2 - 6u_0 t + 6(s - s_0)}{t^3} + \frac{\mu_d g}{t} \ln \left[ \frac{s}{s_0} \right] = 0
\]

\[
3a_u t^2 + 6u_0 t - 6(s - s_0) - \mu_d g t^2 \ln \left[ \frac{s}{s_0} \right] = 0
\]

Grouping terms:

\[
(3a_u - \mu_d g \ln \left[ \frac{s}{s_0} \right]) t^2 + 6u_0 t - 6(s - s_0) = 0
\]

Equation (A.8) is the implicit quadratic equation for time with distance as a coefficient functions.

\[
i^2 A(s) + tB + C(s) = 0
\]

\[
A(s) = [3a_u - \mu_d g \ln(s / s_0)]; \quad B = 6u_0; \quad C(s) = -6(s - s_0);
\]

Using the quadratic formula, time-distance relationship was obtained:

\[
t = \frac{-6u_0 \pm \left\{ 36u_0^2 + 24(s - s_0)[3a_u - \mu_d g \ln(s / s_0)] \right\}^{1/2}}{2[3a_u - \mu_d g \ln(s / s_0)]}
\]

Only one of the two roots satisfies the initial condition: \(s(t = 0) = s_0.\) This gives the final solution for time as:
Subsequent implicit differentiations of Eq. (A.10) lead to velocity and acceleration at the lubricated interface. Differentiating Eq. (A.10) with respect to \( t \) yields the velocity in Eq. (A.11):

\[
\frac{d}{ds} = \frac{1}{N(s)^2} \frac{N'(s)}{N(s)M'(s) - M(s)N'(s)}
\]

\[
\Rightarrow v = \frac{ds}{dt} = \frac{N(s)^2}{N(s)M'(s) - M(s)N'(s)}
\]

\[
M'(s) = \frac{3(3a_0 - \mu_d g \ln(s/s_0) - 3(s-s_0)[\mu_d g / s]}{\left(9u_0^2 + 6(s-s_0)[3a_0 - \mu_d g \ln(s/s_0)]\right)^{1/2}}
\]

\[
N'(s) = -\mu_d g / s
\]

Substituting the differential terms the expression for velocity can be obtained as:

\[
v = \frac{ds}{dt} = \left(\frac{[3a_0 - \mu_d g \ln(s/s_0)]}{[3a_0 - \mu_d g \ln(s/s_0)]} \left(\frac{3[3a_0 - \mu_d g \ln(s/s_0)]}{-3(s-s_0)[\mu_d g / s]} \left(\frac{9u_0^2 + 6(s-s_0)[3a_0 - \mu_d g \ln(s/s_0)]}{\left(9u_0^2 + 6(s-s_0)[3a_0 - \mu_d g \ln(s/s_0)]\right)^{1/2}}\right)\right)\right)
\]

Equation (A.12) satisfies the initial condition for velocity \( v(t=0) = u_0 \). Similarly, second derivative of Eq. (A.10) with respect to time gives the acceleration in Eq. (A.13)
\[ 0 = \frac{NM' - MN'}{N^2} \frac{d^2 s}{dt^2} + \frac{d}{dt} \left( \frac{NM' - MN'}{N^2} \right) \frac{ds}{dt} \]

\[ \Rightarrow \frac{1}{v} \frac{d^2 s}{dt^2} = -\frac{d}{dt} \left( \frac{NM' - MN'}{N^2} \right) \]

\[ \Rightarrow a = \frac{d^2 s}{dt^2} = -v^3 \left( \frac{N (NM'' - MN'') - 2N (NM' - MN')} {N^3} \right) \]  

(A.13)

\[ M^*(s) = \frac{F'G - G'F}{G^2} \]

\[ F'(s) = -6 \left( \mu_d g / s \right) + 3(s - s_o) \left( \mu_d g / s^2 \right) \]

\[ G'(s) = M'; \quad N^*(s) = \mu_d g / s^2 \]

Substituting the differential terms into Eq. (A.13) gives the expression for acceleration as:

\[ a = \frac{1}{v^3} \left[ \left( -6 \left( \mu_d g / s \right) \left[ 9u_o^2 - 3(s - s_o) \left( \mu_d g \ln(s / s_o) \right)^2 \right]^{1/2} \right) \right. \]

\[ \left. - \left[ 3 \left( \mu_d g \ln(s / s_o) \right) - 3(s - s_o) \left( \mu_d g / s \right)^3 \right] \left[ 9u_o^2 + 6(s - s_o) \left( \mu_d g \ln(s / s_o) \right)^3 \right]^{1/2} \right] \]

\[ \left. \left[ 9u_o^2 + 6(s - s_o) \left( \mu_d g \ln(s / s_o) \right) \right] \right] \]

Equation (A.13) satisfies the initial condition for acceleration, \( a(t = 0) = a_0 \).
Appendix B: APT Solution for the Nonlinear Lubrication

Jerk Balance Equation

In chapter 4, the power to energy transfer rate equation produced a jerk balance equation for lubricated sliding using the friction-velocity coupling from powder lubricated sliding. This JBE for lubricated sliding without wear was:

\[
\gamma g \frac{d^3 s}{dt^3} + \gamma g \left( \frac{ds}{dt} \right)^2 - \frac{\mu g}{s} \frac{ds}{dt} \frac{d^3 s}{dt^3} = 0
\]  \hspace{1cm} (4.23)

Or

\[
- \frac{d^3 s}{dt^3} = \lambda_1 \hspace{1cm} (B.2)
\]

\[
\gamma g \left( \frac{ds}{dt} \right)^2 = \lambda_2 \hspace{1cm} (B.3)
\]

\[
\frac{\mu g}{s} \frac{ds}{dt} = \lambda_3 \hspace{1cm} (B.4)
\]

\[
\sum_{i=1}^{4} \lambda_i = 0 \hspace{1cm} (B.5)
\]

We treat Eq. (B.1) as an initial value problem (IVP) with three initial conditions are: \( s(t=0) = s_0; \ s/dt \ (t=0) = u_0 \) and \( d^2 s / dt^2 \ (t=0) = a_0 \). Following the APT procedure, from partitioned Eq. (B.2):

\[
\frac{d^3 s}{dt^3} = -\lambda_1; \text{Integrate once with respect to } t : \frac{d^2 s}{dt^2} = -\lambda_1 t + c_1;
\]

Apply initial condition : \( \frac{d^2 s}{dt^2} (t = 0) = a_0; c_1 = a_0; \Rightarrow \frac{d^2 s}{dt^2} = -\lambda_1 t + a_0; \)

Integrate again : \( \frac{ds}{dt} = -\frac{\lambda_1 t^2}{2} + a_0 t + c_2; \)

Apply initial condition : \( \frac{ds}{dt} (t = 0) = u_0; c_2 = u_0; \Rightarrow \frac{ds}{dt} = -\frac{\lambda_1 t^2}{2} + a_0 t + u_0; \)
Integrate again: $s = -\frac{\lambda_1 t^3}{6} + \frac{a_0 t^2}{2} + u_0 t + c_3$;

Apply initial condition: $s(t = 0) = s_0; c_3 = s_0; s = -\frac{\lambda_1 t^3}{6} + \frac{a_0 t^2}{2} + u_0 t + s_0$; (B.7)

Rearrange terms: $\lambda_1 = \frac{3a_0 t^2 + 6u_0 t - 6(s - s_0)}{t^3}$;

From partitioned Eq. (B.3):

$$\frac{d^2 s}{dt^2} = \frac{\lambda_2}{\gamma g};$$

Integrate once: $\frac{ds}{dt} = \frac{\lambda_2}{\gamma g} t + c_4$;

Apply initial condition: $\frac{ds}{dt}(t = 0) = u_0; c_4 = u_0; \frac{ds}{dt} = \frac{\lambda_2}{\gamma g} t + u_0$;

Integrate again: $s = \frac{\lambda_2}{2\gamma g} t^2 + u_0 t + c_5$; (B.8)

Apply initial condition: $s(t = 0) = s_0; c_5 = s_0; s = \frac{\lambda_2}{2\gamma g} t^2 + u_0 t + s_0$;

Rearrange terms: $\lambda_2 = \frac{2\gamma g (s - u_0 t - s_0)}{t^2}$;

From partitioned Eq. (B.4):

$$\gamma g \left( \frac{ds}{dt} \right)^2 = \lambda_3; \frac{ds}{dt} = \frac{\lambda_3^{1/2}}{\sqrt{\gamma g}};$$

separate variables: $s^{1/2} ds = \frac{\lambda_3^{1/2}}{\sqrt{\gamma g}} dt$;

Integrate on both sides: $2s^{1/2} = \frac{\lambda_3^{1/2} t}{\sqrt{\gamma g}} + c_6$; (B.9)

Apply initial condition: $s(t = 0) = s_0; c_6 = 2s_0^{1/2}; 2s^{1/2} = \frac{\lambda_3^{1/2} t}{\sqrt{\gamma g}} + 2s_0^{1/2}$;

Rearrange terms: $\lambda_3 = \frac{4\gamma g (s^{1/2} - s_0^{1/2})^2}{t^2}$;

From partitioned Eq. (B.5):
- $\frac{\mu_d g}{s} \frac{ds}{dt} = \frac{\lambda_4}{s}$;

separate variables: $\frac{ds}{s} = -\frac{\lambda_4}{\mu_d g} dt$;

Integrate on both sides: $\ln s = -\frac{\lambda_4}{\mu_d g} t + c_7$;

Apply initial condition: $s(t = 0) = s_0$; $c_7 = \ln s_0$

\[
\ln \left[ \frac{s_0}{s} \right] = -\frac{\lambda_4}{\mu_d g} t; \quad \frac{\lambda_4}{\mu_d g} t = \ln \left[ \frac{s_0}{s} \right];
\]

Rearrange terms: $\lambda_4 = \frac{\mu_d g}{t} \ln \left[ \frac{s_0}{s} \right]$

Invoking the algebraic partitioning condition (APC) of Eq. (B.6):

\[
3a_d t^2 + 6a_d t - 6(s - s_0) + 2\gamma g(s - u_d t - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2 + \frac{\mu_d g}{t} \ln \left[ \frac{s_0}{s} \right] = 0
\] (B.11)

Equation (B.11) gives an implicit solution which can be rearranged by multiplying through by $t^2$:

\[
3a_d t^2 + 6a_d t - 6(s - s_0) + 2\gamma g(t(s - u_d t - s_0) + 4\gamma g(t(s^{1/2} - s_0^{1/2}))^2 + \mu_d g t^2 \ln \left[ \frac{s_0}{s} \right] = 0
\] (B.12)

Clearly, the implicit solution in Eq. (B.12) satisfies the I.C. at $t = 0$, where $s = s_0$.

Although Eq. (B.12) is implicit in distance-time it seems explicit in time. This outlook provides an outlet to determine at least an approximate (or probably exact) though nonlinear relation between distance and time. From Eq. (B.12) one can factorize in terms of time:

\[
t^2[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)]
+ t[6u_0 + 2\gamma g(s - s_0) + 4\gamma g(s^{1/2} - s_0^{1/2})^2] - 6(s - s_0) = 0
\] (B.13)

Equation (5.45) is a quadratic equation in time with distance as the coefficient functions as shown in Eq. (B.14).
\[ t^2A(s) + tB(s) + C(s) = 0 \]
\[ A(s) = [3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_o / s)]; \]
\[ B(s) = [6u_0 + 2\gamma g(s - s_o) + 4\gamma g(s^{1/2} - s_o^{1/2})^2]; \]
\[ C(s) = -6(s - s_o); \]

Using the quadratic formula, we obtain time-distance relationship as:

\[
t = \frac{-[6u_0 + 2\gamma g(s - s_o) + 4\gamma g(s^{1/2} - s_o^{1/2})^2] \pm \sqrt{[6u_0 + 2\gamma g(s - s_o) + 4\gamma g(s^{1/2} - s_o^{1/2})^2]^2}}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_o / s)]}
\]

To check whether our initial condition is satisfied explicitly, we observe that setting \( s = s_0 \) Eqn. (B.15) becomes:

\[
t = \frac{-[6u_0]}{2[3a_0 - 2\gamma gu_0]} \pm \frac{\sqrt{[6u_0]^2}}{2[3a_0 - 2\gamma gu_0]} \]

In fact, Eq. (B.16) yields zero initial time only when the positive sign is maintained in the last term on the RHS of Eq. (B.16). Thus, the choice of time that satisfies our initial condition is:

\[
t = \frac{-[6u_0 + 2\gamma g(s - s_o) + 4\gamma g(s^{1/2} - s_o^{1/2})^2] \pm \sqrt{[6u_0 + 2\gamma g(s - s_o) + 4\gamma g(s^{1/2} - s_o^{1/2})^2]^2}}{2[3a_0 - 2\gamma gu_0 + \mu_d g \ln(s_o / s)]}
\]

Equation (B.17) is the desired solution. Before doing any further differentiations simplify Eq. (B.17) further. This facilitates obtaining the velocity and acceleration.

The procedure is as follows: The interfacial velocity from implicit differentiation is given in Eq. (B.18):
\[ t = Q(s) + R(s) \Rightarrow 1 = \left[ Q'(s) + R'(s) \right] \dot{s} = \left[ Q'(s) + R'(s) \right] v \]
\[ v = \frac{1}{\left[ Q'(s) + R'(s) \right]} \]  
(B.18)

Subsequent differentiations of Eq. (B.18) gives the acceleration in Eq.(B.19)

\[ t = Q(s) + R(s) \Rightarrow 1 = \left[ Q'(s) + R'(s) \right] \dot{s}; \]
\[ \Rightarrow 0 = \left[ Q'(s) + R'(s) \right] \ddot{s} + \dot{s}^2 \left[ Q''(s) + R''(s) \right] \]
\[ \ddot{s} = \frac{d^2 s}{dt^2} = -\frac{\dot{s}^2 \left[ Q''(s) + R''(s) \right]}{\left[ Q'(s) + R'(s) \right]} = -\dot{s}^3 \left[ Q''(s) + R''(s) \right] \]  
(B.19)

Both velocity and acceleration can be obtained by differentiating the terms:

\[ A(s) = [3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)]; \]
\[ B(s) = [6u_0 + 2\gamma g (s - s_0) + 4\gamma g (s^{1/2} - s_0^{1/2})^2]; \]
\[ C(s) = -6(s - s_0); \]
\[ Q(s) = \frac{-B}{2A} = \frac{F}{G} \quad \text{and} \quad R(s) = \frac{\sqrt{B^2 - 4AC}}{2A} = \frac{H}{G} \]

Assign variables Q(s), R(s), etc. to simplify the process of obtaining the velocity and acceleration.

\[ Q'(s) = \frac{G F' - F G'}{G^2} \quad \text{and} \quad R'(s) = \frac{G H' - H G'}{G^2} \]
\[ Q''(s) = -\frac{G \left[ G F' - G F \right] - 2G' [G F' - F G]}{G^3} \]
\[ R''(s) = -\frac{G \left[ G H' - G H \right] - 2G' [G H' - H G]}{G^3} \]

\[ A' = \mu_d g / s \quad \text{and} \quad A'' = -\mu_d g / s^2, \quad C' = -6; \quad C'' = 0 \]
\[ B' = 2\gamma g + 4\gamma g (s^{1/2} - s_0^{1/2})s^{-1/2}; \quad B'' = -2\gamma g (s^{1/2} - s_0^{1/2})s^{-3/2} + 2\gamma g / s \]
\[ G' = 2A'; \quad G'' = 2A''; \quad F' = -B'; \quad F'' = -B''; \quad \]
\[ H' = \left( \frac{BB' - 2A'C - 2AC'}{H} \right) \]
\[ H'' = \frac{H \left( BB'' + \left[ B' \right]^2 - 4A'C' - 2A'C - 2AC' \right) - H' \left( BB' - 2A'C - 2AC' \right)}{H^2} \]

Inserting the differential terms obtained in equation
Equation (B.21) satisfies the initial condition: \( s = s_0; v = u_0 \).

Subsequent differentiations of Eq. (B.21) gives the acceleration in Eq. (B.19)

\[
\ddot{s} = \frac{d^2 s}{dt^2} = -s^3 \left[ \frac{Q^r(s) + R^r(s)}{Q'(s) + R'(s)} \right] = -v^3 \left[ Q^r(s) + R^r(s) \right] \tag{5.52} \text{ or (B.19)}
\]

The various terms in Eq. (5.52) are as follows:

\[
Q^r(s) = \left[ 4[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)]^2 \left( -2\gamma g (s^{1/2} - s_0^{1/2})s^{-3/2} + 2\gamma g / s \right) \right]
-2(2\mu_d g / s^2) \left[ 2[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s) \left[ 6u_0 + 2\gamma g (s - s_0) + 4\gamma g (s^{1/2} - s_0^{1/2})^2 \right] \right]
-2(2\mu_d g / s) \left[ -2 \left[ 3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s) \right] \left( 2\gamma g + 4\gamma g (s^{1/2} - s_0^{1/2})s^{-1/2} \right) \right]
+ 8[3a_0 - 2\gamma g u_0 + \mu_d g \ln(s_0 / s)]^3
\]

And \( R^r(s) \):
where the velocity shown in Eq. (B.18). Also, Equation (B.19) satisfies the initial condition for acceleration. \( s = s_0; a = a_0 \).
Appendix C: APT Solution for the Mass Loss from Adhesive Wear during Unlubricated Sliding

The Jerk balance equation with adhesive wear was found to be:

\[
\frac{d^3s}{dt^3} - \frac{\beta s d^2s}{m_0 dt} + \frac{\mu F_n ds}{m_0 s dt} - \frac{3\beta ds d^2s}{2m_0 dt^2} - \frac{\beta}{2m_0} \left( \frac{ds}{dt} \right)^3 + \frac{v_c \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 + \frac{v_c \beta}{m_0 s} \frac{d^2s}{dt^2} - \frac{v_c^2 \beta}{2m_0 s} \frac{ds}{dt} = 0
\]

(6.24)

The following differential transformation was used to simplify Eq. (6.24):

\[
\frac{d^3s^2}{dt^3} = \frac{d}{dt} \left( \frac{d^2s^2}{dt^2} \right) = \frac{d}{dt} \left( \frac{d}{dt} \left( 2s \frac{ds}{dt} \right) \right) = \frac{d}{dt} \left( 2s \frac{d^2s}{dt^2} + 2 \left( \frac{ds}{dt} \right)^2 \right) = 2s \frac{d^3s}{dt^3} + 2 \frac{ds}{dt} \frac{d^2s}{dt^2} + 4 \frac{d^2s}{dt^2} \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

\[
\Rightarrow \frac{d^3s^2}{dt^3} = 2s \frac{d^3s}{dt^3} + 6 \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

\[
\Rightarrow s \frac{d^3s}{dt^3} = \frac{1}{2} \frac{d^3s}{dt^3} - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} = \frac{1}{2} \frac{d^3s}{dt^3} - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} \quad (C.1)
\]

Applying Eq. (C.1), the resulting jerk balance equation for unlubricated sliding with adhesive wear reduces to:

\[
\frac{-d^3s}{dt^3} + \frac{\beta}{2m_0} \frac{d^3s^2}{dt^3} + \frac{\left( v_c^2 \beta - 2\mu F_n \right)}{2m_0} \frac{1ds}{dt} + \frac{\beta}{2m_0} \left( \frac{ds}{dt} \right)^3 \quad (C.2)
\]

or

\[
-\frac{v_c \beta}{m_0 s} \left( \frac{ds}{dt} \right)^2 - \frac{v_c \beta}{m_0 s} \frac{d^2s}{dt^2} = 0 \quad (6.25)
\]

Consider Eq. (C.2) as an initial value problem (IVP) with three initial conditions:

\[
s(t = 0) = s_0, \quad \frac{ds}{dt} (t = 0) = u_0, \quad \text{and} \quad \frac{d^2s}{dt^2} (t = 0) = a_0.
\]

Applying the algebraic partitioning technique, we partition Eq. (C.2) as:

\[
-\frac{d^3s}{dt^3} = \lambda_1 \quad (C.3) \quad \frac{\beta}{2m_0} \frac{d^3s^2}{dt^3} = \lambda_2 \quad (C.4)
\]
\[
\frac{\left(\nu^2 - 2\mu F_n\right)}{2m_0} \frac{1}{s} \frac{ds}{dt} = \lambda_3 \quad \text{(C.5)}
\]

\[
\frac{\beta}{2m_0s} \left( \frac{ds}{dt} \right)^3 = \lambda_4 \quad \text{(C.6)}
\]

\[
- \frac{v_c \beta}{m_0s} \left( \frac{ds}{dt} \right)^2 = \lambda_5 \quad \text{(C.7)}
\]

\[
- \frac{v_c \beta}{m_0} \frac{d^2s}{dt^2} = \lambda_6 \quad \text{(C.8)}
\]

\[
\sum_{i=1}^{6} \lambda_i = 0 \quad \text{(C.9)}
\]

Following the APT procedure, from partitioned Eq. (C.3):

\[
\frac{d^3s}{dt^3} = -\lambda_i;
\]

Integrate once with respect to t:

\[
\frac{d^2s}{dt^2} = -\lambda_i t + c_i;
\]

Apply initial condition: \(\frac{d^2s}{dt^2}(t = 0) = a_{i0}; \Rightarrow c_i = a_{i0};\)

\[
\frac{d^2s}{dt^2} = -\lambda_i t + a_{i0};
\]

Integrate again:

\[
\frac{ds}{dt} = -\frac{\lambda_i t^2}{2} + a_{i0}t + c_2;
\]

Apply initial condition: \(\frac{ds}{dt}(t = 0) = u_{i0}; \Rightarrow c_2 = u_{i0};\)

\[
\frac{ds}{dt} = -\frac{\lambda_i t^2}{2} + a_{i0}t + u_{i0};
\]

Integrate again:

\[
s = -\frac{\lambda_i t^3}{6} + \frac{a_{i0}t^2}{2} + u_{i0}t + c_3;
\]

Apply initial condition: \(s(t = 0) = s_{i0}; c_3 = s_{i0};\)

\[
s = -\frac{\lambda_i t^3}{6} + \frac{a_{i0}t^2}{2} + u_{i0}t + s_{i0};
\]

Rearrange terms:

\[
\lambda_i = \frac{3a_{i0}t^2 + 6u_{i0}t - 6(s - s_{i0})}{t^3} \quad \text{(C.10)}
\]
From Eq. (6.27) or Eq. (C.4):

$$\frac{\beta}{2m_0} \frac{d^3s^2}{dt^3} = \lambda_2 \Rightarrow \frac{d^3s^2}{dt^3} = \frac{2m_0}{\beta} \lambda_2;$$

Integrate once with respect to \( t \):

$$\frac{d^2s^2}{dt^2} = \lambda_2 \frac{2m_0}{\beta} t + d_1;$$

$$\frac{d^2s^2}{dt^2} = \frac{d}{dt} \left( 2s \frac{ds}{dt} \right) = 2s \frac{d^2s^2}{dt^2} + 2 \left( \frac{ds}{dt} \right)^2;$$

Apply initial conditions: \( d_1 = \left( 2a_0s_0 + 2u_0^2 \right); \)

$$\frac{d^2s^2}{dt^2} = \lambda_2 \frac{2m_0}{\beta} t + \left( 2a_0s_0 + 2u_0^2 \right);$$

(6.34) Integrate again:

$$\frac{ds^2}{dt} = \lambda_2 \frac{m_0}{\beta} t^3 + \left( 2a_0s_0 + 2u_0^2 \right) t + d_2;$$

or

Apply initial conditions: \( d_2 = 2u_0 s_0; \)

$$\frac{ds^2}{dt} = \lambda_2 \frac{m_0}{\beta} t^3 + \left( 2a_0s_0 + 2u_0^2 \right) t + 2u_0 s_0;$$

(C.11) Integrate again:

$$s^2 = \lambda_2 \frac{m_0}{3\beta} t^3 + \left( a_0s_0 + u_0^2 \right) t^2 + 2u_0 s_0 t + d_3;$$

Apply initial condition: \( d_3 = s_0^2; \)

$$s^2 = \lambda_2 \frac{m_0}{3\beta} t^3 + \left( a_0s_0 + u_0^2 \right) t^2 + 2u_0 s_0 t + s_0^2;$$

Rearrange terms:

$$\lambda_2 = \frac{3\beta \left( s^2 - s_0^2 \right) - \left( a_0s_0 + u_0^2 \right) t^2 + 2u_0 s_0 t}{m_0 t^3}$$

From Eq. (6.28) or Eq. (C.5):

$$\left( v_e^2 \beta - 2\mu F_n \right) \frac{1}{2m_0} \frac{ds}{dt} = \lambda_3; \Rightarrow \frac{ds}{s} = 2\lambda_3 \frac{m_0}{v_e^2 \beta - 2\mu F_n} \; dt;$$

Integrate both sides:

$$\ln s = \frac{2\lambda_3 m_0}{v_e^2 \beta - 2\mu F_n} t + c_3;$$

Apply initial condition: \( c_3 = \ln s_0 \)

$$\ln \left[ \frac{s}{s_0} \right] = \frac{2\lambda_3 m_0}{v_e^2 \beta - 2\mu F_n} t;$$
Rearrange terms:

\[ \lambda_3 = \frac{(v^2 \beta - 2\mu F)}{2m_0 t} \ln \left( \frac{s}{s_0} \right) \]  

(6.35)

From Eq. (6.29) or Eq. (C.6):

\[ \frac{\beta}{2m_0 s} \left( \frac{ds}{dt} \right)^3 = \lambda_4 \Rightarrow s^{-1/3} ds = \sqrt[3]{\frac{2\lambda_4 m_0}{\beta}} dt; \]

Integrate on both sides:

\[ \frac{3s^{2/3}}{2} = \sqrt[3]{\frac{2\lambda_4 m_0}{\beta}} \cdot t + c_7; \]

(6.36)

Apply initial condition: \( c_7 = \frac{3s_0^{2/3}}{2} \); \( \Rightarrow \frac{3s^{2/3}}{2} = t \sqrt[3]{\frac{2\lambda_4 m_0}{\beta}} + \frac{3s_0^{2/3}}{2}; \)

Or

Rearrange terms:

\[ \lambda_4 = \frac{27 \beta}{16} \left( s^{2/3} - s_0^{2/3} \right)^3; \]

(6.37)

From Eq. (6.30) or Eq. (C.7):

\[ -\frac{v}{m_0} \beta \left( \frac{ds}{dt} \right)^2 = \lambda_5 \Rightarrow s^{-1/2} ds = \sqrt{-\lambda_5 m_0} \sqrt{\frac{s}{v \beta}} dt; \]

Integrate both sides:

\[ 2s^{1/2} = t \sqrt{-\lambda_5 m_0} \sqrt{\frac{s}{v \beta}} + c_6; \]

Or

Apply initial condition: \( c_6 = 2s_0^{1/2} \); \( \Rightarrow 2s^{1/2} = t \sqrt{-\lambda_5 m_0} \sqrt{s_0^{1/2}} + 2s_0^{1/2}; \)

Rearrange terms:

\[ \lambda_5 = \frac{-4v \beta (s^{1/2} - s_0^{1/2})^2}{m_0 t^2} \]

(6.38)

From Eq. (6.31) or Eq. (C.8):

\[ -\frac{v}{m_0} \frac{d^2 s}{dt^2} = \lambda_6 \Rightarrow \frac{d^2 s}{dt^2} = -\frac{\lambda_6 m_0}{v \beta}; \]

Integrate once:

\[ \frac{ds}{dt} = -\frac{\lambda_6 m_0}{v \beta} t + c_9; \]

Apply initial condition: \( c_9 = u_0 \):

\[ \frac{ds}{dt} = \frac{\lambda_6 m_0}{v \beta} t + u_0; \]
Integrate again: \( s = -\frac{\lambda_0 m_0}{2v_\beta} t^2 + u_{t_0} + c_{i_0}; \)  \hspace{1cm} (6.38)

Apply initial condition: \( c_{i_0} = s_0; \)

\[ s = -\frac{\lambda_0 m_0}{2v_\beta} t^2 + u_{t_0} + s_0; \]  \hspace{1cm} (C.15)

Rearrange terms: \( \lambda_0 = \frac{2v_\beta (u_{t_0} - (s - s_0))}{m_0 t^2}; \)

Invoking the algebraic partitioning condition (APC) of Eq.(6.32) or Eq. (C.9):

\[ \frac{3a_{t_0} t^2 + 6u_{t_0} - 6(s - s_0)}{t^3} + \frac{3\beta ((s^2 - s_0^2) - (a_0 s_0 + u_{t_0}) t^2 - 2a_0 s_0 t)}{m_0 t^3} \]
\[ + \frac{(v_\beta^2 - 2\mu F_n)}{2m_{t_0}} \ln \left[ \frac{s}{s_0} \right] + \frac{27\beta (s^{2/3} - s_0^{2/3})^3}{16 m_{t_0} t^3} - \frac{4v_\beta (s^{1/2} - s_0^{1/2})^2}{m_0 t^2} \]
\[ + \frac{2v_\beta (u_{t_0} - (s - s_0))}{m_0 t^2} = 0 \]  \hspace{1cm} (6.39)

Multiply through by \(16m_{t_0} t^3\) and simplify. This gives an implicit time-distance relationship:

\[ \left[ 48m_0 a_0 - 48\beta (a_0 s_0 + u_{t_0}^2) + 8(v_\beta^2 - 2\mu F_n) \ln \left[ \frac{s}{s_0} \right] + 32v_\beta u_{t_0} \right] t^2 \]
\[ + \left[ 96m_0 u_0 - 96\beta u_0 s_0 - 64v_\beta \beta (s^{1/2} - s_0^{1/2})^2 - 32v_\beta \beta (s - s_0) \right] t \]
\[ + 27\beta (s^{2/3} - s_0^{2/3})^3 - 96m_0 (s - s_0) + 48\beta (s^2 - s_0^2) = 0 \]  \hspace{1cm} (C.17)

Using the quadratic root formula:

\[ t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]  \hspace{1cm} (6.41)

\[ A(s) = \left[ 48m_0 a_0 - 48\beta (a_0 s_0 + u_{t_0}^2) + 8(v_\beta^2 - 2\mu F_n) \ln \left[ \frac{s}{s_0} \right] + 32v_\beta u_{t_0} \right] \]
\[ B(s) = \left[ 96m_0 u_0 - 96\beta u_0 s_0 - 64v_\beta \beta (s^{1/2} - s_0^{1/2})^2 - 32v_\beta \beta (s - s_0) \right] \]
\[ C(s) = \left[ 27\beta (s^{2/3} - s_0^{2/3})^3 - 96m_0 (s - s_0) + 48\beta (s^2 - s_0^2) \right] \]  \hspace{1cm} (C.18)
To check for the initial condition: when \( s = s_0 \), it should give \( t = 0 \)

\[
t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]

\( A(s_0) = \left[ 48m_0a_0 - 48\beta \left( a_0s_0 + u_0^2 \right) + 32\nu \beta u_0 \right]; \)

\( B(s_0) = 96u_0 \left[ m_0 - \beta s_0 \right]; \) \( C(s_0) = 0 \)

Clearly, only the positive root satisfies the initial condition.

\[
t = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{(C.19)}
\]

To obtain the velocity and acceleration we differentiate Eq. (C.19) with respect to time. The velocity resulting from the implicit differentiation is in Eq. (C.20).

\[
t = Q(s) + R(s)
\]

\[
Q(s) = -\frac{B(s)}{2A(s)}; \quad R(s) = \frac{\sqrt{\left[ B(s) \right]^2 - 4A(s)C(s)}}{2A(s)}
\]

\[
t = Q(s) + R(s) \Rightarrow 1 = [Q'(s) + R'(s)] \hat{s} = [Q'(s) + R'(s)]v
\]

\[
v = \frac{1}{[Q'(s) + R'(s)]}
\]

Similarly, the acceleration was obtained by differentiating Eq. (C.19) twice with respect to time. The final form of the acceleration becomes:

\[
t = Q(s) + R(s) \Rightarrow 1 = [Q'(s) + R'(s)] \hat{s};
\]

\[
\Rightarrow 0 = [Q'(s) + R'(s)] \hat{s} + \ddot{s} \left[ Q''(s) + R''(s) \right]
\]

\[
\ddot{s} = \frac{d^2s}{dt^2} = -\frac{v^2 \left[ Q''(s) + R''(s) \right]}{[Q'(s) + R'(s)]} = -v^3 \left[ Q''(s) + R''(s) \right]
\]

Here, acceleration is a function of velocity and time. The task of obtaining the velocity and acceleration can be made easy using the assignment of variables. The differentiation process using the assigned variables is shown below:
Using these terms the velocity and acceleration can be obtained and studied parametrically.
Appendix D: APT Solutions for Jerk Balance Equation with Friction, Lubrication and Wear

The Power to energy transfer rate equation for the sliding with wear and friction-velocity coupling can was found as:

\[
\frac{d^3s}{dt^3} - \frac{\mu_s \beta_s}{m_0} \frac{d^3s}{dt^3} - \frac{v_c \gamma \beta_s}{m_0} \frac{d^3s}{dt^3} + 2\beta_c \gamma' \frac{ds}{dt} + \frac{4v_c \gamma \beta_s}{m_0} \frac{ds}{dt} \frac{d^2s}{dt^2} + \frac{2\beta_c \gamma' \frac{ds}{dt}}{m_0} \left( \frac{d^2s}{dt^2} \right)^2
\]

\[
+ \frac{\left( \gamma \beta_c v_c^2 - 2\gamma F_n + 2v_c \mu_s \beta_s \right)}{2m_0} \frac{d^2s}{dt^2} - \frac{3\mu_s \beta_s}{m_0} \frac{ds}{dt} \frac{d^2s}{dt^2} + \frac{2\gamma \beta_s}{m_0} \left( \frac{d^2s}{dt^2} \right)^2
\]

\[
- \frac{\left( \mu_s \beta_s + 2v_c \gamma \beta_s \right)}{2m_0 s} \left( \frac{ds}{dt} \right)^3 + \frac{\gamma \beta_s}{2m_0} \left( \frac{ds}{dt} \right)^4 + \frac{11\gamma \beta_s}{2m_0} \left( \frac{ds}{dt} \right)^2 \frac{d^2s}{dt^2}
\]

\[
+ \frac{\left( 2\mu_s F_n - \mu_s \beta_s v_c^2 \right)}{2m_0} \frac{1}{s} \frac{ds}{dt} + \frac{\left( \gamma \beta_c v_c^2 - 2\gamma F_n + 2v_c \mu_s \beta_s \right)}{2m_0} \frac{1}{s} \frac{ds}{dt} = 0
\]  

(D.1)

Or

\[
\text{(6.53)}
\]

Use following differential transformation to simplify Eq. (D.1):

\[
\frac{d^3s^2}{dt^3} = \frac{d}{dt} \left( \frac{d^2s^2}{dt^2} \right) = \frac{d}{dt} \left( \frac{d}{dt} \left( 2s \frac{ds}{dt} \right) \right)
\]

\[
= \frac{d}{dt} \left( 2s \frac{d^2s}{dt^2} + 2 \left( \frac{ds}{dt} \right)^2 \right) = 2s \frac{d^3s}{dt^3} + 2 \frac{d^2s}{dt^2} + 4 \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

(D.2)

\[
\Rightarrow \frac{d^3s^2}{dt^3} = 2s \frac{d^3s}{dt^3} + 6 \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

\[
\Rightarrow \frac{d^3s}{dt^3} = 1 \frac{d^3s}{dt^3} - 3 \frac{ds}{dt} \frac{d^2s}{dt^2}
\]

Substitute Eq. (D.2) into Eq. (D.1):

\[
\frac{d^3s}{dt^3} - \frac{\mu_s \beta_s}{m_0} \left( 1 \frac{d^3s}{dt^3} - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} \right) - \frac{v_c \gamma \beta_s}{m_0} \left( 1 \frac{d^3s}{dt^3} - 3 \frac{ds}{dt} \frac{d^2s}{dt^2} \right) + 2\beta_c \gamma' \frac{ds}{dt} + \frac{4v_c \gamma \beta_s}{m_0} \frac{ds}{dt} \frac{d^2s}{dt^2} + \frac{2\beta_c \gamma' \frac{ds}{dt}}{m_0} \left( \frac{d^2s}{dt^2} \right)^2
\]

\[
- \frac{\left( \mu_s \beta_s + 2v_c \gamma \beta_s \right)}{2m_0 s} \left( \frac{ds}{dt} \right)^3 + \frac{\gamma \beta_s}{2m_0} \left( \frac{ds}{dt} \right)^4 + \frac{11\gamma \beta_s}{2m_0} \left( \frac{ds}{dt} \right)^2 \frac{d^2s}{dt^2}
\]

\[
+ \frac{\left( 2\mu_s F_n - \mu_s \beta_s v_c^2 \right)}{2m_0} \frac{1}{s} \frac{ds}{dt} + \frac{\left( \gamma \beta_c v_c^2 - 2\gamma F_n + 2v_c \mu_s \beta_s \right)}{2m_0} \frac{1}{s} \frac{ds}{dt} = 0
\]

(D.3)

Simplify further:
\[
\frac{d^3 s}{dt^3} - \left(\mu_d \beta_i + \nu, \gamma \beta_i\right) \frac{d^3 s}{dt^3} + \frac{2 \beta_i \gamma}{m_0} \left(\frac{d s}{dt}\right)^2 \frac{ds}{dt} + s \left(\frac{d s}{dt}\right)^3 + \frac{11 \gamma \beta_i}{2 m_0} \left(\frac{ds}{dt}\right) \frac{d^3 s}{dt^3} + \frac{2 m_0^2}{2m_0} \left(\frac{ds}{dt}\right)^3 + \frac{\gamma \beta_i}{2 m_0} \left(\frac{ds}{dt}\right)^4 \quad (D.3)
\]

Using another transformation:

\[
\frac{d}{dt}\left(s \frac{ds}{dt} \frac{d^2 s}{dt^2}\right) = s \left(\frac{d^2 s}{dt^2}\right)^2 + s \frac{ds}{dt} \frac{d^3 s}{dt^3} + \left(\frac{ds}{dt}\right)^2 \frac{d^3 s}{dt^3} \quad (D.4)
\]

Substitute Eq. (D.4) into Eq. (D.3) and simplify:

\[
\frac{d^3 s}{dt^3} - \left(\mu_d \beta_i + \nu, \gamma \beta_i\right) \frac{d^3 s}{dt^3} + \frac{2 \beta_i \gamma}{m_0} \left(\frac{d s}{dt}\right) \frac{ds}{dt} + \frac{7 \gamma \beta_i}{2 m_0} \left(\frac{ds}{dt}\right) \frac{d^3 s}{dt^3} \quad (D.5)
\]

There are two more transformations:

\[
\frac{d}{dt}\left(\frac{ds}{dt}\right)^3 = 3 \left(\frac{ds}{dt}\right)^2 \frac{d^2 s}{dt^2} = \frac{1}{3} \frac{d}{dt}\left(\frac{ds}{dt}\right)^3 \quad (D.6)
\]

Substitute Eqs. (D.6) and (D.7) into Eq. (D.5) and simplify:

\[
\frac{d^3 s}{dt^3} - \beta_i \left(\mu_d + \nu, \gamma \beta_i\right) \frac{d^3 s}{dt^3} + \left(\nu \frac{d^2 s}{dt^2} + 2 \nu, \mu_d \beta_i - 2 \gamma F_n\right) \frac{d s}{dt} \frac{ds}{dt} \frac{d^3 s}{dt^3} \quad (D.8)
\]
Equation (D.8) is a third order equation showing jerk balance but with \( s \) as unknown.

Rearrange terms:

\[
\frac{d^3 s}{dt^3} + \frac{\beta_1 (\mu_d + \gamma v_e)}{2m_0} \frac{d^3 s^2}{dt^3} + \frac{\left( \mu_d \beta_1 v_e^2 - 2\mu_d F_n \right)}{2m_0 s} \frac{ds}{dt} + \frac{\beta_1 (\mu_d + \gamma v_e)}{2m_0 s} \left( \frac{ds}{dt} \right)^3
\]

(D.9)

\[
- \frac{\left( \gamma v_e^2 \beta_1 + 2v_e \mu_d \beta_1 - 2\gamma F_n \right)}{2m_0 s} \left( \frac{ds}{dt} \right)^2 - \frac{\left( \gamma v_e^2 \beta_1 + 2v_e \mu_d \beta_1 - 2\gamma F_n \right)}{2m_0} \frac{d^2 s}{dt^2} - \frac{\gamma \beta_1}{2m_0 s} \left( \frac{ds}{dt} \right)^4
\]

\[
+ \frac{v \gamma \beta_1}{2m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 - \frac{7\gamma \beta_1}{6m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^3 - \frac{2 \beta_1 \gamma}{m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 = 0
\]

Equation (D.9) can be treated as an IVP with the initial conditions: \( s(t=0) = s_0 \), \( ds/dt(t=0) = u_0 \), and \( d^2 s/dt^2(t=0) = a_0 \). Applying the APT, partition Eq. (D.9) as:

\[
- \frac{d^3 s}{dt^3} = \lambda_1
\]

(D.10)

\[
\frac{\beta_1 (\mu_d + \gamma v_e)}{2m_0} \frac{d^3 s^2}{dt^3} = \lambda_2
\]

(D.11)

\[
\frac{\left( \mu_d \beta_1 v_e^2 - 2\mu_d F_n \right)}{2m_0 s} \frac{1}{s} \frac{ds}{dt} = \lambda_3
\]

(D.12)

\[
\frac{\beta_1 (\mu_d + \gamma v_e)}{2m_0 s} \left( \frac{ds}{dt} \right)^3 = \lambda_4
\]

(D.13)

\[
- \frac{\left( \gamma v_e^2 \beta_1 + 2v_e \mu_d \beta_1 - 2\gamma F_n \right)}{2m_0 s} \left( \frac{ds}{dt} \right)^2 = \lambda_5
\]

(D.14)

\[
- \frac{\left( \gamma v_e^2 \beta_1 + 2v_e \mu_d \beta_1 - 2\gamma F_n \right)}{2m_0} \frac{d^2 s}{dt^2} = \lambda_6
\]

(D.15)

\[
- \frac{\gamma \beta_1}{2m_0 s} \left( \frac{ds}{dt} \right)^4 = \lambda_7
\]

(D.16)

\[
+ \frac{v \gamma \beta_1}{2m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 = \lambda_8
\]

(D.17)

\[
- \frac{7\gamma \beta_1}{6m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^3 = \lambda_9
\]

(D.18)
\[
- \frac{2 \beta \gamma}{m_0} \frac{d}{dt} \left( s \frac{ds}{dt} \frac{d^2 s}{dt^2} \right) = \lambda_{10} \tag{D.19}
\]

\[
\sum_{i=1}^{10} \lambda_i = 0 \tag{D.20}
\]

For partitioned Eq. (D.10):

\[
\frac{d^3 s}{dt^3} = -\lambda_i;
\]

Integrate once: \[
\frac{d^2 s}{dt^2} = -\lambda_i t + c_i;
\]

Apply initial condition: \[
\frac{d^2 s}{dt^2}(t = 0) = a_o; c_1 = a_o; \frac{d^2 s}{dt^2} = -\lambda_i t + a_o;
\]

Integrate again: \[
\frac{ds}{dt} = -\frac{\lambda_i t^2}{2} + a_o t + c_2;
\]

Apply initial condition: \[
\frac{ds}{dt}(t = 0) = u_o; c_2 = u_o; \frac{ds}{dt} = -\frac{\lambda_i t^2}{2} + a_o t + u_o;
\]

Integrate again: \[
s = -\frac{\lambda_i t^3}{6} + \frac{a_o t^2}{2} + u_o t + c_3;
\]

Apply initial condition: \[
s(t = 0) = s_o; c_3 = s_o; s = -\frac{\lambda_i t^3}{6} + \frac{a_o t^2}{2} + u_o t + s_o;
\]

Rearrange terms: \[
\lambda_i = \frac{3a_o t^2 + 6u_o t - 6(s - s_o)}{t^3};
\]

From partitioned Eq. (D.11):

\[
\frac{\beta_i (\mu_d + \gamma v_e)}{2m_0} \frac{d^3 s^2}{dt^3} = \lambda_2 \Rightarrow \frac{d^3 s^2}{dt^3} = \lambda_2 \frac{2m_o}{\beta_i (\mu_d + \gamma v_e)};
\]

Integrate once: \[
\frac{d^2 s^2}{dt^2} = \lambda_2 \frac{2m_0}{\beta_i (\mu_d + \gamma v_e)} t + c_1;
\]

\[
\frac{d^2 s^2}{dt^2} = \frac{d}{dt} \left( 2s \frac{ds}{dt} \right) = 2s \frac{d^2 s}{dt^2} + 2 \left( \frac{ds}{dt} \right)^2;
\]

Apply initial conditions: \[
s(t = 0) = s_o; \frac{ds}{dt}(t = 0) = u_o; \frac{d^2 s^2}{dt^2}(t = 0) = a_0;
\]

\[
c_i = (2a_o s_o + 2u_o); \Rightarrow \frac{d^2 s^2}{dt^2} = \lambda_2 \frac{2m_0}{\beta_i (\mu_d + \gamma v_e)} + \left( 2a_o s_o + 2u_o \right);
\]

Integrate again:
\[
\frac{ds^2}{dt} = \lambda_2 \frac{m_0}{\beta_1 (\mu_d + \gamma v_e)} t^2 + \left(2a_0s_0 + 2u_0^2\right)t + c_2;
\]

Apply initial conditions: \(s(t = 0) = s_0; \frac{ds}{dt}(t = 0) = u_0; c_2 = 2u_0s_0;\)

\[
\Rightarrow \frac{ds^2}{dt} = \lambda_2 \frac{m_0}{\beta_1 (\mu_d + \gamma v_e)} t^2 + \left(2a_0s_0 + 2u_0^2\right)t + 2u_0s_0;
\]

Integrate again: \(s^2 = \lambda_2 \frac{m_0}{3\beta_1 (\mu_d + \gamma v_e)} t^3 + \left(a_0s_0 + u_0^2\right)t^2 + 2u_0s_0 t + c_3;\)

Apply initial condition: \(s(t = 0) = s_0; c_3 = s_0^2;\)

\[
s^2 = \lambda_2 \frac{m_0}{3\beta_1 (\mu_d + \gamma v_e)} t^3 + \left(a_0s_0 + u_0^2\right)t^2 + 2u_0s_0 t + s_0^2;
\]

Rearrange terms: \(\lambda_2 = \frac{3\beta_1 (\mu_d + \gamma v_e) \left((s^2 - s_0^2) - (a_0s_0 + u_0^2)t^2 - 2u_0s_0 t\right)}{m_0 t^3};\)

From partitioned Eq. (D.12):

\[
\left(\mu_d \beta_1 v_e^2 - 2\mu_d F_n\right) \frac{1}{2m_0} \frac{ds}{dt} = \lambda_3;
\]

separate variables: \(\frac{ds}{s} = \frac{2\lambda_3 m_0}{\left(\mu_d \beta_1 v_e^2 - 2\mu_d F_n\right)} dt;\)

Integrate on both sides: \(\ln s = \frac{2\lambda_3 m_0}{\left(\mu_d \beta_1 v_e^2 - 2\mu_d F_n\right)} t + c_3;\)

Apply initial condition: \(s(t = 0) = s_0; c_3 = \ln s_0; \Rightarrow \ln \left[\frac{s}{s_0}\right] = \frac{2\lambda_3 m_0}{\left(\mu_d \beta_1 v_e^2 - 2\mu_d F_n\right)} t;\) \hspace{1cm} (D.23)

Rearrange terms: \(\lambda_3 = \frac{\left(\mu_d \beta_1 v_e^2 - 2\mu_d F_n\right)}{2m_0 t} \ln \left[\frac{s}{s_0}\right];\)

From partitioned Eq. (D.13):
\[ \frac{\beta_i (\mu_d + \gamma v_e)}{2m_0s} \left( \frac{ds}{dt} \right)^3 = \lambda_i \Rightarrow \frac{ds}{dt} = s^{1/3} \sqrt[3]{\frac{2\lambda_im_0}{\beta_i (\mu_d + \gamma v_e)}} ; \]

separate variables: \( s^{-1/3} ds = 3 \sqrt[3]{\frac{2\lambda_im_0}{\beta_i (\mu_d + \gamma v_e)}} dt ; \)

Integrate on both sides: \( \frac{3s^{2/3}}{2} = 3 \sqrt[3]{\frac{2\lambda_im_0}{\beta_i (\mu_d + \gamma v_e)}} t + c_\gamma ; \)

Apply initial condition: \( s(t = 0) = s_0 ; c_\gamma = \frac{3s_0^{2/3}}{2} / 2 = t \sqrt[3]{\frac{2\lambda_im_0}{\beta_i (\mu_d + \gamma v_e)}} + \frac{3s_0^{2/3}}{2} ; \)

Rearrange terms: \( \lambda_i = \frac{27\beta_i (\mu_d + \gamma v_e) (s^{2/3} - s_0^{2/3})^3}{16m_0 t^3} ; \)

From partitioned Eq. (D.14):

\[ - \frac{(\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n)}{2m_0} \frac{1}{s} \left( \frac{ds}{dt} \right)^2 = \lambda_3 \Rightarrow \frac{ds}{dt} = s^{1/2} \sqrt{-\frac{2\lambda_im_0}{\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n}} ; \]

separate variables: \( s^{-1/2} ds = \frac{\sqrt{-2\lambda_im_0}}{\sqrt{\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n}} dt ; \)

Integrate on both sides: \( 2s^{1/2} = t \frac{\sqrt{-2\lambda_im_0}}{\sqrt{\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n}} + c_6 ; \)

Apply initial condition: \( s(t = 0) = s_0 ; c_6 = 2s_0^{1/2} ; \)

\( 2s^{1/2} = t \frac{\sqrt{-2\lambda_im_0}}{\sqrt{\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n}} + 2s_0^{1/2} ; \)

Rearrange terms: \( \lambda_3 = \frac{-2 (\gamma v_e^2 \beta_i + 2v_e \mu_d \beta_i - 2\gamma F_n) (s^{1/2} - s_0^{1/2})^2}{m_0 t^2} ; \)

From partitioned Eq. (D.15):
\[
- \frac{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)}{2m_0} \frac{d^2 s}{dt^2} = \lambda \Rightarrow \frac{d^2 s}{dt^2} = -\frac{2\lambda m_0}{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)};
\]

Integrate once:
\[
\frac{ds}{dt} = -\frac{2\lambda m_0}{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)} t + c_9;
\]

Apply initial condition:
\[
\frac{ds}{dt} (t = 0) = u_0; \quad c_9 = u_0;
\]

\[
\frac{ds}{dt} = -\frac{2\lambda m_0}{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)} t + u_0;
\]

Integrate again:
\[
s = -\frac{\lambda m_0}{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)} t^2 + u_0 t + c_{10};
\]

Apply initial condition:
\[
s (t = 0) = s_0; \quad c_{10} = s_0;
\]

\[
s = -\frac{\lambda m_0}{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)} t^2 + u_0 t + s_0;
\]

Rearrange terms:
\[
\lambda = \frac{(\gamma v^e_\beta + 2v_\gamma \mu \beta_1 - 2\gamma F_n)(u_0 t - (s - s_0))}{m_0 t^2};
\]

From partitioned Eq. (D.16):
\[
- \frac{\gamma \beta_1}{2m_0 s} \left( \frac{ds}{dt} \right)^4 = \lambda; \quad \Rightarrow \frac{ds}{dt} = \left( -\frac{2\lambda m_0}{\gamma \beta_1} \right)^{1/4} s^{3/4};
\]

Separate variables:
\[
s^{-1/4} ds = \left( -\frac{2\lambda m_0}{\gamma \beta_1} \right)^{1/4} dt;
\]

Integrate on both sides:
\[
\frac{4}{3} s^{3/4} = \left( -\frac{2\lambda m_0}{\gamma \beta_1} \right)^{1/4} t + c_{11};
\]

Apply initial condition:
\[
s (t = 0) = s_0; \quad c_{11} = \frac{4}{3} s_0^{3/4} \; \frac{4}{3} s^{3/4} = \left( -\frac{2\lambda m_0}{\gamma \beta_1} \right)^{1/4} + \frac{4}{3} s_0^{3/4};
\]

Rearrange terms:
\[
\lambda = \frac{128\gamma \beta_1 (s^{3/4} - s_0^{3/4})^4}{81m_0 t^2};
\]

From partitioned Eq. (D.17):
\[
\frac{v_i \gamma \beta_i}{2m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 = \lambda_s \Rightarrow \frac{d}{dt} \left( \frac{ds}{dt} \right)^2 = \left( \frac{2 \lambda_s m_0}{v_i \gamma \beta_i} \right);
\]

Integrate once: \[\left( \frac{ds}{dt} \right)^2 = \left( \frac{2 \lambda_s m_0}{v_i \gamma \beta_i} \right) t + c_{12};\]

Apply initial condition: \[\frac{ds}{dt} (t = 0) = u_0; c_{12} = u_0^2;\]

\[\left( \frac{ds}{dt} \right)^2 = \left( \frac{2 \lambda_s m_0}{v_i \gamma \beta_i} \right) t + u_0^2;\]

Partition this equation as follows

\[- \left( \frac{ds}{dt} \right)^2 = \phi_1; \quad \phi_2 = \left( \frac{2 \lambda_s m_0}{v_i \gamma \beta_i} \right) t + u_0^2;\]

\[\phi_1 + \phi_2 = 0\]

From the first partition:

\[- \left( \frac{ds}{dt} \right)^2 = \phi_1 \Rightarrow \frac{ds}{dt} = \sqrt{\phi_1} \Rightarrow s = \sqrt{\phi_1} t + d_i; d_i = s_o;\]

\[\Rightarrow s = \sqrt{\phi_1} t + s_o; \Rightarrow \phi_1 = - \left( \frac{s - s_o}{t} \right)^2\]

Applying APC:

\[\phi_1 + \phi_2 = 0\]

\[- \left( \frac{s - s_o}{t} \right)^2 + \left( \frac{2 \lambda_s m_0}{v_i \gamma \beta_i} \right) t + u_0^2 = 0 \Rightarrow \lambda_s = \frac{v_i \gamma \beta_i \left( \frac{(s - s_o)^2 - u_o^2 t^2}{2m_0 t^3} \right)}{7}\]

From partitioned Eq. (D.18):

\[- \frac{7 \gamma \beta_i}{6m_0} \frac{d}{dt} \left( \frac{ds}{dt} \right)^3 = \lambda_o \Rightarrow \frac{d}{dt} \left( \frac{ds}{dt} \right)^3 = \left( \frac{-6m_0 \lambda_o}{7 \gamma \beta_i} \right);\]

Integrate once: \[\left( \frac{ds}{dt} \right)^3 = \left( \frac{-6m_0 \lambda_o}{7 \gamma \beta_i} \right) t + c_{13};\]

Apply initial condition: \[\frac{ds}{dt} (t = 0) = u_0; c_{13} = u_0^3;\]

\[\left( \frac{ds}{dt} \right)^3 = \left( \frac{-6m_0 \lambda_o}{7 \gamma \beta_i} \right) t + u_0^3;\]

Partition this equation as follows
\[-\left(\frac{ds}{dt}\right)^3 = \phi_3; \quad \phi_4 = \left(\frac{-6m_0\lambda_0}{7\gamma\beta_1}\right)t + u_0^3,\]

\[\phi_1 + \phi_2 = 0\]

From the first partition:

\[-\left(\frac{ds}{dt}\right)^3 = \phi_1 \Rightarrow \frac{ds}{dt} = \sqrt[3]{\phi_1} \Rightarrow s = \sqrt[3]{\phi_1}t + d_1; d_1 = s_0;\]

\[\Rightarrow s = \sqrt[3]{\phi_1}t + s_0; \Rightarrow \phi_1 = -\left(\frac{s - s_0}{t^3}\right)\]

Applying the APC: \(\phi_1 + \phi_2 = 0\)

\[\frac{(s - s_0)^3}{t^3} + \left(\frac{-6m_0\lambda_0}{7\gamma\beta_1}\right)t + u_0^3 = 0 \Rightarrow \lambda_0 = -\frac{7\gamma\beta_1\left[(s - s_0)^3 - u_0^3t^3\right]}{6m_0t^4}\]

From partitioned Eq. (D.19):

\[-\frac{2\beta\gamma}{m_0}\frac{d}{dt}\left(s\frac{ds}{dt}\right)^2 = \lambda_0;\]

Integrate once: \(s\frac{ds}{dt}\frac{d^2s}{dt^2} = -\frac{m_0}{2\beta\gamma}x_0t + e_i;\)

Apply initial conditions: \(s(t = 0) = s_0; \frac{ds}{dt}(t = 0) = u_0; \frac{d^2s}{dt^2}(t = 0) = a_0; e_i = s_0u_0a_0;\)

Use transformation:

\[s\frac{ds}{dt}\frac{d^2s}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(s\left(\frac{ds}{dt}\right)^2\right) - \frac{1}{2}\left(\frac{ds}{dt}\right)^3;\]

Simplify equation: \(\frac{d}{dt}\left(s\left(\frac{ds}{dt}\right)^2\right) - \left(\frac{ds}{dt}\right)^3 = -\frac{m_0}{\beta\gamma}x_0t + 2s_0u_0a_0;\)

Apply APT again:

Partition this equation as follows

\[\phi_1 = \frac{d}{dt}\left(s\left(\frac{ds}{dt}\right)^2\right); \phi_2 = -\left(\frac{ds}{dt}\right)^3; \phi_3 = \frac{m_0}{\beta\gamma}x_0t - 2s_0u_0a_0;\]

From the first partition:

\[\frac{d}{dt}\left(s\left(\frac{ds}{dt}\right)^2\right) = \phi_1; s\left(\frac{ds}{dt}\right)^2 = \phi_1t + d_1; d_1 = s_0u_0^2; s\left(\frac{ds}{dt}\right)^2 = \phi_1t + s_0u_0^2;\]
Partition further: $\pi_1 = -s\left(\frac{ds}{dt}\right)^2; \pi_2 = \phi_t + s_0u_0^2$; and $\pi_1 + \pi_2 = 0$

\[ s\left(\frac{ds}{dt}\right)^2 = -\pi_1; s^{3/2}\frac{ds}{dt} = \sqrt{-\pi_1}; \frac{2}{3}s^{3/2} = t\sqrt{-\pi_1} + d_s; d_s = \frac{2}{3}s_0^{3/2}; \]

\[ \frac{2}{3}(s^{3/2} - s_0^{3/2}) = t\sqrt{-\pi_1}; \pi_1 = -\frac{4}{9}\frac{(s^{3/2} - s_0^{3/2})^2}{t^2} \]

Apply APC: $\pi_1 + \pi_2 = 0$

\[ -\frac{4}{9}\frac{(s^{3/2} - s_0^{3/2})^2}{t^2} + \phi_tv + s_0u_0^2 = 0 \Rightarrow \phi_t = \frac{4(s^{3/2} - s_0^{3/2})^2 - 9u_0^2s_0^2}{9t^3} \]

From the second partition:

\[ -\left(\frac{ds}{dt}\right)^3 = \phi_2 \Rightarrow \frac{ds}{dt} = \frac{1}{3}\phi_2 \Rightarrow s = \frac{1}{3}\phi_2t + d_3; d_3 = s_0; \]

$\Rightarrow s = \frac{1}{3}\phi_2t + s_0; \Rightarrow \phi_2 = -\left(\frac{s - s_0}{t}\right)^3$

Applying APC:

$\phi_1 + \phi_2 + \phi_3 = 0$

\[ \frac{4(s^{3/2} - s_0^{3/2})^2 - 9u_0^2s_0t^2}{9t^3} - \frac{(s - s_0)^3}{t^3} + \frac{m_0}{\beta_i\gamma} \lambda_{10}t - 2s_0v_0a_0 = 0 \quad (D.30) \]

\[ \frac{4(s^{3/2} - s_0^{3/2})^2 - 9u_0^2s_0t^2 - 9(s - s_0)^3 - 18t^3s_0v_0a_0}{9t^3} = \frac{m_0}{\beta_i\gamma} \lambda_{10}t \]

\[ \lambda_{10} = \frac{\gamma\beta_i\left(9u_0^2s_0t^2 + 9(s - s_0)^3 + 18t^3s_0v_0a_0 - 4(s^{3/2} - s_0^{3/2})^2\right)}{9m_0t^4} \]

Applying the APC:
\[
\frac{3a_0 t^2 + 6u_0 t - 6(s - s_0)}{t^3} + \frac{3\beta_i (\mu_d + \gamma v_{e}) \left( \left( s^2 - s_0^2 \right) - \left( a_0 s_0 + u_0^2 \right) t^2 - 2u_0 s_0 t \right)}{m_0 t^3} \\
+ \left( \mu_d \beta_i \nu_{e}^2 - 2 \mu_d \nu_{e} F_n \right) \ln \left[ \frac{s}{s_0} \right] + \frac{27 \beta_i (\mu_d + \gamma v_{e}) \left( s^{2/3} - s_0^{2/3} \right)^3}{16 m_0 t^3} \\
- \frac{2 \left( \gamma v_{e}^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \left( s^{1/2} - s_0^{1/2} \right)^2}{m_0 t^2} + \left( \gamma v_{e}^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \left( u_0 - (s - s_0) \right) \\
- \frac{128 \gamma \beta_i \left( s^{3/4} - s_0^{3/4} \right)^4}{8 m_0 t^2} + \frac{v_e \gamma \beta_i \left( (s - s_0)^2 - u_0 t^2 \right)}{2 m_0 t^3} \\
- \frac{7 \gamma \beta_i \left( s - s_0 \right)^3 - u_0 t^2}{6 m_0 t^4} + \frac{\gamma \beta_i \left( 9 u_0^2 s_0 t^2 + 9 \left( s - s_0 \right)^3 + 18 t^3 s_i u_0 a_0 - 4 \left( s^{3/2} - s_0^{3/2} \right)^2 \right)}{9 m_0 t^4} = 0
\]

Multiply through by \(1296 m_0 t^4\) and simplify:

\[
\begin{align*}
&\left[ 3888 m_0 a_0 + 2592 \gamma \beta_i a_0 a_0 s_0 - 648 v_e \gamma \beta_i u_0^2 - 3888 \beta_i (\mu_d + \gamma v_e) \left( a_0 s_0 + u_0^3 \right) \right] t^3 \\
&+ 1296 u_0 \left( \nu_{e}^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) + 1512 \gamma \beta_i u_0^3 + 648 \left( \mu_d \beta_i \nu_{e}^2 - 2 \mu_d F_n \right) \ln \left[ \frac{s}{s_0} \right] \\
&+ \left[ 7776 m_0 u_0 - 7776 \beta_i (\mu_d + \gamma v_e) u_0 s_0 + 1296 \gamma \beta_i u_0^2 s_0 - 2048 \gamma \beta_i \left( s^{3/4} - s_0^{3/4} \right)^4 \right] t^2 \\
&+ \left[ - \gamma \beta_i \left( \nu_{e}^2 \beta_i + 2 v_e \mu_d \beta_i - 2 \gamma F_n \right) \left[ 2592 (s^{1/2} - s_0^{1/2})^2 + 1296 (s - s_0) \right] \right] t \\
&+ \left[ 7776 m_0 (s - s_0) + (\mu_d + \gamma v_e) \left[ 3888 \beta_i \left( s^2 - s_0^2 \right) + 2187 \beta_i (s^{2/3} - s_0^{2/3})^3 \right] \right] \\
&+ 648 v_e \gamma \beta_i \left( s - s_0 \right)^3 \\
&+ \left[ -216 \gamma \beta_i \left( s - s_0 \right)^3 - 576 \beta_i \left( s^{3/2} - s_0^{3/2} \right)^2 \right] = 0
\end{align*}
\]

Equation (D.31) is a cubic in time with coefficients being the function of \(s\)

\[
A(s) = B(s) = C(s) = D(s) = 0
\]

The roots of this cubic equation can be found using the following procedure:
\[ t^3 + a_1 t^2 + a_2 t + a_3 = 0 \]

\[ Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54} \]

\[ S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}; T = \sqrt[3]{R - \sqrt{Q^3 + R^2}} \]

where \( ST = -Q \) \hspace{1cm} (D.33)

\[ t_1 = S + T - \frac{1}{3} a_1 \]

\[ t_2 = -\frac{1}{2} (S + T) - \frac{1}{3} a_1 + \frac{1}{2} i \sqrt{3} (S - T) \]

\[ t_3 = -\frac{1}{2} (S + T) - \frac{1}{3} a_1 - \frac{1}{2} i \sqrt{3} (S - T) \]

By checking it seems that the first root with

\[ t_1 = S + T - \frac{1}{3} a_1 \] \hspace{1cm} (D.34)

Satisfies the initial condition and is the desired solution. Differentiating Eq. (D.34) gives the velocity in Eq. (D.35).

\[ t = S + T - \frac{1}{3} a_1 \] \hspace{1cm} (D.35)

\[ \Rightarrow 1 = \left[ S'(s) + T'(s) - \frac{a_1}{3} \right] \dot{s} = \left[ S'(s) + T'(s) - \frac{a_1}{3} \right] v \]

\[ v = \frac{1}{ \left[ S'(s) + T'(s) - \frac{a_1}{3} \right] } \]

And the acceleration is obtained by differentiating Eq. (D.34) twice with time to yield the acceleration in Eq. (D.36).

\[ \Rightarrow 0 = \left[ S'(s) + T'(s) - \frac{a_1}{3} \right] \ddot{s} + \dot{s}^2 \left[ S''(s) + T''(s) - \frac{a_1^*}{3} \right] \]

\[ \ddot{s} = \frac{d^2 s}{dt^2} = -\frac{v^2 \left[ S''(s) + T''(s) - \frac{a_1^*}{3} \right]}{\left[ S'(s) + T'(s) - \frac{a_1}{3} \right]} = -\frac{v^3 \left[ S''(s) + T''(s) - \frac{a_1^*}{3} \right]}{\left[ S'(s) + T'(s) - \frac{a_1}{3} \right]} \] \hspace{1cm} (D.36)
The expressions for the velocity and acceleration can be simplified completely by differentiating the terms:

\[ a'_1 = \frac{AB' - BA'}{A^2}; \quad a'_2 = \frac{AC' - CA'}{A^2}; \quad a'_3 = \frac{AD' - DA'}{A^2} \]

\[ a''_1 = \frac{A[AB'' - BA''] - 2A'[AB' - BA']}{A^3}; \]

\[ a''_2 = \frac{A[AC'' - CA''] - 2A'[AC' - CA']}{A^3}; \]

\[ a''_3 = \frac{A[AD'' - DA''] - 2A'[AD' - DA']}{A^3}; \]

\[ Q' = \frac{1}{9}[3a'_2 - 2a'_1]; \quad Q'' = \frac{1}{9}\left[3a''_2 - 2a''_1 - 2(a'_1)^2\right]; \]

\[ R' = \frac{1}{54}\left[9a_1a_2 + 9a_1a'_2 - 27a'_3 - 6a_1^2a'_1\right]; \]

\[ R'' = \frac{1}{54}\left[9a_1a''_2 + 9a_1a''_1 + 18a'_1a'_2 - 27a''_3 - 6a_1^2a''_1 - 12a_1(a'_1)^2\right] \]

The various terms and their differentiations are as follows:

\[ A(s) = \left[3888m_0a_0 + 2592\gamma\beta\mu_0s_0 - 648\nu\beta\mu_0^2 - 3888\beta_1(\mu_d + \gamma\nu\mu)d_0 + u_0^2]\right] \]

\[ A' = 648\left(\mu_d\beta\nu^2 - 2\mu_dF_n\right)/s \]

\[ A'' = -648\left(\mu_d\beta\nu^2 - 2\mu_dF_n\right)/s^2 \]

\[ B(s) = \left[7776m_0u_0 - 7776\beta_1(\mu_d + \gamma\nu\mu)u_0s_0 + 1296\gamma\beta\mu_0s_0^2 - 2048\gamma\beta_1(s^{3/4} - s_0^{3/4})^2\right] \]

\[ B' = -6144\gamma\beta_1(s^{3/4} - s_0^{3/4})^3s^{-1/4} - \left(\gamma\nu^2\beta_1 + 2\nu\mu\beta_1 - 2\gamma\nu\beta_1\right)\left[2592(s^{3/2} - s_0^{1/2})^2 + 1296(s - s_0)\right] \]

\[ B'' = 1536\gamma\beta_1(s^{3/4} - s_0^{3/4})^3s^{-3/4} - 13824\gamma\beta_1(s^{3/4} - s_0^{3/4})^2s^{-1/2} \]

\[ - \left(\gamma\nu^2\beta_1 + 2\nu\mu\beta_1 - 2\gamma\nu\beta_1\right)\left[-1296(s^{3/2} - s_0^{1/2})s^{-3/2} + 1296s^{-1}\right] \]
\[ C(s) = \left[ -7776m_0(s-s_0) + (\mu_d + \gamma v_c) \left( 3888\beta_1(s^2 - s_0^2) + 2187\beta_1(s^{2/3} - s_0^{2/3})^3 \right) \right] + 648\gamma\beta_1(s-s_0)^2 \]
\[ C' = -7776m_0 + 1296\gamma\beta_1(s-s_0) + \beta (\mu_d + \gamma v_c) \left[ 7776s + 4374(s^{2/3} - s_0^{2/3})^2 s^{-1/3} \right] \]
\[ C'' = 1296\gamma\beta_1 + \beta (\mu_d + \gamma v_c) \left[ 7776 - 1458(s^{2/3} - s_0^{2/3})^2 s^{-4/3} + 5832(s^{2/3} - s_0^{2/3})s^{-2/3} \right] \]
\[ D(s) = \left[ -216\gamma\beta_1(s-s_0)^3 - 576\gamma\beta_1(s^{3/2} - s_0^{3/2})^2 \right] \]
\[ D' = -648\gamma\beta_1(s-s_0)^2 - 1728\gamma\beta_1(s^{3/2} - s_0^{3/2})s^{1/2} \]
\[ D'' = -1296\gamma\beta_1(s-s_0) - 864\gamma\beta_1(s^{3/2} - s_0^{3/2})s^{-1/2} - 2592\gamma\beta_1 s \]

\[ S = \sqrt{R+\sqrt{Q^3+R^2}}; T = \sqrt{R-\sqrt{Q^3+R^2}} \]
\[ s = \left( R + \left( R^2 + Q^3 \right)^{1/2} \right)^{1/3} = \left( M + \left( M^2 + N \right)^{1/2} \right)^{1/3} \]
\[ T = \left( R - \left( R^2 + Q^3 \right)^{1/2} \right)^{1/3} = \left( M - \left( M^2 + N \right)^{1/2} \right)^{1/3} \]

where
\[ M = R; M' = R'; M'' = R'' \]
\[ N = Q^3; N' = 3Q^2Q'; N'' = 3Q^2Q'' + 6Q[Q']^2 \]
\[ S' = \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M' + \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right] \]
\[ T' = \frac{1}{3} \left( M - \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M' - \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right] \]
\[ S'' = -\left( \frac{2}{9} \right) \left( M + \left( M^2 + N \right)^{1/2} \right)^{-5/3} \left[ M' + \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right] \]
\[ + \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M'' - \frac{1}{4} \left( M^2 + N \right)^{-3/2} \left( 2MM' + N' \right)^2 \right] \]
\[ + \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM'' + 2[M']^2 + N'' \right) \]
\[ T'' = -\left( \frac{2}{9} \right) \left( M - \left( M^2 + N \right)^{1/2} \right)^{-5/3} \left[ M' - \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM' + N' \right) \right] \]
\[ + \frac{1}{3} \left( M + \left( M^2 + N \right)^{1/2} \right)^{-2/3} \left[ M'' + \frac{1}{4} \left( M^2 + N \right)^{-3/2} \left( 2MM' + N' \right)^2 \right] \]
\[ - \frac{1}{2} \left( M^2 + N \right)^{-1/2} \left( 2MM'' + 2[M']^2 + N'' \right) \]
Appendix E: APT Solution for the Heat Generation at Lubricated Interfaces

Heat to internal energy transfer rate equation for lubricated sliding interface is given as:

\[
-\frac{ds}{dt} + \frac{\gamma}{\mu_d} \left( \frac{ds}{dt} \right)^2 + \frac{m \varepsilon \cdot c_p}{\mu_d F_n} \frac{dT}{dt} = 0
\]  

(7.8) Or (E.1)

Equation (E.1) can be treated as an initial value problem with the given initial conditions: \( s(t = 0) = s_0 \); \( T(t = 0) = T_0 \). To apply the algebraic partitioning technique:

\[ -\frac{ds}{dt} = \lambda_1 \]  

(E.2)

\[ \frac{\gamma}{\mu_d} \left( \frac{ds}{dt} \right)^2 = \lambda_2 \]  

(E.3)

\[ \frac{m \varepsilon \cdot c_p}{\mu_d F_n} \frac{dT}{dt} = \lambda_3 \]  

(E.4)

\[ \lambda_1 + \lambda_2 + \lambda_3 = 0 \]  

(E.5)

From the partitioned Eq. (E.2):

\[ -\frac{ds}{dt} = \lambda_1 \]

Integrate once with respect to \( t \) : \(-s = \lambda_1 t + c_1\)

Apply initial condition : \( s(t = 0) = s_0 \Rightarrow c_1 = -s_0 \)  

\[ \Rightarrow -s = \lambda_1 t - s_0 \]  

Rearrange terms: \( \lambda_1 = \frac{s_0 - s}{t} \)

From Partitioned Eq. (E.3):
\( \frac{\gamma}{\mu_d} \left( \frac{ds}{dt} \right)^2 = \lambda_2 \Rightarrow \frac{ds}{dt} = \sqrt{\frac{\lambda_2 \mu_d}{\gamma}} \)

Integrate once with respect to \( t \):
\[ s = t \sqrt{\frac{\lambda_2 \mu_d}{\gamma}} + c_2 \]

Apply initial condition: \( s(t = 0) = s_0 \Rightarrow c_2 = s_0 \)  \hspace{1cm} (E.7)

\[ s = t \sqrt{\frac{\lambda_2 \mu_d}{\gamma}} + s_0 \]

Rearrange terms:
\[ \lambda_2 = \frac{\gamma}{\mu_d} \left( s - s_0 \right)^2 \]

From Partitioned Eq. (E.4):
\[ \frac{m\varepsilon_m c_p}{\mu_d F_n} \frac{dT}{dt} = \lambda_3 \]

Integrate once with respect to \( t \):
\[ \frac{m\varepsilon_m c_p}{\mu_d F_n} T = \lambda_3 t + c_3 \]

Apply initial condition: \( T(t = 0) = T_0 \Rightarrow c_3 = \frac{m\varepsilon_m c_p}{\mu_d F_n} T_0 \)  \hspace{1cm} (E.8)

\[ \Rightarrow \frac{m\varepsilon_m c_p}{\mu_d F_n} T = \lambda_3 t + \frac{m\varepsilon_m c_p}{\mu_d F_n} T_0 \]

Rearrange terms:
\[ \lambda_3 = \frac{m\varepsilon_m c_p}{\mu_d F_n} \frac{(T - T_0)}{t} \]

Invoking the APC in Eq. (E.5):
\[ \frac{(s_0 - s)}{t} + \frac{\gamma}{\mu_d} \left( s - s_0 \right)^2 + \frac{m\varepsilon_m c_p}{\mu_d F_n} \frac{(T - T_0)}{t} = 0 \]  \hspace{1cm} (E.9)

Multiply through by \( \mu_d F_n t^2 \):
\[ \gamma F_n (s - s_0)^2 - \mu_d F_n t(s - s_0) + m\varepsilon_m c_p t(T - T_0) = 0 \]  \hspace{1cm} (E.10)

Solving for \( (s - s_0) \) using the quadratic formula:
\[ (s - s_0) = \frac{\mu_d t}{2\gamma} \pm \frac{1}{2} \sqrt{\left( \frac{\mu_d t}{\gamma} \right)^2 - \frac{4m\varepsilon_m c_p t(T - T_0)}{\gamma F_n}} \]  \hspace{1cm} (E.11)
Thus, when $T = T_0$:

$$
(s - s_0) = \frac{\mu_d t}{2\gamma} \pm \frac{\mu_d t}{2\gamma}  
$$

(E.12)

The only case when the solution in Eq. (E.11) satisfies the initial condition $s = s_0$ is when the solution becomes:

$$
(s - s_0) = \frac{\mu_d t}{2\gamma} - \frac{1}{2\gamma} \sqrt{\left(\frac{\mu_d t}{\gamma}\right)^2 - \frac{4m\varepsilon_m c_p t (T - T_0)}{\gamma F_n}}  
$$

(E.13)

Rearrange terms and square on both sides:

$$
(s - s_0 - \frac{\mu_d t}{2\gamma})^2 = \frac{1}{4} \left[\left(\frac{\mu_d t}{\gamma}\right)^2 - \frac{4m\varepsilon_m c_p t (T - T_0)}{\gamma F_n}\right]  
$$

Expand and simplify further:

$$
T = T_0 + \frac{\gamma F_n}{4m\varepsilon_m c_p t} \left[\left(\frac{\mu_d t}{\gamma}\right)^2 - 4 \left(s - s_0\right)^2 + \left(\frac{\mu_d t}{2\gamma}\right)^2 - \frac{(s - s_0) \mu_d t}{\gamma}\right]  
$$

(E.15)

Or

$$
T = T_0 + \frac{\gamma F_n}{4m\varepsilon_m c_p t} \left[4(s - s_0) \frac{\mu_d t}{\gamma} - 4(s - s_0)^2\right]  
$$

(7.9)
CURRICULUM VITAE

Divyeshkumar Patel

Place of Birth: Navsari, Gujarat, India

Education:

- Bachelor of Science, Sardar Patel University, Gujarat, India, July 2007
  Major: Mechanical Engineering
- Masters of Science, Gannon University, Erie, Pennsylvania, December 2009
  Major: Mechanical Engineering

Dissertation Title: On the Nonlinear Tribological Jerk Dynamics at Sliding Interfaces

Work experience:

- Graduate Research Assistant, University of Wisconsin-Milwaukee (August 2010 – December 2014)
  - Designed parts of in situ cooling mechanism for rolling element bearings and planetary gear systems using SolidWorks. SolidWorks Simulation tools to test designs.
  - Experiment with universal multi-specimen tribometer.
    - Friction, wear, and lubrication experiments.
    - Temperature and thermal penetration experiments.
  - Experiment with nanoindenter with atomic force microscopy (AFM) capability.
  - Basic Physics of Sliding Contacts.
    - Novel Fundamental Quantification of Friction, Lubrication, and Wear at Sliding Contacts: Nonlinear Tribological Jerk Dynamics.
    - Fundamental Predictions of Kinematic (Distance, Velocity, Acceleration, and Jerk) and Dynamic Forces under a Variety of Tribological Sliding Conditions.
  - General Partitioned Integral Techniques for Solving Differential Equations (Partial and Ordinary).
  - Exact Algebraic Technique for Solving Linear Partial Differential Equations.
  - Deterministic Modeling Tool for Characterizing Rough Surfaces.
  - Designed Green energy systems to recharge battery backup for the house. 3D modeling with Creo Parametric 2.0. Created part drawings and with GD and T. Printed parts using 3D printer and sheet metal operations for a prototype.
• **Energy Engineer** - US Department of Energy Industrial Assessment Center at University of Wisconsin-Milwaukee. (March 2012- December 2014)
  o Conducted 35 on-site energy audits for small and medium sized manufacturing industries
  o Prepared energy audit reports with energy/cost savings, implementation cost of developed conservation measures, and associated payback periods, submitted to the companies and Department of Energy.
  o Perform detailed analysis on electrical demand, lighting, HVAC, air compressors, boilers, chillers, cooling towers, electric motors, VFD’s, process heating/cooling, refrigeration, steam systems, waste heat recovery systems, and building envelopes
  o Analyze collected data, utility rate schedules, and past client utility bills for time of use, demand, and power factor profiles to quantify possible energy conservation measures
  o Employ Onset HOBO data loggers and current transducers at each plant to monitor power consumption, temperature, and humidity; as well as other advanced equipment such as an infrared camera, combustion analyzer, and ultrasonic leak detector
  o Develop a comprehensive training strategy for new team members

• **Teaching Assistant**, University of Wisconsin-Milwaukee
  o Engineering Fundamentals –II (MECHENG -111)
  o How Things Work- Understanding Technology (MECHENG -150)
  o Basic Heat Transfer (MECHENG -321)

• **Graduate Research Assistant**, Gannon University, Erie, PA  (Jan 08– July 10)
  o 3D modeling in Pro-E for different parts of a trailer such as gooseneck, lever, cylinders.
  o Prepared parts and assembly drawings, bill of materials (BOM), order materials and build a prototype
  o Project scheduling and communication with team members and suppliers.
  o Finite element analysis using ANSYS for structural, thermal and fluid flow applications.
  o Technical support with MATLAB and Pro-E and AutoCAD software.
  o Design and set up the fluid mechanics laboratory experiments.
  o Mathematical model for the nano robotic arm.

**Publications:**


Awards

Academic Excellence Award, Gannon University May 2010

Affiliation/Membership

- American Society of Mechanical Engineers (ASME)
- Society of Tribologists and Lubrication Engineers (STLE)
- Wisconsin Association of Energy Engineers (WAEE)