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The Boundedness of Hausdorff Operators on Function Spaces

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ABSTRACT
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For a fixed kernel function Φ, the one dimensional Hausdorff operator is defined in the integral form by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f(\frac{x}{t})\, dt.$$

By the Minkowski inequality, it is easy to check that the Hausdorff operator is bounded on the Lebesgue spaces L^p when $p \geqslant 1$, with some size condition assumed on the kernel functions Φ. However, people discovered that the above boundedness property is quite different on the Hardy space H^p when $0 < p < 1$. To establish the boundedness on the Hardy space for $0 < p < 1$, some smoothness must be assumed on the kernel functions Φ.

In this thesis, we first study the boundedness of h_Φ on the Hardy space H^1, and on the local Hardy space $h^1(\mathbb{R})$. Our work shows that for $\Phi(t) \geqslant 0$, the Hausdorff operator h_Φ is bounded on the Hardy space H^1 if and only if Φ is a Lebesgue integrable function; and h_Φ is bounded on the local Hardy space $h^1(\mathbb{R})$ if and only if the functions $\Phi(t)\chi_{(1,\infty)}(t)$ and $\Phi(t)\chi_{(0,1)(t)}\log(\frac{1}{t})$ are Lebesgue integrable. These results solve an open question posed by the Israeli mathematician Liflyand. We also establish an $H^1(\mathbb{R}) \to H^{1,\infty}(\mathbb{R})$ boundedness theorem for h_Φ. As applications, we obtain many decent properties for the Hardy operator and the kth order Hardy operators. For instance, we know that the Hardy operator $\mathcal{H}$ is bounded from $H^1(\mathbb{R}) \to H^{1,\infty}(\mathbb{R})$, bounded on the atomic space $H^1_A(\mathbb{R}_+)$, but it is not bounded on both $H^1(\mathbb{R})$ and the

local Hardy space $h^1(\mathbb{R})$.

We also extend part of these results to the high dimensional Hausdorff operators. Here, we study two high dimensional extentions on the Hausdorff operator h_Φ:

$$\tilde{H}_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{n-\beta}} f(\frac{x}{|y|})\, dy, \quad n \geqslant \beta \geqslant 0,$$

and

$$H_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y)\, dy, \quad n \geqslant \beta \geqslant 0,$$

where Φ is a local integrable function.

For $0 < p < 1$, we obtain a sufficient condition for the H^p boundedness for the Hausdorff operator in the one dimensional case. This theorem needs less smoothness on the kernel Φ than any other theorems in the literature. Since there is no result involving the boundedness on $H^p(\mathbb{R}^n)$ in the literature for the high dimensional Hausdorff operators, if $0 < p < 1$ and $n \geqslant 2$, it is interesting to study such problems in the high dimensional spaces. We establish several sufficient conditions by using a duality argument.

Additionally, we study boundedness of Hausdorff operators on some Herz type spaces, and some bilinear Hausdorff operators and fractional Hausdorff operators.

CONTENTS

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CHAPTER 1
INTRODUCTION

One of the core problems in harmonic analysis is to study the boundedness of an operator T on some function/distribution spaces

$$\|Tf\|_Y \lesssim \|f\|_X ,$$

where X and Y are two function/distribution spaces with norms or quasi norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively. This question arises from many natural problems in mathematics and the sciences. To illustrate the importance of this problem, we look at the following two examples.

Example 1.1. *Any L^1 function f on $[-\pi, \pi]$ has its Fourier series*

$$f(x) \sim \sum_k c_k e^{ikx}.$$

This means a signal f might be built up from many simple sine and cosine waves with different wave lengths and amplitude. Unfortunately,

$$f(x) = \sum_k c_k e^{ikx}$$

is not always true pointwise, even when f is a continuous function(see [31]). So we need to modify the information using a filter $\{m(\epsilon k)\}$ (called the summation of the Fourier series):

$$T_{m,\epsilon} f(x) = \sum_k c_k m(\epsilon k) e^{ikx}, \quad \epsilon > 0,$$

where m is a suitably nice function and $m(0) = 1$. T_m is called a multiplier (operator) with symbol m. This summation greatly improves the convergence of the Fourier series *in the sense that*

$$\lim_{\epsilon\to 0+}\sum_k c_k m(\epsilon,k)e^{ikx}=f(x)$$

uniformly for a continuous function f *if one chooses a suitable function* m. *For instance, from [31], we know that* $T_{m,\epsilon}(f)$ *is the Abel summation if* $m(\sqrt{\epsilon}k)=e^{-\epsilon|k|^2}$, *and the Riesz summation if* $m(\epsilon k)=(1-\epsilon|k|)_+$.

Furthermore, let F *be some function/distribution space with quasi-norm* $\|\cdot\|_F$, *we are interested in the global convergence*

$$\lim_{\epsilon\to 0+}\|T_{m,\epsilon}g-g\|_F=0,\quad \text{for } g\in F.$$

We suppose that the class S *of Schwartz functions is dense in* F, *and that for* $f\in S$,

$$\lim_{\epsilon\to 0+}\|T_{m,\epsilon}f-f\|_F=0.$$

Then it is easy to check that the boundedness

$$\|T_{m,\epsilon}g\|_F\lesssim\|g\|_F\,,\ \text{for } \epsilon \text{ sufficiently small}$$

implies that for all $g\in F$,

$$\lim_{\epsilon\to 0+}\|T_{m,\epsilon}g-g\|_F=0.$$

To see this fact, for given $g\in F$, *and any* $\delta>0$, *choose* $f\in S$ *such that*

$$\|f-g\|_F<\delta.$$

As a consequence, we have

$$\begin{aligned}\lim_{\epsilon\to 0+}\|T_{m,\epsilon}g-g\|_F&\lesssim\lim_{\epsilon\to 0+}\left(\|f-g\|_F+\|T_{m,\epsilon}(f-g)\|_F+\|T_{m,\epsilon}f-f\|_F\right)\\&\lesssim\delta+\lim_{\epsilon\to 0+}\|T_{m,\epsilon}f-f\|_F\\&\lesssim\delta.\end{aligned}$$

Thus the convergence problem $T_{m,\epsilon}f\to f$ *in the space* F *is reduced to the boundedness* *of* $T_{m,\epsilon}$ *on the space* F.

Example 1.2. *The solution* $u(t,x)$ *of the Cauchy problem of the Schrödinger equation*

$$\begin{cases} i\partial_t u - \Delta u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = u_0(x) \end{cases}$$

is formally given by $u(t,x) = (e^{-it\Delta}u_0)(x)$*, where* $u(t,x) = (e^{-it\Delta}u_0)$ *is defined through its Fourier transform by*

$$(\widehat{e^{-it\Delta}u_0})(\xi) = e^{it|\xi|^2}\widehat{u_0}(\xi).$$

Let X *and* Y *be two function spaces. To study the regularity of the solution, we need to estimate*

$$\left\|e^{-it\Delta}(u_0 - v_0)\right\|_Y = \left\|e^{-it\Delta}u_0 - e^{-it\Delta}v_0\right\|_Y \lesssim \|u_0 - v_0\|_X \,,$$

since the operator $e^{-it\Delta}$ *is linear. Again we face the boundedness inequality*

$$\left\|e^{-it\Delta}f\right\|_Y \lesssim \|f\|_X \,.$$

These examples begin to show the importance of demonstrating the boundedness of operators on function spaces. In this thesis, we mainly study the boundedness of the Hausdorff operators on the Lebesgue spaces and on the Hardy spaces. We begin by recalling the one dimensional Hausdorff operator, defined in the integral form by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f(\frac{x}{t})\, dt, \tag{1.1}$$

where, for simplicity, we initially define h_Φ on the Schwartz space S. Clearly, h_Φ is a linear operator. This integral operator is deeply rooted in the study of 1-dimensional Fourier analysis. Particularly, it is closely related to the summability of the classical Fourier series (see [13]).

The definition of h_Φ is based on the dilation structure of the Euclidean space. Two important geometric transformations on the Euclidean space $\mathbb{R}^n$ are translation and dilation. The translation L_y is a linear operator defined by $L_y(f)(x) = f(x+y)$, so its integral form

$$\int \Phi(y) L_{-y}(f)(x)\, dy = \int \Phi(y) f(x-y)\, dy \tag{1.2}$$

is the convolution operator $\Phi * f(x)$ with kernel Φ, and its L^1 norm satisfies

$$\|\Phi * f\|_{L^1} = \|\Phi\|_{L^1} \|f\|_{L^1}, \quad \text{if} \quad \Phi, f \geqslant 0.$$

The dilation D_t is a linear operator defined by $D_t(f)(x) = f(tx)$, for $t > 0$. The integral

$$\int \Phi(t) D_{t^{-1}}(f)(x)\, dt = \int \Phi(t) f(\frac{x}{t})\, dt \tag{1.3}$$

corresponds to the Hausdorff operator

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} D_{t^{-1}}(f)(x)\, dt. \tag{1.4}$$

Note that $\Phi(t)$ in equation (1.3) is replaced by $\frac{\Phi(t)}{t}$ in equation (1.4). This normalizes the operator, so that

$$\|h_\Phi(f)\|_{L^1} = \|\Phi\|_{L^1} \|f\|_{L^1}, \quad \text{if} \quad \Phi, f \geqslant 0.$$

Many important operators in real and complex analysis are special cases of the Hausdorff operator, by taking suitable choice of Φ. These operators include, among many others:

(1) the Hardy operator

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)\, dt,$$

the average of antiderivative of a function f in the Fundamental Theorem of Calculus, is obtained by choosing $\Phi(t) = \frac{\chi_{(1,\infty)}(t)}{t}$;

(2) the adjoint Hardy operator

$$\mathcal{H}^* f(x) = \int_x^\infty \frac{f(t)}{t}\, dt,$$

which is obtained by choosing $\Phi(t) = \chi_{(0,1)}(t)$;

(3) the Cesàro operator

$$\mathcal{C}_\alpha(f)(x) = \alpha \int_0^1 \frac{(1-t)^{\alpha-1}}{t} f(\frac{x}{t})\, dt,$$

which is obtained by choosing $\Phi(t) = \chi_{(0,1)}(t)(1-t)^{\alpha-1}$;

(4) the Hardy-Littlewood-Pólya operator

$$P(f)(x) = \int_0^\infty \frac{f(t)}{\max\{t,x\}}\, dt = Hf(x) + H^* f(x).$$

Also, the Riemann-Liouville fractional derivatives

$$\tilde{D}_\beta(f)(x) = \int_0^x (x-t)^{\beta-1} f(t)\, dt, \quad x > 0$$

can be derived from the fractional Hausdorff operator. In fact

$$\begin{aligned}\tilde{D}_\beta(f)(x) &= x^{\beta-1} \int_0^x \left(1 - \frac{t}{x}\right)^{\beta-1} f(t)\, dt \\ &\cong x^\beta \int_1^\infty \frac{(1-\frac{1}{t})^{\beta-1}}{t^2} f(\frac{x}{t})\, dt.\end{aligned}$$

Thus,

$$\tilde{D}_\beta(f)(x) \cong x^\beta h_\Phi(f)(x)$$

with

$$\Phi(t) = \frac{(1-\frac{1}{t})^{\beta-1}}{t} \chi_{(1,\infty)}(t).$$

The Hausdorff operator has received extensive study in recent years, particularly its boundedness on the Lebesgue space L^p and the Hardy space H^p (see [9, 14, 18, 28, 30]).

Consider a quasi-normed space X with quasi-norm $\|\cdot\|_X$. We say that X satisfies the norm scaling (N-S) property if there exists a number $\sigma = \sigma(X)$ such that

$$\left\| f(\frac{\cdot}{t}) \right\|_X = t^\sigma \left\| f \right\|_X$$

for any $f \in X, t > 0$. If X satisfies the N-S property, then by Minkowski's inequality

$$\begin{aligned} \| h_\Phi f \|_X &\leqslant \int_0^\infty \frac{|\Phi(t)|}{t} \left\| f(\frac{\cdot}{t}) \right\|_X \, dt \\ &= \left(\int_0^\infty \frac{|\Phi(t)|}{t} t^\sigma \, dt \right) \| f \|_X . \end{aligned}$$

This shows that h_Φ is bounded on the space X if

$$\left(\int_0^\infty \frac{|\Phi(t)|}{t} t^\sigma \, dt \right) < \infty. \tag{1.5}$$

Based on these observations, our main interests in the Hausdorff operator are:

(1) Determine when the condition given in equation (1.5) is sharp, that is, for which spaces X satisfying the N-S property the condition in equation (1.5) is necessary as well as sufficient.

(2) Study the boundedness of $h_\Phi : X \to X$ if X does not satisfy the N-S property.

The Lebesgue spaces L^p when $p \geqslant 1$ satisfy the N-S property. Thus, by the above argument, we have

$$\| h_\Phi f \|_{L^p} \leqslant C_\Phi \| f \|_{L^p} ,$$

where C_Φ is the constant

$$C_\Phi = \left(\int_0^\infty \frac{|\Phi(t)|}{t} t^{\frac{1}{p}} \, dt \right) .$$

It is known that the Hausdorff operator is bounded on the Lebesgue spaces L^p when $p \geqslant 1$ if and only if $C_\Phi < \infty$, provided $\Phi(t)$ is a non-negative valued function. The real Hardy space H^1 is also a space with the N-S property. Again, by a scaling argument together with the Minkowski inequality, we obtain

$$\| h_\Phi f \|_{H^1} \leqslant \| f \|_{H^1}$$

if $\Phi(t)$ is a Lebesgue integrable function on $(0, \infty)$. The Israeli mathematician Liflyand posed the following question.

Question 1 (Liflyand [13]). Determine the sharpness of the condition $\Phi \in L^1(0, \infty)$ to give h_Φ bounded on $H^1(\mathbb{R})$.

In this thesis, we will solve this problem by showing that if $\Phi(t)$ is a non-negative valued function, then

$$\|h_\Phi f\|_{H^1} \lesssim \|f\|_{H^1}$$

if and only if $\Phi(t)$ is a Lebesgue integrable function on $(0, \infty)$. We will also solve the same problem on the local Hardy space h^1. As applications, we obtain some interesting results for the Hardy operator on the spaces near H^1.

The real Hardy space H^p is not a normed space when $0 < p < 1$. It is known that the above boundedness property on L^p or H^1 is quite different on the Hardy space H^p when $0 < p < 1$. To establish the boundedness of h_Φ on the real Hardy space H^p for all $0 < p < 1$, it seems that any reasonable size condition on Φ is not sufficient; some smoothness condition must be included. This phenomeonon was discovered by Kinjin [9] who required a smoothness condition on $\widehat{\Phi}$, the Fourier transforms of Φ. It was further explored by Liflyand and Miyachi in [14], who required a smoothness condition on Φ, and who further found a bounded function Φ supported in a compact set $E \subset (0, \infty)$, such that the operator h_Φ is not bounded on $H^p(\mathbb{R})$ for any $0 < p < 1$.

The original work of Kanjin,Liflyand and Miyachi was motivated by the Cesàro operator

$$\mathcal{C}_\alpha(f)(x) = \alpha \int_0^1 \frac{(1-t)^{\alpha-1}}{t} f(\frac{x}{t})\, dt, \quad \alpha > 0.$$

Kanjin showed in ([9]) that $\mathcal{C}_\alpha$ is bounded on $H^p(\mathbb{R})$ provided $\frac{2}{2\alpha+1} < p < 1$, and that this result actually is an application of the following:

Theorem A ([9])**.** *Let* $0 < p < 1$, $M = [1/p - 1/2] + 1$, *and*

$$A_{\Phi,p} = \int_0^\infty t^{-1+1/p} \, |\Phi(t)| \, dt.$$

Suppose $A_{\Phi,1} + A_{\Phi,2} < \infty$, *and* $\widehat{\Phi} \in C^{2M}(\mathbb{R})$ *with*

$$\sup_{\xi \in \mathbb{R}} |\xi|^M \left(\left| \widehat{\Phi}^{(M)}(\xi) \right| + \left| \widehat{\Phi}^{(2M)}(\xi) \right| \right) < \infty.$$

Then h_Φ *is a bounded operator on* $H^p(\mathbb{R})$.

The proof of Theorem A is based on the Taibleson-Weiss atomic-molecular characterization of the Hardy space (see [28]). Kanjin proved that if Φ satisfies the conditions of Theorem A, then the Hausdorff operator h_Φ maps an atom to an H^p molecule. Later on, using a different method, which includes a modified atomic decomposition of H^p, Miyachi improved Kanjin's result by showing that the Cesàro operator $\mathcal{C}_\alpha$ is bounded on $H^p(\mathbb{R})$ for any $\alpha > 0$ and all $p > 0$ (see [23]). In [14] Liflyand and Miyachi further extended the method used by Miyachi in [23] to study the $H^p(\mathbb{R})$ boundedness of the Hausdorff operator h_Φ. First, they observed that the following Theorem B is a direct corollary of Theorem A.

Theorem B ([14])**.** *Let* $0 < p < 1$ *and* $M = [1/p - 1/2] + 1$. *If* $\Phi \in C^M$ *and its support is a compact set in* $(0, \infty)$, *then the Hausdorff operator* h_Φ *is bounded on* $H^p(\mathbb{R})$.

Liflyand and Miyachi then generalized the main result in [23] and obtained the following two theorems.

Theorem C ([14])**.** *Let* $0 < p < 1$, $M = [1/p - 1/2] + 1$ *and let* ε *be a positive number. If* $\Phi \in C^M(0, \infty)$ *satisfies*

$$\left| \Phi^{(k)}(t) \right| \leqslant \min\{t^\varepsilon, t^{-\varepsilon}\} t^{-1/p-k}$$

for $k = 0, 1, \ldots, M$, *then* h_Φ *is bounded on* $H^p(\mathbb{R})$.

Theorem D ([14]). *Let $0 < p < 1$, $M = [1/p - 1/2] + 1$ and let ε and a be positive numbers. Suppose that Φ is a function on $(0, \infty)$ such that the support of Φ is a compact subset of $(0, \infty)$. If Φ is of class C^M on $(0, a) \cup (a, \infty)$ and satisfies*

$$\left|\Phi^{(k)}(t)\right| \leqslant |t - a|^{\varepsilon - 1 - k} \text{ for } k = 0, 1, ..., M,$$

then h_Φ is bounded on $H^p(\mathbb{R})$.

In this thesis, we will establish another sufficient condition for H^p boundedness of h_Φ with less smoothness on Φ than that in Theorems B to D.

Next, we observe that all methods used to show the boundedness of h_Φ on $H^p(\mathbb{R})$ in the above theorems fail in the high dimensional case, and we notice that there is no result involving the boundedness on $H^p(\mathbb{R}^n)$ in literature for the high dimensional Hausdorff operators, if $0 < p < 1$ and $n \geqslant 2$. Thus we are particularly interested in studying such problems in the high dimensional spaces.

In the one-dimensional case, when $x > 0$, by a changing of variables,

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(\frac{x}{t})}{t} f(t)\, dt.$$

This suggests us to study two different extensions of Hausdorff operator in the high dimensional space:

$$\tilde{H}_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{n-\beta}} f(\frac{x}{|y|})\, dy, \quad n \geqslant \beta \geqslant 0,$$

and

$$H_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y)\, dy, \quad n \geqslant \beta \geqslant 0,$$

where Φ is a locally integrable function. We denote

$$\tilde{H}_{\Phi,0} = \tilde{H}_\Phi \quad \text{and} \quad H_{\Phi,0} = H_\Phi.$$

We note that operators of the form H_Φ include the high dimensional Hardy operator

$$\frac{1}{|x|^n}\int_{|y|\leqslant|x|} f(y)dy$$

and the adjoint Hardy operator

$$\int_{|y|\geqslant|x|} \frac{f(y)}{|y|^n}dy.$$

In [10, 12, 15, 18], Liflyand and Mòricz addressed the following theorem.

Theorem E (Liflyand and Mòricz)**.** *$\tilde{H}_\Phi$ is bounded on $H^1(\mathbb{R}^n), n \geqslant 2$, if $\Phi \in L^1(\mathbb{R}^n)$.*

Again, we face the following question.

Question 2 (Liflyand [13])**.** Determine the sharpness of the condition on Φ given in Theorem E. That is, determine whether $\tilde{H}_\Phi$ is bounded on $H^1(\mathbb{R}^n), n \geqslant 2$, if *and only if* $\Phi \in L^1(\mathbb{R}^n)$.

Also, we notice that there is no research in the literature addressing $H^p(\mathbb{R}^n)$ boundedness of either the operator $\tilde{H}_\Phi$ or the operator H_Φ. All methods of treating the one dimensional operator used in Theorems A to D fail to establish a similar theorem on the high dimensional case. This raises the following question.

Question 3. Establish some $H^p(\mathbb{R}^n)$ boundedness theorems for the operators $\tilde{H}_\Phi$ or H_Φ for $n \geqslant 2$ and $0 < p < 1$.

As usual, the notation, $A \lesssim B$ means that there is a constant $C > 0$ that is independent of all essential variables such that $A \leqslant CB$. Similarly, we use the notation $A \cong B$ if there exist positive constants C and c, independent of all essential variables, such that

$$cB \leqslant A \leqslant CB.$$

The structure of this thesis is as follows.

Chapter 2 contains some preliminary knowledge. This summary contains no new results, but several facts which are required for later chapters.

Chapter 3 presents some results on $H^p(\mathbb{R})$ for $0 < p < 1$. In particular, Chapter 3 improves on the smoothness conditions described in Theorems B to D.

Chapter 4 includes several results on $H^1(\mathbb{R})$ and related spaces. It includes an answer to Question 1.

Chapter 5 concerns $H^p(\mathbb{R}^n)$, $0 < p \leqslant 1$, $n \geqslant 2$. This chapter addresses Questions 2 and 3.

Chapter 6 explores extending the techniques used for the previous results to study Hausdorff operators on the related Herz-type Hardy spaces, and also to study bilinear Hausdorff operators.

The following are the new theorems that we obtained.

In Chapter 3, we prove a boundedness result for Hausdorff operators on one-dimensional Hardy spaces:

Theorem 3.1. *Let Φ be a Lebesgue integrable function. Denote $\phi(t) = \frac{\Phi(\frac{1}{t})}{t}$ for $t > 0$, and $\phi(t) = 0$ for $t \leqslant 0$. Assume $0 < p < 1$ and $\alpha = \frac{1}{p} - 1$. If $\phi \in \Lambda_\alpha$ then*

$$\|h_\Phi(f)\|_{H^{p,\infty}(\mathbb{R})} \precsim \|f\|_{H^p(\mathbb{R})}.$$

If there exists a small $\epsilon > 0$, such that $\alpha - \epsilon > 0$ and $\phi \in \Lambda_{\alpha+\epsilon} \cap \Lambda_{\alpha-\epsilon}$, then

$$\|h_\Phi(f)\|_{H^p(\mathbb{R})} \precsim \|f\|_{H^p(\mathbb{R})}.$$

In Chapter 4, we study the boundedness of Hausdorff operators on the Hardy space $H^1(\mathbb{R})$, obtaining the following main theorems:

Theorem 4.2. *Let Φ be a nonnegative valued locally integrable function.*

(1) h_Φ is bounded on $H^1(\mathbb{R})$ if and only if $\Phi \in L^1(0,\infty)$.

(2) h_Φ is bounded on the local Hardy space $h^1(\mathbb{R})$ if and only if

$$\int_1^\infty \Phi(t)\,dt + \int_0^1 \Phi(t)\left(1+\log(\tfrac{1}{t})\right)\,dt < \infty.$$

Theorem 4.3. *Let $\phi(t) = t^{-1}\Phi_0(t)$. Suppose $\phi \in \mathrm{BMO}(\mathbb{R})$. Then h_Φ extends to a bounded operator from $H^1(\mathbb{R})$ to $H^{1,\infty}(\mathbb{R})$.*

Corollary 4.6. *The Hardy operator $\mathcal{H}$ has the following properties*

(1) $\mathcal{H}$ is not bounded on $H^1(\mathbb{R})$.

(2) $\mathcal{H}$ is not bounded on $h^1(\mathbb{R})$.

(3) $\mathcal{H}$ is bounded from $H^1(\mathbb{R})$ to $H^{1,\infty}(\mathbb{R})$.

(4) $\mathcal{H}$ is bounded on $H^1_A(0,\infty)$.

We also apply our results to some generalizations of the Hardy operator.

In Chapter 5, we begin by generalizing Theorem 3.1 to obtain

Theorem 5.1. *Let Φ be a nonnegative valued locally integrable function.*

(1) $\tilde{H}_\Phi$ is bounded on $H^1(\mathbb{R}^n)$ if and only if $\Phi \in L^1(\mathbb{R}^n)$.

(2) $\tilde{H}_\Phi$ is bounded on the local Hardy space $h^1(\mathbb{R}^n)$ if and only if

$$\int_{|y|\geqslant 1} \Phi(y)\,dy + \int_{|y|\leqslant 1} \Phi(y)\left(1+\log\left(\frac{1}{|y|}\right)\right)\,dy < \infty.$$

We next show some boundedness results for the power-weight Lebesgue spaces, particularly,

Theorem 5.4. *Let $1 \leqslant p, q < \infty$, $0 < \beta < n$, $\gamma > \beta p - n$ and*

$$\frac{1}{p} - \frac{\beta}{n+\gamma} = \frac{1}{q}.$$

In addition, let

$$C_{p,\epsilon} = \int_0^\infty |\Phi(t)|^{\frac{p}{p-1}} t^{\frac{\gamma-\beta p+n-p+1}{p-1}+\epsilon} dt \text{ for } p>1, \qquad C_{1,\epsilon} = \||\cdot|^{n-\beta+\gamma+\epsilon}\Phi(\cdot)\|_{L^\infty}.$$

For any $p \geqslant 1$*, if, for arbitrarily small positive* ϵ*,* $C_{p,\pm\epsilon} < \infty$*, then*

$$\|H_{\Phi,\beta}(f)(x)\|_{L^q(|x|^\gamma dx)} \preceq \|f\|_{L^p(|x|^\gamma dx)}.$$

Finally, we show sufficient conditions for boundedness of operators on high-dimensional Hardy spaces.

Lemma 5.5. *Let* $0 \leqslant \beta < n$ *and*

$$\psi(y) = \frac{\Phi(1/|y|)}{|y|^{n-\beta}}.$$

Assume $0 < p < 1$ *and* $\alpha = n\left(\frac{1}{p} - 1\right)$*. If* $\psi \in \Lambda_\alpha$*, then*

$$\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}$$

where q *satisfies*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

Theorem 5.6. *Let* $\beta, p, \alpha, q, \psi$ *be as in Lemma 5.5. If for some* $\epsilon > 0$ *small enough that* $\alpha - \epsilon > 0$*,* $\psi \in \Lambda_{\alpha+\epsilon} \cap \Lambda_{\alpha-\epsilon}$*, then*

$$\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}.$$

Theorem 5.11. *Suppose* $0 < p < 1$*,* $0 < \beta < n$*. Let* $\widehat{\Phi}$ *denote the Fourier transform of* Φ *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M = n(\frac{1}{p}-1)$*, suppose that* $\widehat{\Phi}$ *is a function in* $C^{2M+n}(\mathbb{R}^n)$ *with compact support in the set* $\mathbb{R}^n \backslash \{0\}$*. Then*

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}.$$

Theorem 5.13. *Suppose* $0 < p < 1$*,* $0 < \beta < n$*. Let* $\widehat{\Phi}$ *denote the Fourier transform* *of* Φ *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M > n(\frac{1}{p} - 1)$*, suppose that* $\widehat{\Phi}$ *is a function in* $C^{2M+n}(\mathbb{R}^n)$ *with compact support in the set* $\mathbb{R}^n \backslash \{0\}$*. Then*

$$\|H_{\Phi,\beta}(f)\|_{H^q(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}.$$

In Chapter 6, we establish a size condition for boundedness of Hausdorff operators on the Herz-type Hardy spaces:

Theorem 6.1. *Let* $0 < p \leqslant 1 < q < \infty$*, and* $n(1 - \frac{1}{q}) \leqslant \alpha < \infty$*.*

(1) For $0 < p < 1$*, let*

$$C_{p,\sigma} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{\alpha} |y|^{\frac{n}{q}} (1 + \log |y|)^{\sigma} \, dy.$$

If, for some $\sigma > \frac{1-p}{p}$*,* $C_p := C_{p,\sigma} < \infty$*, then*

$$\|H_{\Phi}(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

(2) For $p = 1$*, let*

$$C_1 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{\alpha} |y|^{\frac{n}{q}} \, dy.$$

If $C_1 < \infty$*, then*

$$\|H_{\Phi}(f)\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)}.$$

We further provide a condition for a modified Hausdorff operator to map the Herz-type Hardy spaces into a weak Hardy space. Here, the modified Hausdorff operator $\widetilde{h}_{\Phi}$ is defined by

$$\widetilde{h}_{\Phi}(f)(x) = \int_{\mathbb{R}} \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) \, dt$$

and we let

$$\phi(v) = \begin{cases} \frac{\Phi\left(\frac{1}{v}\right)}{v} & \text{if } v \neq 0 \\ \lim\limits_{v\to 0} \frac{\Phi\left(\frac{1}{v}\right)}{v} & \text{if } v = 0. \end{cases}$$

Theorem 6.3. *Let $0 < p \leqslant 1 < q < \infty$, and $(1 - \frac{1}{q}) \leqslant \alpha < \infty$. For*

$$r = \frac{1}{\alpha + \frac{1}{q}}, \quad N = [\alpha + \frac{1}{q} - 1] = \left[\frac{1}{r} - 1\right], \quad \frac{1}{q'} + \frac{1}{q} = 1.$$

Let

$$\gamma = \frac{1}{r} - 1 - N = \frac{1}{r} - 1 - \left[\frac{1}{r} - 1\right].$$

If $\phi \in C^N$ and

$$\int_{-\rho}^{\rho} \left|\phi^{(N)}(t) - \phi^{(N)}(0)\right|^{q'} \, dt \leqslant \rho^{1+\gamma q'}$$

uniformly for $\rho > 0$, then we have

$$\left\|\widetilde{h}_\Phi(f)\right\|_{H^{r,\infty}} \leqslant \|f\|_{H\dot{K}_q^{\alpha,p}} .$$

Finally, we prove some preliminary boundedness results for bilinear Hausdorff operators on one-dimensional Lebesgue spaces.

Theorem 6.5. *Let $m, k = 1, 2, \ldots$. For any $p, p_1, p_2, r, p' \geqslant 1$ satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \qquad \frac{1}{r} = \frac{m}{p_1} + \frac{k}{p_2}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

if

$$\left(\int_{-\infty}^{\infty} \left|\frac{\Phi\left(\frac{1}{t}\right)}{t}\right|^{p'} \, dt\right)^{\frac{1}{p'}} < \infty$$

then

$$\|H_{\Phi,m,k}(f,g)\|_{L^{r,\infty}} \leqslant \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} .$$

CHAPTER 2

PRELIMINARIES

2.1 Lebesgue and Hardy Spaces

2.1.1 Lorentz Spaces

Let dx be the Lebesgue measure on $\mathbb{R}^n$. The Lebesgue space L^p, $0 < p < \infty$, is the set of all measurable functions f satisfying

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \; dx \right)^{\frac{1}{p}} < \infty.$$

The power weight Lebesgue space $L^p(|x|^\gamma \, dx)$, $1 \leqslant p \leqslant \infty$, is the set of all measurable functions f satisfying

$$\|f\|_{L^p(|x|^\gamma dx)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, |x|^\gamma \; dx \right)^{\frac{1}{p}} < \infty.$$

For a measurable function f, we define the set $E_f(t)$, for any $t > 0$, by

$$E_f(t) = \{x : |f(x)| \geqslant t\}.$$

The distribution function of f is $\mu_f(t) = |E_f(t)|$, the Lebesgue measure of $E_f(t)$. We define the weak Lebesgue space by

$$L^{p,\infty} = \{f : \|f\|_{L^{p,\infty}}\} < \infty,$$

where

$$\|f\|^p_{L^{p,\infty}} = \sup_{t>0} \left(t^p(\mu_f(t))\right).$$

When $0 < p < \infty$ and $0 < q < \infty$, we define the norm (or quasi-norm)

$$\|f\|_{L^{p,q}} = p^{\frac{1}{q}} \left(\int_0^\infty t^q \, |\mu_f(t)|^{\frac{q}{p}} \, \frac{dt}{t} \right)^{\frac{1}{q}}.$$

The Lorentz space $L^{p,q}$ is the set of all measurable function f satisfying

$$L^{p,q} = \{f : \|f\|_{L^{p,q}} < \infty\}.$$

The Lebesgue space L^p and its weak version $L^{p,\infty}$ are special cases of the Lorentz space $L^{p,q}$. (It is easy to see that $L^{p,p} = L^p$.)

2.1.2 Hardy Spaces

When $0 < p < 1$, the dual space $(L^p)'$ of L^p contains only the zero function, which makes the structure of L^p difficult to study in this case. Instead, we study a closely related space, the Hardy space H^p.

Let $S(\mathbb{R}^n)$ be the set of Schwartz functions and $\Psi \in S(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \Psi(y)\, dy = 1,$$

and denote $\Psi_s(y) = \frac{1}{s^n}\Psi(\frac{y}{s})$

The Hardy space $H^p(\mathbb{R}^n)$ is the space of all distributions f satisfying

$$\|f\|_{H^p(\mathbb{R}^n)} := \left\| \sup_{0<s<\infty} |\Psi_s * f| \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

We similarly define the weak Hardy space $H^{p,\infty}$ to be the set of distributions f satisfying

$$\|f\|_{H^{p,\infty}(\mathbb{R}^n)} := \left\| \sup_{0<s<\infty} |\Psi_s * f| \right\|_{L^{p,\infty}(\mathbb{R}^n)} < \infty$$

and more generally the Hardy-Lorentz space $H^{p,q}$ the set of distributions f satisfying

$$\|f\|_{H^{p,q}(\mathbb{R}^n)} := \left\| \sup_{0<s<\infty} |\Psi_s * f| \right\|_{L^{p,q}(\mathbb{R}^n)} < \infty.$$

The local Hardy space $h^p(\mathbb{R}^n)$ is the space of all distributions f satisfying

$$\|f\|_{h^p(\mathbb{R}^n)} := \left\| \sup_{0<s<1} |\Psi_s * f| \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

The definitions above are independent (up to equivalence of norms) of the choices of the function Ψ. From the definition, it is easy to see that

$$\|f\|_{h^p} \leqslant \|f\|_{H^p},$$

so that we have an embedding

$$H^p \subset h^p.$$

We note that $\|f\|_{H^p}$ and $\|f\|_{h^p}$ are norms when $p \geqslant 1$, and they are quasi-norms when $0 < p < 1$. When $p > 1$, it is well known that $H^p = L^p$.

2.2 Duality

One motivation for studying H^p rather than L^p when $0 < p < 1$ is that the dual of L^p is degenerate in this case, while the dual of H^p is well-developed. The dual of H^p is an appropriately chosen Companato space; for a good treatment of this space and its duality with H^p, see for instance [21]. However, in computation, it is easier to pass to a related space, the Lipschitz space Λ_α, introduced below.

2.2.1 Lipschitz Spaces

For $\alpha \geqslant 0$ we define the Lipschitz space Λ_α as follows.

- For $\alpha = 0, \Lambda_0 = L^\infty, \|f\|_{\Lambda_0} = \|f\|_{L^\infty}$.
- For $0 < \alpha < 1$, $\Lambda_\alpha = \{f : \exists c > 0 \text{ s.t. } \forall x, y,\ |f(x) - f(y)| \leqslant c\,|x-y|^\alpha\}$,
  $\|f\|_{\Lambda_\alpha} = \inf\{c : \forall x, y,\ |f(x) - f(y)| \leqslant c\,|x-y|^\alpha\}$.
- For $\alpha \geqslant 1$ write $\alpha = k + \beta$, $k \in \mathbb{N}$, $0 \leqslant \beta < 1$.
  $\Lambda_\alpha = \{f : \forall \kappa \text{ multiindex s.t. } |\kappa| = k, f^{(\kappa)} \in \Lambda_\beta\}$,
  $\|f\|_{\Lambda_\alpha} = \max_{|\kappa|=k} \left\|f^{(\kappa)}\right\|_{\Lambda_\beta}$.

Intuitively, we can think of $f \in \Lambda_\alpha$ as meaning that f is "α times" differentiable with bounded derivative. (This certainly makes sense when $\alpha \in \mathbb{N}$.) This intuition suggests that for $\alpha < \beta$, $\Lambda_\beta \subseteq \Lambda_\alpha$; but this is not true in general. We do, however, have the following inclusions.

- If $\alpha < \gamma < \beta$, then $\Lambda_\alpha \cap \Lambda_\beta \subseteq \Lambda_\gamma$.
- If $\alpha < \beta$ and K is compact, then $\Lambda_\beta(K) \subseteq \Lambda_\alpha(K)$.

We don't have to look very far to find an exception to the expected inclusion. For example, $f(x) = x^2 \in \Lambda_2(\mathbb{R})$ but not in $\Lambda_1(\mathbb{R})$.

When $\alpha = n(\frac{1}{p} - 1)$, for any $f \in H^p(\mathbb{R}^n)$, $g \in \Lambda_\alpha(\mathbb{R}^n)$, an easy computation shows the pairing inequality

$$|\langle f, g\rangle| = \|f\|_{H^p} \|g\|_{\Lambda_\alpha}. \tag{2.1}$$

2.2.2 BMO

The dual space of $H^1(\mathbb{R}^n)$ is BMO($\mathbb{R}^n$), the space of bounded mean oscillation. Here, we recall that BMO($\mathbb{R}^n$) is the space of all locally integrable functions f satisfying

$$\|f\|_{\mathrm{BMO}} := \sup \frac{1}{|B|} \int_B \left| f(x) - \frac{1}{|B|} \int_B f(t)\, dt \right| dx < \infty,$$

where the *sup* is taken over all balls B in $\mathbb{R}^n$. It is known that the space L^∞ is a proper subspace of BMO and

$$\log|x| \in \mathrm{BMO} \setminus L^\infty.$$

2.3 Alternate Characterizations of H^p

The Hardy spaces have several equivalent characterizations. In this thesis, we will invoke the atomic decomposition and the Hilbert/Riesz transform characterization.

2.3.1 Atomic Characterization

Let $0 < p \leqslant 1 \leqslant q \leqslant \infty, p \neq q, s \geqslant [n(\frac{1}{p} - 1)]$, the integer part of $[n(\frac{1}{p} - 1)]$. We say that a function $a(x) \in L^q(\mathbb{R}^n)$ is a (p, q, s) atom with the center at x_0, if it satisfies the following conditions:

Support Condition

$\quad$ supp $(a) \subset B(x_0, \rho)$, $\rho > 0$, Cancellation Condition

$$\int_{\mathbb{R}^n} y^k a(y)\, dy = 0 \text{ for all multi-indices } k \text{ such that } |k| \leqslant s,$$

Size Condition

$$\|a\|_{L^q} \leqslant \rho^{(\frac{n}{q}-\frac{n}{p})}.$$

A function b is called a small (p,q) block if b satisfies the support and size conditions with $\rho < 1$. A function B is called a big (p,q) block if B satisfies the support and size conditions with $\rho \geqslant 1$.

A well-known theorem by Coifman [3] says that any $f \in H^p(\mathbb{R})$ has an atomic decomposition

$$f = \sum \lambda_j a_j,$$

where $\{\lambda_j\} \in \ell^p$, and that

$$\|f\|_{H^p}^p \cong \inf \left\{ \sum |\lambda_j|^p : f = \sum \lambda_j a_j,\ a_j \text{ are } (p,q,s) \text{ atoms} \right\}.$$

It is also well known

$$\|a\|_{L^p} \leqslant \|a\|_{H^p} \lessapprox 1$$

uniformly for all (p,q,s) atoms if $s > n(\frac{1}{p} - 1)$.

The space h^p has an similar decomposition that was discovered by Goldberg in [6]. Namely, any $f \in h^p(\mathbb{R})$ has a decomposition

$$f = \sum \lambda_j a_j,$$

where

$$\|f\|_{h^p}^p \cong \inf \left\{ \sum |\lambda_j|^p : f = \sum \lambda_j a_j \right\}$$

and each a_j is a (p, q, s) atom or a big (p, q) block. A simple computation (or see [6]) shows

$$\|B\|_{h^p} \leqslant 1$$

uniformly for all big (p, q) blocks B. We establish a boundedness result for some small blocks as Lemma 4.1.

2.3.2 Hilbert and Riesz Transform

When $n = 1$, another important characterization of $H^p(\mathbb{R})$ is the one involving the Hilbert transform

$$\mathcal{R}f(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

The Hilbert transform is a very important operator in analysis. From [27] we know that a non-identity linear transform T commutes with dilation, translation and reflection if and only if T is the Hilbert transform, up to a constant multiple. It is known that the H^p space can be characterized by the Hilbert transform in the sense that, for all $f \in H^p \cap L^2$,

$$\|f\|_{H^p} \cong \|\mathcal{R}f\|_{L^p} + \|f\|_{L^p}.$$

By checking the Fourier transform, it is easy to see that

$$\widehat{\mathcal{R}f(\xi)} = i\,\mathrm{sgn}(\xi)\widehat{f}(\xi),$$

so $-\mathcal{R}^2 = \mathrm{Id}$ is the identity operator. The relationship between the Hilbert transform and the Hausdorff operator was studied in [17] by Liflyand and Móricz . Particularly, they obtained the identity

$$\mathcal{R}(h_\Phi f)(x) = h_\Phi(\mathcal{R}f)(x).$$

When $n \geqslant 2$, one can similarly characterize the space H^p by using the Riesz transform. For $j \in \{1, 2, \ldots, n\}$, the jth Riesz transform $\mathcal{R}_j$ is defined by

$$\mathcal{R}_j(f)(x) = p.v. \int_{\mathbb{R}^n} c_n \frac{y_j}{|y|^{n+1}} f(x-y)\, dy,$$

where the constant c_n is given by

$$c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{\frac{n+1}{2}}.$$

The Fourier transform of the kernel of $\mathcal{R}_j$ is called the symbol of $\mathcal{R}_j$. From [25], we know that the symbol of $\mathcal{R}_j$ is $i\xi_j / |\xi|$. From this, we can easily see that

$$\sum_{j=1}^{n} \mathcal{R}_j^2 = -\mathrm{Id},$$

where Id is the identity map. We also denote

$$\mathcal{R}_0(f)(x) = f(x).$$

For an integer $L \geqslant 0$, and a multi-index $J = \{j_1, \ldots, j_L\} \in \{0, 1, 2, \ldots, n\}^L$, let $\mathcal{R}_J(f)$ denote the generalized Riesz transform $\mathcal{R}_J(f) = \mathcal{R}_{j_1} \ldots \mathcal{R}_{j_L} f$. It is known that for L such that $p > \frac{n-1}{n-1+L}$ and all $f \in L^2 \cap H^p(\mathbb{R}^n)$,

$$\sum_J \|\mathcal{R}_J(f)\|_{L^p(\mathbb{R}^n)} \cong \|f\|_{H^p(\mathbb{R}^n)},$$

where the sum is taken over all $J \in \{0, 1, 2, \ldots, n\}^L$. It follows easily that Riesz transforms are bounded on H^p for all $0 < p \leqslant 1$.

We have an analogous result for the weak Hardy spaces:

$$\sum_J \|\mathcal{R}_J(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \cong \|f\|_{H^{p,\infty}(\mathbb{R}^n)}.$$

Particularly, for $n = 1$, we have

$$\|f\|_{H^{p,\infty}} \cong \|\mathcal{R} f\|_{L^{p,\infty}} + \|f\|_{L^{p,\infty}}$$

for $f \in H^{p,\infty} \cap L^2$.

2.4 Interpolation

One major tool we will use to generate boundedness results is interpolation, which allows us to obtain strong boundedness from weak boundedness. In particular, we will use two interpolation theorems: Stein-Weiss analytic interpolation [27] and Marcinkiewisz interpolation.

2.4.1 Stein-Weiss Analytic Interpolation

Let $(M, \mathcal{A}, \mu)$ and $(N, \mathcal{B}, \nu)$ be two measure spaces and $D(M)$ and $D(N)$ be the sets of all simple functions on $(M, \mathcal{A}, \mu)$ and $(N, \mathcal{B}, \nu)$ respectively. Define the set Δ to be the strip

$$\Delta = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant 1\}$$
$$\mathring{\Delta} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}.$$

Definition 2.1. *Let T_z be a linear operator $T_z : D(M) \to D(N)$ for each $z \in \Delta$. If for any $f \in D(M), g \in D(N)$,*

$$h(z) = \int_N T_z(f) g \, dv \tag{2.2}$$

is analytic on $\mathring{\Delta}$ and continuous on Δ, and there exists a constant $a < \pi$, such that

$$e^{-a|y|} \log |h(z)|, \quad z = x + iy$$

has an upper bound on Δ, then we call the family of the operators $\{T_z\}$ admissible.

Theorem F (Stein-Weiss)**.** *Let $\{T_z\}$ be an admissible family and $z = x + iy$. If*

$$\|T_{iy}(f)\|_{L^{q_0,\infty}(N)} \leqslant M_0(y) \|f\|_{L^{p_0}(M)}$$
$$\|T_{1+iy}(f)\|_{L^{q_1,\infty}(N)} \leqslant M_1(y) \|f\|_{L^{p_1}(M)}.$$

for all $f \in D(M)$, and if there exists $b < \pi$ for which

$$\sup_{y \in \mathbb{R}} e^{-b|y|} \log M_j(y) < \infty, \quad j \in \{1, 2\}, \tag{2.3}$$

then for $t \in (0,1)$*, and* p_t, q_t *be defined by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|T_t(f)\|_{L^{q_t}(N)} \leqslant M_t \|f\|_{L^{p_t}(M)}$$

for all $f \in D(M)$*, where* M_t *is a bounded constant that can be computed using the three circle theorem (a theorem in Complex Analysis).*

We will be using Stein-Weiss analytic interpolation to prove results about spaces of Schwartz distributions. In general measure spaces, we have no derivatives, no Schwartz space, no C^∞, etc. For our purposes, however, we will fix $M = N = \mathbb{R}^n$. Further, since in Stein-Weiss analytic interpolation we need the transformations T_z to work on dense subspaces of our distribution spaces, we will replace $D(M)$, $D(N)$ by

$$C_c^\infty(\mathbb{R}^n) = \{f \in C^\infty : f \text{ has compact support}\}.$$

In this context, we can replace equation (2.2) by

$$h(z) = \int_{\mathbb{R}^n} T_z(f)(x)g(x)\, d\varpi(x), \tag{2.4}$$

where $d\varpi(x)$ is some measure on $\mathbb{R}^n$. We will use the following corollary of Theorem F:

Corollary 2.2. *Let* $\{T_z\}$ *be an admissible family and* $z = x + iy$*. If*

$$\|T_{iy}(f)\|_{L^{q_0,\infty}(\mathbb{R}^n, d\varpi)} \leqslant M_0(y) \|f\|_{L^{p_0}(\mathbb{R}^n, d\varpi)}$$

$$\|T_{1+iy}(f)\|_{L^{q_1,\infty}(\mathbb{R}^n, d\varpi)} \leqslant M_1(y) \|f\|_{L^{p_1}(\mathbb{R}^n, d\varpi)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$*, and there exists* $b < \pi$ *for which*

$$\sup_{y \in \mathbb{R}} e^{-b|y|} \log M_j(y) < \infty, \quad j \in \{1,2\}$$

then for $t \in (0,1)$, *and* p_t, q_t *be defined by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|T_t(f)\|_{L^{q_t}(\mathbb{R}^n, d\varpi)} \leqslant M_t \|f\|_{L^{p_t}(\mathbb{R}^n, d\varpi)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, *where* M_t *is a bounded constant, and* $d\varpi$ *is some measure on* $\mathbb{R}^n$.

2.4.2 Marcinkiewisz Interpolation

T is called a quasilinear operator if there exists a constant $C > 0$ such that T satisfies

$$|T(f+g)(x)| \leqslant C\left(|Tf(x)| + |Tg(x)|\right)$$

for almost every x. An operator T (possibly quasilinear) satisfying an estimate of the form

$$\|Tf\|_{L^{q,\infty}} \leqslant C \|f\|_{L^p}$$

is said to be of weak type (p,q). An operator is simply of type (p,q) if T is a bounded transformation from L^p to L^q:

$$\|Tf\|_{L^q} \leqslant C \|f\|_{L^p}.$$

Now we are ready to recall the Marcinkiewicz interpolation theorem (see [27] for more details).

Lemma 2.3 (Marcinkiewicz Interpolation)**.** *If* T *is a quasilinear operator of weak type* (p_0, q_0) *and of weak type* (p_1, q_1) *where* $q_0 \neq q_1$, *then for each* $\theta \in (0,1)$, T *is of type* (p,q), *for* p *and* q *with* $p \leqslant q$ *of the form*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

2.5 Herz Type Spaces

Many of the results we would like to obtain for higher dimensional Hardy spaces can be more easily obtained on a subspace, the so-called Herz-type Hardy space. Herz type spaces are important function spaces in harmonic analysis. Lu and Yang have made tremendous contributions on these spaces. Their book [20] (joint with Hu) is the unique research book on this topic.

For each $k \in \mathbb{Z}$, define

$$B_k = \left\{x \in \mathbb{R}^n : |x| \leqslant 2^k\right\}, \quad E_k = B_k \backslash B_{k-1}$$

and let χ_k denote the characteristic function of E_k.

Definition 2.4 (Homogeneous Herz Space)**.** *Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$\dot{K}_q^{\alpha,p} = \left\{f \in L^q_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{p}}.$$

Definition 2.5 (Herz-type Hardy Space)**.** *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$. The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in S'(\mathbb{R}^n) : Gf \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\right\},$$

where Gf is the grand maximal function of f and

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Similar to the Hardy spaces, the space $H\dot{K}_q^{\alpha,p}$ can be decomposed into atoms.

Definition 2.6 (Central Atom)**.** *Suppose $1 < q < \infty$, $n(1-\frac{1}{q}) \leqslant \alpha < \infty$, and $s \geqslant [\alpha + n(\frac{1}{q} - 1)]$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central (α, q) atom if* *(i)* *supp* $a \subset B(0,\rho)$ *,*

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leqslant |B(0,\rho)|^{-\alpha/n}$,

(iii) $\int_{\mathbb{R}^n} a(x)x^\beta\, dx = 0$ *for any multi-index* β *with* $|\beta| \leqslant s$.

It is known that, for $0 < p < \infty$, $1 < q < \infty$, and $n(1-\frac{1}{q}) \leqslant \alpha < \infty$, $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if there exist a sequence of numbers $\{\lambda_k\} \in \ell^p$ and a sequence of central (a,q) atoms $\{a_k\}$ with the support in B_k such that

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$$

in S'. Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \cong \inf\left\{\left(\sum_k |\lambda_k|^p\right)^{\frac{1}{p}} : f = \sum_k \lambda_k a_k \text{ is an atomic decomposition of } f\right\}.$$

By the definition, it is not difficult to see

$$\dot{K}_p^{\alpha,p} = L^p \text{ if } \alpha = 0 \text{ and } p \geqslant 1.$$

When $0 < p \leqslant 1$, and $\frac{n}{p} = \alpha + \frac{n}{q}$, $H\dot{K}_q^{\alpha,p}$ is a subspace of H^p.

CHAPTER 3

ONE DIMENSIONAL HAUSDORFF OPERATORS: THE CASE $0 < p < 1$

In this chapter, we obtain a boundedness result for Hausdorff operators

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right)\, dt$$

on one-dimensional Hardy spaces H^p, $0 < p < 1$, which requires a relatively simple smoothness condition on the function Φ. First, we recall the following result, due to Liflyand and Miyachi:

Theorem B ([14])**.** *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. If $\Phi \in C^M$ and its support is a compact set in $(0, \infty)$, then the Hausdorff operator h_Φ is bounded on $H^p(\mathbb{R})$.*

Liflyand and Miyachi obtained further generalizations of Theorem B (Theorems C and D, see page 8), which imply the boundedness of the Cesàro operator C_α for all $\alpha, p > 0$. However, the main purpose of this chapter is to use a different method from those in [9] and [14] to obtain a new sufficient condition on Φ to ensure the $H^p(\mathbb{R})$ boundedness of h_Φ. We will establish the following result.

Theorem 3.1. *Let Φ be a Lebesgue integrable function. Denote $\phi(t) = \frac{\Phi(\frac{1}{t})}{t}$ for $t > 0$, and $\phi(t) = 0$ for $t \leqslant 0$. Assume $0 < p < 1$ and $\alpha = \frac{1}{p} - 1$. If $\phi \in \Lambda_\alpha$ then*

$$\|h_\Phi(f)\|_{H^{p,\infty}(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

If there exists a small $\epsilon > 0$, such that $\alpha - \epsilon > 0$ and $\phi \in \Lambda_{\alpha+\epsilon} \cap \Lambda_{\alpha-\epsilon}$, then

$$\|h_\Phi(f)\|_{H^p(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

Clearly, Theorem 3.1 is an improvement of Theorem B. But we should point out that the theorem is mutually independent to Theorems C and D. First, Theorems C and D do not imply Theorem 3.1, since Theorem 3.1 assumes less smoothness condition. On the other hand, although in Theorems C and D, Φ is assumed more smoothness on the set $(0,a) \cup (a,\infty)$, Φ is allowed to have very little smoothness at a single point a. With this advantage, one easily sees that Theorems C and D imply the $H^p(\mathbb{R})$ boundedness of the Cesàro operator C_α for any $a > 0$, while the $H^p(\mathbb{R})$ boundedness of the Cesàro operator can not be deduced from Theorem 3.1 if $\alpha < 1/p$.

Proof of Theorem 3.1. The Hardy space H^p is a distribution space when $0 < p < 1$. However, it suffices to show the theorem for functions f in the space $H^p \cap L^2$, since this space is dense in H^p. Recall that the H^p space can be characterized by the Hilbert transform in the sense of

$$\|f\|_{H^p} \cong \|\mathcal{R}f\|_{L^p} + \|f\|_{L^p}$$

for all $f \in H^p \cap L^2$. By Theorem 3 in [17],

$$\mathcal{R} \circ h_\Phi(f)(x) = h_\Phi(\mathcal{R}f)(x) = \int_{-\infty}^{\infty} \frac{\Phi(t)}{t} \left(\mathcal{R}f\right)(\frac{x}{t})dt.$$

Changing variables $\frac{1}{t} = v$, by the definition of ϕ, we have

$$\begin{aligned} |\mathcal{R} \circ h_\Phi(f)(x)| &= \left| \int_{-\infty}^{\infty} \frac{\Phi(\frac{1}{v})}{v} \left(\mathcal{R}f\right)(xv)dv \right| \\ &= \left| \int_{-\infty}^{\infty} \phi(v) \left(\mathcal{R}f\right)(xv)dv \right|. \end{aligned}$$

Thus, by duality and scaling,

$$\begin{aligned} |\mathcal{R} \circ h_\Phi(f)(x)| &\leqslant \|\phi\|_{\Lambda_\alpha} \|\mathcal{R}f(x\cdot)\|_{H^p} \\ &= |x|^{-1/p} \|\phi\|_{\Lambda_\alpha} \|\mathcal{R}f\|_{H^p} \\ &\preceq |x|^{-1/p} \|\phi\|_{\Lambda_\alpha} \|f\|_{H^p}, \end{aligned}$$

where the last inequality is true because the Hilbert transform is bounded on H^p for any $p > 0$. Now, for any $\lambda > 0$, it is easy to see that

$$\begin{aligned} &|\{x : |\mathcal{R} \circ h_\Phi(f)(x)| > \lambda\}| \\ &\preceq \left| \{x \in \mathbb{R} : |x|^{-1/p} \|\phi\|_{\Lambda_\alpha} \|f\|_{H^p} > \lambda\} \right| \end{aligned}$$

$$\begin{aligned} &= \left|\{x \in \mathbb{R} : |x|^{1/p} < \frac{\|\phi\|_{\Lambda_\alpha} \|f\|_{H^p}}{\lambda}\}\right| \\ &\simeq \left|\{x \in \mathbb{R} : |x| < \left\{\frac{\|\phi\|_{\Lambda_\alpha} \|f\|_{H^p}}{\lambda}\right\}^p\}\right|. \end{aligned}$$

This shows that for $\alpha = \frac{1}{p} - 1$,

$$\|\mathcal{R} \circ h_\Phi(f)\|_{L^{p,\infty}} \preceq \|f\|_{H^p}.$$

Similarly we can show that

$$\|h_\Phi(f)\|_{L^{p,\infty}} \preceq \|f\|_{H^p}$$

if $\alpha = \frac{1}{p} - 1$. These two inequalities imply, by the Hilbert transform characterization of $H^{p,\infty}$, that

$$\|h_\Phi(f)\|_{H^{p,\infty}(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

Put

$$p_1 = \frac{1}{\alpha + \epsilon + 1}, \quad p_2 = \frac{1}{\alpha - \epsilon + 1}.$$

By the above weak H^p estimate, we have that for $j = 1, 2$,

$$\begin{aligned} \|\mathcal{R} \circ h_\Phi(f)\|_{L^{p_j,\infty}(\mathbb{R})} &\preceq \|f\|_{H^{p_j}(\mathbb{R})}, \\ \|h_\Phi(f)\|_{L^{p_j,\infty}(\mathbb{R})} &\preceq \|f\|_{H^{p_j}(\mathbb{R})}. \end{aligned}$$

Then it follows from the Marcinkiewicz interpolation theorem that,

$$\begin{aligned} \|\mathcal{R} \circ h_\Phi(f)\|_{L^p(\mathbb{R})} &\preceq \|f\|_{H^p(\mathbb{R})} \\ \|h_\Phi(f)\|_{L^p(\mathbb{R})} &\preceq \|f\|_{H^p(\mathbb{R})}. \end{aligned}$$

Therefore, we have

$$\|h_\Phi(f)\|_{H^p(\mathbb{R})} \simeq \|\mathcal{R} \circ h_\Phi(f)\|_{L^p(\mathbb{R})} + \|h_\Phi(f)\|_{L^p(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

This completes the proof of Theorem 3.1. $\square$

We have the following corollary:

Corollary 3.2. *Suppose* $\alpha > \frac{1}{p} - 1$ *and* α *is a positive integer. Assume that* $\Phi \in C^{\alpha} \cap L^{1}$ *and satisfies*

$$\begin{aligned} \chi_{(1,\infty)}(t)\left|\Phi^{(k)}(t)\right| &\preceq t^{-2k-1-\alpha} \text{ for all } k = 0, 1, 2, ..., \alpha, \\ \chi_{(0,1)}(t)\left|\Phi^{(k)}(t)\right| &\preceq t^{-k-1} \text{ for all } k = 0, 1, 2, ..., \alpha. \end{aligned}$$

Then

$$\|h_{\Phi}(f)\|_{H^p(\mathbb{R})} \preceq \|f\|_{H^p(\mathbb{R})}.$$

Proof. It is easy to see that if $t \leqslant 1$,

$$\left|\frac{d^{\alpha}}{dt^{\alpha}}\left(\frac{\Phi(\frac{1}{t})}{t}\right)\right| \preceq \sum_{k=0}^{\alpha}\left|\frac{\Phi^{k}(\frac{1}{t})}{t^{1+2k}}\right| \preceq 1;$$

and if $t > 1$

$$\left|\frac{d^{\alpha}}{dt^{\alpha}}\left(\frac{\Phi(\frac{1}{t})}{t}\right)\right| \preceq \sum_{k=0}^{\alpha}\left|\frac{\Phi^{k}(\frac{1}{t})}{t^{\alpha+1}}\right| \preceq 1.$$

Hence, the function Φ satisfies the conditions of Theorem 3.1. □

In the capacity of Φ in Theorem 3.1, one may take a number of usual Fourier multipliers, for example, $e^{-|t|}, e^{-t^2/2}$, which correspond to the Poisson and Gaussian kernels, respectively. It is also worthwhile to investigate the Riesz multiplier

$$\Phi(t) = (1 - t^2)_{+}^{\delta},$$

where $f(t)_+$ is the function that is equal to 0 if $f(t) \leqslant 0$ and is equal to $f(t)$ if $f(t) > 0$. It is easy to check that $(1-t^2)_+^{\delta}$ satisfies the conditions in Theorem 1 if $\delta > \frac{1}{p} - 1$. We observe that Stein, Taibleson and Weiss in [26] proved that the Bochner-Riesz operator $B_{\delta} * f$ is bounded on $H^p(\mathbb{R}^n)$ if $\delta > \frac{n}{p} - \frac{n+1}{2}$, where

$$(\widehat{B_{\delta} * f})(\xi) = (1 - |\xi|^2)_{+}^{\delta}\widehat{f}(\xi).$$

Therefore, when $n = 1$, our result matches the critical index obtained by Stein, Taibleson and Weiss. With this observation, it will be very interesting to extend Theorem 3.1 to the higher dimensional Hausdorff operator. This generalization is the subject of Chapter 5.

CHAPTER 4

ONE DIMENSIONAL HAUSDORFF OPERATORS: THE CASE $p = 1$

4.1 Introduction

This chapter will show that, for $\Phi(t) \geqslant 0$, the Hausdorff operator h_Φ is bounded on the Hardy space $H^1(\mathbb{R})$ if and only if Φ is a Lebesgue integrable function; and h_Φ is bounded on the local Hardy space $h^1(\mathbb{R})$ if and only if the function $\Phi(t)\chi_{(0,1)}(t)\log(\frac{1}{t})$ is Lebesgue integrable. We also establish a weak type $H^1(\mathbb{R})$ boundedness theorem for h_Φ. As an application, we conclude that the Hardy operator $\mathcal{H}$ is not bounded on either $H^1(\mathbb{R})$ or $h^1(\mathbb{R})$, but it is bounded from $H^1(\mathbb{R})$ to the weak space $H^{1,\infty}(\mathbb{R})$. We also study the boundedness property for the kth order Hardy operator $\mathcal{H}_{(k)}$ and fractional Hardy operator $\mathcal{H}_{(k),\alpha}$ on spaces $H^p(\mathbb{R})$ for $k \geqslant \frac{1}{p} - 1 \geqslant 0,\ 0 \leqslant \alpha < 1$.

One operator in particular will serve as a model of Hausdorff operators acting on $H^1(\mathbb{R})$. We begin with a discussion of the Hardy operator.

4.1.1 The Hardy Operator and its Generalizations

The Hardy operator $\mathcal{H}$ is defined by

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)\, dt.$$

It is known that the operator $\mathcal{H}$ is bounded on $L^p(\mathbb{R})$, for all $1 < p \leqslant \infty$ with the best bound constant $\frac{p}{p-1}$ (or 1 for $p = \infty$), and $\mathcal{H}$ is not bounded on $L^1(\mathbb{R})$(see [2], also [7]). It is also known that some important operators are bounded on $H^1(\mathbb{R})$ but not on $L^1(\mathbb{R})$. Thus a natural question is whether the Hardy operator is bounded on $H^1(\mathbb{R})$.

After a changing of variables, we have

$$\mathcal{H}f(x) = \int_0^\infty \frac{\chi_{(1,\infty)}(t)\ t^{-1}}{t} f(\frac{x}{t})\, dt.$$

Thus, $\mathcal{H}$ is a Hausdorff operator with $\Phi = \frac{\chi_{(1,\infty)}(t)}{t}$.

Two generalizations of the Hardy operator will also be of interest: the kth Hardy operator $\mathcal{H}_{(k)}$ $(k = 0, 1, 2, \ldots)$

$$\begin{aligned}\mathcal{H}_{(k)}(f)(x) &= \frac{1}{x^{k+1}} \int_0^x t^k f(t)\, dt \\ &= \int_1^\infty \frac{1}{t^{k+2}} f(\frac{x}{t})\, dt\end{aligned}$$

and the kth fractional Hardy operator $\mathcal{H}_{(k),\alpha}$ $(k = 0, 1, 2, \ldots,\ 0 \leqslant \alpha < 1)$

$$\mathcal{H}_{(k),\alpha}(f)(x) = \frac{1}{x^{k+1-\alpha}} \int_0^x t^k f(t)\, dt.$$

It is easy to see that $\mathcal{H}_{(0)} = \mathcal{H}$ and $\mathcal{H}_{(k),0} = \mathcal{H}_{(k)}$.

4.1.2 A Lemma in Local Hardy Spaces

Recall from that the local Hardy space h^1 is characterized by decomposition into atoms and big blocks. In order to demonstrate a bound on Hausdorff operators on h^1, it is useful to establish bounds on the h^1 norm of both big and small blocks.

A simple computation shows (or see [6])

$$\|B\|_{h^p} \lesssim 1$$

uniformly for all big (p, q) blocks B. For small blocks, we have the following estimate.

Lemma 4.1. *For a small* $(1, q)$ *block* b *with support in* $(x_0 - r, x_0 + r)$*, we have*

$$\|b\|_{h^1} \lesssim 1 + \log \frac{1}{r}$$

uniformly on r *and* x_0*.*

Proof. By a change of variables, we may assume that the support of b is in the interval $(-r, r)$. Let $B(u) = rb(ru)$. It is easy to see that supp $B(u) \subseteq (-1, 1)$ and (after a simple computation) $\|B\|_{L^q} \lesssim 1$; so $B(u)$ is a big $(1, q)$-block. Choose $\Psi(y) = e^{-|y|^2}$

in the definition of $h^1(\mathbb{R})$. We have

$$\begin{aligned}
\|b\|_{h^1(\mathbb{R})} &= \int_{\mathbb{R}} \sup_{0<s\leqslant 1} \left| \int_{\mathbb{R}} \Psi_s(x-u)b(u)\,du \right| \,dx \\
&= \int_{\mathbb{R}} \sup_{0<s\leqslant 1} \left| \int_{\mathbb{R}} \Psi_s(x-ru)rb(ru)\,du \right| \,dx \\
&= \int_{\mathbb{R}} \sup_{0<s\leqslant 1} r\left| \int_{\mathbb{R}} \Psi_s(r(x-u))B(u)\,du \right| \,dx \\
&= \int_{\mathbb{R}} \sup_{0<s\leqslant 1} \left| \int_{\mathbb{R}} \frac{1}{(s/r)}\Psi\left(\frac{x-u}{s/r}\right)B(u)\,du \right| dx \\
&= \int_{\mathbb{R}} \sup_{0<s\leqslant \frac{1}{r}} \left| \int_{\mathbb{R}} \Psi_s(x-u)B(u)\,du \right| \,dx \\
&= \int_{\mathbb{R}} \sup_{0<s\leqslant \frac{1}{r}} |\Psi_s * B(x)|\,dx.
\end{aligned}$$

Let

$$N = \left[\log_2 \frac{1}{r}\right] + 1,$$

then (recalling that $\frac{1}{r} > 1$)

$$\begin{aligned}
\sup_{0<s\leqslant \frac{1}{r}} |\Psi_s * B(x)| &\leqslant \sup_{0<s\leqslant 1} |\Psi_s * B(x)| + \sum_{k=0}^{N} \sup_{2^k<s\leqslant 2^{k+1}} |\Psi_s * B(x)| \\
&\leqslant \sup_{0<s\leqslant 1} |\Psi_s * B(x)| + \sum_{k=0}^{N} \sup_{2^k<s\leqslant 2^{k+1}} |\Psi_s| * |B(x)|, \\
&\leqslant \sup_{0<s\leqslant 1} |\Psi_s * B(x)| + \sum_{k=0}^{N} (\tilde{\Psi}_k * |B|)(x)
\end{aligned}$$

where

$$\tilde{\Psi}_k(u) = \frac{1}{2^k}\Psi(\frac{u}{2^{k+1}}).$$

This gives

$$\int_{\mathbb{R}} \sup_{0<s\leqslant \frac{1}{r}} |\Psi_s * B(x)| \,dx \leqslant \left(\left\| \sup_{0<s<1} |\Psi_s * B| \right\|_{L^1(\mathbb{R})} + \sum_{k=0}^{N} \left\| \tilde{\Psi}_k * |B| \right\|_{L^1(\mathbb{R})} \right).$$

It is easy to check that there is a constant C independent of k such that

$$\left\| \tilde{\Psi}_k * |B| \right\|_{L^1(\mathbb{R})} \leqslant C \|B\|_{L^1(\mathbb{R})} \leqslant C.$$

Hence,

$$\|b\|_{h^1(\mathbb{R})} \lesssim N\,\|B\|_{L^1} + \|B\|_{h^1(\mathbb{R})} \lesssim 1 + \log_2 \frac{1}{r}. \qquad \square$$

4.2 Necessary and Sufficient Conditions for H^1 Boundedness

In [15, 18], Liflyand and Móricz proved that the Hausdorff operator has the same behavior on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$, in the following sense:

Theorem G ([15])**.** *If $\Phi \in L^1(0,\infty)$, then h_Φ is bounded on $H^1(\mathbb{R})$.*

Motivated by the known L^p results, our first aim is to show that $\Phi \in L^1(0,\infty)$ is also a necessity condition in Theorem G, if Φ is nonnegative valued. Precisely, we establish the following result.

Theorem 4.2. *Let Φ be a nonnegative valued locally integrable function.*

(1) h_Φ is bounded on $H^1(\mathbb{R})$ if and only if $\Phi \in L^1(0,\infty)$.

(2) h_Φ is bounded on the local Hardy space $h^1(\mathbb{R})$ if and only if

$$\int_1^\infty \Phi(t)\,dt + \int_0^1 \Phi(t)\left(1 + \log(\tfrac{1}{t})\right)\,dt < \infty.$$

Proof of Part 1. To prove the first part of the theorem, by Theorem G, it suffices to show the only if part. Suppose $\Phi \notin L^1(0,\infty)$, we use the atomic characterization of the space H^1. Let b be a C^∞ odd function with support on $[-1,1]$ which satisfies

$$b(x) = \tfrac{1}{2} \text{ for } x \in [\tfrac{1}{4}, \tfrac{1}{2}], \text{ and } \tfrac{1}{2} \geqslant b(x) \geqslant 0 \text{ for } x \in [0,1].$$

Then b is a $(1,\infty,0)$ atom. Thus, $b \in H^1$. Let $a = -\mathcal{R}(b)$. Recalling that $-\mathcal{R}^2 = \mathrm{Id}$ and that the Hilbert transform is bounded on H^1, we know $a \in H^1$ and $b = \mathcal{R}(a)$.

Now

$$
\begin{aligned}
\|h_\Phi(a)\|_{H^1(\mathbb{R})} &\geq \|\mathcal{R}h_\Phi(a)\|_{L^1(\mathbb{R})} \\
&= \left\| \int_0^\infty \frac{\Phi(t)}{t} \mathcal{R}a(\frac{x}{t})\ dt \right\|_{L^1(\mathbb{R},\ dx)} \\
&= \left\| \int_0^\infty \frac{\Phi(t)}{t} b(\frac{x}{t})\ dt \right\|_{L^1(\mathbb{R},\ dx)}.
\end{aligned}
$$

The last equality holds because the Hilbert transform commutes with the dilation $\frac{1}{t}$. So,

$$
\begin{aligned}
\|h_\Phi(a)\|_{H^1(\mathbb{R})} &\geq \int_0^\infty \left| \int_0^\infty \frac{\Phi(t)}{t} b(\frac{x}{t})\ dt \right|\ dx \\
&= \int_0^\infty \Phi(t)\ dt \int_0^\infty b(x)\ dx = \infty.
\end{aligned}
$$

If h_Φ were bounded on H^1, it could be

$$\|h_\Phi(a)\|_{H^1(\mathbb{R})} \leqslant \|a\|_{H^1} \leqslant 1,$$

which leads to a contradiction. □

Proof of Part 2. To show the second part, we first show sufficiency. For any $f \in h^1(\mathbb{R})$, we may write

$$f(x) = \sum_j \lambda_j a_j(x) + \sum_j \mu_j B_j(x)$$

where each a_j is a $(1, \infty, 0)$ atom and each B_j is a big $(1, \infty)$ block, and

$$\sum_j (|\lambda_j| + |\mu_j|) \cong \|f\|_{h^1(\mathbb{R})}.$$

By the Minkowski inequality,

$$\|h_\Phi(f)\|_{h^1(\mathbb{R})} \leqslant \sum_j |\lambda_j|\, \|h_\Phi(a_j)\|_{h^1(\mathbb{R})} + \sum_j |\mu_j|\, \|h_\Phi(B_j)\|_{h^1(\mathbb{R})}.$$

Since

$$\frac{a_j(\frac{x}{t})}{t} = A_j(x)$$

is again a $(1,\infty,0)$ atom, by the Minkowski inequality

$$\|h_\Phi(a_j)\|_{h^1(\mathbb{R})} \leqslant \int_0^\infty \Phi(t)\,\|A_j\|_{h^1(\mathbb{R})}\,dt \leqslant \int_0^\infty \Phi(t)\,dt.$$

Similarly,

$$\|h_\Phi(B_j)\|_{h^1(\mathbb{R})} \leqslant \int_0^1 \Phi(t)\left\|\frac{1}{t}B_j(\frac{\cdot}{t})\right\|_{h^1(\mathbb{R})} dt + \int_1^\infty \Phi(t)\left\|\frac{1}{t}B_j(\frac{\cdot}{t})\right\|_{h^1(\mathbb{R})} dt.$$

Note that $\frac{1}{t}B_j(\frac{x}{t})$ is again a big $(1,\infty)$ block if $t \geqslant 1$, and it may become a small $(1,\infty)$ block when $t < 1$. Thus, by Lemma 4.1 we have

$$\|h_\Phi(B_j)\|_{h^1(\mathbb{R})} \leqslant \int_0^1 \Phi(t)(1+\log_2(\frac{1}{t}))\,dt + \int_1^\infty \Phi(t)\,dt.$$

Combining all estimates, we obtain

$$\|h_\Phi(f)\|_{h^1(\mathbb{R})} \leqslant C\,\|f\|_{h^1(\mathbb{R})},$$

where

$$C = \int_0^1 \Phi(t)(1+\log_2(\frac{1}{t}))\,dt + \int_1^\infty \Phi(t)\,dt.$$

Conversely, suppose

$$\int_0^1 \Phi(t)(1+\log_2(\frac{1}{t}))\,dt = \infty.$$

Let $\Psi_s(x) = \frac{1}{s}e^{\frac{-x^2}{s^2}}$ and $B(y) = \chi_{[0,\frac{1}{2}]}(y)$. Then $B(y)$ is a big $(1,\infty)$ block. We have

$$\int_{\mathbb{R}} \sup_{0<s\leqslant 1} |\Psi_s * h_\Phi(B)(x)|\,dx \geqslant \int_0^1 \sup_{0<s\leqslant 1} |\Psi_s * h_\Phi(B)(x)|\,dx \geqslant \int_0^1 |\Psi_x * h_\Phi(B)(x)|\,dx.$$

Here

$$\begin{aligned}|\Psi_x * h_\Phi(B)(x)| &= \int_0^\infty \frac{\Phi(t)}{t}\int_{\mathbb{R}} \frac{1}{x}e^{-(\frac{x-y}{x})^2}B(\frac{y}{t})\,dy\,dt \\ &\geqslant \int_0^1 \frac{\Phi(t)}{t}\int_0^{\frac{t}{2}} \frac{1}{x}e^{-(\frac{x-y}{x})^2}\,dy\,dt.\end{aligned}$$

Thus

$$\int_{\mathbb{R}} \sup_{0<s\leqslant 1} |\Psi_s * h_\Phi(B)(x)|\ dx \geqslant \int_0^1 \int_0^1 \frac{\Phi(t)}{t} \int_0^{\frac{t}{2}} \frac{1}{x} e^{-(\frac{x-y}{x})^2}\ dy\ dt\ dx$$
$$\geqslant \int_0^1 \frac{\Phi(t)}{t} \left(\int_t^1 \int_0^{\frac{t}{4}} \frac{1}{x} e^{-(\frac{x-y}{x})^2}\ dy\ dx \right)\ dt$$
$$\geqslant \frac{e^{-4}}{4} \int_0^1 \Phi(t) \left(\int_t^1 \frac{1}{x}\ dx \right)\ dt$$
$$\cong -\frac{e^{-4}}{4} \int_0^1 \Phi(t) \log_2(t)\ dt = \infty.$$

On the other hand, if $h_\Phi(f)$ were bounded on $h^1(\mathbb{R})$, it would be

$$\int_{\mathbb{R}} \sup_{0<s\leqslant 1} |\Psi_s * h_\Phi(B)(x)|\ dx = \|h_\Phi(B)\|_{h^1(\mathbb{R})} \leqslant \|B\|_{h^1(\mathbb{R})} \leqslant 1,$$

which leads to a contradiction. Next, assume

$$\int_1^\infty \Phi(t)\ dt = \infty,$$

and let B be as before. We have

$$\int_{\mathbb{R}} \sup_{0<s\leqslant 1} |\Psi_s * h_\Phi(B)(x)|\ dx \geqslant \int_{\mathbb{R}} |\Psi_1 * h_\Phi(B)(x)|\ dx$$
$$\geqslant \int_1^\infty \frac{\Phi(t)}{t} \int_0^{\frac{t}{2}} \left(\int_{\mathbb{R}} e^{-(x-y)^2}\ dx \right)\ dy\ dt$$
$$\geq \int_1^\infty \Phi(t)\ dt = \infty,$$

so again h_Φ is unbounded. □

Our second aim is to establish conditions for $H^1(\mathbb{R}) \to H^{1,\infty}(\mathbb{R})$ boundedness. To do this, we use an approach similar to that in Chapter 3. We rewrite the Hausdorff operator as

$$h_\Phi f(x) = \int_{-\infty}^\infty \frac{\Phi_0(t)}{t} f\left(\frac{x}{t}\right)\ dt,$$

where $\Phi_0(t)$ is a function that is equal to Φ on $(0, \infty)$ and is equal to 0 on $(-\infty, 0)$.

Denote

$$\phi(t) = \frac{\Phi_0(\frac{1}{t})}{t},$$

Then we have the following result:

Theorem 4.3. *Let* $\phi(t) = t^{-1}\Phi_0(t)$*. Suppose* $\phi \in \mathrm{BMO}(\mathbb{R})$*. Then* h_Φ *extends to a bounded operator from* $H^1(\mathbb{R})$ *to* $H^{1,\infty}(\mathbb{R})$*.*

Proof. It suffices to show the theorem for functions f in the space $H^1 \cap L^2$, since this space is dense in H^1.

By Theorem 3 in [17],

$$\mathcal{R} \circ h_\Phi(f)(x) = h_\Phi(\mathcal{R}f)(x) = \int_{-\infty}^{\infty} \frac{\Phi_0(t)}{t} (\mathcal{R}f)(\frac{x}{t})\, dt.$$

Changing variables $\frac{1}{t} = v$, we have

$$\begin{aligned} |\mathcal{R} \circ h_\Phi(f)(x)| &= \left| \int_{-\infty}^{\infty} \frac{\Phi_0(\frac{1}{v})}{v} (\mathcal{R}f)(xv)\, dv \right| \\ &= \left| \int_{-\infty}^{\infty} \phi(v)(\mathcal{R}f)(xv)\, dv \right|. \end{aligned}$$

Thus, by duality and scaling,

$$\begin{aligned} |\mathcal{R} \circ h_\Phi(f)(x)| &\leqslant \|\phi\|_{\mathrm{BMO}} \|(\mathcal{R}f)(x\cdot)\|_{H^1} \\ &= |x|^{-1} \|\phi\|_{\mathrm{BMO}} \|(\mathcal{R}f)\|_{H^1} \\ &\leqslant |x|^{-1} \|\phi\|_{\mathrm{BMO}} \|f\|_{H^1}. \end{aligned}$$

The last inequality is because that the Hilbert transform is bounded on H^1. Note, for any $\lambda > 0$, it is easy to see that

$$\begin{aligned} &|\{x \in \mathbb{R} : |\mathcal{R} \circ h_\Phi(f)(x)| > \lambda\}| \\ &\leqslant \left| x \in \mathbb{R} : |x|^{-1} \|\phi\|_{\mathrm{BMO}} \|f\|_{H^1} > \lambda \right| \\ &= \left| \left\{ x \in \mathbb{R} : |x| \leqslant \frac{\|\phi\|_{\mathrm{BMO}} \|f\|_{H^1}}{\lambda} \right\} \right| \\ &\cong \frac{\|\phi\|_{\mathrm{BMO}} \|f\|_{H^1}}{\lambda}. \end{aligned}$$

So by the definition,

$$\|\mathcal{R} \circ h_\Phi(f)\|_{L^{1,\infty}} \leqslant \|\phi\|_{\mathrm{BMO}} \|f\|_{H^1}.$$

Similarly, we can show

$$\|h_\Phi(f)\|_{L^{1,\infty}} \leqslant \|\phi\|_{\mathrm{BMO}} \|f\|_{H^1}.$$

These two inequalities imply

$$\|h_\Phi(f)\|_{H^{1,\infty}} \leqslant \|f\|_{H^1(\mathbb{R})}. \qquad \square$$

4.3 Boundedness of the Hardy Operator Near H^1

So far we have considered Hardy spaces defined on the whole real line $\mathbb{R}$. If we wish to consider a Hardy space only on the half line $\mathbb{R}_+ = (0, \infty)$, we might use the atomic Hardy space $H^1_A(\mathbb{R}_+)$ studied by Coifman and Weiss (see [4]).

The space $\mathbb{R}_+ = (0, \infty)$ is a space of homogeneous type. Based on the study by Coifman and Weiss [4], we consider the atomic Hardy space $H^1_A(\mathbb{R}_+)$. Applying the definition of a $(1, \infty, 0)$ atom $a(x)$ (or 1-atom, for the sake of brevity) to the current case, we obtain

Support Condition

$\text{supp } (a) \subset (\alpha, \beta) \subset (0, \infty).$

Cancellation Condition

$\int_{\mathbb{R}_+} a(y)\, dy = 0.$

Size Condition

$\|a\|_{L^\infty} \leqslant (\beta - \alpha)^{-1}.$

The space $H^1_A(\mathbb{R}_+)$ is the collection of all $f \in L^1(\mathbb{R}_+)$ such that

$$f(x) = \sum \lambda_j a_j(x),$$

where each a_j is a 1-atom and $\{\lambda_j\} \in \ell^1$. We define

$$\|f\|_{H^1_A(\mathbb{R}_+)} = \inf\left\{\sum|\lambda_j| : f(x) = \sum\lambda_j a_j(x)\right\}.$$

It is a little surprising to note that the Hardy operator $\mathcal{H}$ is bounded on $H^1_A(\mathbb{R}_+)$.

Theorem 4.4. *For the Hardy operator* $\mathcal{H}$

$$\|\mathcal{H}(f)\|_{H^1_A(\mathbb{R}_+)} \leqslant \|f\|_{H^1_A(\mathbb{R}_+)}.$$

Proof. We need only show that, for any 1-atom a, $\mathcal{H}(a)$ is also a 1-atom. We proceed by showing that $\mathcal{H}(a)$ satisfies the three conditions.

Support Condition

Suppose a has support in (α,β). We show that if $x \notin (\alpha,\beta)$, then $\mathcal{H}(a)(x) = 0$, so supp $(\mathcal{H}(a)) \subseteq (\alpha,\beta)$. Recall

$$\mathcal{H}(a)(x) = \frac{1}{x}\int_0^x a(t)\ dt.$$

Now, if $x \leqslant \alpha$, then $a(t) = 0$ for all $t \in (0,x)$, so $\mathcal{H}(a)(x) = \frac{1}{x}\int_0^x a(t)\ dt = 0$. On the other hand, of $x \geqslant \beta$, then $a(t) = 0$ for $t > x$, and so

$$\begin{aligned}\mathcal{H}(a)(x) &= \frac{1}{x}\int_0^x a(t)\ dt \\ &= \frac{1}{x}\int_0^\infty a(t)\ dt = 0\end{aligned}$$

with the last equality holding by the cancellation condition.

Size Condition

This is easily seen by a computation.

$$\|\mathcal{H}(a)\|_{L^\infty} \leqslant \frac{1}{x}\int_0^x \|a\|_{L^\infty}\ dy = \|a\|_{L^\infty} \leqslant \frac{1}{\beta-\alpha}.$$

Cancellation Condition

An easy computation with the support condition on $\mathcal{H}(a)$ gives

$$\int_0^\infty \mathcal{H}(a)(x)\ dx = \int_\alpha^\beta\int_1^\infty \frac{1}{t^2}a\left(\frac{x}{t}\right)\ dt\ dx.$$

Since the above double integral is well defined, by the Fubini theorem we obtain

$$\begin{aligned}\int_0^\infty \mathcal{H}(a)(x)\,dx &= \int_1^\infty \frac{1}{t^2}\int_0^\infty a(\frac{x}{t})\,dx\,dt \\ &= \lim_{N\to\infty}\int_1^N \frac{1}{t^2}\int_0^\infty a(\frac{x}{t})\,dx\,dt \\ &= \lim_{N\to\infty}\int_1^N \frac{1}{t}\left(\int_0^\infty a(x)\,dx\right)\,dt = 0.\end{aligned}$$

Thus $\mathcal{H}(a)$ is a 1-atom, and so for $f \in H^1_A(0,\infty)$,

$$\mathcal{H}(f) = \mathcal{H}\left(\sum \lambda_j a_j(x)\right) = \sum \lambda_j \mathcal{H}(a_j)(x)$$

expresses $\mathcal{H}(f)$ as a sum of atoms, and so

$$\|\mathcal{H}(f)\|_{H^1_A(0,\infty)} \leqslant \sum |\lambda_j| = \|f\|_{H^1_A(0,\infty)}$$

which is the desired bound. □

In [5], García-Cuerva, and Rubio De Francia discuss a connection between $H^1_A(\mathbb{R}_+)$ and even elements of $H^1(\mathbb{R})$(see [5, lemmas 7.39 and 7.40] especially.) This suggests that the following result is a corollary to Theorem 4.4, and indeed, this is what we see.

Corollary 4.5. *Let*

$$f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$$

be the even part of f. If $f \in H^1$, then we have

$$\|\mathcal{H}(f_e)\|_{H^1} \leqslant \|f\|_{H^1}.$$

Particularly, we have

$$\|\mathcal{H}(f)\|_{H^1} \leqslant \|f\|_{H^1}$$

for all even functions $f \in H^1$.

Proof. We write

$$f(x) = \sum \lambda_j a_j(x)$$

with each a_j being a $(1, \infty, 0)$ atom and

$$\sum |\lambda_j| \cong \|f\|_{H^1}.$$

As in the proof of Lemma 7.39 on page 353 in [5], we may write

$$f_e(x) = \sum \lambda_j A_j(x)$$

where

$$A_j(x) = \frac{a_j(x) + a_j(-x)}{2}.$$

If $a_j(x)$ has support in $[0, \infty)$, then $a_j^-(x) = a_j(-x)$ has support in $(-\infty, 0]$. Following the proof of Theorem 4.4, we know both $\mathcal{H}(a_j)$ and $\mathcal{H}(a_j^-)$ are $(1, \infty, 0)$ atoms so that

$$\|\mathcal{H}(A_j)\|_{H^1} \leqslant 1.$$

If 0 is interior to the interval-support (α_j, β_j) of a_j, then without loss of generality, we assume $\beta_j > |\alpha_j|$. Thus A_j is supported in the interval $(-\beta_j, \beta_j)$, and

$$\frac{\beta_j - \alpha_j}{2} \leqslant \beta_j \leqslant \beta_j - \alpha_j.$$

On the other hand, since A_j is an even function, with the cancellation condition we have

$$0 = \int_{-\beta_j}^{\beta_j} A_j(x)\, dx = 2\int_0^{\beta_j} A_j(x)\, dx = 2\int_{-\beta_j}^{0} A_j(x)\, dx.$$

This indicates that, without loss of generality, both $\chi_{(0,\infty)}(x)A_j(x)$ and $\chi_{(-\infty,0)}(x)A_j(x)$ are $(1, \infty, 0)$ atoms. Thus, again we have

$$\|\mathcal{H}(A_j)\|_{H^1} \leqslant \left\|\mathcal{H}(\chi_{(-\infty,0)}A_j)\right\|_{H^1} + \left\|\mathcal{H}(\chi_{(0,\infty)}A_j)\right\|_{H^1} \leqslant 1.$$

This completes the proof. □

Corollary 4.6. *The Hardy operator $\mathcal{H}$ has the following properties*

(1) $\mathcal{H}$ is not bounded on $H^1(\mathbb{R})$.

(2) $\mathcal{H}$ is not bounded on $h^1(\mathbb{R})$.

(3) $\mathcal{H}$ is bounded from $H^1(\mathbb{R})$ to $H^{1,\infty}(\mathbb{R})$.

(4) $\mathcal{H}$ is bounded on $H^1_A(0,\infty)$.

Proof. Recalling that $\mathcal{H} = h_\Phi$ with $\Phi = \frac{\chi_{(1,\infty)}(t)}{t}$, and noting that $\Phi \notin L^1(0,\infty)$, we see parts (1) and (2) from Theorem 4.2. However, noting that in this case $\phi(t) = \chi_{(0,1)}(t) \in$ BMO, Theorem 4.3 gives part (3). Part (4) is the result in Theorem 4.4, above. □

4.4 Generalizations of the Hardy Operator $(0 < p < 1)$

We now turn to the kth order Hardy operator

$$\mathcal{H}_{(k)}(f)(x) = \frac{1}{x^{k+1}} \int_0^x t^k f(t)\, dt,$$

Theorem 4.7. *Let $0 < p < 1$.*

(1) $\mathcal{H}_{(k)}$ is bounded from $H^p(\mathbb{R})$ to $H^{p,\infty}(\mathbb{R})$ if $k = \frac{1}{p} - 1$.

(2) $\mathcal{H}_{(k)}$ is bounded on $H^p(\mathbb{R})$ if $k > \frac{1}{p} - 1$.

Proof. We begin by showing the first part. Then, with the easy fact that $\mathcal{H}_{(k)}$ is weakly bounded on H^1 (by Theorem 4.3), we complete the proof of the theorem by using an interpolation argument.

Choose p such that $\frac{1}{p} - 1 = k$. It suffices to show that $\left\|\mathcal{H}_{(k)}(f)\right\|_{H^{p,\infty}} \lesssim \|f\|_{H^p}$ for $f \in H^p \cap L^2$, since $H^p \cap L^2$ is dense in H^p. Using the Hilbert transform, we can show this by showing

$$\left\|\mathcal{R} \circ \mathcal{H}_{(k)}(f)\right\|_{L^{p,\infty}} + \left\|\mathcal{H}_{(k)}(f)\right\|_{L^{p,\infty}} \lesssim \|f\|_{H^p} .$$

We show $\left\|\mathcal{R}\circ\mathcal{H}_{(k)}(f)\right\|_{L^{p,\infty}} \lesssim \|f\|_{H^p}$; the proof that $\left\|\mathcal{H}_{(k)}(f)\right\|_{L^{p,\infty}} \lesssim \|f\|_{H^p}$ is the same.

As in the proof of Theorem 4.3, the Hilbert Transform satisfies

$$\mathcal{R}\circ\mathcal{H}_{(k)}(f)(x) = \mathcal{H}_{(k)}(\mathcal{R}f)(x) = \int_1^\infty \frac{1}{t^{k+2}}(\mathcal{R}f)(\frac{x}{t})\ dt.$$

For any $f \in H^p(\mathbb{R})$, $\mathcal{R}f \in H^p$ since the Hilbert transform is bounded on H^p. Thus, we have an atomic decomposition

$$\mathcal{R}f = \sum_j \lambda_j a_j,$$

where each a_j is a (p,∞,k) atom and

$$\sum |\lambda_j|^p \cong \|\mathcal{R}f\|_{H^p}^p .$$

Hence

$$\mathcal{R}\circ\mathcal{H}_{(k)}(f) = \sum_{j\in\mathbb{Z}} \lambda_j \int_1^\infty \frac{1}{t^{k+2}} a_j(\frac{\cdot}{t})\ dt = \sum \lambda_j \mathcal{H}_{(k)}(a_j),$$

and therefore,

$$\left\|\mathcal{R}\circ\mathcal{H}_{(k)}(f)\right\|_{L^{p,\infty}}^p = \left\|\sum \lambda_j \mathcal{H}_{(k)}(a_j)\right\|_{L^{p,\infty}}^p .$$

By [26, Lemma 1.8], showing that

$$\left\|\sum \lambda_j \mathcal{H}_{(k)}(a_j)\right\|_{L^{p,\infty}}^p \lesssim \sum |\lambda_j|^p$$

is equivalent to show that

$$\left\|\mathcal{H}_{(k)}(a_j)\right\|_{L^{p,\infty}}^p \lesssim 1$$

for any (p,∞,k)-atom a_j. Since $\sum |\lambda_j|^p \cong \|\mathcal{R}f\|_{H^p}^p \lesssim \|f\|_{H^p}^p$, this will give the desired result.

If a_j is a (p,∞,k) atom with support in $(\alpha_j,\beta_j) \subset (0,\infty)$ or $(\alpha_j,\beta_j) \subset (-\infty,0)$, then by a method similar to that in the proof of Theorem 4.4, it easy to see that

$\mathcal{H}_{(k)}(a_j)$ is again a (p,∞,k) atom with support in (α_j,β_j). Thus, we have

$$\left\|\mathcal{H}_{(k)}(a_j)\right\|_{H^{p,\infty}} \lesssim \left\|\mathcal{H}_{(k)}(a_j)\right\|_{H^p} \lesssim 1$$

uniformly on $j \in \mathbb{Z}$.

If the support of a (p,∞,k) atom a_j contains the origin, without loss of generality, we may assume

$$\text{supp }(a_j) \subset (-r_j, r_j)$$

and

$$\|a_j\|_{L^\infty} \lesssim r_j^{-\frac{1}{p}}.$$

For each fixed $x \neq 0$, the support of $a_j(tx)$ is in $(-\frac{r_j}{|x|}, \frac{r_j}{|x|})$. Write

$$\mathcal{H}_{(k)}(a_j)(x) = \frac{1}{|x|^{\frac{1}{p}}} \int_{-\infty}^{\infty} \chi_{(0,1)}(t) \left(|x|^{\frac{1}{p}}\, t^k \chi_{(-\frac{r_j}{|x|}, \frac{r_j}{|x|})}(t) a_j(tx)\right) dt.$$

We now claim that, for $x \neq 0$, the function

$$A_{j,x}(t) = |x|^{\frac{1}{p}}\, t^k \chi_{(-\frac{r_j}{|x|}, \frac{r_j}{|x|})}(t) a_j(tx)$$

is a $(1,\infty,0)$ atom supported in the interval $(-\frac{r_j}{|x|}, \frac{r_j}{|x|})$. In fact, the support condition is obvious from the definition. Also,

$$\int_{-\infty}^{\infty} A_{j,x}(t)\, dt \cong \int_{-r_j}^{r_j} t^k a_j(t)\, dt = 0,$$

by the cancellation condition on a_j, and

$$\|A_{j,x}\|_{L^\infty} \lesssim \|a\|_{L^\infty} \left|\frac{r_j}{x}\right|^k |x|^{\frac{1}{p}} \lesssim \left|\frac{x}{r_j}\right|.$$

This shows that $A_{j,x}$ also satisfies the cancellation and size conditions. Thus, by the duality,

$$\left|\mathcal{H}_{(k)}(a_j)(x)\right| \lesssim \frac{1}{|x|^{\frac{1}{p}}} \left\|\chi_{(0,1)}\right\|_{\text{BMO}} \|A_{j,x}\|_{H^1} \lesssim \frac{1}{|x|^{\frac{1}{p}}}$$

uniformly on j and $x \in \mathbb{R}\backslash\{0\}$. Following the proof of Theorem 4.3 we obtain that, for any $\lambda > 0$,

$$\left|\left\{x : \left|\mathcal{H}_{(k)}(a_j)(x)\right| > \lambda\right\}\right| \lesssim \frac{1}{\lambda^p}.$$

Thus, by the same proof as in Theorem 4.3, we obtain that

$$\left\|\mathcal{H}_{(k)}(a_j)(x)\right\|_{L^{p,\infty}} \lesssim 1.$$

as desired.

Now we show part (2). Given k, $0 < p < 1$ such that $k > \frac{1}{p} - 1$, we fix p_0 such that $k = \frac{1}{p_0} - 1$. Then $p_0 < p < 1$. The above argument shows that $\mathcal{H}_k$ is weakly bounded on H^{p_0}, and Theorem 4.3 shows that $\mathcal{H}_k$ is weakly bounded on H^1. Thus by the Marcinkiewicz interpolation, we obtain the result in part (2). □

Remark 4.8. The first part of Theorem 4.7 can also be proved by using the result in [14].

This result can be further generalized to the kth order fractional Hardy operators

$$\mathcal{H}_{(k),\alpha}(f)(x) = \frac{1}{x^{k+1-\alpha}} \int_0^x t^k f(t)\, dt, \quad 0 \leqslant \alpha < 1.$$

We first establish an easy $L^p \to L^q$ estimate.

Lemma 4.9. *Choose q so that $\frac{1}{p} - \alpha = \frac{1}{q}$.*

(1) For $1 \leqslant p < \infty$, $\left\|\mathcal{H}_{(k),\alpha}(f)\right\|_{L^{q,\infty}} \lesssim \|f\|_{L^p}$.

(2) For $1 < p < \infty$, $\left\|\mathcal{H}_{(k),\alpha}(f)\right\|_{L^q} \lesssim \|f\|_{L^p}$.

Proof. Clearly, we only need to show the weak boundedness. The strong $L^p \to L^q$ boundedness then follows by an interpolation. By Hölder's inequality,

$$\left|\mathcal{H}_{(k),\alpha}(f)(x)\right| = |x|^\alpha \left|\int_0^1 t^k f(xt)\, dt\right| \leqslant \frac{1}{|x|^{\frac{1}{q}}} \|f\|_{L^p}.$$

Hence, for any $\lambda > 0$,

$$\left|\left\{x \in \mathbb{R} : \left|\mathcal{H}_{(k),\alpha}(f)(x)\right| \geqslant \lambda\right\}\right| \leqslant \left(\frac{\|f\|_{L^p}}{\lambda}\right)^q .$$

This proves the weak $L^p \to L^{q,\infty}$ boundedness. ☐

Theorem 4.10. *Let $0 < p < 1$, and $\frac{1}{p} - \alpha = \frac{1}{q}$.*

(1) $\mathcal{H}_{(k),\alpha}$ is bounded from $H^p(\mathbb{R})$ to $L^{q,\infty}(\mathbb{R})$ if $k = \frac{1}{p} - 1$.

(2) $\mathcal{H}_{(k),\alpha}$ is bounded from $H^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if $k > \frac{1}{p} - 1$.

Proof. With Lemma 4.9 and interpolation, it suffices to show the first part of the theorem. For $\frac{1}{p} = k + 1$ and any $f \in H^p(\mathbb{R})$, as in the proof of Theorem 4.7, we write an atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

where each a_j is a (p, ∞, k) atom. Thus,

$$\mathcal{H}_{(k),\alpha}(f) = \sum_{j \in \mathbb{Z}} \lambda_j \mathcal{H}_{(k),\alpha}(a_j) .$$

Similar to the proof of Theorem 4.7, we only need to show

$$\left\|\mathcal{H}_{(k),\alpha}(a_j)\right\|_{L^{q,\infty}} \leqslant 1,$$

uniformly for all (p, ∞, k) atoms a_j.

If a_j is (p, ∞, k) atom with support in $(\alpha_j, \beta_j) \subset (0, \infty)$ or $(\alpha_j, \beta_j) \subset (-\infty, 0)$, $\mathcal{H}_{(k),\alpha}(a_j)$ is also supported in $(\alpha_j, \beta_j) = I_j$. Choose $p', q' > 1$ so that $\frac{1}{p'} - \alpha = \frac{1}{q'}$. Then we have the following bound uniformly on j.

$$\begin{aligned}
\left\|\mathcal{H}_{(k),\alpha}(a_j)\right\|_{L^q}^q &= \left\|\mathcal{H}_{(k),\alpha}(a_j)^q \cdot \chi_{I_j}\right\|_{L^1} \\
&\leqslant \left\|\mathcal{H}_{(k),\alpha}(a_j)^q\right\|_{L^{\frac{q'}{q}}} \cdot \left\|\chi_{I_j}\right\|_{L^{\frac{q'}{q'-q}}} \\
&= \left(\int \left|\mathcal{H}_{(k),\alpha}(a_j)(x)\right|^{q \cdot \frac{q'}{q}} \, dx\right)^{\frac{1}{q'} \cdot q} \left|I_j\right|^{1 - \frac{q}{q'}}
\end{aligned}$$

$$= \left\|\mathcal{H}_{(k),\alpha}(a_j)\right\|_{L^{q'}}^{q} |I_j|^{1-\frac{q}{q'}}$$

$$\leqslant \|a_j\|_{L^{p'}}^{q} |I_j|^{1-\frac{q}{q'}} \leqslant 1.$$

If the support of a (p, ∞, k) atom a_j contains the origin, with an easy modification of the proof for Theorem 4.7, we obtain

$$\left\|\mathcal{H}_{(k),\alpha}(a_j)\right\|_{L^{q,\infty}} \leqslant 1,$$

uniformly for all (p, ∞, k) atom a_j. This completes the proof of part (1).

To show part (2), take $p_1 = 1$, p_2 such that $p_2 = \frac{1}{1+k}$, and q_1, q_2 such that $\frac{1}{q_i} = \frac{1}{p_i} - \alpha$ $(i = 1, 2)$. Then by part (1), $\left\|\mathcal{H}_{(k),\alpha}(f)\right\|_{L^{q_2,\infty}} \leqslant \|f\|_{H^{p_2}}$, and by Lemma 4.9, $\left\|\mathcal{H}_{(k),\alpha}(f)\right\|_{L^{q_1,\infty}} \leqslant \|f\|_{H^1}$. Applying the Marcinkiewicz interpolation in this case gives $\left\|\mathcal{H}_{(k),\alpha}(f)\right\|_{L^q} \leqslant \|f\|_{H^p}$, as desired. □

Remark 4.11. We do not proceed to show $H^p \to H^q$ boundedness because $\mathcal{H}_{(k),\alpha}$ fails to commute with the Hilbert transform.

CHAPTER 5

HIGH DIMENSIONAL HAUSDORFF OPERATORS

In this chapter, we study two extensions of the Hausdorff operator in $\mathbb{R}^n$. For one, we obtain a sufficient and necessary condition for its boundedness on the real Hardy space $H^1(\mathbb{R}^n)$. For the other, we study its boundedness on the real Hardy space $H^p(\mathbb{R}^n)$ for $0 < p < 1$.

Recall the definition of the one dimensional Hausdorff operator: Let Φ be a locally integrable function on the positive real line. The one dimensional Hausdorff operator h_Φ with the generating function Φ is defined in the integral form by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f(\frac{x}{t})\, dt,$$

where, for simplicity, we initially assume that the operator h_Φ is defined on the class of all Schwartz functions f. For positive values x, a change of variables gives an equivalent form of h_Φ by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(\frac{x}{t})}{t} f(t)\, dt.$$

This suggests two different extensions of the Hausdorff operator on high dimensional space,

$$\tilde{H}_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{n-\beta}} f(\frac{x}{|y|})\, dy, \quad n > \beta \geqslant 0$$

and

$$H_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y)\, dy, \quad n > \beta \geqslant 0,$$

where Φ is a locally integrable function. Each of these definitions gives a so-called fractional Hausdorff operator. We denote

$$\tilde{H}_{\Phi,0} = \tilde{H}_\Phi, \qquad H_{\Phi,0} = H_\Phi.$$

When $\Phi(x) = \Phi(|x|)$, and $\Phi(t) = \chi_{(1,\infty)}(t)t^{-n+\beta}$, $H_{\Phi,\beta}$ is the fractional Hardy operator if $0 < \beta < n$, and $H_{\Phi,\beta}$ is the Hardy operator if $\beta = 0$ (see [2]).

By the Minkowski inequality and a scaling argument, it is easy to see that $\tilde{H}_\Phi$ is bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ for all $1 \leqslant p \leqslant \infty$, if Φ satisfies the size condition

$$\int_{\mathbb{R}^n} |\Phi(y)|\, |y|^{-n+\frac{n}{p}}\ dy < \infty.$$

Similarly, $\tilde{H}_\Phi$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ if Φ is Lebesgue integrable (see [10, 12, 15, 18]). In [13], Liflyand posed an open question (among others) to establish the sharpness of this condition on Φ to assure the $H^1(\mathbb{R}^n)$ boundedness for $\tilde{H}_\Phi$. Motivated by his question, the first aim of this chapter is to solve the problem by showing that, for $\Phi \geqslant 0$, $\tilde{H}_\Phi$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ if and only if Φ is a Lebesgue integrable function.

In Chapter 3, we established some results which used duality to provide a smoothness condition on Φ to ensure boundedness of h_Φ on H^p. The second aim of this chapter is to apply this technique to give an $H^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ boundedness result for $H_{\Phi,\beta}$ in the case where Φ is a radial function. We also establish boundedness on $L^p(\mathbb{R}^n, |x|^\gamma\, dx) \to L^q(\mathbb{R}^n, |x|^\gamma\, dx)$. With these results, we naturally expect to establish an $H^p(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$ boundedness theorem for the operator $H_{\Phi,\beta}(f)$. However, by checking the proofs of the theorems in the earlier chapter for the one dimensional Hausdorff operator, we find that all methods fail to establish a sufficient condition for $H^p(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$ boundedness for the operator $H_{\Phi,\beta}$. Also, in the high dimensional case, we do not find in the literature any high dimensional $H^p(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$, $0 < p < 1$, boundedness results for the (fractional) Hausdorff operator, although the $H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ boundedness has been studied extensively (see [10, 12, 16, 18, 17, 16, 24, 30]). Hence, as the third purpose of this chapter, we will establish a sufficient condition for the $H^p(\mathbb{R}^n) \to H^q(\mathbb{R}^n)$ boundedness for $H_{\Phi,\beta}$.

5.1 $H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ Boundedness of $\tilde{H}_\Phi$

Theorem 5.1. *Let Φ be a nonnegative valued locally integrable function.*

(1) $\tilde{H}_\Phi$ is bounded on $H^1(\mathbb{R}^n)$ if and only if $\Phi \in L^1(\mathbb{R}^n)$.

(2) $\tilde{H}_\Phi$ is bounded on the local Hardy space $h^1(\mathbb{R}^n)$ if and only if

$$\int_{|y|\geqslant 1} \Phi(y)\, dy + \int_{|y|\leqslant 1} \Phi(y) \left(1 + \log\left(\frac{1}{|y|}\right)\right)\, dy < \infty.$$

Proof. The "if" part of (1) and (2) was proved in [24] and [1], respectively. We need only show the "only if" part. Let a be a function with support on $[-1,1]^n$ which satisfies.

(i) $a(x) = \frac{1}{2}$ for $x \in [\frac{1}{4}, \frac{1}{2}]^n$.

(ii) $\frac{1}{2} \geqslant a(x) \geqslant 0$ for $\sum_{i=1}^n x_i \geqslant 0$.

(iii) $a(x) = -a(-x)$ if $\sum_{i=1}^n x_i \leqslant 0$.

Clearly,

$$\int_{\mathbb{R}^n} a(x)\, dx = 0.$$

Thus, a is a $(1, \infty, 0)$ atom (see [28] or [26]). By the same easy computation as in the one-dimensional case we know

$$\|a\|_{H^1(\mathbb{R}^n)} \leqslant 1.$$

Suppose $\Phi \notin L^1$. If $\tilde{H}_\Phi$ were bounded on H^1, we have,

$$\left\|\tilde{H}_\Phi(a)\right\|_{H^1} \leqslant \|a\|_{H^1} \leqslant 1.$$

On the other hand, by the Riesz transform characterization of H^1, we have

$$\left\|\tilde{H}_\Phi(a)\right\|_{H^1} \succeq \left\|\tilde{H}_\Phi(a)\right\|_{L^1}$$

$$\geqslant \int_{\sum x_i \geqslant 0} \left| \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} a(\frac{x}{|y|})\, dy \right| \, dx$$
$$= \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \left(\int_{\sum x_i \geqslant 0} a(\frac{x}{|y|})\, dx \right) \, dy$$
$$= \int_{\mathbb{R}^n} \Phi(y) \left(\int_{\sum x_i \geqslant 0} a(x)\, dx \right) \, dy = \infty.$$

This leads to a contradiction.

Next, we show the "only if" part of (2). Suppose

$$\int_{|y|<1} \Phi(y)(1 - \log_2 |y|)\, dy = \infty.$$

For notational simplicity, use the ℓ^∞ norm on vectors, i.e. $|x| = \max\{|x_i|\}$. Let $\Psi_s = \frac{1}{s^n} e^{-\frac{|x|^2}{s^2}}$ and $B(x) = \chi_{[0,\frac{1}{2}]^n}(x)$. An easy computation similar to that for one dimension shows

$$\|B\|_{h^1(\mathbb{R}^n)} \leqslant 1.$$

On the other hand,

$$\begin{aligned}
\left\| \tilde{H}_\Phi(B) \right\|_{h^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \sup_{0<s\leqslant 1} \left| \Psi_s * \tilde{H}_\Phi(B)(x) \right| \, dx \\
&\geqslant \int_{[0,1]^n} \sup_{0<s\leqslant 1} \left| \Psi_s * \tilde{H}_\Phi(B)(x) \right| \, dx \\
&\geqslant \int_{[0,1]^n} \left| \Psi_{|x|} * \tilde{H}_\Phi(B)(x) \right| \, dx.
\end{aligned}$$

Here

$$\begin{aligned}
\left| \Psi_{|x|} * \tilde{H}_\Phi(B)(x) \right| &= \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} \int_{\mathbb{R}^n} \frac{1}{|x|^n} e^{-(\frac{|x-z|}{|x|})^2} B\left(\frac{z}{|y|} \right) \, dz \, dy \\
&\geqslant \int_{|y|\leqslant 1} \frac{\Phi(y)}{|y|^n} \int_{|z|\leqslant \frac{|y|}{4}} \frac{1}{|x|^n} e^{-(\frac{|x-z|}{|x|})^2} \, dz \, dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{0<s\leqslant 1} \left| \Psi_s * \tilde{H}_\Phi(B)(x) \right| \, dx \\
&\geqslant \int_{|y|\leqslant 1} \frac{\Phi(y)}{|y|^n} \int_{|z|\leqslant \frac{|y|}{4}} \int_{|y|\leqslant |x| \leqslant 1} \frac{1}{|x|^n} e^{-(\frac{|x-z|}{|x|})^2} \, dx \, dz \, dy
\end{aligned}$$

$$\geqslant -\int_{|y|\leqslant 1} \Phi(y)\log_2 |y| \; dy = \infty.$$

This leads to a contradiction.

Next, suppose

$$\int_{|y|\geqslant 1} \Phi(y)\, dy = \infty.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^n} \sup_{0<s\leqslant 1} \left|\Psi_s * \tilde{H}_\Phi(B)(x)\right| \; dx &\geqslant \int_{\mathbb{R}^n} \left|\Psi_1 * \tilde{H}_\Phi(B)(x)\right| \; dx \\ &\geqslant \int_{|y|\geqslant 1} \frac{\Phi(y)}{|y|^n} \int_{|z|\leqslant \frac{|y|}{4}} \left(\int_{\mathbb{R}^n} e^{-(|x-z|)^2} \; dx\right) \; dz\, dy \\ &\geqslant \int_{|y|\geqslant 1} \Phi(y)\, dy = \infty.\end{aligned}$$

□

5.2 Boundedness on Lebesgue spaces

The duality techniques we have been using to show boundedness of Hausdorff operators on the Hardy spaces also yield the following nice results on power weight Lebesgue spaces.

Theorem 5.2. *Let $1 \leqslant p, q < \infty$, $0 < \beta < n$, $\gamma > \beta p - n$ and*

$$\frac{1}{p} - \frac{\beta}{n+\gamma} = \frac{1}{q}.$$

Finally, let

$$C_p = \int_0^\infty |\Phi(t)|^{\frac{p}{p-1}} t^{\frac{\gamma-\beta p+n-p+1}{p-1}} dt \text{ for } p>1, \qquad C_1 = \| |\cdot|^{n-\beta+\gamma}\Phi(\cdot)\|_{L^\infty}.$$

For any $p \geqslant 1$, if $C_p < \infty$ then

$$\|H_{\Phi,\beta}(f)(x)\|_{L^{q,\infty}(|x|^\gamma dx)} \preceq \|f\|_{L^p(|x|^\gamma dx)}.$$

Proof. We write

$$H_{\Phi,\beta}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{|x|}{|y|})}{|y|^{n-\beta+\frac{\gamma}{p}}} f(y)|y|^{\frac{\gamma}{p}} dy.$$

By Hölder's inequality, we have

$$|H_{\Phi,\beta}(f)(x)| \leqslant \left\{ \int_{\mathbb{R}^n} |\frac{\Phi(\frac{|x|}{|y|})}{|y|^{n-\beta+\frac{\gamma}{p}}}|^{p'} dy \right\}^{\frac{1}{p'}} \|f\|_{L^p(|y|^\gamma dy)}.$$

Using polar coordinates and changing variables, we have

$$\left\{ \int_{\mathbb{R}^n} |\frac{\Phi(\frac{|x|}{|y|})}{|y|^{n-\beta+\frac{\gamma}{p}}}|^{p'} dy \right\}^{1/p'} = |S^{n-1}|^{\frac{1}{p'}} \left(\int_0^\infty |\frac{\Phi(t)}{t^{\beta-n-\gamma/p}}|^{p'} t^{-n-1} dt \right)^{\frac{1}{p'}} |x|^{n/p'+\beta-n-\gamma/p}.$$

If $p = 1$,

$$\begin{aligned} |H_{\Phi,\beta}(f)(x)| &\leqslant \|f\|_{L^1(|x|^\gamma dx)} \| \frac{\Phi(\frac{|x|}{|\cdot|})}{|\cdot|^{n-\beta+\gamma}} \|_{L^\infty} \\ &= \|f\|_{L^1(|x|^\gamma dx)} |x|^{-n+\beta-\gamma} \| |\cdot|^{n-\beta+\gamma} \Phi(\cdot) \|_{L^\infty}. \end{aligned}$$

Let

$$K_p = \left|S^{n-1}\right|^{\frac{1}{p'}} C_p^{\frac{1}{p'}} \text{ for } p > 1 \qquad K_1 = C_1$$

Then for $p \geqslant 1$ and any $\lambda > 0$, we have

$$\begin{aligned} &|\{x \in \mathbb{R}^n : \quad |H_{\Phi,\beta}(f)(x)| > \lambda\}| \\ \leqslant &|\{x \in \mathbb{R}^n : \quad K_p |x|^{n/p'+\beta-n-\gamma/p} \|f\|_{L^p(|x|^\gamma dx)} > \lambda\}| \\ \leqslant &|\{x \in \mathbb{R}^n : \quad |x|^{n+\gamma/p-n/p'-\beta} \leqslant \frac{K_p \|f\|_{L^p(|x|^\gamma dx)}}{\lambda}\}| \\ \leqslant &|\{x \in \mathbb{R}^n : \quad (|x|^{\frac{1}{p}-\frac{\beta}{n+\gamma}})^{n+\gamma} \leqslant \frac{K_p \|f\|_{L^p(|x|^\gamma dx)}}{\lambda}\}| \\ = &|\{x \in \mathbb{R}^n : \quad |x|^{n+\gamma} \leqslant \frac{{K_p}^q \|f\|^q_{L^p(|x|^\gamma dx)}}{\lambda^q}\}|. \end{aligned}$$

Note that in the last equality, we have used the condition

$$\frac{1}{p} - \frac{\beta}{n+\gamma} = \frac{1}{q}.$$

This shows

$$\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}(|x|^\gamma dx)} \preceq \|f\|_{L^p(|x|^\gamma dx)}$$

as desired. □

Recall that if $\Phi(t) = \chi_{(1,\infty)}(t)t^{-n+\beta}$, then $H_{\Phi,\beta}$ is the fractional Hardy operator. Let $\mathcal{H}_\beta$ denote the fractional Hardy operator. Applying Theorem 5.2, we have the following corollary:

Corollary 5.3. *Let* $1 < p, q < \infty$, $0 < \beta < n$, $np > \gamma > \beta p - n$ *and*

$$\frac{1}{p} - \frac{\beta}{n+\gamma} = \frac{1}{q}.$$

We have

$$\|\mathcal{H}_\beta(f)(x)\|_{L^{q,\infty}(|x|^\gamma dx)} \preceq \|f\|_{L^p(|x|^\gamma dx)}.$$

Proof. If $\gamma < np$, then $\Phi = \chi_{(1,\infty)}(t)t^{-n+\beta}$ satisfies the conditions of Theorem 5.2. □

Theorem 5.4. *Let* $1 \leqslant p, q < \infty$, $0 < \beta < n$, $\gamma > \beta p - n$ *and*

$$\frac{1}{p} - \frac{\beta}{n+\gamma} = \frac{1}{q}.$$

In addition, let

$$C_{p,\epsilon} = \int_0^\infty |\Phi(t)|^{\frac{p}{p-1}} t^{\frac{\gamma-\beta p+n-p+1}{p-1}+\epsilon} dt \text{ for } p > 1, \qquad C_{1,\epsilon} = \||\cdot|^{n-\beta+\gamma+\epsilon}\Phi(\cdot)\|_{L^\infty}.$$

For any $p \geqslant 1$, *if, for arbitrarily small positive* ϵ, $C_{p,\pm\epsilon} < \infty$, *then*

$$\|H_{\Phi,\beta}(f)(x)\|_{L^q(|x|^\gamma dx)} \preceq \|f\|_{L^p(|x|^\gamma dx)}.$$

Proof. For $p > 1$, this result follows from Theorem 5.2 and the Marcinkiewicz Interpolation Theorem. To prove the result for $p = 1$, we define an analytic family:

$$H_{\Phi,z}(f)(x) = \int \frac{\Phi(\frac{|x|}{|y|})}{|y|^{n-z}} f(y)dy.$$

Following the proof of Theorem 5.2, it is easy to check that there exist β_1, β_2 satisfying $0 < \beta_1 < \beta < \beta_2 < n$, such that

$$\|H_{\Phi,z_1}(f)\|_{L^{q_1,\infty}(|x|^\gamma dx)} \preceq \|f\|_{L^1(|x|^\gamma dx)}$$

and

$$\|H_{\Phi,z_2}(f)(x)\|_{L^{q_2,\infty}(|x|^\gamma dx)} \preceq \|f\|_{L^1(|x|^\gamma dx)}$$

where Re $z_1 = \beta_1$, Re $z_2 = \beta_2$ and $1 < q_1 < q < q_2 < \infty$. Thus we obtain the result by using the Stein-Weiss analytic Interpolation Theorem (see Corollary 2.2). □

5.3 Boundedness on $H^p(\mathbb{R}^n)$

5.3.1 $H^p \to L^q$ Boundedness of $H_{\Phi,\beta}$

In this section, we suppose that Φ is radial, that is, if $|x_1| = |x_2|$, then $\Phi(x_1) = \Phi(x_2)$. Thus we can think of Φ as having domain $[0,\infty)$,where $\Phi(t) = \Phi(x)$ when $|x| = t$.

Lemma 5.5. *Let $0 \leqslant \beta < n$ and*

$$\psi(y) = \frac{\Phi(1/|y|)}{|y|^{n-\beta}}.$$

Assume $0 < p < 1$ and $\alpha = n\left(\frac{1}{p} - 1\right)$. If $\psi \in \Lambda_\alpha$, then

$$\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}$$

where q satisfies

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

Proof. Changing variables, we compute

$$\frac{\Phi(|x|/|y|)}{|y|^{n-\beta}} = |x|^{-n+\beta}\,\psi(\frac{y}{|x|}).$$

By duality and scaling,

$$|H_\Phi(f)(x)| \leqslant |x|^{-n+\beta} \|f\|_{H^p(\mathbb{R}^n)} \left\| \psi\left(\frac{\cdot}{|x|}\right) \right\|_{\Lambda_\alpha} = |x|^{-n/p+\beta} \|f\|_{H^p(\mathbb{R}^n)} \|\psi\|_{\Lambda_\alpha}.$$

This shows that for all $\lambda > 0$,

$$\begin{aligned} &|\{x \in \mathbb{R}^n : |H_\Phi(f)(x)| > \lambda\}| \\ \leqslant &\left|\left\{x \in \mathbb{R}^n : |x|^{\frac{n}{p}-\beta} < \|f\|_{H^p(\mathbb{R}^n)} \|\psi\|_{\Lambda_\alpha} \lambda^{-1}\right\}\right| \\ \leqslant &\left|\left\{x \in \mathbb{R}^n : |x|^n \leqslant \{\|f\|_{H^p(\mathbb{R}^n)} \|\psi\|_{\Lambda_\alpha} \lambda^{-1}\}^q\right\}\right|. \end{aligned}$$

The last inequality holds because

$$\frac{n}{\frac{n}{p} - \beta} = q.$$

Thus, we obtain

$$\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}$$

as desired. □

Theorem 5.6. *Let $\beta, p, \alpha, q, \psi$ be as in Lemma 5.5. If for some $\epsilon > 0$ small enough that $\alpha - \epsilon > 0$, $\psi \in \Lambda_{\alpha+\epsilon} \cap \Lambda_{\alpha-\epsilon}$, then*

$$\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}.$$

Proof. Choose p_1, p_2, s.t. $\alpha + \epsilon = n(\frac{1}{p_1} - 1)$, $\alpha - \epsilon = n(\frac{1}{p_2} - 1)$, and choose q_1, q_2 satisfying

$$\frac{1}{p_i} = \frac{1}{q_i} + \frac{\beta}{n}, \quad i = 1, 2.$$

An easy computation shows that p_1, p_2, p, q_1, q_2, q satisfy the requirement for the Marcinkiewisz interpolation, and by the Lemma 5.5, $\|H_{\Phi,\beta}(f)\|_{L^{q_i,\infty}(\mathbb{R}^n)} \preceq \|f\|_{H^{p_i}(\mathbb{R}^n)}$. So by Marcinkiewisz Interpolation, $\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n)} \preceq \|f\|_{H^p(\mathbb{R}^n)}$. □

Unfortunately, we cannot continue in the manner shown in Theorem 3.1, since $\mathcal{R}_J \circ H_{\Phi,\beta}(f) \neq H_{\Phi,\beta}(\mathcal{R}_J f)$, as was in the case in one dimension. Instead, we see the following.

Lemma 5.7. *Suppose* $\Phi, \widehat{\Phi} \in L^1(\mathbb{R}^n)$*, and* $0 < \beta < n$*. Then for any* $J = \{j_1, \ldots, j_L\} \in \{0, 1, 2, \ldots, n\}^L$*,* $f \in S$*, we have*

$$\mathcal{R}_J H_{\Phi,\beta}(f) = H_{\mathcal{R}_J \Phi,\beta}(f).$$

Proof. Without loss of generality, we suppose $L = 1$. ($L > 1$ may be shown similarly.) Fix $J = j$. We show the desired equality by taking Fourier transform and inverse Fourier transform. By definition,

$$\widehat{H_{\Phi,\beta}(f)}(\xi) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\beta}} \Phi\left(\frac{x}{|y|}\right) \, dy \right) e^{-i\langle x,\xi\rangle} \, dx.$$

Noting that $\Phi \in L^1$, f is a Schwartz function, and $\beta > 0$, we may use Fubini's theorem to obtain

$$\begin{aligned}
\widehat{H_{\Phi,\beta}(f)}(\xi) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\beta}} \Phi(\frac{x}{|y|}) e^{-i\langle x,\xi\rangle} \, dx \right) \, dy \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |y|^\beta \, f(y) \Phi(u) e^{-i\langle u,|y|\xi\rangle} \, du \right) \, dy \\
&= \int_{\mathbb{R}^n} |y|^\beta \, f(y) \hat{\Phi}(|y| \, \xi) \, dy.
\end{aligned}$$

Thus, we see that

$$\mathcal{R}_j \widehat{H_{\Phi,\beta}}(f)(\xi) \cong \frac{\xi_j}{|\xi|} \widehat{H_{\Phi,\beta}}(\xi) = \frac{\xi_j}{|\xi|} \int_{\mathbb{R}^n} |y|^\beta \, f(y) \hat{\Phi}(|y| \, \xi) \, dy.$$

Next, we let $\mathcal{F}^{-1}$ denote the inverse Fourier transform, and compute

$$\mathcal{F}^{-1} \left(\mathcal{R}_j \widehat{H_{\Phi,\beta}(f)} \right)(x) = \int_{\mathbb{R}^n} \left(\frac{\xi_j}{|\xi|} \int_{\mathbb{R}^n} |y|^\beta \, f(y) \hat{\Phi}(|y| \, \xi) \, dy \right) e^{i\langle x,\xi\rangle} \, d\xi.$$

Noting that $\widehat{\Phi} \in L^1$ and again that $\beta > 0$, we again use Fubini's theorem to obtain

$$\mathcal{F}^{-1} \left(\mathcal{R}_j \widehat{H_{\Phi,\beta}(f)} \right)(x) = \int_{\mathbb{R}^n} |y|^\beta \, f(y) \left(\int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} \widehat{\Phi}(|y| \, \xi) e^{i\langle x,\xi\rangle} \, d\xi \right) \, dy$$

$$= \int_{\mathbb{R}^n} |y|^{\beta-n} f(y)\mathcal{R}_j\Phi\left(\frac{x}{|y|}\right)\, dy$$

$$= H_{\mathcal{R}_j\Phi,\beta}$$

as desired. □

5.3.2 $H^p \to H^q$ Boundedness of $H_{\Phi,\beta}$

The condition of Lemma 5.5, that $\psi \in \Lambda_\alpha$, can be rewritten, when Φ is radial, as

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} \Phi(\frac{|x|}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)} \leqslant |x|^{-\alpha}.$$

In this form, we can omit the condition that Φ is radial, obtaining

Lemma 5.8. *Let $0 < p < 1$, $0 \leqslant \beta < n$, and $\alpha = n(\frac{1}{p} - 1)$. If Φ satisfies, for all $x \in \mathbb{R}^n$,*

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} \Phi(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)} \leqslant |x|^{-\alpha}$$

then

$$\|H_\Phi(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \leqslant \|f\|_{H^p(\mathbb{R}^n)}$$

with

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

Proof. Noting the pairing inequality (equation (2.1)) relating the Lipschitz space Λ_α and the Hardy space H^p, and multiplying and dividing by $|x|^{n-\beta}$ in the definition of $H_{\Phi,\beta}$, we obtain

$$|H_{\Phi,\beta}(f)(x)| \leqslant \|f\|_{H^p(\mathbb{R}^n)} |x|^{-n+\beta} \left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} \Phi(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)}$$

$$\leqslant |x|^{\frac{-n}{p}+\beta} \|f\|_{H^p(\mathbb{R}^n)}.$$

This shows that for all $\lambda > 0$,

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : |H_{\Phi,\beta}(f)(x)| > \lambda\}| \\
&\leqslant \left|\left\{x \in \mathbb{R}^n : |x|^{\frac{-n}{p}+\beta} \|f\|_{H^p(\mathbb{R}^n)} \geqslant \lambda\right\}\right| \\
&= \left|\left\{x \in \mathbb{R}^n : |x| \leqslant \left(\frac{\|f\|_{H^p(\mathbb{R}^n)}}{\lambda}\right)^{\frac{q}{n}}\right\}\right| \\
&\leqslant \left(\frac{\|f\|_{H^p(\mathbb{R}^n)}}{\lambda}\right)^q .
\end{aligned}$$

Thus, $\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}} \leqslant \|f\|_{H^p}$, as desired. □

In many important cases, even when Φ is radial, the condition of Lemma 5.8 is easier to check than the one give in Lemma 5.5. For example, we see this in the proof of the following corollary.

Corollary 5.9. *Let $\Phi(y)$ be the Gaussian function $e^{-|y|^2}$ or the Poisson function $e^{-|y|}$. The Hausdorff operator $H_{\Phi,\beta}$ is bounded from $H^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $0 < p < \infty$, $0 \leqslant \beta < n$, and $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}$.*

Proof. We will show the corollary for the Poisson function $e^{-|y|}$, since the proof for the Gaussian function is similar. By Lemma 5.8, we need to check

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} e^{-\frac{|x|}{|\cdot|}} \right\|_{\Lambda_\alpha} \leqslant |x|^{-\alpha}$$

for all integer $\alpha = 0, 1, 2, \ldots$, where by convention, we denote $\Lambda_0 = L^\infty$. First, when $\alpha = 0$,

$$\begin{aligned}
\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} e^{-\frac{|x|}{|\cdot|}} \right\|_{L^\infty} &= \sup_{|y|>0} \left| \frac{|x|^{n-\beta}}{|y|^{n-\beta}} e^{-\frac{|x|}{|y|}} \right| \\
&= \sup_{|y|>0} \left| |y|^{n-\beta} e^{-|y|} \right| \leqslant 1.
\end{aligned}$$

For integers $\alpha > 0$, we recall the definition of the Λ_α norm, namely

$$\|f\|_{\Lambda_\alpha} = \sup_{|I|=\alpha} \left\{ \left\|\partial^I f\right\|_{L^\infty} \right\}$$

where $I = (i_1, i_2, \cdots, i_n) \in \mathbb{N}^n$ is a multi-index and ∂^I denotes the derivative

$$\left(\frac{\partial}{\partial y_1}\right)^{i_1}\left(\frac{\partial}{\partial y_2}\right)^{i_2}\cdots\left(\frac{\partial}{\partial y_n}\right)^{i_n}.$$

So we must show that, for each multi-index I such that $|I| = \alpha$,

$$\left\|\partial^I\left(\frac{|x|^{n-\beta}}{|y|^{n-\beta}}e^{-\frac{|y|}{|x|}}\right)\right\|_{L^\infty} \lesssim |x|^{-\alpha}.$$

This can easily be shown by computation. □

We can also see how the condition of Lemma 5.8 might be extended to allow us to obtain a boundedness result on Hardy spaces.

Theorem 5.10. *Let $n \geqslant 2$, $\alpha = n(\frac{1}{p} - 1)$, $0 < \beta < n$, and $\Phi, \widehat{\Phi} \in L^1$. If, for some L large enough that $p > \frac{n-1}{n-1+L}$, all generalized Riesz transforms $\mathcal{R}_J(\Phi) = \mathcal{R}_{j_1}\cdots\mathcal{R}_{j_L}\Phi$ satisfy*

$$\left\|\frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}}(\mathcal{R}_J\Phi)(\frac{x}{|\cdot|})\right\|_{\Lambda_\alpha(\mathbb{R}^n)} \lesssim |x|^{-\alpha}$$

then

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}$$

with

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

Proof. Applying Lemma 5.8 to each $\mathcal{R}_J\Phi$, we obtain for each $\mathcal{R}_J$,

$$\left\|H_{\mathcal{R}_J(\Phi),\beta}\right\|_{L^{q,\infty}} \lesssim \|f\|_{H^p}.$$

Lemma 5.7 tells us that for each $\mathcal{R}_J$, $\mathcal{R}_J H_{\Phi,\beta} = H_{\mathcal{R}_J(\Phi),\beta}(f)$. Applying the Reisz transform characterization of $H^{q,\infty}$,

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}} \cong \sum_J \|\mathcal{R}_J H_{\Phi,\beta}(f)\|_{L^{q,\infty}} \lesssim \|f\|_{H^p}$$

as desired. □

The condition of Theorem 5.10 is somewhat laborious to check. The main result of this chapter is to provide a sufficient (but not necessary) condition on Φ to meet the requirement of Theorem 5.10.

Theorem 5.11. *Suppose* $0 < p < 1$, $0 < \beta < n$. *Let* $\widehat{\Phi}$ *denote the Fourier transform of* Φ *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M = n(\frac{1}{p} - 1)$, *suppose that* $\widehat{\Phi}$ *is a function in* $C^{2M+n}(\mathbb{R}^n)$ *with compact support in the set* $\mathbb{R}^n \backslash \{0\}$. *Then*

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^p(\mathbb{R}^n)}.$$

Proof. Let $M = n(\frac{1}{p} - 1)$. By Theorem 5.10, we need to check

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)} \preccurlyeq |x|^{-\alpha},$$

with $\alpha = M$. Taking Fourier transforms, we may write

$$\begin{aligned} &\frac{|x|^{n-\beta}}{|y|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|y|}) \\ &= |x|^{n-\beta} \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi(\xi)} \, |y|^{-(n-\beta)} \, e^{i\frac{1}{|y|}\langle \xi, x\rangle} \, d\xi. \end{aligned}$$

For a multi-index $I = (i_1, i_2, \cdots, i_n) \in \mathbb{N}^n$, we will use the notation

$$\partial^I = \left(\frac{\partial}{\partial y_1}\right)^{i_1} \left(\frac{\partial}{\partial y_2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial y_n}\right)^{i_n},$$

and

$$|I| = i_1 + i_2 + \cdots + i_n.$$

From easy computations

$$\frac{\partial}{\partial y_j} e^{i\frac{1}{|y|}\langle \xi, x\rangle} \cong \langle \xi, x\rangle \frac{y_j}{|y|^3} e^{i\frac{1}{|y|}\langle \xi, x\rangle}$$

and

$$\frac{\partial^2}{\partial y_j^2} e^{i\frac{1}{|y|}\langle \xi, x\rangle} = c_1 \langle \xi, x\rangle^2 \frac{y_j^2}{|y|^6} e^{i\frac{1}{|y|}\langle \xi, x\rangle} + c_2 \langle \xi, x\rangle \frac{1}{|y|^3} e^{i\frac{1}{|y|}\langle \xi, x\rangle} + c_3 \langle \xi, x\rangle \frac{y_j^2}{|y|^5} e^{i\frac{1}{|y|}\langle \xi, x\rangle},$$

where c_1, c_2, c_3 are constants, in general, it is easy to see that for any multi-index I,

$$\partial^I e^{i\frac{1}{|y|}\langle \xi, x\rangle} = e^{i\frac{1}{|y|}\langle \xi, x\rangle} \sum_{\substack{|I| \leqslant l_1 \leqslant 2|I| \\ l_1 - l_2 = |I|}} Q_{l_1}(y) P_{l_2}(\langle \xi, x\rangle),$$

where $Q_{l_1}(y)$ is a $C^\infty(\mathbb{R}^n \backslash \{0\})$ function satisfying

$$|Q_{l_1}| \leqslant \frac{1}{|y|^{l_1}}$$

and P_{l_2} is a homogeneous polynomial of degree l_2.

Also, it is easy to check

$$\partial^I \left(|y|^{-n+\beta} \right) = \frac{S_I(y)}{|y|^{n-\beta+2|I|}},$$

where $S_I(y)$ is a $C^\infty(\mathbb{R}^n \backslash \{0\})$ function satisfying

$$|S_I(y)| \leqslant |y|^{|I|}.$$

So, by the generalized Leibniz rule, for any multi-index I with $|I| \geqslant 1$, we have

$$\begin{aligned} \partial^I \left(\frac{e^{i\frac{1}{|y|}\langle \xi, x\rangle}}{|y|^{n-\beta}} \right) &= \sum_{K_1 + K_2 = I} C^I_{K_1} \partial^{K_1}(|y|^{-n+\beta}) \partial^{K_2}(e^{i\frac{1}{|y|}\langle \xi, x\rangle}) \\ &= \sum_{K_1 + K_2 = I} C^I_{K_1} \frac{S_{K_1}(y)}{|y|^{n-\beta+2|K_1|}} \left(\sum_{\substack{|K_2| \leqslant l_1 \leqslant 2|K_2| \\ l_1 - l_2 = |K_2|}} Q_{l_1}(y) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle) \right) \end{aligned}$$

where $C^I_{K_1}$ are constants.

It is easy to see that

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)}$$

$$= \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \left| \partial^I \left(|x|^{n-\beta} \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi)\, |y|^{-n+\beta}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle}\ d\xi \right) \right| \right\}$$

$$= \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} |x|^{n-\beta} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) \partial^I \left(|y|^{-n+\beta}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle} \right)\ d\xi \right| \right\}.$$

So, let us consider the value of

$$|x|^{n-\beta} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) \partial^I \left(|y|^{-n+\beta}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle} \right)\ d\xi \right|$$

$$= |x|^{n-\beta} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) \sum_{K_1+K_2=I} C^I_{K_1} \frac{S_{K_1}(y)}{|y|^{n-\beta+2|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} Q_{l_1}(y) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle) \right) d\xi \right|$$

$$= |x|^{n-\beta} \left| \sum_{K_1+K_2=I} C^I_{K_1} \frac{S_{K_1}(y)}{|y|^{n-\beta+2|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} Q_{l_1}(y) \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\ d\xi \right) \right|$$

$$\leqslant |x|^{n-\beta} \sum_{K_1+K_2=I} \frac{|S_{K_1}(y)|}{|y|^{n-\beta+2|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} |Q_{l_1}(y)| \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\ d\xi \right| \right)$$

$$\leqslant |x|^{n-\beta} \sum_{K_1+K_2=I} \frac{|y|^{|K_1|}}{|y|^{n-\beta+2|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\ d\xi \right| \right)$$

$$= \left(\frac{|x|}{|y|} \right)^{n-\beta} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\ d\xi \right| \right).$$

We now consider two possible cases. If

$$\frac{|x|}{|y|} \leqslant 1,$$

then

$$\sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \left| \partial^I \left(|x|^{n-\beta} \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi)\, |y|^{-(n-\beta)}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle}\ d\xi \right) \right| \right\}$$

$$\leqslant \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right| \right) \right\}$$

$$\leqslant \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \frac{1}{|y|^{|K_2|}} \frac{|x|^{l_2}}{|y|^{l_2}} \int_{\mathbb{R}^n} \left| \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^{L-l_2}} \widehat{\Phi}(\xi) \right| \, d\xi \right) \right\}$$

$$\leqslant \frac{1}{|x|^{\alpha}} \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \sum_{K_1+K_2=I} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \int_{\mathbb{R}^n} \left| \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^{L-l_2}} \widehat{\Phi}(\xi) \right| \, d\xi \right) \right\}$$

$$\leqslant |x|^{-\alpha}.$$

On the other hand, if

$$\frac{|x|}{|y|} > 1,$$

for fixed x, without loss of generality, we assume $|x_1| \geqslant \frac{|x|}{n}$. For each integral

$$\frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right|,$$

using integration by parts on the ξ_1 variable k times, where

$$k = 2M + n \geqslant |I| + l_2 + n - \beta.$$

Recalling $l_1 - l_2 = |K_2|$, and using the fact that $\widehat{\Phi} \in C^{2M+n}(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n \setminus \{0\}$, we obtain

$$\begin{aligned} &\frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right| \\ \leqslant &\frac{1}{|y|^{l_1-l_2}} \left(\frac{|y|}{|x|}\right)^k \frac{|x|^{l_2}}{|y|^{l_2}} \\ \leqslant &\frac{1}{|y|^{|K_2|}} \left(\frac{|y|}{|x|}\right)^{|I|+n-\beta}. \end{aligned}$$

This shows that

$$\sup_{y\in\mathbb{R}^n\setminus\{0\}}\left\{\sum_{|I|=\alpha}\left|\partial^I\left(|x|^{n-\beta}\int_{\mathbb{R}^n}\frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L}\widehat{\Phi}(\xi)\,|y|^{-n+\beta}\,e^{i\frac{1}{|y|}\langle\xi,x\rangle}\,d\xi\right)\right|\right\}$$

$$\preccurlyeq \sup_{y\in\mathbb{R}^n\setminus\{0\}}\left\{\sum_{|I|=\alpha}\left(\frac{|x|}{|y|}\right)^{n-\beta}\sum_{K_1+K_2=I}\frac{1}{|y|^{|K_1|}}\frac{1}{|y|^{|K_2|}}\left(\frac{|y|}{|x|}\right)^{|I|+n-\beta}\right\}$$

$$\cong \sup_{y\in\mathbb{R}^n\setminus\{0\}}\left\{\sum_{|I|=\alpha}\left(\frac{|x|}{|y|}\right)^{n-\beta}\frac{1}{|y|^{|I|}}\left(\frac{|y|}{|x|}\right)^{|I|+n-\beta}\right\}\preccurlyeq |x|^{-\alpha}.$$

This proves that for each J, when $M = n(\frac{1}{p}-1)$,

$$\left\|\frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}}(\mathcal{R}_J\Phi)(\frac{x}{|\cdot|})\right\|_{\Lambda_\alpha(\mathbb{R}^n)}\preccurlyeq |x|^{-\alpha},$$

and thus by Theorem 5.10, $\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)}\preccurlyeq\|f\|_{H^p(\mathbb{R}^n)}$. □

We show the following $H^1(\mathbb{R}^n)$-boundedness result in order to provide another endpoint for interpolation. Note that the results obtained in Section 5.1 were about $\widetilde{H}_{\Phi,\beta}$, and so unrelated to this one.

Lemma 5.12. *Suppose* $0<\beta<n$. *Let* $\widehat{\Phi}$ *denote the Fourier transform of* Φ *and*

$$1=\frac{1}{q}+\frac{\beta}{n}.$$

Suppose $\widehat{\Phi}$ *is a function in* $C^n(\mathbb{R}^n)$ *with compact support. Then*

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}}\preccurlyeq\|f\|_{H^1}.$$

Proof. Define the analytic family of operators

$$H_{\Phi,z}(f)(x)=\int_{\mathbb{R}^n}\frac{\Phi\left(\frac{x}{|y|}\right)}{|y|^{n-z}}f(y)\,dy.$$

Fix z_1,z_2 such that $\operatorname{Re}z_1=\beta_1$, $\operatorname{Re}z_2=\beta_2$, with $0<\beta_1<\beta<\beta_2<n$. Using Hölder's inequality, for $i=1,2$, we have

$$|H_{\Phi,z_i}(f)(x)|=\left|\int_{\mathbb{R}^n}\frac{\Phi(\frac{x}{|y|})}{|y|^{n-z_i}}f(y)\,dy\right|$$

$$\leqslant \|f\|_{L^1} \sup_{|y|>0} \left| \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta_i}} \right| .$$

Clearly, Φ is a bounded function, by assumption. Thus, if $\frac{|x|}{|y|} \leqslant 1$, we can write,

$$|H_{\Phi,z_i}(f)(x)| \leqslant \|f\|_{L^1} |x|^{-n+\beta_i} .$$

If $\frac{|x|}{|y|} > 1$, we use integration by parts n times, using $\widehat{\Phi} \in C^n$ with compact support to obtain

$$\left| \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta_i}} \right| = \left| \frac{1}{|y|^{n-\beta_i}} \int_{\mathbb{R}^n} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} \, d\xi \right| \leqslant |x|^{-n+\beta_i} .$$

This gives

$$\begin{aligned} |\{x : |H_{\Phi,z_i}(f)(x)| > \lambda\}| &\leqslant \left| \left\{ x : \|f\|_{L^1} |x|^{-n+\beta_i} \geqslant \lambda \right\} \right| \\ &\cong \left| \left\{ x : |x| \leqslant \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{\frac{1}{n-\beta_i}} \right\} \right| \\ &\cong \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{\frac{n}{n-\beta_i}} \\ &= \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{q_i} . \end{aligned}$$

where q_1, q_2 are such that

$$\frac{1}{q_i} + \frac{\beta_i}{n} = 1 \qquad i = 1, 2.$$

By definition of the weak type spaces, then, the last inequality implies

$$\|H_{\Phi,z_i}(f)\|_{L^{q_i,\infty}(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} .$$

Thus, by Stein-Weiss analytic interpolation, we obtain, for any q_0 such that $q_1 < q_0 < q_2$,

$$\|H_{\Phi,\beta}(f)\|_{L^{q_0}(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \leqslant \|f\|_{H^1(\mathbb{R}^n)} .$$

In particular, this holds when $q_0 = q$, so

$$\|H_{\Phi,\beta}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}\,.$$

But $q > 1$, so $H^q = L^q$, and thus

$$\|H_{\Phi,\beta}(f)\|_{H^q(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}\,.$$

Further, we know $\|\cdot\|_{H^{q,\infty}} \lesssim \|\cdot\|_{H^q}$, so

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)}$$

as desired. □

Theorem 5.13. *Suppose* $0 < p < 1$, $0 < \beta < n$. *Let* $\widehat{\Phi}$ *denote the Fourier transform of* Φ *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M > n(\frac{1}{p}-1)$, *suppose that* $\widehat{\Phi}$ *is a function in* $C^{2M+n}(\mathbb{R}^n)$ *with compact support in the set* $\mathbb{R}^n \backslash \{0\}$. *Then*

$$\|H_{\Phi,\beta}(f)\|_{H^q(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}\,.$$

Proof. Since $M > n(\frac{1}{p} - 1)$ we have a $p_1 < p$ satisfying

$$M = n\left(\frac{1}{p_1} - 1\right).$$

By Theorem 5.11, we have

$$\|H_{\Phi,\beta}(f)\|_{H^{q_1,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p_1}(\mathbb{R}^n)}\,,$$

where

$$\frac{1}{p_1} = \frac{1}{q_1} + \frac{\beta}{n}.$$

Further, by Lemma 5.12,

$$\|H_{\Phi,\beta}(f)\|_{H^{q_2,\infty}(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^1(\mathbb{R}^n)}$$

where

$$1 = \frac{1}{q_2} + \frac{\beta}{n}.$$

Thus, by an interpolation argument, we obtain

$$\|H_{\Phi,\beta}(f)\|_{H^q(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^p(\mathbb{R}^n)}$$

as desired. □

The techniques used to prove Theorems 5.11 and 5.13 can also be used to show the following result. The specific Φ considered here does not meet the conditions on Theorems 5.11 and 5.13, which may suggest in general a method of relaxing these conditions.

Corollary 5.14. *Suppose* $0 < p < 1$, $0 < \beta < n$. *Let* $\Psi \in S(\mathbb{R}^n)$, *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M \geqslant n(\frac{1}{p} - 1)$, *let*

$$\Phi = \begin{cases} |\Delta|^{M+\frac{n}{2}}\,\Psi & \textit{if } n \textit{ is even} \\ |\Delta|^{M+\frac{n+1}{2}}\,\Psi & \textit{if } n \textit{ is odd} \end{cases},$$

where Δ *is the Laplacian and* $|\Delta|^{\gamma}$ *is the fractional Laplacian defined for any function* F *by*

$$(\widehat{|\Delta|^{\gamma} F})(\xi) = |\xi|^{2\gamma}\,\widehat{F}(\xi).$$

If $M = n(\frac{1}{p} - 1)$, *then*

$$\|H_{\Phi,\beta}(f)\|_{H^{q,\infty}(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^p(\mathbb{R}^n)},$$

If $M > n(\frac{1}{p} - 1)$, *then*

$$\|H_{\Phi,\beta}(f)\|_{H^q(\mathbb{R}^n)} \lesssim \|f\|_{H^p(\mathbb{R}^n)}.$$

Proof. Recall that according to Theorem 5.10, we must show that

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)} \lesssim |x|^{-\alpha}$$

for for all $J \in \{0, ..., n\}^L$, where L is large enough that $p > \frac{n-1}{n-1+L}$. Note that we can choose L even such that $2M + n > L$.

In the proof of Theorem 5.11, we have shown that

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)}$$

$$\lesssim \sup_{y \in \mathbb{R}^n \setminus \{0\}} \left\{ \sum_{|I|=\alpha} \left(\frac{|x|}{|y|}\right)^{n-\beta} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2| \leqslant l_1 \leqslant 2|K_2| \\ l_1 - l_2 = |K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \frac{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}}{|\xi|^L} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right| \right) \right\},$$

where P_{l_2} is a homogeneous polynomial of degree l_2. In this case, from the definition of Φ, we can rewrite the above as

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} (\mathcal{R}_J \Phi)(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha(\mathbb{R}^n)}$$

$$\lesssim \sup_{y \in \mathbb{R}^n \setminus \{0\}} \left\{ \sum_{|I|=\alpha} \left(\frac{|x|}{|y|}\right)^{n-\beta} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2| \leqslant l_1 \leqslant 2|K_2| \\ l_1 - l_2 = |K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} P(\xi) \widehat{\Psi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right| \right) \right\},$$

where $P(\xi)$ is a function of ξ with polynomial growth, namely,

$$P(\xi) = \begin{cases} \xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}|\xi|^{2M+n-L} & \text{if } n \text{ is even} \\ \xi_{j_1}\xi_{j_2}\cdots\xi_{j_L}|\xi|^{2M+n+1-L} & \text{if } n \text{ is odd} \end{cases}.$$

An integration by parts, as used in the proof of Theorem 5.11, works as expected. □

We can replace the support condition in Theorems 5.11 and 5.13 by the following condition, to show $H^p \to L^q$ boundedness.

Corollary 5.15. *Suppose* $0 < p < 1$, $0 \leqslant \beta < n$. *Let* $\widehat{\Phi}$ *denote the Fourier transform of* Φ *and*

$$\frac{1}{p} = \frac{1}{q} + \frac{\beta}{n}.$$

For an integer $M \geqslant n(\frac{1}{p} - 1)$, *suppose* $\widehat{\Phi}$ *is a function in* $C^{2M+n}(\mathbb{R}^n)$ *and satisfies, for all* I *with* $|I| \leqslant 2M + n$,

$$\partial^I(\widehat{\Phi})(\xi)\, |\xi|^k \in L^1(\mathbb{R}^n), \text{ for } k = 0, 1, 2, \ldots |I|.$$

Assume also

$$\lim_{\xi \to \infty} \left| \partial^I(\widehat{\Phi})(\xi) \right| |\xi|^M = 0, \text{ for all } I \text{ satisfying } |I| \leqslant 2M + n.$$

If $M = n(\frac{1}{p} - 1)$, *then*

$$\|H_{\Phi,\beta}(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^p(\mathbb{R}^n)}.$$

If $M > n(\frac{1}{p} - 1)$, *then*

$$\|H_{\Phi,\beta}(f)\|_{L^{q}(\mathbb{R}^n)} \preccurlyeq \|f\|_{H^p(\mathbb{R}^n)}.$$

Proof. The proof follows the same argument as the proof of Theorem 5.11. We prove the case $M = n(\frac{1}{p} - 1)$. To prove the first inequality in the corollary, by Lemma 5.8, we need to check

$$\left\| \frac{|x|^{n-\beta}}{|\cdot|^{n-\beta}} \Phi(\frac{x}{|\cdot|}) \right\|_{\Lambda_\alpha} \preccurlyeq |x|^{-\alpha},$$

where $\alpha = M$. Here, by Fourier transforms, we may write

$$\frac{|x|^{n-\beta}}{|y|^{n-\beta}} \Phi(\frac{x}{|\cdot|}) = |x|^{n-\beta} \int_{\mathbb{R}^n} \widehat{\Phi}(\xi)\, |y|^{-n+\beta}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle}\, d\xi.$$

Thus, as in the proof of Theorem 5.11,

$$\sup_{y \in \mathbb{R}^n \setminus \{0\}} \left\{ \sum_{|I|=\alpha} \left| \partial^I \left(|x|^{n-\beta} \int_{\mathbb{R}^n} \widehat{\Phi}(\xi)\, |y|^{-(n-\beta)}\, e^{i\frac{1}{|y|}\langle \xi, x\rangle}\, d\xi \right) \right| \right\}$$

$$\leqslant \sup_{y\in\mathbb{R}^n\setminus\{0\}} \left\{ \sum_{|I|=\alpha} \left(\frac{|x|}{|y|}\right)^{n-\beta} \sum_{K_1+K_2=I} \frac{1}{|y|^{|K_1|}} \left(\sum_{\substack{|K_2|\leqslant l_1 \leqslant 2|K_2| \\ l_1-l_2=|K_2|}} \frac{1}{|y|^{l_1}} \left| \int_{\mathbb{R}^n} \widehat{\Phi}(\xi) e^{i\frac{1}{|y|}\langle \xi, x\rangle} P_{l_2}(\langle \xi, x\rangle)\, d\xi \right| \right) \right\}.$$

The conditions in the theorem are sufficient to allow the integration by parts in Theorem 5.11 to proceed, which allows us to prove the first inequality. An argument similar to the one used to prove Lemma 5.12 shows that the condition is also sufficient to give the conclusion to that lemma. Thus, we obtain the second inequality by an interpolation argument. □

CHAPTER 6

ADDITIONAL SPACES AND OPERATORS

6.1 Motivation

In studying the H^p boundedness of the Hausdorff operator, we have so far obtained only results which require some smoothness condition on Φ.

Question 4. If we don't assume any smoothness condition on Φ, and only assume Φ satisfies some size condition, can we obtain the H^p boundedness of Φ?

To gain insight into this question, let us review the argument showing that h_Φ is bounded on $H^1(\mathbb{R})$ when $\Phi \in L^1$. When $p = 1$, H^1 is a normed space, and so we may use the Minkowski inequality.

Recall that for $f \in H^p$, $0 < p \leqslant 1$, we may write f as an atomic decomposition,

$$f = \sum \lambda_j a_j$$

with

$$\sum |\lambda_j|^p \cong \|f\|_{H^p}^p ,$$

where each $a_j(x)$ is a (p, ∞, s) atom, with s an integer such that $s \geqslant \left[n\left(\frac{1}{p} - 1\right)\right]$. We study the H^p boundedness of the Hausdorff operator.

So, for $f \in H^1$, take an atomic decomposition $f = \sum \lambda_j a_j$. Using the Minkowski inequality, we have

$$\|h_\Phi(f)\|_{H^1} \leqslant \sum |\lambda_j| \, \|h_\Phi(a_j)\|_{H^1}.$$

where $\sum \lambda_j a_j$ is an atomic decomposition of f. In the above inequality, by the Minkowski inequality again,

$$\|h_\Phi(a_j)\|_{H^1} = \left\| \int_0^\infty \frac{\Phi(t)}{t} a_j(\frac{x}{t}) \, dt \right\|_{H^1}$$

$$\leqslant \int_0^\infty \frac{|\Phi(t)|}{t} \left\| a_j(\frac{\cdot}{t}) \right\|_{H^1} dt$$
$$= \int_0^\infty |\Phi(t)| \left\| \frac{1}{t} a_j(\frac{\cdot}{t}) \right\|_{H^1} dt.$$

Since, for fixed $t > 0$, $\frac{1}{t} a_j(\frac{x}{t})$ is also a $(1, \infty, s)$ atom, we have

$$\left\| \frac{1}{t} a_j(\frac{\cdot}{t}) \right\|_{H^1} \leqslant 1$$

uniformly for t and j. Thus we obtain that

$$\|h_\Phi(f)\|_{H^1} \leqslant \sum |\lambda_j| \int_0^\infty |\Phi(t)| \ dt \cong \|f\|_{H^1} \int_0^\infty |\Phi(t)| \ dt.$$

Therefore, we obtain that h_Φ is bounded on H^1 provided

$$\int_0^\infty |\Phi(t)| \ dt < \infty.$$

Now we try to extend the above argument to the H^p boundedness for $0 < p < 1$. First we have

$$\|h_\Phi(f)\|_{H^p}^p \leqslant \sum |\lambda_j|^p \|h_\Phi(a_j)\|_{H^p}^p . \tag{6.1}$$

If we can show, by assuming certain size condition on Φ, that

$$\|h_\Phi(a_j)\|_{H^p}^p \leqslant 1 \tag{6.2}$$

uniformly on j, then from equation (6.1) we obtain the H^p boundedness of h_Φ. So, showing the H^p boundedness of h_Φ is reduced to showing equation (6.2). Unfortunately, when $p < 1$, H^p is not a normed space, so we cannot apply the Minkowski inequality to obtain, for an atom a,

$$\|h_\Phi(a)\|_{H^p} \leqslant \int_0^\infty \frac{|\Phi(t)|}{t} \left\| a(\frac{\cdot}{t}) \right\|_{H^p} dt$$

as we did for H^1 norm. To overcome this difficulty, we write

$$h_\Phi(a)(x) = \int_0^\infty \frac{\Phi(t)}{t} a(\frac{x}{t}) \ dt = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} a(\frac{x}{t}) \ dt,$$

and study each

$$\int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} a(\frac{x}{t})\ dt.$$

We introduce a normalizing factor, and define a $(p, \infty, 0)$ atom A_k by

$$A_k(x) = \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} a(\frac{x}{t})\ dt.$$

This is possible if supp a contains the origin. To show this fact, without loss of generality, we may assume the support of a is $(-\rho, \rho)$. As t runs over the interval $(2^k, 2^{k+1})$, we can view that the support of $a(\frac{x}{t})$ is contained in the interval$\{x : |x| \leqslant \rho 2^{k+1}\}$. Thus $A_k(x)$ is supported in $(-\rho 2^{k+1}, \rho 2^{k+1})$.

Next, by applying Fubini's theorem, we can show the cancellation condition, as follows. Fix $u \leqslant s$. Then

$$\begin{aligned}\int_{\mathbb{R}} x^u A_k(x) &= \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} \int_{\mathbb{R}} x^u a(\frac{x}{t})\ dx\ dt \\ &= \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \int_{2^k}^{2^{k+1}} \Phi(t) t^u \int_{\mathbb{R}} x^u a(x)\ dx\ dt = 0.\end{aligned}$$

Also, we have the size condition with radius $\rho 2^{k+1}$

$$\begin{aligned}\|A_k\|_{L^\infty} &\leqslant \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \int_{2^k}^{2^{k+1}} \frac{|\Phi(t)|}{t} \|a\|_{L^\infty}\ dt \\ &\leqslant \rho^{-\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{|\Phi(t)|}{t}\ dt \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \\ &= 2^{\frac{1}{p}} (2^{k+1}\rho)^{-\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{2^{\frac{k}{p}}\,|\Phi(t)|}{t}\ dt \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right)^{-1} \\ &\cong (2^{k+1}\rho)^{-\frac{1}{p}}.\end{aligned}$$

Thus, we have proved that each A_k is a (p, ∞, s) atom. We write

$$\int_0^\infty \frac{\Phi(t)}{t} a(\frac{x}{t})\ dt = \sum_{k=-\infty}^{\infty} \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}}\,|\Phi(t)|}{t}\ dt \right) A_k(x).$$

Now, if we assume that for some $\sigma > \frac{1-p}{p}$,

$$\int_{\mathbb{R}} \frac{t^{\frac{1}{p}} |\Phi(t)|}{t}(1 + |\log_2 |t||)^{\sigma} \, dt \leqslant C$$

then by an easy computation, we obtain

$$\begin{aligned} \|h_\Phi(a)\|_{H^p}^p &\leqslant \sum_{k=-\infty}^{\infty} \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}} |\Phi(t)|}{t} \, dt \right)^p \|A_k\|_{H^p}^p \\ &\leqslant \sum_{k=-\infty}^{\infty} \left(2^{\frac{1}{p}} \int_{2^k}^{2^{k+1}} \frac{t^{\frac{1}{p}} |\Phi(t)|}{t} \, dt \right)^p \\ &\leqslant \left(\int_{\mathbb{R}} \frac{t^{\frac{1}{p}} |\Phi(t)|}{t}(1 + |\log_2 |t||)^{\sigma} \, dt \right)^p \leqslant 1. \end{aligned}$$

Thus equation (6.2) is proved, if a_j is supported in an interval containing 0.

However, we will see that the above argument fails if the support of a does not contain the origin. If supp $(a) \subset (\alpha, \beta)$, and $\beta > \alpha > 0$, then

$$\int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} a(\frac{x}{t}) \, dt$$

is supported in $(2^k\alpha, 2^{k+1}\beta)$. Now the length of $(2^k\alpha, 2^{k+1}\beta)$ is

$$2^{k+1}\beta - 2^k\alpha = 2^{k+1}(\beta - \alpha) + 2^k\alpha.$$

This length can not be compared to $2^{k+1}(\beta-\alpha)$, since the number $2^k\alpha$ may be arbitrarily large. Thus, we can not choose a normalizing factor F_k so that $F_k \int_{2^k}^{2^{k+1}} \frac{\Phi(t)}{t} a(\frac{x}{t}) \, dt$ satisfies the size condition in general.

Based on these observations, we wish to restrict our discussion to only those atoms whose support includes the origin. In other words, we wish to consider a space whose elements f have the central atomic decompositions, that is, each atom a_j in a decomposition of f is supported in $(-\rho_j, \rho_j)$. Such spaces are called Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, introduced in section 2.5.

6.2 Boundedness of Hausdorff Operator on $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$

Recall from page 26 the definitions of the homogeneous Herz space and the Herz-type Hardy space.

Definition 2.4 (Homogeneous Herz Space)**.** *Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$\dot{K}_q^{\alpha,p} = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

Definition 2.5 (Herz-type Hardy Space)**.** *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$. The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : Gf \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \right\},$$

where Gf is the grand maximal function of f and

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Recall also from page 26 that the Herz-type Hardy space can be decomposed into central atoms.

Definition 2.6 (Central Atom)**.** *Suppose $1 < q < \infty$, $n(1 - \frac{1}{q}) \leqslant \alpha < \infty$, and $s \geqslant [\alpha + n(\frac{1}{q} - 1)]$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central (α, q) atom if*

(i) *supp* $a \subset B(0, \rho)$,

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leqslant |B(0,\rho)|^{-\alpha/n}$,

(iii) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\beta \, dx = 0$ *for any multi-index β with $|\beta| \leqslant s$.*

Following logic similar to that in Section 6.1, we have the following result for Herz-type Hardy spaces.

Theorem 6.1. *Let* $0 < p \leqslant 1 < q < \infty$, *and* $n(1 - \frac{1}{q}) \leqslant \alpha < \infty$.

(1) For $0 < p < 1$, *let*

$$C_{p,\sigma} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^\alpha |y|^{\frac{n}{q}} (1 + \log |y|)^\sigma \, dy.$$

If, for some $\sigma > \frac{1-p}{p}$, $C_p := C_{p,\sigma} < \infty$, *then*

$$\|H_\Phi(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

(2) For $p = 1$, *let*

$$C_1 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^\alpha |y|^{\frac{n}{q}} \, dy.$$

If $C_1 < \infty$, *then*

$$\|H_\Phi(f)\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,1}(\mathbb{R}^n)}.$$

Proof. We prove part (1) only; the argument for part (2) is identical. For $f \in H\dot{K}_q^{\alpha,p}$, take a central atomic decomposition.

$$f = \sum_k \lambda_k a_k$$

where

$$\sum_{k \in \mathbb{Z}} |\lambda_k|^p \cong \|f\|^p_{H\dot{K}_q^{\alpha,p}}$$

and each a_k is an (α, q) central atom supported in $B(0, \rho_k)$. Now we have

$$H_\Phi(f) = \sum_{k \in \mathbb{Z}} \lambda_k H_\Phi(a_k).$$

To prove the theorem, it suffices to show that

$$H_\Phi(a_k) = \sum_{j \in \mathbb{Z}} c_{k,j} a_{k,j},$$

where each $a_{k,j}$ again is a central (α, q) atom and

$$\sum_{j\in\mathbb{Z}} |c_{k,j}|^p \leqslant 1$$

uniformly on $k \in \mathbb{Z}$.

We write

$$b_{k,j}(x) = \int_{2^{-j}\leqslant|y|\leqslant 2^{-j+1}} \frac{\Phi(y)}{|y|^n} a_k\left(\frac{x}{|y|}\right)\, dy, \quad j \in \mathbb{Z}.$$

So

$$H_\Phi(a_k)(x) = \sum_{j\in\mathbb{Z}} b_{k,j}(x).$$

It is easy to check that each $b_{k,j}$ satisfies the same cancellation condition as a_k. By the Minkowski inequality, the size of $b_{k,j}$ is

$$\begin{aligned} \|b_{k,j}\|_{L^q} &\leqslant \int_{2^{-j}\leqslant|y|\leqslant 2^{-j+1}} \frac{|\Phi(y)|}{|y|^n} \left\| a_k\left(\frac{\cdot}{|y|}\right)\right\|_{L^q} dy \\ &\leqslant \rho^{-\alpha} \int_{2^{-j}\leqslant|y|\leqslant 2^{-j+1}} \frac{|\Phi(y)|}{|y|^n} |y|^{\frac{n}{q}}\, dy. \end{aligned}$$

If $|x| > 2^{-j+1}\rho_k$ then

$$\frac{|x|}{|y|} \geqslant 2^{j-1}|x| > \rho_k,$$

which means $a_k(\frac{x}{|y|}) = 0$ for all $2^{-j} \leqslant |y| \leqslant 2^{-j+1}$. This tells us that

$$\text{supp}\,(b_{k,j}) \subset B(0, 2^{-j+1}\rho_k).$$

Now we write

$$H_\Phi(a_k)(x) = \sum_{j\in\mathbb{Z}} c_{k,j} a_{k,j}$$

with

$$c_{k,j} = 2^{(-j+1)\alpha} \int_{2^{-j}\leqslant|y|\leqslant 2^{-j+1}} \frac{|\Phi(y)|}{|y|^n} |y|^{\frac{n}{q}}\, dy,$$

and

$$a_{k,j} = \frac{1}{c_{k,j}} b_{k,j}.$$

It is easy to check that $a_{k,j}$ is a central (α, q) atom and

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |c_{k,j}|^p &\leqslant \sum_{j\in\mathbb{Z}} \left(\int_{2^{-j}\leqslant|y|\leqslant 2^{-j+1}} \frac{|\Phi(y)|}{|y|^n} |y|^{\alpha+\frac{n}{q}} \ dy \right)^p \\ &\leqslant \left(\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{\alpha+\frac{n}{q}} (1+|log\,|y||)^\sigma \ dy \right)^p = C_p.\end{aligned}$$

This shows

$$\begin{aligned}H_\Phi(f) &= \sum_{k\in\mathbb{Z}} \lambda_k H_\Phi(a_k) \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \lambda_k c_{k,j} a_{k,j}.\end{aligned}$$

By the atomic decomposition, we obtain

$$\begin{aligned}\|H_\Phi(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\leqslant \left(\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |\lambda_k c_{k,j}|^p \right)^{\frac{1}{p}} \\ &\leqslant \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p \right)^{\frac{1}{p}} \\ &\leqslant \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},\end{aligned}$$

as desired. □

We now return to the example of the Cesàro operator $\mathcal{C}_\beta$.

Corollary 6.2. *Let* $0 < p \leqslant 1 < q < \infty$, *and* $n(1-\frac{1}{q}) \leqslant \alpha < \infty$. *For any* $\beta > 0$, *we have*

$$\|\mathcal{C}_\beta(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant \|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Proof. Note $\mathcal{C}_\beta = H_\Phi$, where $\Phi = \chi_{|y|<1}(y)(1-|y|)^{\beta-1}$. It is easy to check that this Φ meets the conditions of Theorem 6.1. □

6.3 $H\dot{K}_q^{\alpha,p}(\mathbb{R}) \to H^r(\mathbb{R})$ Boundedness

In this section, we study the $H\dot{K}_q^{\alpha,p}(\mathbb{R}) \to H^{r,\infty}(\mathbb{R})$ boundedness for the one dimensional modified Hausdorff operator

$$\widetilde{h}_\Phi(f)(x) = \int_{\mathbb{R}} \frac{\Phi(t)}{t} f(\frac{x}{t})\, dt.$$

Note that here $\Phi : \mathbb{R} \to \mathbb{R}$ is a locally integrable function. As before, we denote

$$\phi(v) = \begin{cases} \frac{\Phi\left(\frac{1}{v}\right)}{v} & \text{if } v \neq 0 \\ \lim\limits_{v\to 0} \frac{\Phi\left(\frac{1}{v}\right)}{v} & \text{if } v = 0. \end{cases}$$

Theorem 6.3. *Let* $0 < p \leqslant 1 < q < \infty$, *and* $(1 - \frac{1}{q}) \leqslant \alpha < \infty$. *For*

$$r = \frac{1}{\alpha + \frac{1}{q}}, \quad N = [\alpha + \frac{1}{q} - 1] = \left[\frac{1}{r} - 1\right], \quad \frac{1}{q'} + \frac{1}{q} = 1.$$

Let

$$\gamma = \frac{1}{r} - 1 - N = \frac{1}{r} - 1 - \left[\frac{1}{r} - 1\right].$$

If $\phi \in C^N$ *and*

$$\int_{-\rho}^{\rho} \left|\phi^{(N)}(t) - \phi^{(N)}(0)\right|^{q'}\, dt \lesssim \rho^{1+\gamma q'}$$

uniformly for $\rho > 0$, *then we have*

$$\left\|\widetilde{h}_\Phi(f)\right\|_{H^{r,\infty}} \lesssim \|f\|_{H\dot{K}_q^{\alpha,p}}.$$

Proof. For any $f \in H\dot{K}_q^{\alpha,p}$, we write

$$f = \sum_k \lambda_k a_k,$$

where

$$\sum_{k\in\mathbb{Z}} |\lambda_k|^p \cong \|f\|^p_{H\dot{K}_q^{\alpha,p}},$$

and each a_k is an (α, q) central atom supported in the ball $B(0, \rho_k)$. Now we have

$$\widetilde{h}_\Phi(f) = \sum_{k\in\mathbb{Z}} \lambda_k \widetilde{h}_\Phi(a_k).$$

Then as in the proof of Theorem 3.1, for the Hilbert transform $\mathcal{R}$, we can write

$$\mathcal{R} \circ \widetilde{h}_\Phi(f) = \sum_{k\in\mathbb{Z}} \lambda_k (\mathcal{R} \circ \widetilde{h}_\Phi)(a_k)$$

and

$$(\mathcal{R} \circ \widetilde{h}_\Phi)(a_k)(x) = \int_{-\infty}^{\infty} \frac{\Phi(t)}{t} (\mathcal{R}a_k)(\frac{x}{t})\, dt.$$

Changing variables $\frac{1}{t} = v$, by the definition of ϕ, we have

$$\begin{aligned} \left|\mathcal{R} \circ \widetilde{h}_\Phi(a_k)(x)\right| &= \left|\int_{-\infty}^{\infty} \frac{\Phi(\frac{1}{v})}{v} (\mathcal{R}a_k)(xv)\, dv\right| \\ &= \left|\int_{-\infty}^{\infty} \phi(v)(\mathcal{R}a_k)(xv)\, dv\right|. \end{aligned}$$

It is known [20] that the Hilbert transform is bounded on the space $H\dot{K}_q^{\alpha,p}$. That is,

$$\|(\mathcal{R}a_k)\|_{H\dot{K}_q^{\alpha,p}} \leqslant \|a_k\|_{H\dot{K}_q^{\alpha,p}} \leqslant 1$$

uniformly for all atoms a_k. Thus $\mathcal{R}a_k$ is an element in $H\dot{K}_q^{\alpha,p}$, so we may write

$$(\mathcal{R}a_k) = \sum_j c_{k,j} a_{k,j},$$

where each $a_{k,j}$ is again a central (α, q) atom and

$$\sum_j |c_{k,j}|^p \leqslant 1$$

uniformly on k. This shows that

$$\left|\mathcal{R} \circ \widetilde{h}_\Phi(a_k)(x)\right| = \left|\sum_j c_{k,j} \int_{-\infty}^{\infty} \phi(v) a_{k,j}(xv)\, dv\right|.$$

We further note that for each fixed $x \neq 0$,

$$b_{k,j}(v) = |x|^{\frac{1}{r}}\, a_{k,j}(xv)$$

is again an (α, q) central atom. Thus,

$$\left|\mathcal{R} \circ \widetilde{h}_{\Phi}(a_k)(x)\right| = |x|^{-\frac{1}{r}} \left|\sum_j c_{k,j} \int_{-\infty}^{\infty} \phi(v) b_{k,j}(v)\, dv\right|.$$

For simplicity of notation, we write

$$b(v) = b_{k,j}(v),$$

and assume the support of b is $(-\rho, \rho)$. Thus,

$$\int_{-\infty}^{\infty} \phi(v) b_{k,j}(v)\, dv = \int_{-\rho}^{\rho} \phi(v) b(v)\, dv.$$

First we assume $N \geqslant 1$. Using the Taylor expansion of ϕ, we write

$$\phi(v) = \sum_{k=0}^{N-1} \frac{1}{k!} \phi^{(k)}(0) v^k + \frac{1}{(N-1)!} v^N \int_0^1 (1-s)^{N-1} \phi^{(N)}(sv)\, ds.$$

By the cancellation condition on b, we now have

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(v) b(v)\, dv &= \frac{1}{(N-1)!} \int_{-\rho}^{\rho} v^N \left(\int_0^1 (1-s)^{N-1} \phi^{(N)}(sv)\, ds \right) b(v)\, dv \\
&\cong \int_{-\rho}^{\rho} \left\{ \int_0^1 (1-s)^{N-1} [\phi^{(N)}(sv) - \phi^{(N)}(0)]\, ds \right\} v^N b(v)\, dv \\
&= \int_0^1 (1-s)^{N-1} \int_{-\rho}^{\rho} [\phi^{(N)}(sv) - \phi^{(N)}(0)] v^N b(v)\, dv\, ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \phi(v) b(v)\, dv\right| &\lesssim \rho^N \int_0^1 (1-s)^{N-1} \int_{-\rho}^{\rho} \left|\phi^{(N)}(sv) - \phi^{(N)}(0)\right| |b(v)|\, dv\, ds \\
&\lesssim \rho^N \|b\|_{L^q} \int_0^1 (1-s)^{N-1} \left(\int_{-\rho}^{\rho} \left|\phi^{(N)}(sv) - \phi^{(N)}(0)\right|^{q'} dv \right)^{\frac{1}{q'}} ds \\
&\lesssim \rho^N \|b\|_{L^q} \int_0^1 (1-s)^{N-1} s^{-\frac{1}{q'}} \left(\int_{-s\rho}^{s\rho} \left|\phi^{(N)}(v) - \phi^{(N)}(0)\right|^{q'} dv \right)^{\frac{1}{q'}} ds \\
&\lesssim \rho^{N-\alpha} \rho^{\gamma + \frac{1}{q'}} \\
&= \rho^{\frac{1}{r} - 1 + \frac{1}{q'} - \alpha} \\
&= \rho^{\frac{1}{r} - \frac{1}{q} - \alpha} = 1
\end{aligned}$$

uniformly for all atoms b. On the other hand, if $N = 0$,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \phi(v)b(v)\ dv\right| &= \left|\int_{-\rho}^{\rho} (\phi(v) - \phi(0))b(v)\ dv\right| \\ &\leqslant \left(\int_{-\rho}^{\rho} |\phi(v) - \phi(0)|\ dv\right)^{\frac{1}{q'}} \|b\|_{L^q} \leqslant 1\end{aligned}$$

uniformly for all atoms b.

This indicates

$$\left|\mathcal{R} \circ \widetilde{h}_\Phi(a_k)(x)\right| = |x|^{-\frac{1}{r}} \left|\sum_j c_{k,j} \int_{-\infty}^{\infty} \phi(v)b_{k,j}(v)\ dv\right| \leqslant |x|^{-\frac{1}{r}}$$

uniformly on central atoms a_k.

For any $\lambda > 0$, now we have

$$\begin{aligned}&\left|\left\{x \neq 0 : \left|\mathcal{R} \circ \widetilde{h}_\Phi(a_k)(x)\right| > \lambda\right\}\right| \\ &\leqslant \left|\left\{x \neq 0 : |x|^{-\frac{1}{r}} \geqslant \lambda\right\}\right| \\ &= \left|\left\{x \neq 0 : |x|^{\frac{1}{r}} \leqslant \frac{1}{\lambda}\right\}\right| \\ &\cong \left(\frac{1}{\lambda}\right)^r.\end{aligned}$$

We can show by a similar method

$$\left|\left\{x : \left|\widetilde{h}_\Phi(a_k)(x)\right| > \lambda\right\}\right| \leqslant \left(\frac{1}{\lambda}\right)^r.$$

This gives

$$\begin{aligned}\left\|\widetilde{h}_\Phi(f)\right\|_{H^{r,\infty}} &\cong \left\|\mathcal{R} \circ \widetilde{h}_\Phi(f)\right\|_{L^{r,\infty}} + \left\|\widetilde{h}_\Phi(f)\right\|_{L^{r,\infty}} \\ &\leqslant \left(\sum_k |\lambda_k|^p\right)^{\frac{1}{p}} \leqslant \|f\|_{H\dot{K}_q^{\alpha,p}}\end{aligned}$$

as desired. □

Remark 6.4. An analogous strong boundedness result can be shown by interpolating on r and q.

6.4 Bilinear Hausdorff Operators

Next we study the bilinear Hausdorff operator

$$H_{\Phi,m,k}(f,g)(x) = \int_{-\infty}^{\infty} \frac{\Phi(t)}{t} f(\frac{x^m}{t}) g(\frac{x^k}{t})\, dt.$$

Note this is defined for all $x \in \mathbb{R}\backslash\{0\}$ when $m, k \in \mathbb{Z}$.

Theorem 6.5. *Let $m, k = 1, 2, \ldots$. For any $p, p_1, p_2, r, p' \geqslant 1$ satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \qquad \frac{1}{r} = \frac{m}{p_1} + \frac{k}{p_2}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

if

$$\left(\int_{-\infty}^{\infty} \left|\frac{\Phi\left(\frac{1}{t}\right)}{t}\right|^{p'} dt\right)^{\frac{1}{p'}} < \infty$$

then

$$\|H_{\Phi,m,k}(f,g)\|_{L^{r,\infty}} \leqslant \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

Proof. By Hölder's inequality and a scaling argument, for $x > 0$

$$\begin{aligned}
|H_{\Phi,m,k}(f,g)(x)| &= \left|\int_{-\infty}^{\infty} \frac{\Phi(\frac{1}{t})}{t} f(tx^m) g(tx^k)\, dt\right| \\
&\leqslant \left(\int_{-\infty}^{\infty} \left|\frac{\Phi(\frac{1}{t})}{t}\right|^{p'} dt\right)^{\frac{1}{p'}} \left\|f(\cdot x^m) g(\cdot x^k)\right\|_{L^p} \\
&\leqslant \left(\int_{-\infty}^{\infty} \left|\frac{\Phi(\frac{1}{t})}{t}\right|^{p'} dt\right)^{\frac{1}{p'}} \|f(\cdot x^m)\|_{L^{p_1}} \left\|g(\cdot x^k)\right\|_{L^{p_2}} \\
&= |x|^{-\frac{m}{p_1} - \frac{k}{p_2}} \left(\int_{-\infty}^{\infty} \left|\frac{\Phi(\frac{1}{t})}{t}\right|^{p'} dt\right)^{\frac{1}{p'}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \\
&= C_\Phi |x|^{-\frac{m}{p_1} - \frac{k}{p_2}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},
\end{aligned}$$

where

$$C_\Phi = \left(\int_{-\infty}^{\infty} \left|\frac{\Phi(\frac{1}{t})}{t}\right|^{p'} dt\right)^{\frac{1}{p'}}.$$

Therefore,

$$\begin{aligned}
&|\{x : |H_\Phi(f,g)(x)| > \lambda\}| \\
&\leqslant \left|\left\{x : C_\Phi \left|x\right|^{-\frac{m}{p_1}-\frac{k}{p_2}} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} > \lambda\right\}\right| \\
&\cong \left|\left\{x : |x|^{\frac{m}{p_1}+\frac{k}{p_2}} \leqslant \frac{\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}}{\lambda}\right\}\right| \\
&\cong \left(\frac{\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}}{\lambda}\right)^{\frac{1}{\frac{m}{p_1}+\frac{k}{p_2}}},
\end{aligned}$$

where by assumption $\frac{m}{p_1} + \frac{k}{p_2} < 1$.

Recalling $r = \frac{1}{\frac{m}{p_1}+\frac{k}{p_2}} \geqslant 1$, we have

$$\begin{aligned}
&|\{x : |H_\Phi(f,g)(x)|^r > \lambda\}| \\
&= \left|\left\{x : |H_\Phi(f,g)(x)| > \lambda^{\frac{1}{r}}\right\}\right| \\
&\leqslant \left(\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}\right)^{\frac{1}{\frac{m}{p_1}+\frac{k}{p_2}}} \lambda^{-1}.
\end{aligned}$$

This gives

$$\|H_\Phi(f,g)\|_{L^{r,\infty}} \leqslant \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

as desired. □

If we restrict the domains of f, g, Φ to $[0,\infty)$, we can extend the definition of the bilinear Hausdorff operator as: for $f, g \in S([0,\infty)), \alpha, \beta \in \mathbb{R}$,

$$H_{\Phi,\alpha,\beta}(f,g)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x^\alpha}{t}\right) g\left(\frac{x^\beta}{t}\right) \, dt,$$

in which case, the method of proof used for Theorem 6.5 can also be used to show the analogous result.

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Xiaoying Lin

Curriculum Vitae

Personal Data

Place of Birth	Xiamen, China
Nationality	Chinese Citizen, US Permanent Resident
Language	Native Chinese speaker, Fluent in English

Education

2013 **PhD in Mathematics** from the University of Wisconsin—Milwaukee
Thesis: Boundedness of Hausdorff Operator in Various Function Spaces
Advisor: Dashan Fan

2009 **MS in Mathematics** from the University of Wisconsin—Milwaukee

2007 **BS in Mathematics Education** from the University of Wisconsin—Milwaukee

2005 **AA in Math Teacher Education** from Milwaukee Area Technical College

1995 **AAS in Financial Accounting of Commercial Enterprise** from Beijing Social Correspondence University of China

1992 **AAS in Sugar Refinery Technology** from Jimei Light Industry School (China)

Employment

2010 – 2013 Graduate Teaching Assistant in the Mathematical Sciences Department at UW—Milwaukee

2009 – 2010 Adjunct Faculty in the Mathematical Science Department at Milwaukee Area Technical College

2009 – 2010 High School Teacher at Milwaukee Montessori IB High School

2007 – 2009 Graduate Teaching Assistant in the Mathematical Sciences Department at UW—Milwaukee

2007 Math Student Teacher at Riverside University High School, Milwaukee

2006 Math Student Teacher at I.D.E.A.L Charter School, Milwaukee

1997 – 2001 Marketing Sales Representative, Futis Oversea Limited, Xiamen, China

1992 – 1997 Assistant Engineer, Xiamen Sugar Factory, Xiamen, China

Teaching Experience

2007 – 2013: Teaching Fellow

UWM Mathematical Sciences Department.
Had various levels of responsibility for a variety of college level courses.

Fall 2012 Complex Analysis—Grader. Graduate-level course in Complex Analysis.

Fall 2011 Advanced Calculus—Grader. Upper-level undergraduate course, which treats calculus from an abstract math viewpoint. This is a first proof-writing oriented course for many students. Grading involves evaluation of student writing as well as computation.

Fall 2010 Calculus and Analytic Geometry I—Primary instructor. Limits, derivatives, and graphs of algebraic, trigonometric, exponential, and logarithmic functions; antiderivatives, the definite integral, and the fundamental theorem of calculus, with applications. Responsible for setting syllabus to prepare students to reach university benchmarks to proceed to Calculus II.

Fall 2008 Survey in Calculus and Analytic Geometry—Discussion Leader. This one semester survey course covers applications for business administration, economics, and health sciences. Topics include coordinate systems, equations of curves, limits, differentiation in one and several variables, basic techniques of integration, and many applications.

Fall 2007 Discrete Probability and Statistics for Elementary Education Majors (Topics for K – 8 teachers)—Classroom Assistant. This course covers basic concepts in probability and statistics, with a focus on hands-on activities and the understanding of mathematics as fundamentally comprehensible study. Taught in cooperation with a professor and a visiting teacher from Milwaukee Public Schools. Responsible for grading, assistance with classroom activities, and occasional lecture.

2009 – 2010: Part Time Faculty

Milwaukee Area Technical College Math Department
Had full responsibility for a variety of math courses.

Summer 2010 Calculus and Analytic Geometry I—Primary Instructor. This is an intensive eight-week summer course. Students come from various other colleges to take summer credits for transfer to a primary institution. Course introduces the basic properties of limits, rate of change of functions, continuity, derivatives of algebraic, trigonometric and elementary and transcendental functions and their applications of derivatives, the indefinite integral and its applications including areas, derivatives and integrals involving logarithmic exponential, inverse trigonometric and hyperbolic functions, curve sketching, finding maxima and minima. Responsible for ensuring consistency of material covered with what is listed in transfer materials, in order to provide a comfortable transfer experience.

Spring 2010 Calculus and Analytic Geometry III—Primary Instructor. Taught in multiple locations via simultaneous video feed as part of school's long-distance learning program. Communication with students was through email, and written work was submitted and returned by fax. The challenges of this teaching style included communicating clearly about mathematics in plain-text emails, monitoring understanding of remote students, ensuring proper organization for materials that needed to be sent to several campuses, and scheduling test proctors for remote locations. This course is targeted toward applied science students. Full responsibility for designing syllabus and selecting course topics. Topics include vectors, geometry of space, vector valued functions, partial derivatives, multiple integrals, and vector analysis.

Fall 2009 College Mathematics—Primary Instructor. A first course in college level algebra. Taught in a technical college setting to students from diverse backgrounds, including many returning students and students with work and family responsibilities. Many students come to this class with very limited mathematical knowledge. One central idea this course tries to teach is learning math through understanding rather than rote memorization of procedures. Topics covered include solving polynomial equations, polynomial arithmetic, introduce exponentials and logarithms.

2009 – 2010: High School Teacher
Montessori IB High School, Milwaukee

Followed International Baccalaureate (IB) program guidelines to teach mathematics to high school juniors and seniors. Students came from low-income families in a majority-minority area. The school's goal was to help these students be well-prepared for college in mathematics and other areas. Working with the challenges provided by limited school resources, I worked to provide individualized instruction in a classroom setting. Material included polynomials, working with functions, inequalities, sequences and patterns, and basic trigonometry.

Spring 2007: Math Student Teacher
Riverside University High School, Milwaukee

Worked with senior teacher to develop hands-on lessons for a diverse population of high school students. This school's focus was on fostering student creativity and investigation, rather than rote training of individual mathematical techniques.

Fall 2006: Math Student Teacher
I.D.E.A.L. Charter School, Milwaukee

Worked with senior teacher to develop lessons for a diverse population of middle school students. A major focus at this level was to emphasize relevance of material to the world beyond the classroom. Worked to involve students' parents in their learning. Focused on hands-on activities and investigation.

2004 – 2007: Teacher of Chinese Language and Culture
Cricket Academy, Wauwatosa

Saturday enrichment class for children aged 5 – 7, covering reading, writing and speaking of basic Chinese, as well as Chinese culture.

Training and Professional Development

2010 – 2012 Progress toward Wisconsin Community/Technical College Teaching Certification, Milwaukee Area Technical College, Milwaukee, WI

2010 IB (Internal Baccalaureate) Program Teacher Training, Salt Lake City, UT

Publications and Preprints

L. Sun and X. Lin *Some estimates on Hausdorff Operator*, Acta Sci.Math.(Szeged), accepted for publication May 2012.

Honors and Distinctions

2007 – 2012 Recipient of *Graduate Assistantship in Areas of National Need Fellowship*. This fellowship is awarded to students with excellent records who demonstrate financial need and plan to pursue the highest degree available in their course of study in a field designated as an area of national need.

2010 – 2012 Recipient of *Advanced Opportunity Program Fellowship*. This fellowship is awarded to graduate students who are members of groups underrepresented in graduate study.

2007 Recipient of *Greater Milwaukee Foundation's Alice C. Helland Scholarship*, an award to female students planning to re-enter the workforce, enrolled in a science, math, engineering, or computer science curriculum.

Service

Spring 2005 Volunteer Teacher Assistant, Grand View High School, Milwaukee, WI
Assisted in math classroom in alternative school for at-risk students. Conveyed my own learning experiences to students to encourage success.

Fall 2006 Volunteer Math Tutor, University of Wisconsin—Milwaukee

2003 – 2004 Volunteer Math Tutor, Milwaukee Area Technical College
Provided individual instruction to college students in algebra and general math. Introduced the use of graphing calculators. Focused on building self confidence and increasing motivation. Provided encouragement with immediate praise and verbal reinforcement.

Licenses and Certifications

Teaching Certification, Wisconsin Early Adolescence to Adolescence (ages 10 – 21) in Mathematics

Memberships

National Council of Teachers of Mathematics
Wisconsin Mathematics Council
American Mathematical Society