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Associated Hypotheses in Linear Models for Unbalanced Data

Carlos J. Soto
University of Wisconsin-Milwaukee

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Carlos J. Soto

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ABSTRACT

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Carlos J. Soto

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When looking at factorial experiments there are several natural hypotheses that can be tested. In a two-factor or $a \times b$ design, the three null hypotheses of greatest interest are the absence of each main effect and the absence of interaction. There are two ways to construct the numerator sum of squares for testing these, namely either adjusted or sequential sums of squares (also known as type I and type III in SAS). Searle has pointed out that, for unbalanced data, a sequential sum of squares for one of these hypotheses is equal (with probability 1) to an adjusted sum of squares for a non-standard associated hypothesis. In his view, then, sequential sums of squares may test the wrong hypotheses. We give an exposition of this topic to show how to derive the hypothesis associated to a given sequential sum of squares.

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1 Introduction

When looking at factorial experiments there are several natural hypotheses that can be tested. In a two-factor or $a \times b$ design, the three null hypotheses of greatest interest are the absence of each main effect and the absence of interaction. We are under the assumption that no cell in our experiment is empty and further the cells could be unbalanced, that is that, they do not necessarily have the same sample size in each cell.

To test our hypothesis we first fit a linear model. Then there are two ways to construct the "numerator" sum of squares for testing these hypotheses, namely either "adjusted" or "sequential" sums of squares (also known as type I and type III in SAS).

In *Linear Models* [2, Section 7.2(f)] Searle derives the hypotheses associated with various sequential sum of squares in a $a \times b$ design. He has pointed out that, for unbalanced data, a sequential sum of squares for one of these hypotheses is equal (with probability 1) to an adjusted sum of squares for a non-standard "associated hypothesis." In his view, then, sequential sums of squares may test the wrong hypotheses. Some of the hypotheses are not directly given in terms of the cell means μ_{ij}, and the computations are somewhat complicated and ad hoc. We then give an exposition to build up to Theorem 3.3 and Corollary 3.1, and then derive from them hypotheses associated with sequential sum of squares in a $a \times b$ design.

Linear models are first explored and the usual methods of testing hypotheses are given. Several theorems that are presented without proofs are from *Linear Models and Design* [1].

2 Linear Models

For testing hypotheses we will be fitting a linear full model of $E(Y) = X\beta$ and comparing it to a restricted model as that under our null hypothesis, H_0. To fit our model the method of least squares will be used. For the full and restricted models the sum of squared errors are denoted as SSE and SSE_R respectively. If SSE and SSE_R are significantly close to each other, then we will not reject H_0. It's important to note that $SSE_R \geq SSE$ since a restricted model's error can't be less than a full model. Thus, if SSE_R is significantly greater than SSE we will reject H_0 which is equivalent to SSE_R/SSE being significantly large.

2.1 Notation

Working with an $a \times b$ design there is several useful notation which we'll be using. First, let $p = ab$ which is the number of cells. Let μ_{ij} denote the mean of the cell ij. Next, $n_{i\cdot}$ is used to denote the sum of observations in the ith row. That is $n_{i\cdot} = \sum_{j=1}^{b} n_{ij}$. Similarly for columns $n_{\cdot j} = \sum_{i=1}^{a} n_{ij}$. Thus the total number of observations is $N = n_{\cdot\cdot} = \sum_{j=1}^{b}\sum_{i=1}^{a} n_{ij} = \sum_{j=1}^{b} n_{\cdot j} = \sum_{i=1}^{a} n_{i\cdot}$

In general when the lower and upper bounds of summations are obvious the following notation will be used: $\sum_i = \sum_{i=1}^{a}$ and $\sum_j = \sum_{j=1}^{b}$.

2.2 Least Squares

Since the method of least squares will be used, a few notes about the method will be made. However, since the method is quite standard, derivations of most equations are omitted. Let X denote the $N \times p$ design matrix. Let Y represent the vector of observed values and let $\hat{Y}$ represents the vector of fitted values, both of which are $N \times 1$. The method of least squares is that which minimizes $SSE := \|Y - \hat{Y}\|^2 = \sum_{j=1}^{N}(Y_j - \hat{Y}_j)^2$

To minimize the SSE, $\hat{Y} = X(X'X)^{-1}X'Y$ and since $\hat{Y} = X\hat{\beta}$ then $\hat{\beta} = (X'X)^{-1}X'Y$. Throughout this paper, the "hat" matrix is defined to be $P = X(X'X)^{-1}X'$. The matrix P is a projection matrix from $\mathbb{R}^N$ to $\mathbb{R}^N$ where N is the total number observations as defined in the previous section. It is useful to think of $\mathbb{R}^N$ as the observation space. $\hat{Y} = PY$ is the orthogonal projection of Y onto $V := R(X)$, the column space of X.

2.3 Hypotheses

In a $a \times b$ design there are six main hypotheses of interest: "only A present", "only B present", absence of each main effect and absence of interaction.

The hypotheses of "only A present" consists of the equations

$$\mu_{11} = \cdots = \mu_{1b}$$

$$\mu_{21} = \cdots = \mu_{2b}$$

$$\vdots$$

$$\mu_{a1} = \cdots = \mu_{ab}$$

Let ρ_i denote the ith row mean. That is $\rho_i = \frac{1}{b}\sum_{j=1}^{b}\mu_{ij}$. A natural hypotheses would be $H_0 : \rho_1 = \rho_2 = \cdots = \rho_a$, that is all rows have the same mean. This is referred to as "A effect absent."

The hypotheses of "only B present" consists of the equations

$$\mu_{11} = \cdots = \mu_{a1}$$

$$\mu_{12} = \cdots = \mu_{a2}$$

$$\vdots$$

$$\mu_{1b} = \cdots = \mu_{ab}$$

Let γ_j denote the jth column mean. That is $\gamma_j = \frac{1}{a}\sum_{i=1}^{a} \mu_{ij}$. A natural hypotheses would be $H_0 : \gamma_1 = \gamma_2 = \cdots = \gamma_b$, that is all columns have the same mean. This is referred to as "B effect absent."

Another main hypothesis tested is that of "interaction" of A and B, defined to be lack of additivity. The factors A and B are considered to be additive if a change from row i to i^* can be achieved by adding the same constant to each cell mean of row i. Note that the constant need not be the same amongst different pairs of rows.

Whilst it is true that additivity holds between any pair of rows it's more useful to equivalently consider additivity between row 1 and row i, $i = 2, \ldots, a$. The hypothesis of additivity consists of the $(a-1)(b-1)$ equations

$$\mu_{ij} - \mu_{1j} = \mu_{i1} - \mu_{11}, i = 2, \ldots, a, j = 2, \ldots, b$$

The last main hypothesis tested is that of no effect present, that is, $H_0 : \mu_{ij}$ equal for all ij.

2.4 Restrictions

In our hypothesis of interest some sort of restriction or *linear constraint* is made on β where $\beta \in \mathbb{R}^p$. It is useful to think of $\mathbb{R}^p$ as the parameter space of the model. A hypothesis which makes a linear constraint is considered a *linear hypothesis*. There are several ways in which a linear constraint can be written and few are given below.

Lemma 2.1. *The following are equivalent*

(i) There exists a subspace $U \subset \mathbb{R}^p$ *such that*

$$\beta \perp U$$

(ii) There exists a subspace $W \subset \mathbb{R}^p$ *such that*

$$\beta \in W$$

(iii) There exists a matrix W *and a vector* β_0 *such that*

$$\beta = W\beta_0$$

U is a set of contrast vectors. From this Lemma it's important to note that W and U are orthogonal complements of each other.

Since the hypotheses we will be testing are defined by a linear constraint then it's important to know how the model acts under such a linear constraint. It seems natural that a linear model under a linear constraint should still be a linear model. We thus have the following theorem

Theorem 2.1. *If* $E(Y) = X\beta$ *and if* β *is subject to a linear constraint* $\beta \in W$*, then the constrained model is also a linear model*

$$E(Y) = X_0\beta_0$$

where $X_0 = XW_0$ *and appropriate* β_0*.*

We will be assuming that X has full rank and thus $W \cap N(X) = (0)$, where $N(X)$ is the nullspace of X, that is $N(X) = \{c|Xc = 0\}$. Therefore X_0 has full rank as well.

3 Hypothesis in Linear Models

3.1 Linear Hypothesis

We have already shown how to fit the unrestricted model and now that we know how to restrict our β then fitting a restricted model is similar in method.

Let $H_0 : \beta \in W$ be our null hypothesis, where W is a subspace of $\mathbb{R}^p$ and let $V_0 := X(W)$, a subspace of V. To fit the restricted model, we find the value of $\hat{Y} \in V_0$ that minimizes the distance to Y. Thus, $\hat{Y}$ is the orthogonal projection of Y onto V_0, and we define

$$SSE_R := \|Y - \hat{Y}_0\|^2.$$

At this point, we have Y, $\hat{Y}$, and $\hat{Y}_0$ and since $\hat{Y}, \hat{Y}_0 \in V = R(X)$ then we can see that these three vectors form a right triangle. Thus we have,

$$\|Y - \hat{Y}_0\|^2 = \|Y - \hat{Y}\|^2 + \|\hat{Y} - \hat{Y}_0\|^2.$$

If we denote $SS(H_0) = \|\hat{Y} - \hat{Y}_0\|^2$ (notation will become apparent shortly) then

$$SSE_R = SSE + SS(H_0) \Rightarrow \frac{SSE_R}{SSE} = 1 + \frac{SS(H_0)}{SSE}$$

Recall that we will reject H_0 when SSE_R/SSE is significantly large. Thus this is equivalent to rejecting when $\frac{SS(H_0)}{SSE}$ is large. Thus, we call $SS(H_0)$ the *sum of squares for testing* H_0.

3.2 Nested Hypothesis and Sequential Sum of Squares

Thus far, we have been interested in testing a single linear hypothesis. Generally an analysis of variance tests several hypotheses. Several interesting things occur when there are more than one hypothesis and we'll begin to explore this.

Definition 3.1. A hypothesis $H^{(1)}$ is *nested* in hypothesis $H^{(2)}$ if $H^{(1)}$ implies $H^{(2)}$.

Since in our linear model $\beta \in \mathbb{R}^p$, then from Lemma 2.1 a general hypothesis $H^{(i)}$ may be expressed as $\beta \in W_{(i)}$ for an appropriate subspace $W_{(i)} \subset \mathbb{R}^p$. Thus $H^{(1)}$ is nested in $H^{(2)}$ if and only if $W_1 \subset W_2$.

Applying Lemma 2.1, $H^{(i)}$ may be expressed as $\beta \perp U_{(i)}$. Thus $H^{(1)}$ is nested in $H^{(2)}$ if and only if $U_2 \subset U_1$.

The choice of H_0 is the simplest possible hypothesis model. Generally this is the mean of all cells being equal.

In general, suppose $W_0 \subset W_1$ which thus means the hypothesis $\beta \in W_0$ is nested in $\beta \in W_1$. Let $V_i = X(W_i)$ be the image of W_i so thus $V = R(X)$ the column space of X. Thus $V_0 \subset V_1 \subset V$

Theorem 3.1. *We have* $\hat{Y} - \hat{Y}_1 \in V \ominus V_1$ *and* $\hat{Y}_1 - \hat{Y}_0 \in V_1 \ominus V_0$. *In particular,* $Y - \hat{Y}$, $\hat{Y} - \hat{Y}_1$ *and* $\hat{Y}_1 - \hat{Y}_0$ *are pairwise orthogonal.*

Suppose we wanted to test $\beta \in W_0$, given that $\beta \in W_1$. Note that

$$\|Y - \hat{Y}_0\|^2 = \|Y - \hat{Y}\|^2 + \|\hat{Y} - \hat{Y}_1\|^2 + \|\hat{Y}_1 - \hat{Y}_0\|^2$$

$$= \|Y - \hat{Y}_1\|^2 + \|\hat{Y}_1 - \hat{Y}_0\|^2$$

As when we defined $SS(H_0)$, if we divided both sides by $SSE = \|Y - \hat{Y}_1\|^2$ then,

$$\frac{\|Y - \hat{Y}_0\|^2}{SSE} = 1 + \frac{\|\hat{Y}_1 - \hat{Y}_0\|^2}{SSE}$$

Thus $\|\hat{Y}_1 - \hat{Y}_0\|^2$ is appropriate for testing $\beta \in W_0$, given that $\beta \in W_1$. We thus denote this as

$$SS(\beta \in W_0 | \beta \in W_1) = \|\hat{Y}_1 - \hat{Y}_0\|^2$$

Keeping with this notation

$$\|\hat{Y} - \hat{Y}_0\|^2 = \|\hat{Y} - \hat{Y}_1\|^2 + \|\hat{Y}_1 - \hat{Y}_0\|^2$$

$$\Rightarrow \|\hat{Y}_1 - \hat{Y}_0\|^2 = \|\hat{Y} - \hat{Y}_0\|^2 - \|\hat{Y} - \hat{Y}_1\|^2$$

$$\Rightarrow SS(\beta \in W_0|\beta \in W_1) = SS(\beta \in W_0) - SS(\beta \in W_1)$$

Since $SS(\beta \in W_0) = \|\hat{Y} - \hat{Y}_0\|^2$ is the simplest model, it is generally referred to as the *sum of squares for the model*. Thus since $SS(\beta \in W_0) = S(\beta \in W_0|\beta \in W_1) + SS(\beta \in W_1)$, then $S(\beta \in W_0|\beta \in W_1)$ and $SS(\beta \in W_1)$ are referred to as *sequential sums of squares*. This is so because if we started with $\|Y - \hat{Y}\|^2$ and add $SS(\beta \in W_1) = \|\hat{Y} - \hat{Y}_1\|^2$ and $SS(\beta \in W_0|\beta \in W_1) = \|\hat{Y}_1 - \hat{Y}_0\|^2$ sequentially, then we build the sum of squares for the model.
To generalize Theorem 3.1 we can consider $H_0 : \beta \in W_i$ where $W_0 \subset W_1 \subset \cdots W_k \subset \mathbb{R}^p$ and the W_i are distinct. Then $V_0 \subset V_1 \subset \cdots \subset V$.

Theorem 3.2. *We have $\hat{Y} - \hat{Y}_k \in V \ominus V_k$ and $\hat{Y}_j - \hat{Y}_{j-1} \in V_j \ominus V_{j-1}$. In particular, $Y - \hat{Y}$, $\hat{Y} - \hat{Y}_k$, $\hat{Y}_k - \hat{Y}_{k-1}$,..., $\hat{Y}_1 - \hat{Y}_0$ are pairwise orthogonal.*

3.3 Associated Hypothesis

Definition 3.2. Given a hypothesis $H^{(1)}$ nested in $H^{(2)}$, the hypothesis H^* is *associated* to $SS(H^{(1)}|H^{(2)})$ and $SS(H^{(1)}|H^{(2)}) = SS(H^*)$.

Theorem 3.3. *Consider the model $E(Y) = X\beta$, where $X_{N\times p}$ has full rank, and let $W_1 \subset W_2 \subset \mathbb{R}^p$. Then there is a unique subspace W^* satisfying $SS(\beta \in W^*) = SS(\beta \in W_1|\beta \in W_2)$, and we have*

$$df(\beta \in W^*) = df(\beta \in W_1|\beta \in W_2).$$

The subspace is given by $W^* = R(TP^*)$*, where* $T = (X'X)^{-1}X'$ *and* P^* *is defined as follows.*

Let $V_i = X(W_i)$*,* $V = R(X)$ *be the column space of* X*, and let* P *be* P_i *be the orthogonal projections of* $\mathbb{R}^N$ *on* V *and on* V_i*, respectively. Then* $P^* = P - P_2 + P_1$*.*

Corollary 3.1. *The subspace* U^* *is given by* $U^* = N(P^*T)$

4 Some Associated Hypotheses

Throughout the rest of this paper instead of working with an $a \times b$ design, we'll be working with a 2×3 design.

4.1 Rows

Theorem 4.1. *Consider the hypotheses* $H^{(0)}$ *:all* μ_{ij} *equal and* $H^{(1)}$ *:Only* $\boldsymbol{A}$ *present. Clearly* $H^{(0)}$ *is nested in* $H^{(1)}$*. The hypothesis associated with the sequential sum of squares* $SS(H^{(0)}|H^{(1)})$ *is* $H^* : \rho'_1 = \rho'_2 = \cdots = \rho'_a$ *where* $\rho'_i = \frac{1}{n_{i\cdot}} \sum_j n_{ij}\mu_{ij}$*.*

Thus let $H^{(0)} : \mu_{11} = \mu_{12} = \mu_{13} = \mu_{21} = \mu_{22} = \mu_{23}$ and $H^{(1)} : \mu_{11} = \mu_{12} = \mu_{13}$ and $\mu_{21} = \mu_{22} = \mu_{23}$. The hypothesis associated with $SS(H^{(0)}|H^{(1)})$ of is $H^* : \rho'_1 = \rho'_2$.

Proof. We need $P^* = P - P_1 + P_0$ by Theorem 3.3.
Let 1_k be the $k \times 1$ vector of 1's, let $J_{n\times m}$ be the $n \times m$ matrix of $1's$, and J_n the $n \times n$ matrix of 1's.

For our unrestricted full model we have

$$X = \begin{bmatrix} 1_{n_{11}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{n_{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{n_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n_{22}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n_{23}} \end{bmatrix}$$

Thus $P = X(X'X)^{-1}X' = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{bmatrix}$ where

$A = \frac{1}{n_{11}} J_{n_{11}}$ $B = \frac{1}{n_{12}} J_{n_{12}}$ $C = \frac{1}{n_{13}} J_{n_{13}}$

$D = \frac{1}{n_{21}} J_{n_{21}}$ $E = \frac{1}{n_{22}} J_{n_{22}}$ $F = \frac{1}{n_{23}} J_{n_{23}}$

Under our null hypothesis of equality of means in rows we want $\mu_{11} = \mu_{12} = \mu_{13}$ and $\mu_{21} = \mu_{22} = \mu_{23}$. We have two equations so we have two free parameters, say μ_{11} and μ_{21}. So by Theorem 2.1 we want a matrix W_1 such that

$$\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} = W_1 \begin{bmatrix} \mu_{11} \\ \mu_{21} \end{bmatrix}$$

We can see that $W_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ since $\begin{bmatrix} \mu_{11} \\ \mu_{11} \\ \mu_{11} \\ \mu_{21} \\ \mu_{21} \\ \mu_{21} \end{bmatrix} = W_1 \begin{bmatrix} \mu_{11} \\ \mu_{21} \end{bmatrix}$.

Therefore, we have $X_1 = XW_1$.

Thus

$$X_1 = \begin{bmatrix} 1_{n_{1\cdot}} & 0 \\ 0 & 1_{n_{2\cdot}} \end{bmatrix}$$

and

$$P_1 = X_1(X_1'X_1)^{-1}X_1' = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where $A = \frac{1}{n_{1\cdot}} J_{n_{1\cdot}}$ and $B = \frac{1}{n_{2\cdot}} J_{n_{2\cdot}}$.

For the simplest model we have $\mu_{11} = \mu_{21} = \mu_{12} = \mu_{22} = \mu_{13} = \mu_{23}$. We have one free parameter, say μ_{11}. By Theorem 2.1 so we need a W_0 such that $\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} = W_0 \begin{bmatrix} \mu_{11} \end{bmatrix}$.

It is clear that, $W_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} \mu_{11} \\ \mu_{11} \\ \mu_{11} \\ \mu_{11} \\ \mu_{11} \\ \mu_{11} \end{bmatrix} = W_0 \begin{bmatrix} \mu_{11} \end{bmatrix}$

Therefore, we have $X_0 = XW_0 = 1_{N\times 1}$. Further $P_0 = X_0(X_0'X_0)^{-1}X_0' = \frac{1}{N}J_{N\times N}$ Since P is a projection matrix from $\mathbb{R}^N$ to $\mathbb{R}^N$, all P matrices are $N \times N$ matrices. We have $P^* = P - P_1 + P_0$. The only matrix we need now is T' where $T = (X'X)^{-1}X' =$.

$$T' = \begin{bmatrix} \frac{1}{n_{11}}1_{n_{11}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n_{12}}1_{n_{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{n_{13}}1_{n_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_{21}}1_{n_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{n_{22}}1_{n_{22}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{n_{23}}1_{n_{23}} \end{bmatrix}_{N\times 6}$$

Remark 4.1. The very next step is to calculate P^*T' and then according to Corollary 3.1. find its nullspace. Let us introduce the notation $C(P^*T')$ which we'll think of as the "core" of our matrix. The matrix P^*T' has several repeated rows, in fact only 6 unique rows. Thus $N - 6$ rows can be eliminated when finding the nullspace. The 6 unique rows compose $C(P^*T')$ and thus $C(P^*T')$ is 6×6. To picture P^*T', the first row of $C(P^*T')$ is repeated n_{11} times, the second row is repeated n_{12} times, the third is repeated n_{13} times, etc. Note that $N(P^*T') = N(C(P^*T'))$ since eliminated repeated rows do not effect the nullspace.

$$C(P^*T') = \begin{bmatrix} A & \frac{1}{N}J_3 \\ \frac{1}{N}J_3 & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} \frac{1}{n_{11}} - \frac{1}{n_{1\cdot}} + \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{1\cdot}} & \frac{1}{N} - \frac{1}{n_{1\cdot}} \\ \frac{1}{N} - \frac{1}{n_{1\cdot}} & \frac{1}{n_{12}} - \frac{1}{n_{1\cdot}} + \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{1\cdot}} \\ \frac{1}{N} - \frac{1}{n_{1\cdot}} & \frac{1}{N} - \frac{1}{n_{1\cdot}} & \frac{1}{n_{13}} - \frac{1}{n_{1\cdot}} + \frac{1}{N} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{1}{n_{21}} - \frac{1}{n_{2\cdot}} + \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{2\cdot}} & \frac{1}{N} - \frac{1}{n_{2\cdot}} \\ \frac{1}{N} - \frac{1}{n_{2\cdot}} & \frac{1}{n_{22}} - \frac{1}{n_{2\cdot}} + \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{2\cdot}} \\ \frac{1}{N} - \frac{1}{n_{2\cdot}} & \frac{1}{N} - \frac{1}{n_{2\cdot}} & \frac{1}{n_{23}} - \frac{1}{n_{2\cdot}} + \frac{1}{N} \end{bmatrix}$$

The next step is to put the matrix in reduced row echelon form. It's important to note that the Gaussian elimination method is not efficient and should be avoided if replicating this result. The reduced matrix is as follows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{n_{11}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 1 & 0 & 0 & 0 & \frac{n_{12}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 0 & 1 & 0 & 0 & \frac{n_{13}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 0 & 0 & 1 & 0 & -\frac{n_{21}}{n_{23}} \\ 0 & 0 & 0 & 0 & 1 & -\frac{n_{22}}{n_{23}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next to find the nullspace we need to solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{n_{11}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 1 & 0 & 0 & 0 & \frac{n_{12}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 0 & 1 & 0 & 0 & \frac{n_{13}n_{2\cdot}}{n_{1\cdot}n_{23}} \\ 0 & 0 & 0 & 1 & 0 & -\frac{n_{21}}{n_{23}} \\ 0 & 0 & 0 & 0 & 1 & -\frac{n_{22}}{n_{23}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus we have

$$\begin{cases} c_{11} = -\frac{n_{11}n_{2\cdot}}{n_{1\cdot}n_{23}}c_{23} \\ c_{12} = -\frac{n_{12}n_{2\cdot}}{n_{1\cdot}n_{23}}c_{23} \\ c_{13} = -\frac{n_{13}n_{2\cdot}}{n_{1\cdot}n_{23}}c_{23} \\ c_{21} = \frac{n_{21}}{n_{23}}c_{23} \\ c_{22} = \frac{n_{22}}{n_{23}}c_{23} \end{cases}$$

If we let $c_{23} = -\frac{n_{23}}{n_{2\cdot}}$, then

$$\begin{cases} c_{11} = \frac{n_{11}}{n_{1\cdot}} \\ c_{12} = \frac{n_{12}}{n_{1\cdot}} \\ c_{13} = \frac{n_{13}}{n_{1\cdot}} \\ c_{21} = -\frac{n_{21}}{n_{2\cdot}} \\ c_{22} = -\frac{n_{22}}{n_{2\cdot}} \end{cases}.$$

Thus

$$\frac{n_{11}}{n_{1\cdot}}\mu_{11} + \frac{n_{12}}{n_{1\cdot}}\mu_{12} + \frac{n_{13}}{n_{1\cdot}}\mu_{13} - \frac{n_{21}}{n_{2\cdot}}\mu_{21} - \frac{n_{22}}{n_{2\cdot}}\mu_{22} - \frac{n_{23}}{n_{2\cdot}}\mu_{23} = 0$$

$$\frac{n_{11}}{n_{1\cdot}}\mu_{11} + \frac{n_{12}}{n_{1\cdot}}\mu_{12} + \frac{n_{13}}{n_{1\cdot}}\mu_{13} = \frac{n_{21}}{n_{2\cdot}}\mu_{21} + \frac{n_{22}}{n_{2\cdot}}\mu_{22} + \frac{n_{23}}{n_{2\cdot}}\mu_{23}$$

$$\frac{1}{n_{1\cdot}}[n_{11}\mu_{11} + n_{12}\mu_{12} + n_{13}\mu_{13}] = \frac{1}{n_{2\cdot}}[n_{21}\mu_{21} + n_{22}\mu_{22} + n_{23}\mu_{23}]$$

Recall that $\rho_i' = \frac{1}{n_{i\cdot}}\sum_j n_{ij}\mu_{ij}$. Therefore, the associated hypothesis is $\rho_1' = \rho_2'$, as claimed. □

Example 4.1. Lets see the hypothesis $H_0 : \rho_1 = \rho_2$ as applied to the following experiment. Consider the following data: Searle claims that the sequential sum of

	Variety		
Soil	1	2	3
1	6, 10, 11	13, 15	14, 22
2	12, 15, 19, 18	31	18, 9, 12

Table 4.1: This is a table of soil and variety

squares for Soil only tests the hypothesis $\frac{1}{7}(3\mu_{11}+2\mu_{12}+2\mu_{13}) = \frac{1}{8}(4\mu_{21}+\mu_{22}+3\mu_{23})$.

Using the previous theorem to derive this hypothesis we have

$$X = \begin{bmatrix} 1_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_3 \end{bmatrix}$$

Using statistical software we get

$$P^*T' = \begin{bmatrix} \frac{9}{35} & \frac{-8}{105} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{9}{35} & \frac{-8}{105} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{9}{35} & \frac{-8}{105} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{-8}{105} & \frac{89}{210} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{-8}{105} & \frac{89}{210} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{-8}{105} & \frac{-8}{105} & \frac{89}{210} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{-8}{105} & \frac{-8}{105} & \frac{89}{210} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{23}{120} & \frac{-7}{120} & \frac{-7}{120} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{23}{120} & \frac{-7}{120} & \frac{-7}{120} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{23}{120} & \frac{-7}{120} & \frac{-7}{120} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{23}{120} & \frac{-7}{120} & \frac{-7}{120} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{-7}{120} & \frac{113}{120} & \frac{-7}{120} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{-7}{120} & \frac{-7}{120} & \frac{11}{40} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{-7}{120} & \frac{-7}{120} & \frac{11}{40} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{-7}{120} & \frac{-7}{120} & \frac{11}{40} \end{bmatrix}$$

As we can see the first row is repeated $n_{11} = 3$ times, the fourth row is repeated $n_{12} = 2$ times and etc. Since our total sample size is $N = 15$ we ended up with a 15×6 matrix. The core of P^*T' is

$$C(P^*T') = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{8}{7} \\ 0 & 1 & 0 & 0 & 0 & \frac{16}{21} \\ 0 & 0 & 1 & 0 & 0 & \frac{16}{21} \\ 0 & 0 & 0 & 1 & 0 & \frac{-4}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{-1}{3} \end{bmatrix}$$

Thus we have

$$\begin{cases} c_{11} = -\frac{8}{7}c_{23} \\ c_{12} = -\frac{16}{21}c_{23} \\ c_{13} = -\frac{16}{21}c_{23} \\ c_{21} = \frac{4}{3}c_{23} \\ c_{22} = \frac{1}{3}c_{23} \end{cases}$$

If we let $c_{23} = -\frac{n_{23}}{n_{2\cdot}} = -\frac{3}{8}$ then

$$\begin{cases} c_{11} = -\frac{8}{7}(-\frac{3}{8}) = \frac{3}{7} \\ c_{12} = -\frac{16}{21}(-\frac{3}{8}) = \frac{2}{7} \\ c_{13} = -\frac{16}{21}(-\frac{3}{8}) = \frac{2}{7} \\ c_{21} = \frac{4}{3}(-\frac{3}{8}) = -\frac{4}{8} \\ c_{22} = \frac{1}{3}(-\frac{3}{8}) = -\frac{1}{8} \end{cases}$$

Therefore

$H^* : \frac{3}{7}\mu_{11} + \frac{2}{7}\mu_{12} + \frac{2}{7}\mu_{13} - \frac{4}{8}\mu_{21} - \frac{1}{8}\mu_{22} - \frac{3}{8}\mu_{23} = 0$. So,

$H^* : \frac{3}{7}\mu_{11} + \frac{2}{7}\mu_{12} + \frac{2}{7}\mu_{13} = \frac{4}{8}\mu_{21} + \frac{1}{8}\mu_{22} + \frac{3}{8}\mu_{23}$, or

$H^* : \frac{1}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13}) = \frac{1}{8}(4\mu_{21} + \mu_{22} + 3\mu_{23})$, as Searle claims.

4.2 Columns

Theorem 4.2. *Consider the hypotheses $H^{(0)}$:all μ_{ij} equal and $H^{(1)}$:Only $\boldsymbol{B}$ present. Clearly $H^{(0)}$ is nested in $H^{(1)}$. The hypothesis associated with the sequential sum of squares $SS(H^{(0)}|H^{(1)})$ is $H^* : \gamma'_1 = \gamma'_2 = \cdots = \gamma'_b$ where $\gamma'_i = \frac{1}{n_{\cdot j}}\sum_i n_{ij}\mu_{ij}$.*

Thus let $H^{(0)} : \mu_{11} = \mu_{12} = \mu_{13} = \mu_{21} = \mu_{22} = \mu_{23}$ and $H^{(1)} : \mu_{11} = \mu_{21}$, $\mu_{12} = \mu_{22}$, and $\mu_{13} = \mu_{23}$. The hypothesis associated with $SS(H^{(0)}|H^{(1)})$ of is $H_0^* : \gamma'_1 = \gamma'_2 = \gamma'_3$.

Proof. First we will note that the full model and simplest model are the same as in the hypothesis of row equality in the previous section. Thus P, P_0, and T are the same as in the previous section.

Under our null hypothesis of equality of means in columns we want $\mu_{11} = \mu_{21}$, $\mu_{12} = \mu_{22}$, and $\mu_{13} = \mu_{23}$. We have three equations so we have three free parameters, say μ_{11}, μ_{12}, and μ_{13}. So by Theorem 2.1 we want a matrix W_1 such that

$$\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} = W_1 \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \end{bmatrix}$$

Thus we can see that

$$W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since

$$\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{11} \\ \mu_{12} \\ \mu_{13} \end{bmatrix} = W_1 \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \end{bmatrix}.$$

Thus we have

$$X_1 = XW_1 = \begin{bmatrix} 1_{n_{11}} & 0 & 0 \\ 0 & 1_{n_{12}} & 0 \\ 0 & 0 & 1_{n_{13}} \\ 1_{n_{21}} & 0 & 0 \\ 0 & 1_{n_{22}} & 0 \\ 0 & 0 & 1_{n_{23}} \end{bmatrix}.$$

Further

$$P_1 = X_1(X_1'X_1)^{-1}X_1' = \begin{bmatrix} \frac{1}{n_{\cdot 1}}J_{n_{11}} & 0 & 0 & \frac{1}{n_{\cdot 1}}J_{n_{21}} & 0 & 0 \\ 0 & \frac{1}{n_{\cdot 2}}J_{n_{12}} & 0 & 0 & \frac{1}{n_{\cdot 2}}J_{n_{22}} & 0 \\ 0 & 0 & \frac{1}{n_{\cdot 3}}J_{n_{13}} & 0 & 0 & \frac{1}{n_{\cdot 3}}J_{n_{23}} \\ \frac{1}{n_{\cdot 1}}J_{n_{11}} & 0 & 0 & \frac{1}{n_{\cdot 1}}J_{n_{21}} & 0 & 0 \\ 0 & \frac{1}{n_{\cdot 2}}J_{n_{12}} & 0 & 0 & \frac{1}{n_{\cdot 2}}J_{n_{22}} & 0 \\ 0 & 0 & \frac{1}{n_{\cdot 3}}J_{n_{13}} & 0 & 0 & \frac{1}{n_{\cdot 3}}J_{n_{23}} \end{bmatrix}$$

Again, P^*T' has 6 unique rows and $C(P^*T') = \begin{bmatrix} A & B \end{bmatrix}$

where

$$A = \begin{bmatrix} \frac{1}{n_{11}} - \frac{1}{n_{\cdot 1}} + \frac{1}{N} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{n_{12}} - \frac{1}{n_{\cdot 2}} + \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{n_{13}} - \frac{1}{n_{\cdot 3}} + \frac{1}{N} \\ \frac{1}{N} - \frac{1}{n_{\cdot 1}} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{\cdot 2}} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{\cdot 3}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{1}{N} - \frac{1}{n_{\cdot 1}} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{\cdot 2}} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} - \frac{1}{n_{\cdot 3}} \\ \frac{1}{n_{21}} - \frac{1}{n_{\cdot 1}} + \frac{1}{N} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{n_{22}} - \frac{1}{n_{\cdot 2}} + \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{n_{23}} - \frac{1}{n_{\cdot 3}} + \frac{1}{N} \end{bmatrix}$$

The matrix $C(P^*T')$ in reduced row echelon form is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{n_{\cdot 2}n_{11}}{n_{\cdot 1}n_{22}} & \frac{n_{\cdot 3}n_{11}}{n_{\cdot 1}n_{23}} \\ 0 & 1 & 0 & 0 & -\frac{n_{12}}{n_{22}} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{n_{13}}{n_{23}} \\ 0 & 0 & 0 & 1 & \frac{n_{\cdot 2}n_{21}}{n_{\cdot 1}n_{22}} & \frac{n_{\cdot 3}n_{21}}{n_{\cdot 1}n_{23}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus to find the nullspace of $C(P^*T')$ we need to solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{n_{\cdot 2}n_{11}}{n_{\cdot 1}n_{22}} & \frac{n_{\cdot 3}n_{11}}{n_{\cdot 1}n_{23}} \\ 0 & 1 & 0 & 0 & -\frac{n_{12}}{n_{22}} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{n_{13}}{n_{23}} \\ 0 & 0 & 0 & 1 & \frac{n_{\cdot 2}n_{21}}{n_{\cdot 1}n_{22}} & \frac{n_{\cdot 3}n_{21}}{n_{\cdot 1}n_{23}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus we have

$$\begin{cases} c_{11} = -\frac{n_{\cdot 2}n_{11}}{n_{\cdot 1}n_{22}}c_{22} - \frac{n_{\cdot 3}n_{11}}{n_{\cdot 1}n_{23}}c_{23} \\ c_{12} = \frac{n_{12}}{n_{22}}c_{22} \\ c_{13} = \frac{n_{13}}{n_{23}}c_{23} \\ c_{21} = -\frac{n_{\cdot 2}n_{21}}{n_{\cdot 1}n_{22}}c_{22} - \frac{n_{\cdot 3}n_{21}}{n_{\cdot 1}n_{23}}c_{23} \end{cases}$$

If we let $c_{22} = \frac{n_{22}}{n_{\cdot 2}}$ and $c_{23} = \frac{n_{23}}{n_{\cdot 2}}$, then

$$\begin{cases} c_{11} = -\frac{2n_{11}}{n_{\cdot 1}} \\ c_{12} = \frac{n_{12}}{n_{\cdot 2}} \\ c_{13} = \frac{n_{13}}{n_{23}} \\ c_{21} = -\frac{2n_{21}}{n_{\cdot 1}} \end{cases} .$$

So

$$-\frac{2n_{11}}{n_{\cdot 1}}\mu_{11} + \frac{n_{12}}{n_{\cdot 2}}\mu_{12} + \frac{n_{13}}{n_{\cdot 3}}\mu_{13} - \frac{2n_{21}}{n_{\cdot 1}}\mu_{21} + \frac{n_{22}}{n_{\cdot 2}}\mu_{22} + \frac{n_{23}}{n_{\cdot 2}}\mu_{23} = 0$$

$$[(\frac{n_{12}}{n_{\cdot 2}}\mu_{12} + \frac{n_{22}}{n_{\cdot 2}}\mu_{22}) - (\frac{n_{11}}{n_{\cdot 1}}\mu_{11} + \frac{n_{21}}{n_{\cdot 1}}\mu_{21})] - [(\frac{n_{11}}{n_{\cdot 1}}\mu_{11} + \frac{n_{21}}{n_{\cdot 1}}\mu_{21}) - (\frac{n_{13}}{n_{\cdot 3}}\mu_{13} + \frac{n_{23}}{n_{\cdot 2}}\mu_{23})] = 0$$

$$\frac{1}{n_{\cdot 2}}(n_{12}\mu_{12}+n_{22}\mu_{22}) - \frac{1}{n_{\cdot 1}}(n_{11}\mu_{11}+n_{21}\mu_{21}) = \frac{1}{n_{\cdot 1}}(n_{11}\mu_{11}+n_{21}\mu_{21}) - \frac{1}{n_{\cdot 3}}(n_{13}\mu_{13}+n_{23}\mu_{23})$$

Recall that $\gamma_i' = \frac{1}{n_{\cdot j}} \sum_i n_{ij}\mu_{ij}$. Thus

$$\gamma_2' - \gamma_1' = \gamma_1' - \gamma_3'.$$

Similarly, if we go back to our system of equations and let $c_{22} = \frac{n_{22}}{n_{\cdot 2}}$ and $c_{23} = 0$ then $\gamma_2' - \gamma_1' = 0$

Thus $\gamma_1' = \gamma_2'$ and by our previous equations, $\gamma_1' = \gamma_2' = \gamma_3'$, as claimed.

□

Example 4.2. Continuing using the data from Example 4.1, we will be examining the hypothesis of of variety only.

$$C(P^*T') = \begin{bmatrix} \frac{9}{35} & \frac{1}{15} & \frac{1}{15} & \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{7}{30} & \frac{1}{15} & \frac{1}{15} & \frac{-4}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{11}{30} & \frac{1}{15} & \frac{1}{15} & \frac{-2}{15} \\ \frac{-8}{105} & \frac{1}{15} & \frac{1}{15} & \frac{73}{420} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{-4}{15} & \frac{1}{15} & \frac{1}{15} & \frac{11}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{-2}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{5} \end{bmatrix}.$$

After reducing the matrix we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{9}{7} & \frac{5}{7} \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{-2}{3} \\ 0 & 0 & 0 & 1 & \frac{12}{7} & \frac{20}{21} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we have

$$\begin{cases} c_{11} = -\frac{9}{7}c_{22} - \frac{5}{7}c_{23} \\ c_{12} = 2c_{22} \\ c_{13} = \frac{2}{3}c_{23} \\ c_{21} = -\frac{12}{7}c_{22} - \frac{20}{21}c_{23} \end{cases}$$

If we let $c_{22} = \frac{n_{22}}{n_{\cdot 2}} = \frac{1}{3}$ and $c_{23} = \frac{n_{23}}{n_{\cdot 3}} = \frac{3}{5}$ then

$$\begin{cases} c_{11} = -\frac{6}{7} \\ c_{12} = \frac{2}{3} \\ c_{13} = \frac{2}{5} \\ c_{21} = -\frac{8}{7} \end{cases}$$

Thus

$$-\frac{6}{7}\mu_{11} + \frac{2}{3}\mu_{12} + \frac{2}{5}\mu_{13} - \frac{8}{7}\mu_{21} + \frac{1}{3}\mu_{22} + \frac{3}{5}\mu_{23} = 0$$

$$\frac{2}{3}\mu_{12} + \frac{1}{3}\mu_{22} - \frac{3}{7}\mu_{11} - \frac{4}{7}\mu_{21} = \frac{3}{7}\mu_{11} + \frac{4}{7}\mu_{21} - \frac{2}{5}\mu_{13} - \frac{3}{5}\mu_{23}$$

$$\frac{1}{3}(2\mu_{12} + \mu_{22}) - \frac{1}{7}(3\mu_{11} + 4\mu_{21}) = \frac{1}{7}(3\mu_{11} + 4\mu_{21}) - \frac{1}{5}(2\mu_{13} + 3\mu_{23})$$

Recall that $\gamma_i' = \frac{1}{n_{\cdot j}} \sum_i n_{ij}\mu_{ij}$, so,

$$\gamma_2' - \gamma_1' = \gamma_1' - \gamma_3'.$$

Similarly if we let $c_{22} = \frac{n_{22}}{n_{\cdot 2}} = \frac{1}{3}$ and $c_{23} = 0$ then we get

$$\begin{cases} c_{11} = -\frac{3}{7} \\ c_{12} = \frac{2}{3} \\ c_{13} = 0 \\ c_{21} = -\frac{4}{7} \end{cases}$$

Thus

$-\frac{3}{7}\mu_{11} + \frac{2}{3}\mu_{12} - \frac{4}{7}\mu_{21} + \frac{1}{3}\mu_{22} = 0$, or

$\frac{2}{3}\mu_{12} + \frac{1}{3}\mu_{22} = \frac{3}{7}\mu_{11} + \frac{4}{7}\mu_{21}$, so

$\gamma'_2 = \gamma'_1$ and thus $\gamma'_1 = \gamma'_2 = \gamma'_3$.

4.3 Additivity

Theorem 4.3. *Consider the hypotheses $H^{(1)}$:Only $\boldsymbol{A}$ present and $H^{(2)}$:Additivity. $H^{(1)}$ is nested in $H^{(2)}$. Then the hypothesis associated with the sequential sum of squares $SS(H^{(1)}|H^{(2)})$ is $H^* : \gamma'_j = \frac{1}{n_{\cdot j}} \sum_i n_{ij}\rho'_i \forall j$ where $\gamma'_i = \frac{1}{n_{\cdot j}} \sum_i n_{ij}\mu_{ij}$.*

Thus for our 2×3 case we have $H^{(1)} : \mu_{11} = \mu_{12} = \mu_{13}$ and $\mu_{21} = \mu_{22} = \mu_{23}$ and $H^{(2)} : \mu_{22} - \mu_{12} = \mu_{21} - \mu_{11}$ and $\mu_{23} - \mu_{13} = \mu_{21} - \mu_{11}$. The hypothesis associated with $SS(H^{(1)}|H^{(2)})$ is $H_0^* : \gamma'_1 = \frac{1}{n_{\cdot 1}}(n_{11}\rho'_1 + n_{21}\rho'_2)$ and $\gamma'_2 = \frac{1}{n_{\cdot 2}}(n_{12}\rho'_1 + n_{22}\rho'_2)$.

Proof. This hypothesis actually ending up being the most challenging to calculate, so instead of doing this with unbalanced data we will be using balanced data.

For the unrestriced model we have

$$X = \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

and

$$P = X(X'X)^{-1}X' = \begin{bmatrix} \frac{1}{n}J_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{n}J_n & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{n}J_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n}J_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{n}J_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{n}J_n \end{bmatrix}.$$

For the model of “A only” in the balanced case we have

$$P_1 = X_1(X_1'X_1)^{-1}X_1' = \begin{bmatrix} \frac{1}{3n}J_n & 0 \\ 0 & \frac{1}{3n}J_n \end{bmatrix}.$$

Under our hypothesis of additivity we have $\mu_{22} - \mu_{12} = \mu_{21} - \mu_{11}$ and $\mu_{23} - \mu_{13} = \mu_{21} - \mu_{11}$. Thus we want W_2 such that

$$\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} = W_2 \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \end{bmatrix}$$

Thus we want

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

Therefore, we have

$$X_2 = XW_2 = \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 1_n \\ -1_n & 1_n & 0 & 1_n \\ -1_n & 0 & 1_n & 1_n \end{bmatrix}$$

and

$$P_2 = X_2(X_2'X_2)^{-1}X_2' = \begin{bmatrix} \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{-1}{6n}J_n & \frac{-1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{-1}{6n}J_n & \frac{1}{3n}J_n & \frac{-1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{-1}{6n}J_n & \frac{-1}{6n}J_n & \frac{1}{3n}J_n \\ \frac{1}{3n}J_n & \frac{-1}{6n}J_n & \frac{-1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n \\ \frac{-1}{6n}J_n & \frac{1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n \\ \frac{-1}{6n}J_n & \frac{-1}{6n}J_n & \frac{1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n \end{bmatrix}.$$

Thus we have

$$P^* = P - P_2 + P_1 = \begin{bmatrix} \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{-1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{-1}{3n}J_n & \frac{1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{-1}{3n}J_n \\ \frac{-1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{-1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n & \frac{1}{6n}J_n \\ \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{-1}{3n}J_n & \frac{1}{6n}J_n & \frac{1}{6n}J_n & \frac{2}{3n}J_n \end{bmatrix}$$

and

$$C(P^*T') = \begin{bmatrix} \frac{2}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{-1}{3n} & \frac{1}{6n} & \frac{1}{6n} \\ \frac{1}{6n} & \frac{2}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{-1}{3n} & \frac{1}{6n} \\ \frac{1}{6n} & \frac{1}{6n} & \frac{2}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{-1}{3n} \\ \frac{-1}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{2}{3n} & \frac{1}{6n} & \frac{1}{6n} \\ \frac{1}{6n} & \frac{-1}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{2}{3n} & \frac{1}{6n} \\ \frac{1}{6n} & \frac{1}{6n} & \frac{-1}{3n} & \frac{1}{6n} & \frac{1}{6n} & \frac{2}{3n} \end{bmatrix}.$$

After some algebra we have the reduced form of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find the nullspace of this matrix, so we want to solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have

$$\begin{cases} c_{11} = -c_{22} - c_{23} \\ c_{12} = c_{22} \\ c_{13} = c_{23} \\ c_{21} = -c_{22} - c_{23} \end{cases}.$$

In this case, deriving the associated hypothesis isn't straightforward. So instead, we can verify that the associated hypothesis is correct. Consider $\gamma_1' = \frac{1}{2}(\rho_1' + \rho_2')$ then

$$\frac{1}{2n}(n\mu_{11} + n\mu_{21}) = \frac{1}{2}(\frac{1}{3n}(n\mu_{11} + n\mu_{12} + n\mu_{13}) + \frac{1}{3n}(n\mu_{21} + n\mu_{22} + n\mu_{23}))$$

$$\frac{1}{2}(\mu_{11}+\mu_{21}) = \frac{1}{2}(\frac{1}{3}(\mu_{11}+\mu_{12}+\mu_{13}) + \frac{1}{3}(\mu_{21}+\mu_{22}+\mu_{23}))$$

$$\mu_{11}+\mu_{21} = \frac{1}{3}(\mu_{11}+\mu_{12}+\mu_{13}+\mu_{21}+\mu_{22}+\mu_{23})$$

$$\frac{2}{3}\mu_{11} - \frac{1}{3}\mu_{12} - \frac{1}{3}\mu_{13} + \frac{2}{3}\mu_{21} - \frac{1}{3}\mu_{22} - \frac{1}{3}\mu_{23} = 0$$

Thus if we let $c_{22} = \frac{-1}{3}$ and $c_{23} = \frac{-1}{3}$, then

$$\begin{cases} c_{11} = \frac{2}{3} \\ c_{12} = \frac{-1}{3} \\ c_{13} = \frac{-1}{3} \\ c_{21} = \frac{2}{3} \end{cases},$$

thus verifying the associated hypothesis. □

5 Conclusion

We have shown to how use Theorem 3.1 and Corollary 3.1 [1] to derive certain associated hypotheses, namely those for sequential sum of squares of the form $SS(no\ effect|Only\ A\ present)$. The corresponding computations for sequential sum of squares such as $SS(Only\ A\ present|No\ interaction)$ turn out to be much more difficult for unbalanced data. This needs to be carried out to make the theorem truly usable.

REFERENCES

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