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The Fattened Davis Complex and the Weighted L^2-(Co)Homology of Coxeter Groups

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ABSTRACT

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Associated to a Coxeter system (W, S) there is a contractible simplicial complex Σ called the Davis complex on which W acts properly and cocompactly by reflections. Given a positive real multiparameter $\mathbf{q}$ indexed by S, one can define the weighted L^2–(co)homology groups of Σ and associate to them a nonnegative real number called the weighted L^2–Betti number. Unfortunately, not much is known about the behavior of these groups when $\mathbf{q}$ lies outside a certain restricted range, and weighted L^2–Betti numbers have proven difficult to compute. We propose a program to compute the weighted L^2–(co)homology of Σ by introducing a thickened version of this complex which we call the fattened Davis complex. A salient feature of this complex is that our construction produces a homology manifold with boundary possessing Σ as a W–equivariant retract. This allows us to use many standard algebraic topology tools such as Poincaré duality for computing the L^2–(co)homology of Σ, and we successfully perform computations for many examples of Coxeter groups.

Within the spectrum of weighted L^2–(co)homology there is a conjecture of interest called the Weighted Singer Conjecture. The conjecture claims that if Σ is an n–manifold (equivalently, the nerve of the corresponding Coxeter group is an $(n-1)$–sphere), then the weighted L^2–(co)homology groups of Σ vanish above dimension $\frac{n}{2}$

whenever $\mathbf{q} \leq \mathbf{1}$. We present a proof of the conjecture in dimension three that encompasses all but nine Coxeter groups. Then, under some restrictions on the nerve of the Coxeter group, we obtain partial results whenever $n = 4$ (in particular, the conjecture holds for $n = 4$ if the nerve of the corresponding Coxeter group is a flag complex). We also prove a version of this conjecture in dimensions three and four whenever Σ is a manifold with (nonempty) boundary, and then extend our results in dimension four to prove a general version of the conjecture for the case where the nerve of the Coxeter group is assumed to be a flag triangulation of a 3–manifold.

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Chapter 1

Introduction

A recurring theme in geometric group theory is the investigation of properties of a group by studying a space on which the group acts on (or vice-versa). This space is usually reasonably nice (for example, a metric space), hence geometric group theory closely interacts with algebraic topology, discrete geometry, and geometric topology. The groups that this thesis focuses on are Coxeter groups, and a construction by Davis equips us with a natural simplicial complex called the Davis complex (denoted by Σ) on which the Coxeter group acts on.

Within the spectrum of Coxeter groups is the theory of weighted L^2–(co)homology of Coxeter groups, which is the central subject of this thesis. The main idea is to use the Davis complex to attach to the Coxeter group an equivariant cohomology theory where the objects are Hilbert spaces. Let S denote the generating set of the Coxeter group W and begin with an S–tuple $\mathbf{q} = (q_s)_{s \in S}$ of positive real numbers, where $q_s = q_t$ if s and t are conjugate in W. One uses the S–tuple $\mathbf{q}$ to assign weights (real numbers) to each of the cells of Σ in a way that is compatible with the W–action. Elements of $L^2_{\mathbf{q}}C_k(\Sigma)$ are then infinite W–equivariant real valued k–chains that are square-summable with respect to the weights. In other words, they are real valued functions on the k–cells that are square-summable when taking into account the weights on the k–cells. These spaces are Hilbert spaces, and there is a weighted boundary operator which is adjoint to the ordinary coboundary operator with respect to the inner product (these operators are bounded). One then defines the reduced L^2–(co)homology spaces $L^2_{\mathbf{q}}H_*(\Sigma)$ (the homology and cohomology spaces are isomorphic as Hilbert spaces).

These cohomology groups are generally infinite dimensional when nonzero, but a striking feature of this cohomology theory is that one can assign to these groups a nonnegative real number called the weighted L^2–Betti number (hence they are distinguishable). The predominant goal of this subject is to completely determine the weighted L^2–Betti numbers $L^2_{\mathbf{q}}b_*(\Sigma)$ for any Coxeter group. Weighted L^2–Betti numbers not only tie the theory of weighted L^2–(co)homology to algebraic properties of the Coxeter group, but they also intertwine it with many other topics such as Hecke algebras, growth series, the Euler characteristic conjecture, and operator theory. Outside of the ties to these topics, one of the most important applications of weighted L^2–Betti numbers is that they can be used to compute the ordinary L^2–Betti numbers of buildings of finite thickness.

Weighted L^2–Betti numbers are also notoriously difficult to compute, very little being known when $\mathbf{q} \notin \bar{\mathcal{R}} \cup \bar{\mathcal{R}}^{-1}$, where $\mathcal{R}$ denotes the region of convergence of the growth series of the Coxeter group. In fact, just computing the ordinary L^2–Betti numbers of Coxeter groups still proves troublesome to this day. To illustrate the difficult nature of these invariants, a formula of Atiyah shows that ordinary L^2–Betti numbers can be used to compute the orbihedral Euler characteristic. So, if one considers the fundamental group of a closed aspherical n–manifold, with n even, then knowing the vanishing of the L^2–(co)homology groups of the universal cover outside of the middle dimension implies the Hopf conjecture on the sign of the Euler characteristic of that manifold. Thus L^2–(co)homology theory can be thought of as a formidable attack on the Euler characteristic conjecture and has proven to be successful in many situations (for example, in the case of locally symmetric manifolds).

This thesis is structured as follows. In Chapter 2, we introduce some preliminaries and definitions needed for the content of the thesis. For example, we discuss Coxeter groups, growth series, and explain how to construct the Davis complex. To construct the Davis complex, one needs a notion due to Davis called the basic construction. The idea is to start with a certain type of space X called a mirrored space and use a Coxeter group W to build a new space on which the Coxeter group acts on. An important result of this chapter is that we give new conditions on X so that the basic construction produces a homology manifold with boundary with a proper and cocompact Coxeter group action. This result will be especially important in Chapter 6.

Chapter 3 is dedicated to weighted L^2–(co)homology theory. We first introduce the weighted L^2–(co)homology groups and define weighted L^2–Betti numbers. We then introduce an alternate definition of weighted L^2–Betti numbers and discuss some previous results for the weighted L^2–(co)homology theory of the Davis complex. The new results of this chapter are as follows. We first observe that it is possible to use any acyclic complex on which the Coxeter group acts properly and cocompactly by reflections to compute the weighted L^2–Betti numbers of Coxeter groups. In particular, we can use a $\operatorname{vcd} W$–dimensional complex of Bestvina to compute weighted L^2–Betti numbers. An immediate consequence of this is that $L^2_{\mathbf{q}}b_k(\Sigma) = 0$ for $k > \operatorname{vcd} W$, and in many cases this allows us to obtain vanishing of high-dimensional $L^2_{\mathbf{q}}$–Betti numbers, as Bestvina's complex is usually of much lower dimension than the Davis complex. We then show that top-dimensional $L^2_{\mathbf{q}}$–Betti numbers behave monotonically in $\mathbf{q}$. More precisely, we show that if the top-dimensional $L^2_{\mathbf{q}}$–Betti vanishes for $\mathbf{q} = \mathbf{1}$, then it must have been zero for all $\mathbf{q} \leq \mathbf{1}$. This later allows us to "push" many of our computations (as well as previously known computations) from $\mathbf{q} = \mathbf{1}$ to $\mathbf{q} \leq \mathbf{1}$.

Chapter 4 focuses on specific W–stable subcomplexes of the Davis complex called ruins, which were used in proofs by Davis, Dymara, Januszkiewicz, and Okun in [8]. By considering a particular exact sequence for the $L^2_{\mathbf{q}}$–(co)homology involving these complexes (also used in [8]), we are able to show new concentration theorems for the $L^2_{\mathbf{q}}$–(co)homology of ruins. Using a spectral sequence appearing in [10], we are then able to show that for a certain range of $\mathbf{q}$, $L^2_{\mathbf{q}}H_*(\Sigma, \Sigma^{(k)})$ is concentrated in dimension $k+1$ (here $\Sigma^{(k)}$ denotes the k–skeleton of Σ). We then proceed derive some consequences, one being that we are able to generalize a theorem of Dymara

[12, Theorem 10.3] which states that if $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}H_*(\Sigma)$ is concentrated in dimension zero.

In Chapter 5 we consider the Weighted Singer Conjecture, which was formulated in [8]. It states that if Σ is an n–manifold and $\mathbf{q} \leq \mathbf{1}$, then $L^2_{\mathbf{q}}H_k(\Sigma)$ vanishes for $k > \frac{n}{2}$. We first discuss progress on this conjecture, and then we use the results of Chapter 3 to prove that $L^2_{\mathbf{q}}H_k(\Sigma) = 0$ whenever $k \geq n-1$ and Σ is an n–manifold with (nonempty) boundary. For the case when $n = 3, 4$, this proves a version of the Weighted Singer Conjecture whenever Σ is an n–manifold with (nonempty) boundary. We then adapt an argument appearing in [8] and combine it with our results to prove the Weighted Singer Conjecture in dimension three. Then, we prove the conjecture in dimension four under some additional restrictions on the nerve of the corresponding Coxeter group. A consequence of this is that the conjecture holds in dimension four if the nerve is assumed to be flag complex. Lastly, we generalize the result in dimension four and show that $L^2_{\mathbf{q}}H_k(\Sigma) = 0$ for $k > 2$ whenever the nerve is a flag triangulation of any 3–manifold.

In Chapter 6, we construct a complex that we call the fattened Davis complex. The idea is to "fatten" the Davis complex to a (homology) manifold with boundary so that we have standard algebraic topology tools (such as Poincaré duality) at our disposal. We carefully perform this fattening in such a way so that we can understand the weighted L^2–(co)homology of the boundary. In fact, understanding the weighted L^2–(co)homology of the boundary will simply amount to understanding the weighted L^2–(co)homology of certain infinite special subgroups of W. A large portion of this chapter is dedicated to studying the structure and algebraic topology of the fattened Davis complex. In Chapter 7, we then use the fattened Davis complex (combined with results from previous chapters) to perform new computations of $L^2_{\mathbf{q}}$–Betti numbers for many examples of Coxeter groups. Of note is that mostly all of the computations are performed for $\mathbf{q} \geq \mathbf{1}$, and hence they compute the ordinary L^2–(co)homology of buildings associated to these Coxeter groups with integer thickness vector $\mathbf{q}$.

Chapter 2

Coxeter Groups and Preliminaries

2.1 Coxeter systems and Coxeter groups

A *Coxeter matrix* $M = (m_{st})$ on a set S is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1 & \text{if } s = t \\ \geq 2 & \text{otherwise.} \end{cases}$$

One can associate to M a presentation for a group W as follows. Let S be the set of generators and let $\mathcal{I} = \{(s,t) \in S \times S \mid m_{st} \neq \infty\}$. The set of relations for W is

$$R = \{(st)^{m_{st}}\}_{(s,t) \in \mathcal{I}}.$$

The group defined by the presentation $\langle S, R \rangle$ is a *Coxeter group* and the pair (W, S) is a *Coxeter system*. If all off-diagonal entries of M are either 2 or ∞, then W is *right-angled*.

Given a subset $T \subset S$, define W_T to be the subgroup of W generated by the elements of T. Then (W_T, T) is a Coxeter system. Subgroups of this form are *special subgroups*. W_T is a *spherical subgroup* if W_T is finite and, in this case, T is a *spherical subset*. If W_T is infinite, then T is *non-spherical*. We will let $\mathcal{S}$ denote the poset of spherical subsets (the partial order being inclusion).

Given $w \in W$, call an expression $w = s_1 s_2 \cdots s_n$ *reduced* if there exists no integer $k < n$ with $w = s_1' s_2' \cdots s_k'$. We define the *length* of w, denoted by $l(w)$, to be the integer n so that $w = s_1 s_2 \cdots s_n$ is a reduced expression for w. Given a subset $T \subset S$

and an element $w \in W$, the special coset wW_T contains a unique element of shortest length. This element is said to be $(\varnothing, T)$*–reduced.*

2.2 Growth series

Suppose that (W, S) is a Coxeter system. Let $\mathbf{t} := (t_s)_{s \in S}$ denote an S–tuple of indeterminates, where $t_s = t_{s'}$ if s and s' are conjugate in W. If $s_1 s_2 \cdots s_n$ is a reduced expression for w, define t_w to be the monomial

$$t_w := t_{s_1} t_{s_2} \cdots t_{s_n}.$$

Note that t_w is independent of choice of reduced expression due to Tits' solution to the word problem for Coxeter groups (see the discussion at the beginning of [6, Chapter 17]). The *growth series* of W is the power series in $\mathbf{t}$ defined by

$$W(\mathbf{t}) = \sum_{w \in W} t_w.$$

The *region of convergence* $\mathcal{R}$ for $W(\mathbf{t})$ is defined to be

$$\mathcal{R} := \{\mathbf{t} \in (0, +\infty)^S \mid W(\mathbf{t}) \text{ converges}\}.$$

For each $T \subset S$, we denote the growth series of the special subgroup W_T by $W_T(\mathbf{t})$, the respective region of convergence by $\mathcal{R}_T$, and define $\mathbf{t}^{-1} := (t_s^{-1})_{s \in S}$. We record the following formula for later computations.

Theorem 2.2.1 ([6, Theorem 17.1.9])**.**

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(\mathbf{t}^{-1})}.$$

Note that if W is finite, then $W(\mathbf{t})$ is a polynomial with integral coefficients. Thus an immediate consequence of the above formula is that $W(\mathbf{t})$ is a rational function in $\mathbf{t}$.

2.3 Homology manifolds

A space X is a *homology n–manifold* if it has the same local homology groups as $\mathbb{R}^n$, i.e. that for each $x \in X$

$$H_k(X, X - x) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

The pair $(X, \partial X)$ with ∂X closed in X is a *homology n–manifold with boundary* if it has the same local homology groups as does a manifold with boundary, i.e., the following conditions hold:

- $X - \partial X$ is a homology n–manifold,
- ∂X is a homology $(n-1)$–manifold,
- for each $x \in \partial X$, the local homology groups $H_*(X, X - x)$ all vanish.

X is a *generalized homology n–sphere*, abbreviated GHS^n, if it is a homology n–manifold with the same homology as S^n. Similarly, the pair $(X, \partial X)$ is a *generalized homology n–disk*, abbreviated GHD^n, if it is a homology n–manifold with boundary with the same homology as the pair (D^n, S^{n-1}). Note that the cone on a generalized homology sphere is a generalized homology disk.

2.4 Mirrored spaces

A *mirror structure* over a set S on a space X is a family of subspaces $(X_s)_{s \in S}$ indexed by S. Then X is a *mirrored space over S*. Put $X_\varnothing = X$, and for each nonempty subset $T \subseteq S$, define the following subspaces of X:

$$X_T := \bigcap_{s \in T} X_s, \quad X^T := \bigcup_{s \in T} X_s.$$

If (W, S) is a Coxeter system and X is a mirrored space over S, then the mirror structure $(X_s)_{s \in S}$ is *W–finite* if $X_T = \varnothing$ for all non-spherical $T \subseteq S$.

2.4.1 Mirrored homology manifolds with corners

Suppose that X is a mirrored space over a finite set S. X is an *S–mirrored homology n–manifold with corners* if every nonempty X_T is a homology $(n-|T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supsetneq T} X_U$. By taking $T = \varnothing$, this definition implies that the pair $(X, \partial X)$ is a homology n–manifold with boundary.

Given a Coxeter system (W, S), we set $S' = S \cup \{e\}$, where e is the identity element of W. We now say that $T \subseteq S'$ is spherical if and only if $T - \{e\}$ is spherical. A mirrored space X over the set S' with W–finite mirror structure $(X_s)_{s \in S'}$ is a *partially S–mirrored homology n–manifold with corners* if every nonempty X_T is a homology $(n - |T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supsetneq T} X_U$. To summarize, we simply have defined the non–S–mirrored part of X to be an auxiliary mirror corresponding to the identity element of W.

2.5 Basic construction

Suppose that (W, S) is a Coxeter system and that X is a mirrored space over S. As before, for each nonempty subset $T \subset S$, let W_T be the subgroup of W generated by $T \subset S$. Put $S(x) := \{s \in S \mid x \in X_s\}$. Define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', y)$ if and only if $x = y$ and $w^{-1}w' \in W_{S(x)}$. Give $W \times X$ the product topology and let $\mathcal{U}(W, X)$ denote the quotient space:

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

$\mathcal{U}(W, X)$ is the *basic construction* and X is the *fundamental chamber*. There is a natural W–action on $W \times X$, and this action respects the equivalence relation, hence the W–action on $W \times X$ descends to a W–action on $\mathcal{U}(W, X)$.

We will be interested in conditions on X which guarantee that the basic construction produces a homology n–manifold with boundary. But first, we consider the following proposition, as the proof is similar to the main result of this subsection.

Proposition 2.5.1 (Compare [6, Proposition 10.7.5]). *Suppose that (W, S) is a Coxeter system and that X is an S–mirrored homology n–manifold with corners* *with W–finite mirror structure. Then $\mathcal{U}(W,X)$ is a homology n–manifold.*

Proof. Without loss of generality suppose that $x \in X$. By excision, we need to show that the local homology groups $H_*(U, U - x)$ are correct for some neighborhood U of x in $\mathcal{U}(W,X)$. If $x \in X - \partial X$ then we are done since $X - \partial X$ is a homology n–manifold and x does not lie in any mirror. As before, set $S(x) = \{s \in S \mid x \in X_s\}$ and suppose that $|S(x)| \geq 1$.

Let V be a neighborhood of x in X. For each $s \in S(x)$, set $V_s = V \cap X_s$, and give V the mirror structure $\{V_s\}_{s \in S(x)}$. Note that, for each $T \subseteq S(x)$, $V_T = V \cap X_T$, where as before, $X_T = \bigcap_{s \in T} X_s$. Now, $x \in X_{S(x)}$, so for each $T \subset S(x)$, $x \in \partial X_T$ (X_T is by assumption a homology $(n - |T|)$–manifold with boundary and $X_{S(x)} \subseteq \partial X_T$). Furthermore, x does not lie in $\partial X_{S(x)}$. Therefore by excision, it follows that for each $T \subset S(x)$, the local homology groups $H_*(V_T, V_T - x)$ vanish, and $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$ and $\mathbb{Z}$ in that dimension.

Now, define

$$Z := V \cup \mathrm{Cone}(V - x)$$
$$Z_s := V_s \cup \mathrm{Cone}(V_s - x)$$

So, Z has the mirror structure $\{Z_s\}_{s \in S(x)}$. Since V is a neighborhood of x in X, and $x \in \partial X$, it follows that the local homology groups $H_*(V, V - x)$ vanish. In particular, $H_*(V) \cong H_*(V - x)$, and the Mayer–Vietoris sequence, along with the five lemma, implies that Z is acyclic. Similarly, for each $T \subset S(x)$, since the local homology groups $H_*(V_T, V_T - x)$ vanish, it follows that Z_T is acyclic. Since $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$ and $\mathbb{Z}$ in that dimension, that Mayer–Vietoris sequence again implies that the same is true for $H_*(Z_{S(x)})$. In particular, $Z_{S(x)}$ has the same homology as $S^{n-|S(x)|}$.

We now finish the proof by applying the following lemma:

Lemma 2.5.2 ([6, Corollary 8.2.5])**.** *$\mathcal{U}(W_{S(x)}, Z)$ has the same homology as S^n if and only if there is a unique spherical subset $R \subseteq S(x)$ satisfying the following three conditions:*

(a) $W_{S(x)}$ *decomposes as* $W_{S(x)} = W_R \times W_{S(x)-R}$.

(b) *For all spherical* $T' \subsetneq S(x)$ *with* $T' \neq R$, $(Z, Z^{T'})$ *is acyclic.*

(c) (Z, Z^R) *has the same homology as* (D^n, S^{n-1}).

We apply the lemma to $R = S(x)$. Condition (a) is then satisfied vacuously, so we wish to show (b) and (c). For $T \subsetneq R$, consider the cover of Z^T by the mirrors $\{Z_s\}_{s\in T}$. Note that for each $U \subset R$, the intersection of mirrors Z_U is acyclic. The nerve of this cover is a simplex on U, and in particular is contractible. The Acyclic Covering Lemma [3, Theorem 4.4, Ch VII] then implies that Z^U is acyclic. Note that Z_R has the same homology as $S^{n-|R|}$, so a similar spectral sequence argument also implies that Z^R has the same homology as S^{n-1}.

Now, set $U = \mathcal{U}(W_R, V)$. Since $\mathcal{U}(W_R, Z) = U \cup \mathrm{Cone}(U - x)$ and $\mathcal{U}(W_R, Z)$ has the same homology as S^n, it follows that $H_*(U, U - x)$ is concentrated in dimension n and $\mathbb{Z}$ in that dimension. Therefore U is our desired neighborhood. ∎

Proposition 2.5.3. *Suppose that* (W, S) *is a Coxeter system and suppose that* X *is a partially* S*–mirrored homology* n*–manifold with corners. Set* $Y = X_e$ *and give* Y *the induced mirror structure* $(Y_s)_{s\in S}$*, where* $Y_s := Y \cap X_s$*. Then* $\mathcal{U}(W, X)$ *is a homology* n*–manifold with boundary* $\partial\mathcal{U}(W, X) = \mathcal{U}(W, Y)$.

Proof. Set $\mathcal{U} = \mathcal{U}(W, X)$ and $\partial\mathcal{U} = \mathcal{U}(W, Y)$. Proposition 2.5.1 guarantees that $\partial\mathcal{U}$ is a homology $(n-1)$–manifold. This is because $Y = X_e$, and X_e (with its induced S–mirror structure) is an S–mirrored homology $(n-1)$–manifold with corners. Similarly, Proposition 2.5.1 implies that $\mathcal{U} - \partial\mathcal{U}$ is a homology n–manifold, since $\mathcal{U} - \partial\mathcal{U} = U(W, Z)$, where $Z = X - Y$ (with its induced S–mirror structure) is an S–mirrored homology n–manifold with corners. It remains to show that for each $x \in \partial\mathcal{U}$, the local homology groups $H_*(\mathcal{U}, \mathcal{U} - x)$ vanish.

Suppose that $x \in \partial\mathcal{U}$. Without loss of generality, we can assume that $x \in Y \subset \partial X$. If x does not lie in any mirror $(X_s)_{s\in S}$, then we are done by excision. So, suppose $|S(x)| \geq 1$ (recall $S(x) = \{s \in S \mid x \in X_s\}$) and let V be a neighborhood of x in X. We now give V the S–mirror structure as in the proof of Proposition 2.5.1, noting that the only difference between that proof and the current situation is the fact that the local homology groups $H_*(V_{S(x)}, V_{S(x)} - x)$ vanish. This is because, since $x \in Y$ and $|S(x)| \geq 1$, it follows that $x \in \partial X_{S(x)}$. Now, following the proof of Proposition 2.5.1 line by line, the only difference now is that $Z_{S(x)}$ is acyclic (as opposed to having the homology of S^{n-1} as before). This then implies that $\mathcal{U}(W_{S(x)}, Z)$ is acyclic [6, Corollary 8.2.8], which in turn implies that the local homology groups $H_*(\mathcal{U}, \mathcal{U} - x)$ vanish. ∎

2.6 Posets, abstract simplicial complexes, and geometric realizations

A *poset* is a partially ordered set. Given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, set

$$\mathcal{P}_{\geq p} := \{x \in \mathcal{P} \mid x \geq p\}.$$

Define $\mathcal{P}_{\leq}$, $\mathcal{P}_{<}$, and $\mathcal{P}_{>}$ similarly. The *opposite* or *dual* poset to $\mathcal{P}$ is the poset $\mathcal{P}^{op}$ with the same underlying set but with the order relation reversed.

An *abstract simplicial complex* consists of a set S (called the *vertex set*) and a collection $\mathcal{S}$ of finite subsets of S such that

(i) for each $s \in S$, $\{s\} \in \mathcal{S}$ and

(ii) if $T \in \mathcal{S}$ and if $T' \subset T$, then $T' \in \mathcal{S}$.

An abstract simplicial complex $\mathcal{S}$ is a poset, the partial order being inclusion. An element of $\mathcal{S}$ is called a *simplex*. If T is a simplex of $\mathcal{S}$ and $T' \leq T$, then we call T' a *face* of T. The *dimension* of a simplex T is defined by

$$\dim T := \mathrm{Card}(T) - 1.$$

A subset $\mathcal{S}'$ of an abstract simplicial complex $\mathcal{S}$ is a *subcomplex* if it is also an abstract simplicial complex. The subcomplex $\mathcal{S}'$ is a *full subcomplex* if whenever $T \in \mathcal{S}$ such that the vertices of T are contained in $\mathcal{S}'$, then $T \in \mathcal{S}'$.

2.6.1 Flag complexes

An *incidence relation* is a symmetric and reflexive relation. Suppose that $\mathcal{P}$ is a poset. We can symmetrize the partial order to obtain an incidence relation on $\mathcal{P}$ as follows: two elements $p, q \in \mathcal{P}$ are incident if and only if $p \le q$ or $q \le p$. Any set of incident elements in a poset is totally ordered. A *flag* in $\mathcal{P}$ is a finite chain, i.e., a totally ordered subset. When $\mathcal{P}$ is a poset, $\mathrm{Flag}(\mathcal{P})$ denotes the abstract simplicial complex of all flags in $\mathcal{P}$. It is called the *flag complex* of $\mathcal{P}$.

2.6.2 Geometric realizations

Suppose that $\mathcal{S}$ is an abstract simplicial complex with vertex set S. Let $\mathbb{R}^S$ denote the vector space of all finitely supported functions $S \to \mathbb{R}$. For each $s \in S$ let e_s denote the characteristic function of $\{s\}$. The *standard simplex* on S, denoted by Δ^S, is the convex hull of the standard basis $\{e_s\}_{s\in S}$ of $\mathbb{R}^S$.

For each nonempty finite subset $T \subset S$, let σ_T denote the face of Δ^S spanned by T. Define a subcomplex $\mathrm{Geom}(\mathcal{S})$ of Δ^S by

$$\sigma_T \in \mathrm{Geom}(\mathcal{S}) \text{ if and only if } T \in \mathcal{S}_{>\varnothing}.$$

The simplicial complex $\mathrm{Geom}(\mathcal{S})$ is called the *standard geometric realization of* $\mathcal{S}$.

The *geometric realization of a poset* $\mathcal{P}$ is now defined to be the geometric realization of the simplicial complex $\mathrm{Flag}(\mathcal{P})$. We use the notation

$$|\mathcal{P}| := \mathrm{Geom}(\mathrm{Flag}(\mathcal{P})).$$

2.7 The (Λ, S)–chamber

A *cell* is the convex hull of finitely many points in $\mathbb{R}^n$. A *cell complex* is a collection of cells Λ where

(i) if $C \in \Lambda$ and F is a face of C, then $F \in \Lambda$,

(ii) for any two cells $C_1, C_2 \in \Lambda$, either $C_1 \cap C_2 = \varnothing$ or $C_1 \cap C_2$ is a common face of C_1 and C_2,

(iii) Λ is locally finite, i.e. each cell in Λ is contained in only finitely many other cells of Λ.

Suppose that Λ is a cell complex with vertex set S and let $\mathcal{F}(\Lambda)$ denote the poset of cells of Λ, including the empty set. Let $P := |\mathcal{F}(\Lambda)|$ denote the geometric realization of the poset $\mathcal{F}(\Lambda)$. For each $T \in \mathcal{F}(\Lambda)$, define $P_T := |\mathcal{F}(\Lambda)_{\geq T}|$ and $\partial P_T := |\mathcal{F}(\Lambda)_{>T}|$, so each P_T is the cone on $b\mathrm{Link}(T, \Lambda)$, the barycentric subdvision of $\mathrm{Link}(T, \Lambda)$. In particular, taking $T = \varnothing$, we have that P is the cone on $b\Lambda$, with cone point corresponding to $\varnothing$. For each $s \in S$, put $P_s := P_{\{s\}}$. This endows P with the mirror structure $(P_s)_{s \in S}$. P is the (Λ, S)*–chamber*.

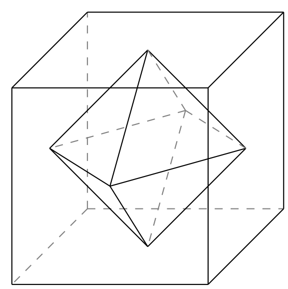


Figure 2.1: (Λ, S)–chamber when Λ is the boundary complex of an octahedron

Note that if Λ is a GHS^{n-1}, then the link of every cell σ in Λ is a $GHS^{n-\dim\sigma-2}$. It follows that P is a GHD^n and that for each $T \in \mathcal{F}(\Lambda)$, the pair $(P_T, \partial P_T)$ is a $GHD^{n-\dim\sigma_T-1}$, where σ_T is the geometric cell in Λ spanned by T.

2.7.1 Neighborhoods of faces

Let σ_T denote the geometric cell spanned by the vertex set T in Λ, and let $b\sigma_T$ denote its barycentric subdivision. By definition, $b\sigma_T$ is the $(\partial\sigma_T, T)$–chamber, and in particular, σ_T has a natural mirror structure over T.

P is itself a flag simplicial complex, and for each $T \in \mathcal{F}(\Lambda)$, P_T is a subcomplex of P. Hence $P_T - \bigcup_{U \supset T} P_U$ has a neighborhood of the form $\sigma_T * P_T$, the join of σ_T

and P_T. Following the join lines for a little while, it follows that $P_T - \bigcup_{U \supset T} P_U$ has neighborhoods of the form $\mathrm{Cone}(\sigma_T) \times P_T$. We record this fact, as we will use it in an upcoming construction.

2.8 The Davis complex

Suppose that (W, S) is a Coxeter system and, as before, denote by $\mathcal{S}$ the poset of all spherical subsets of S, partially ordered by inclusion. $\mathcal{S}$ is an abstract simplicial complex with vertex set S. Let L be the geometric realization of the abstract simplicial complex $\mathcal{S}$ and K be the (L, S)–chamber. In this special situation, K is called the *Davis chamber* and L is called the *nerve* of (W, S).

For each $s \in S$ define

$$K_s := |\mathcal{S}_{\geq \{s\}}|.$$

So, K_s is the union of simplices in K with minimum vertex $\{s\}$. The family $(K_s)_{s \in S}$ is a mirror structure on K.

The *Davis complex* Σ_L associated to the nerve L is now defined to be $\Sigma_L := \mathcal{U}(W, K)$.

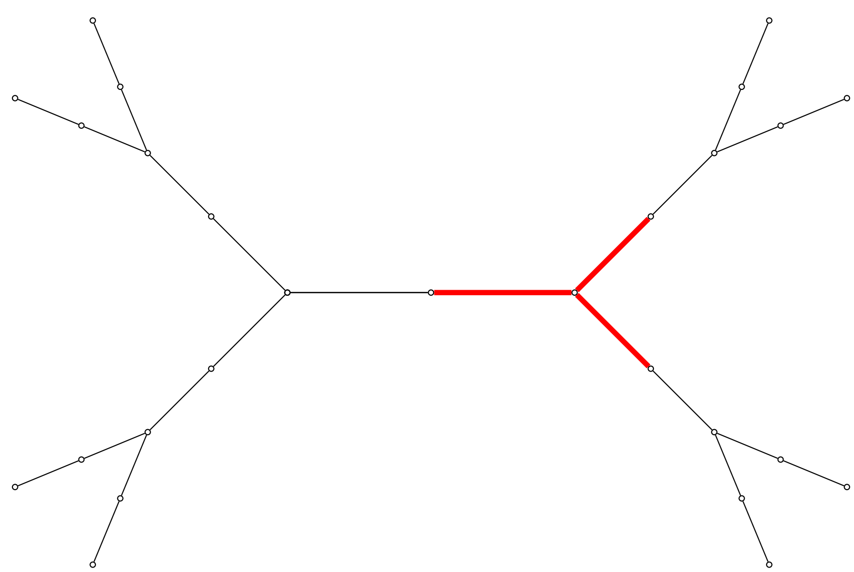


Figure 2.2: Σ_L whenever $W = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

Note that Σ_L is naturally a simplicial complex, the simplicial structure of K inducing a simplicial structure on Σ_L, and moreover, it is proved in [7] that Σ_L is contractible. Furthermore, if L is a triangulation of an $(n-1)$–sphere, then Σ_L is an n–manifold.

2.8.1 The labeled nerve

There is a natural way to label the edges of L so that the Coxeter system (W,S) can be recovered (up to isomorphism) from L. Let $E(L)$ denote the set of edges of L. We define the labeling map $m : E(L) \to \{2,3,...\}$ by sending $\{s,t\} \to m_{st}$, where $m_{st} \in \mathbb{N}$ and $(st)^{m_{st}} = 1$. L with this labeling map is the *labeled nerve.*

2.8.2 Right-angled cones and suspensions

Let c denote a point and let L be the labeled nerve. Consider the join $L' = c * L$, where all of the new edges are labeled by 2. L' is called the *right-angled cone on* L. Note that the corresponding Coxeter system to L' is $(W \times \mathbb{Z}_2, S \cup \{c\})$, and $\Sigma_{L'} = \Sigma_L \times [-1,1]$.

If P is a collection of two points, not joined by an edge, then we call the right-angled join $P * L$ the *right-angled suspension of* L. If the points of P are c_1 and c_2, then the corresponding Coxeter system to the right-angled suspension of L is $(W \times D_\infty, S \cup \{c_1, c_2\})$, where D_∞ is the infinite dihedral group.

2.8.3 The Coxeter cellulation

The Davis complex also admits a decomposition into *Coxeter cells.* For each $T \in \mathcal{S}$, let v_T denote the corresponding barycenter in K. Let c_T denote the union of simplices $c \subset \Sigma_L$ such that $c \cap K_T = v_T$. The boundary of c_T is then cellulated by wc_U, where $w \in W_T$ and $U \subset T$. With its simplicial structure, the boundary ∂c_T is the Coxeter complex corresponding to the Coxeter system (W_T, T), which is a sphere since W_T is finite. It follows that c_T and its translates are disks, which are called *Coxeter cells of type* T. We denote Σ_L with this decomposition into Coxeter cells by Σ_{cc}. Note that Σ_{cc} is a regular CW complex with with poset of cells that can be identified with $W\mathcal{S} := \{wW_U \mid w \in W, T \in \mathcal{S}\}$. The simplicial structure on Σ_L is the geometric realization of the poset $W\mathcal{S}$, hence Σ_L is the barycentric subdivision of Σ_{cc}. The properties of the Coxeter cellulation can be summarized as follows:

Proposition 2.8.1 ([6, p.130, Proposition 7.3.4]). Σ_{cc} *has the following properties:*

(i) its vertex set is W, its 1–skeleton is the Cayley graph, Cay(W, S), and its 2–skeleton is a Cayley 2–complex,

(ii) the link of each vertex is isomorphic to L,

(iii) a subset of W is the vertex set of a cell if and only if it is a coset of a spherical subgroup,

(iv) the poset of cells is $W\mathcal{S}$.

2.8.4 Twisted products

Suppose that H acts on Y and that H is a subgroup of G. The *twisted product* $G \times_H Y$ is the quotient space of $G \times Y$ by the action $h(g, x) = (gh^{-1}, hx)$. The natural G–action on $G \times Y$ descends to a G–action on $G \times_H Y$. Hence one can view $G \times_H Y$ as a G–bundle over G/H, and if G/H is discrete, then it follows that $G \times_H Y$ is just a disjoint union of copies of Y, one for each element of G/H.

Now, suppose that (W, S) is a Coxeter system with Davis complex Σ_L and that $T \subset S$. Let Σ_T denote the Davis complex corresponding to the subgroup W_T. It follows that the subcomplex of Σ_L corresponding to W_T is $W\Sigma_T := W \times_{W_T} \Sigma_T$. In particular, Σ_L contains a copy of Σ_T for every coset of W_T.

2.9 Virtual cohomological dimension

The *cohomological dimension* of a group Γ is

$$\operatorname{cd}\Gamma := \sup\{n \mid H^n(\Gamma; M) \neq 0 \text{ for some } \mathbb{Z}\Gamma-\text{module } M\}.$$

If Γ is virtually torsion free, then its *virtual cohomological dimension*, denoted by $\operatorname{vcd}\Gamma$, is the cohomological dimension of any torsion-free subgroup of finite index.

Since Coxeter groups are virtually torsion free, it makes sense to talk about their virtual cohomological dimension, denoted by $\operatorname{vcd} W$. In fact, given a Coxeter system (W, S), one can determine $\operatorname{vcd} W$ simply by looking at the nerve. Given a spherical $T \in \mathcal{S}$, let σ_T denote the corresponding closed simplex in the nerve L.

Proposition 2.9.1 ([6, Corollary 8.5.5])**.**

$$\operatorname{vcd} W = \max\{n \mid \overline{H}^{n-1}(L - \sigma_T) \neq 0, \ \textit{for some } T \in \mathcal{S}\}$$

Note that the dimension of Σ_L is usually much larger than $\operatorname{vcd} W$. For example, when W is finite, $\dim \Sigma_L = |S|$, while $\operatorname{vcd} W = 0$.

Chapter 3

Weighted L^2–(co)homology

In this chapter we present a brief introduction to weighted L^2–(co)homology. Further details can be found in [6, 8, 12]. We then compile some new and old results pertaining to the weighted L^2–(co)homology of the Davis complex Σ_L.

Let (W, S) be a Coxeter system. For the remainder of this thesis, let $\mathbf{q} = (q_s)_{s\in S}$ denote an S–tuple of positive real numbers satisfying $q_s = q_{s'}$ whenever s and s' are conjugate in W. Set $\mathbf{q}^{-1} = (q_s^{-1})_{s\in S}$. If $w = s_1 \cdots s_n$ is a reduced expression for $w \in W$, we define $q_w := q_{s_1} \cdots q_{s_n}$.

3.1 Hecke–von Neumann algebras

Let $\mathbb{R}(W)$ denote the group algebra of W and let $\{e_w\}_{w\in W}$ denote the standard basis on $\mathbb{R}(W)$ (here e_w denotes the characteristic function of $\{w\}$). Given a multiparameter $\mathbf{q}$ of positive real numbers as above, we deform the standard inner product on $\mathbb{R}(W)$ to an inner product

$$\langle e_w, e_{w'} \rangle_{\mathbf{q}} = \begin{cases} q_w & \text{if } w = w' \\ 0 & \text{otherwise.} \end{cases}$$

Using the multiparameter $\mathbf{q}$, one can give $\mathbb{R}(W)$ the structure of a *Hecke algebra*. We will denote $\mathbb{R}(W)$ with this inner product and Hecke algebra structure by $\mathbb{R}_{\mathbf{q}}(W)$, and $L^2_{\mathbf{q}}(W)$ will denote the Hilbert space completion of $\mathbb{R}_{\mathbf{q}}(W)$ with respect to $\langle\ ,\ \rangle_{\mathbf{q}}$. There is a natural anti-involution on $\mathbb{R}_{\mathbf{q}}(W)$, which implies that there is an associated *Hecke–von Neumann algebra* $\mathcal{N}_{\mathbf{q}}(W)$ acting on the right on $L^2_{\mathbf{q}}(W)$. It is the algebra of all bounded linear endomorphisms of $L^2_{\mathbf{q}}(W)$ which commute with the left $\mathbb{R}_{\mathbf{q}}(W)$–action.

Define the *von Neumann trace* of $\phi \in \mathcal{N}_{\mathbf{q}}(W)$ by $\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\phi) := \langle e_1\phi, e_1\rangle_{\mathbf{q}}$, and similarly for an $(n \times n)$–matrix with coefficients in $\phi \in \mathcal{N}_{\mathbf{q}}(W)$ by taking the sum of the von Neumann traces of elements on the diagonal. This allows us to attribute an non-negative real number called the *von Neumann dimension* for any closed subspace of an n–fold direct sum of copies of $L^2_{\mathbf{q}}(W)$ which is stable under the $\mathbb{R}_{\mathbf{q}}(W)$–action, called a *Hilbert $\mathcal{N}_{\mathbf{q}}$–module.* If $V \subseteq (L^2_{\mathbf{q}}(W))^n$ is a Hilbert $\mathcal{N}_{\mathbf{q}}$–module, and $p_V : (L^2_{\mathbf{q}}(W))^n \to (L^2_{\mathbf{q}}(W))^n$ is the orthogonal projection onto V (note that $p_V \in \mathcal{N}_{\mathbf{q}}(W)$), then define

$$\dim_{\mathcal{N}_{\mathbf{q}}} V := \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(p_V).$$

3.1.1 Induced Hilbert $\mathcal{N}_{\mathbf{q}}$–modules

Suppose that $T \subset S$ and that V_T is a Hilbert $\mathcal{N}_{\mathbf{q}}(W_T)$–module. The *induced Hilbert $\mathcal{N}_{\mathbf{q}}$–module* V is defined to be the completion of the tensor product

$$L^2_{\mathbf{q}}(W) \otimes_{\mathbb{R}_{\mathbf{q}}(W_T)} V_T.$$

A standard fact is that its dimension is given by

$$\dim_{\mathcal{N}_{\mathbf{q}}} V = \dim_{\mathcal{N}_{\mathbf{q}}(W_T)} V_T.$$

3.2 Weighted L^2–(co)homology

Suppose (W, S) is a Coxeter system and that X is a mirrored finite CW complex over S. Set $\mathcal{U} = \mathcal{U}(W, X)$. We first orient the cells of X and extend this orientation to $\mathcal{U}$ in such a way so that if σ is a positively oriented cell of X, then $w\sigma$ is positively oriented for each $w \in W$.

We define a measure on the W–orbit of an i–cell $\sigma \in X$ by

$$\mu_{\mathbf{q}}(w\sigma) = q_u,$$

where u is $(\varnothing, S(\sigma))$–reduced and $S(\sigma) := \{s \in S | \sigma \subseteq X_s\}$. This extends to a measure on the i–cells $\mathcal{U}^{(i)}$, which we also denote by $\mu_{\mathbf{q}}$.

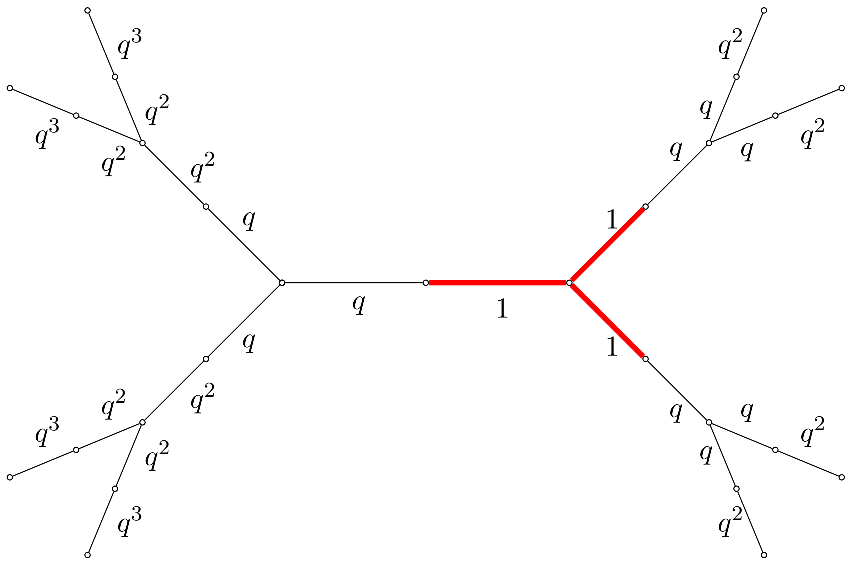


Figure 3.1: Weights on the 1–cells of Σ_L whenever $W = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and $\mathbf{q} = q$, a positive real number

Define the $\mathbf{q}$*–weighted* i*–dimensional* L^2*–(co)chains* on $\mathcal{U}$ to be the Hilbert space:

$$L^2_{\mathbf{q}} C_i(\mathcal{U}) = L^2_{\mathbf{q}} C^i(\mathcal{U}) = L^2(\mathcal{U}^{(i)}, \mu_{\mathbf{q}}).$$

These are infinite W–equivariant square summable (with respect to $\mu_{\mathbf{q}}$) real-valued i–chains. The inner product is given by

$$\langle f, g \rangle_{\mathbf{q}} = \sum_{\sigma} f(\sigma) g(\sigma) \mu_{\mathbf{q}}(\sigma),$$

and we denote the induced norm by $|| \ ||_{\mathbf{q}}$.

The boundary map $\partial_i : L^2_{\mathbf{q}} C_i(\mathcal{U}) \to L^2_{\mathbf{q}} C_{i-1}(\mathcal{U})$ and coboundary map $\delta^i : L^2_{\mathbf{q}} C_i(\mathcal{U}) \to L^2_{\mathbf{q}} C_{i+1}(\mathcal{U})$ are defined by the usual formulas, however there is one caveat: they are not adjoints with respect to this inner product whenever $\mathbf{q} \neq \mathbf{1}$. Thus one remedies this issue by perturbing the boundary map ∂_i to $\partial_i^{\mathbf{q}}$:

$$\partial_i^{\mathbf{q}}(f)(\sigma^{i-1}) = \sum_{\sigma^{i-1} \subset \alpha^i} [\sigma : \alpha] \mu_{\mathbf{q}}(\alpha) \mu_{\mathbf{q}}^{-1}(\sigma) f(\alpha).$$

A simple computation shows that $\partial_i^{\mathbf{q}}$ is the adjoint of δ^i with respect to the weighted inner product, hence $\left(L^2_{\mathbf{q}}C_*(\mathcal{U}), \partial_i^{\mathbf{q}}\right)$ is a chain complex. We now define the *reduced* $\mathbf{q}$*–weighted* L^2*–(co)homology* by

$$L^2_{\mathbf{q}}H_i(\mathcal{U}) = \mathrm{Ker}\partial_i^{\mathbf{q}}/\overline{\mathrm{Im}\partial_{i+1}^{\mathbf{q}}},$$
$$L^2_{\mathbf{q}}H^i(\mathcal{U}) = \mathrm{Ker}\delta^i/\overline{\mathrm{Im}\delta^{i-1}}.$$

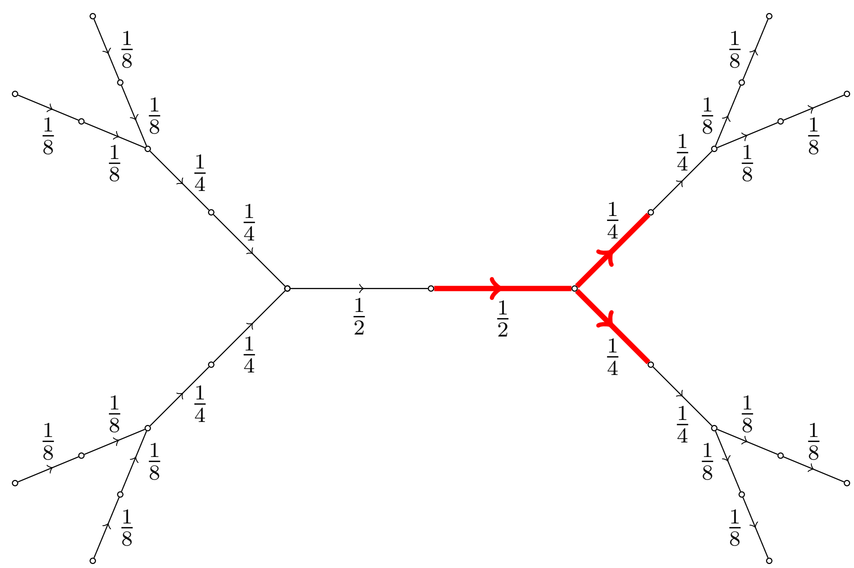


Figure 3.2: An element of $L^2_{\mathbf{1}}H_1(\Sigma_L)$ whenever $W = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

The Hodge Decomposition implies that $L^2_{\mathbf{q}}H^i(\mathcal{U}) \cong L^2_{\mathbf{q}}H_i(\mathcal{U}) \cong \ker \partial_i^{\mathbf{q}} \cap \ker \delta^i$ and versions of Eilenberg–Steenrod axioms hold for this homology theory. There is also a weighted version of Poincaré duality: If $\mathcal{U}$ is a locally compact homology n–manifold with boundary $\partial\mathcal{U}$, then

$$L^2_{\mathbf{q}}H_i(\mathcal{U}) \cong L^2_{\mathbf{q}^{-1}}H_{n-i}(\mathcal{U}, \partial\mathcal{U}).$$

One can also assign the von Neumann dimension to each of the Hilbert spaces $L^2_{\mathbf{q}}H_i(\mathcal{U})$ (as they are Hilbert $\mathcal{N}_{\mathbf{q}}$–modules). We denote this by $L^2_{\mathbf{q}}b_i(\mathcal{U})$ and call it the *i–th $L^2_{\mathbf{q}}$–Betti number of* $\mathcal{U}$. We then define the *weighted Euler characteristic of* $\mathcal{U}$:

$$\chi_{\mathbf{q}}(\mathcal{U}) = \sum(-1)^i L^2_{\mathbf{q}}b_i(\mathcal{U}).$$

3.2.1 $L^2_{\mathbf{q}}$–Betti numbers and twisted products

Suppose that X is a mirrored finite CW complex and and let $T \subset S$. Recall the twisted product $W \times_{W_T} \mathcal{U}(W_T, X)$. It follows that the $\mathcal{N}_{\mathbf{q}}$–module

$$L^2_{\mathbf{q}} H_*(W \times_{W_T} \mathcal{U}(W_T, X))$$

is induced from the $\mathcal{N}_{\mathbf{q}}(W_T)$–module $L^2_{\mathbf{q}} H_*(\mathcal{U}(W_T, X))$. Thus

$$L^2_{\mathbf{q}} b_*(W \times_{W_T} \mathcal{U}(W_T, X)) = L^2_{\mathbf{q}} b_*(\mathcal{U}(W_T, X)).$$

3.2.2 An alternate definition of $L^2_{\mathbf{q}}$–Betti numbers

As discussed in [10, §6], there is an alternate approach in defining $L^2_{\mathbf{q}}$–Betti numbers using the ideas of Lück [15]. The main point is that there is an equivalence of categories between the category of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules and projective modules of $\mathcal{N}_{\mathbf{q}}$. Hence one can define $\dim_{\mathcal{N}_{\mathbf{q}}} M$ for a finitely generated projective $\mathcal{N}_{\mathbf{q}}$–module M which agrees with the dimension of the corresponding Hilbert $\mathcal{N}_{\mathbf{q}}$–module. So, $\dim_{\mathcal{N}_{\mathbf{q}}} M$ for an arbitrary $\mathcal{N}_{\mathbf{q}}$–module is then defined to be the dimension of its projective part.

As before, suppose (W, S) is a Coxeter system and that X is a mirrored finite CW complex over S. Set $\mathcal{U} = \mathcal{U}(W, X)$. As in [15], define $H^W_*(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W))$ to be the homology of the $\mathcal{N}_{\mathbf{q}}(W)$–chain complex

$$C^W_*(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W)) := \mathcal{N}_{\mathbf{q}}(W) \otimes_{\mathbb{R}_{\mathbf{q}}(W)} C_*(\mathcal{U}),$$

where $C_*(\mathcal{U})$ is the cellular chain complex of $\mathcal{U}$ with the induced $\mathbb{R}_{\mathbf{q}}(W)$–structure. Similarly, define the cohomology groups $H^*_W(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W))$ to be the cohomology of the complex

$$C^*_W(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W)) := \mathrm{Hom}_W(C_*(\mathcal{U}), \mathcal{N}_{\mathbf{q}}(W)).$$

It then follows that

$$L^2_{\mathbf{q}} b_i(\mathcal{U}) = \dim_{\mathcal{N}_{\mathbf{q}}} H^W_i(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W)) = \dim_{\mathcal{N}_{\mathbf{q}}} H^i_W(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W)).$$

The advantage of these definitions is that we do no need to take closures of images as in the definition of reduced $\mathbf{q}$–weighted L^2–(co)homology (this is particularly useful when dealing with spectral sequences).

3.3 New and old results for Σ_L

In this section we begin by stating some previous results on the weighted L^2–(co)homology of Σ_L. We start with the following result of Dymara, which explicitly computes $L^2_{\mathbf{q}}b_0(\Sigma_L)$.

Proposition 3.3.1 ([12, Theorem 7.1, Theorem 10.3])**.** *$L^2_{\mathbf{q}}b_0(\Sigma_L) \neq 0$ if and only if $\mathbf{q} \in \mathcal{R}$. Moreover, when $\mathbf{q} \in \mathcal{R}$, $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ for $k > 0$.*

Dymara also computes the weighted Euler characteristic of Σ_L, revealing the connection between weighted L^2–(co)homology of Σ_L and the growth series of the corresponding Coxeter group W.

Proposition 3.3.2 ([12, Corollary 3.4])**.**

$$\chi_{\mathbf{q}}(\Sigma_L) = \frac{1}{W(\mathbf{q})}.$$

Recall that Σ_{cc} denotes Σ_L with the Coxeter cellulation (see Section 2.8.3). The following proposition states that if we compute the weighted L^2–(co)homology with respect to either cellulation, then we get the same answer.

Proposition 3.3.3 ([12, Theorem 5.5])**.**

$$L^2_{\mathbf{q}}H_*(\Sigma_L) \cong L^2_{\mathbf{q}}H_*(\Sigma_{cc}).$$

In conjunction with Proposition 3.3.2, the following theorem explicitly computes the weighted L^2–(co)homology of Coxeter groups which act properly and cocompactly by reflections on Euclidean space.

Theorem 3.3.4 ([8, Corollary 14.5])**.** *Suppose that W is a Euclidean reflection group with nerve L.*

- *If $\mathbf{q} \leq \mathbf{1}$, then $L^2_{\mathbf{q}}H_*(\Sigma_L)$ is concentrated in dimension 0.*
- *If $\mathbf{q} \geq \mathbf{1}$, then $L^2_{\mathbf{q}}H_*(\Sigma_L)$ is concentrated in dimension n.*

The following lemma says that we can compute the weighted L^2–Betti numbers of any acyclic complex of the form $\mathcal{U}(W,X)$, with X finite, on which W acts properly, and get the same answer. Thus we will sometimes write $L^2_{\mathbf{q}}b_k(W)$ instead of $L^2_{\mathbf{q}}b_k(\Sigma_L)$ to denote the *k–th $L^2_{\mathbf{q}}$–Betti number of W*.

Lemma 3.3.5. *Let (W,S) be a Coxeter system and suppose that X and X' are finite mirrored CW complexes with $\mathcal{U}(W,X)$ and $\mathcal{U}(W,X')$ both acyclic and both admitting proper W–action. Then for every $k \geq 0$,*

$$L^2_{\mathbf{q}}b_k(\mathcal{U}(W,X)) = L^2_{\mathbf{q}}b_k(\mathcal{U}(W,X')).$$

Proof. Set $\mathcal{U} = \mathcal{U}(W,X)$ and $\mathcal{U}' = \mathcal{U}(W,X')$. Since $\mathcal{U}$ and $\mathcal{U}'$ are both acyclic, it follows that the respective cellular chain complexes $C_*(\mathcal{U})$ and $C_*(\mathcal{U}')$ are are chain homotopic. This chain homotopy induces a chain homotopy of the chain complexes $C^W_*(\mathcal{U},\mathcal{N}_{\mathbf{q}}(W))$ and $C^W_*(\mathcal{U}',\mathcal{N}_{\mathbf{q}}(W))$. ∎

In fact, Bestvina constructed such a complex for any finitely generated Coxeter group.

Theorem 3.3.6 ([2])**.** *Let W be a finitely generated Coxeter group. Then W acts properly and cocompactly on an acyclic* $\operatorname{vcd} W$*–dimensional complex of the form $\mathcal{U}(W,X)$.*

Corollary 3.3.7. *Let (W,S) be a Coxeter system. Then*

$$L^2_{\mathbf{q}}b_k(W) = 0 \text{ for } k > \operatorname{vcd} W.$$

Proof. We can use the acyclic $\operatorname{vcd} W$–dimensional complex of Theorem 3.3.6 to compute the weighted L^2–Betti numbers of W. Lemma 3.3.5 now completes the proof. ∎

We now prove a lemma which is crucial for later computations.

Lemma 3.3.8. *Let $n = \operatorname{vcd} W$ and suppose and that $L^2_{\mathbf{1}}b_n(W) = 0$. Then*

$$L^2_{\mathbf{q}}b_k(W) = 0 \text{ for } k \geq n \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof. By Corollary 3.3.7, we obtain vanishing for $k > n$. Now, suppose for a contradiction that $L^2_{\mathbf{q}}b_n(W) \neq 0$ for $\mathbf{q} < \mathbf{1}$. Let B_W denote the complex of Theorem 3.3.6. Lemma 3.3.5 says that we can compute weighted L^2–Betti numbers of W with respect to the complex B_W. In particular, $L^2_{\mathbf{q}}b_n(W) = L^2_{\mathbf{q}}b_n(B_W)$ and we can choose a nontrivial element $\psi \in L^2_{\mathbf{q}}H_n(B_W)$. Thus ψ is a cycle under the weighted boundary map $\partial^{\mathbf{q}}$. Consider the isomorphism of Hilbert spaces

$$m_{\mathbf{q}} : L^2_{\mathbf{q}}C_n(B_W) \to L^2_{\mathbf{q}^{-1}}C_n(B_W)$$

defined by $m_{\mathbf{q}}(f(\sigma)) = \mu_{\mathbf{q}}(\sigma)f(\sigma)$. In particular, $m_{\mathbf{q}}\psi \in L^2_{\mathbf{q}^{-1}}C_n(B_W)$ and since $\mathbf{q}^{-1} > \mathbf{1}$,

$$\|m_{\mathbf{q}}\psi\|_{\mathbf{1}} \leq \|m_{\mathbf{q}}\psi\|_{\mathbf{q}^{-1}} < \infty.$$

Hence $m_{\mathbf{q}}\psi \in L^2_{\mathbf{1}}C_n(B_W)$.

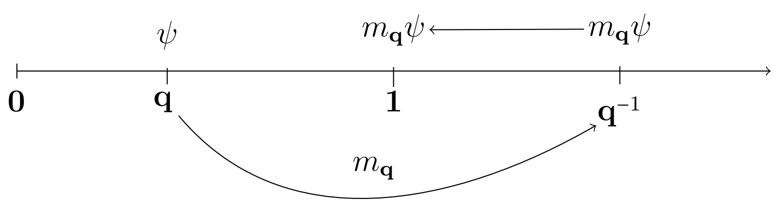


Figure 3.3: Schematic for the proof of Lemma 3.3.8

Now, a simple computation shows that $\partial = m_{\mathbf{q}}\partial^{\mathbf{q}}m_{\mathbf{q}}^{-1}$ and since ψ is a cycle under $\partial^{\mathbf{q}}$, $m_{\mathbf{q}}\psi$ is a cycle under ∂, the standard L^2–boundary operator. Moreover, since B_W is n–dimensional, $m_{\mathbf{q}}\psi$ is trivially a cocycle. Thus we have produced a nontrivial element of $L^2_{\mathbf{1}}H_n(B_W)$, a contradiction. ∎

Remark 3.3.9. Note that the statement of Lemma 3.3.8 holds in the more general setting for $L^2_{\mathbf{q}}b_n(\mathcal{U}(W,X))$ (here X is finite and $n = \dim X$). In fact, we obtain the same statement for relative $L^2_{\mathbf{q}}$–(co)homology as long as we are working in the top dimension.

Chapter 4

Ruins and Weighted L^2–(co)homology

4.1 Some Hilbert $\mathcal{N}_{\mathbf{q}}(W)$–submodules of $L^2_{\mathbf{q}}(W)$

We begin by considering the following self-adjoint idempotents in $\mathcal{N}_{\mathbf{q}}(W)$:

Lemma 4.1.1 ([6, Lemma 19.2.6])**.** *Given a subset $T \subset S$ and and $\mathbf{q} \in \mathcal{R}_T^{-1}$, there is an idempotent $h_T \in \mathcal{N}_{\mathbf{q}}(W)$ defined by*

$$h_T := \frac{1}{W_T(\mathbf{q}^{-1})} \sum_{w \in W_T} \varepsilon_w q_w^{-1} e_w,$$

where $\varepsilon_w = (-1)^{l(w)}$.

Thus the maps defined by $x \to h_T x$ are orthogonal projections (whenever h_T is defined) from $L^2_{\mathbf{q}}(W)$ onto Hilbert $\mathcal{N}_{\mathbf{q}}(W)$–submodules, denoted by H_T. Note that, by [6, Lemma 19.2.13],

$$H_T = \bigcap_{s \in T} H_s.$$

Using these submodules, we define a chain complex as follows. For a spherical subset of cardinality k, $T \in \mathcal{S}^{(k)}$, put

$$C_i(H_T) := \bigoplus_{U \in (\mathcal{S}_{\geq T})^{(i+k)}} H_U.$$

Fix some ordering of $\{s \in S - T \mid T \cup \{s\} \in \mathcal{S}\}$. Whenever $U \subset V$, we have an inclusion $i_V^U : H_V \hookrightarrow H_U$, and thus the boundary map $\partial : C_{i+1}(H_T) \to C_i(H_T)$ corresponds to a matrix (∂_{UV}), where $\partial_{UV} = 0$ unless $U \subset V$, and is equal to $(-1)^j i_V^U$ if U is obtained by deleting the j^{th} element of V. This turns $C_*(H_T)$ into a chain complex of Hilbert $\mathcal{N}_{\mathbf{q}}(W)$–modules. Similarly, whenever $U \subset V$ we have the projection $p_V^U : H_U \to H_V$. Thus we have a coboundary map where the matrix entries consist of projections, and we get a cochain complex $C^*(H_T)$ of Hilbert $\mathcal{N}_{\mathbf{q}}(W)$–modules.

4.2 Ruins

As before, Σ_{cc} is Σ_L with the Coxeter cellulation. Let (W, S) be a Coxeter system and for $U \subset S$, set $\mathcal{S}(U) := \{T \in \mathcal{S} \mid T \subset U\}$. Define $\Sigma(U)$ to be the subcomplex of Σ_{cc} consisting of all (closed) Coxeter cells of type T with $T \in \mathcal{S}(U)$. Note that $\Sigma(U) = W \times_{W_U} \Sigma_U$, where Σ_U is the Davis complex corresponding to the group W_U. Given $T \in \mathcal{S}(U)$, we define the following subcomplexes of $\Sigma(U)$:

$$\begin{aligned} \Omega_{UT} &: \text{ the union of closed cells of type } T', \text{ with } T' \in \mathcal{S}(U)_{\geq T}, \\ \partial\Omega_{UT} &: \text{ the cells of } \Omega_{UT} \text{ of type } T'', \text{ with } T'' \notin \mathcal{S}(U)_{\geq T}. \end{aligned}$$

The pair $(\Omega_{UT}, \partial\Omega_{UT})$ is the (U,T)*–ruin*. Note that if $T = \varnothing$, then $\Omega_{UT} = \Sigma(U)$ and $\partial\Omega_{UT} = \varnothing$. Ruins can also be expressed in terms of the basic construction. Define $K(U,T) := \Omega_{UT} \cap K$ and $\partial K(U,T) := \partial\Omega_{UT} \cap K$, where K is the Davis chamber. Then $K(U,T)$ and $\partial K(U,T)$ have an induced mirror structure, and it follows that

$$\Omega_{UT} = \mathcal{U}(W, K(U,T)),$$

and

$$\partial\Omega_{UT} = \mathcal{U}(W, \partial K(U,T)).$$

The (S,T)–ruin has a chain complex that looks like this:

Proposition 4.2.1 ([6, Lemma 20.6.21]). *For $T \in \mathcal{S}^{(k)}$, the chain complexes $C_*(H_T)$ and $L^2_{\mathbf{q}}C_{*+k}(\Omega_{ST}, \partial\Omega_{ST})$ of $\mathcal{N}_{\mathbf{q}}(W)$–modules are isomorphic. In particular,*

$$L^2_{\mathbf{q}}C_m(\Omega_{ST}, \partial\Omega_{ST}) = 0 \text{ for } m < k.$$

For brevity, we write (Ω_{UT}, ∂). For $s \in T$, set $U' = U - s$ and $T' = T - s$. As in [8, Proof of Theorem 8.3], we have the following weak exact sequence:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_*(\Omega_{U'T'}, \partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega_{UT'}, \partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega_{UT}, \partial) \longrightarrow \cdots \tag{4.1}$$

For the special case when $U = S$ and $T = \{s\}$ the above sequence becomes:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(S - s)) \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(S)) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega_{S\{s\}}, \partial) \longrightarrow \cdots$$

4.3 $L^2_{\mathbf{q}}$–(co)homology of ruins

Given a Coxeter system (W, S), for $T \in \mathcal{S}$ and $T \subseteq V \subseteq S$, define

$$St(T, V) \coloneqq \bigcup_{\substack{U \subseteq V \\ U \cup T \in \mathcal{S}}} U,$$

and

$$Lk(T, V) \coloneqq St(T, V) \smallsetminus T.$$

If $V = S$, then we write $St(T)$ and $Lk(T)$ instead of $St(T, S)$ and $Lk(T, S)$. If $T = \varnothing$, we make the convention that $S(T, U) = U$.

Lemma 4.3.1. *Suppose that (W, S) is right-angled. Then for $T \in \mathcal{S}$,*

$$\Omega_{ST} = \Sigma(Lk(T)) \times \Sigma(T),$$

$$\partial\Omega_{ST} = \Sigma(Lk(T)) \times \partial\Sigma(T).$$

Proof. For $U \in \mathcal{S}$, recall the Coxeter cell c_U from Section 2.8.3. As W is right-angled, all Coxeter cells are cubes of appropriate dimension, and thus it follows that c_U is just a direct product of all the Coxeter cells c_s with $s \in U$.

Now, by definition,

$$K(S,T) = \Omega_{ST} \cap K = \bigcup_{U \in \mathcal{S}_{\geq T}} c_U \cap K,$$

$$K(Lk(T)) = \Sigma(Lk(T)) \cap K = \bigcup_{U \in \mathcal{S}(Lk(T))} c_U \cap K,$$

and

$$K(T) = \Sigma(T) \cap K = c_T \cap K.$$

Now, let $U \in \mathcal{S}_{\geq T}$. Then

$$c_U = c_T \times c_{U-T}.$$

Since $U \in \mathcal{S}_{\geq T}$, it follows that $U \subset St(T)$, as $U \cup T = U \in \mathcal{S}$. Thus $U - T \subset \mathcal{S}(Lk(T))$. Therefore

$$K(S,T) \subseteq K(Lk(T)) \times K(T) = \bigcup_{U \in \mathcal{S}(Lk(T))} (c_T \times c_U) \cap K.$$

For the reverse inclusion, let $U \in \mathcal{S}(Lk(T))$. Then $U \cup T \in \mathcal{S}_{\geq T}$. This is because the only way that $U \cup T$ could fail to be spherical is if there would exist $u \in U$ and $t \in T$ with $m_{tu} = \infty$ (W is right-angled), and if this happened then it would contradict the fact that $U \subset Lk(T)$. Therefore we have shown that

$$K(S,T) = K(Lk(T)) \times K(T),$$

and thus

$$\Omega_{ST} = \mathcal{U}(W, K(S,T)) = \mathcal{U}(W, K(Lk(T))) \times \mathcal{U}(W, K(T)) = \Sigma(Lk(T)) \times \Sigma(T).$$

The proof for $\partial\Omega_{ST}$ follows a similar unwinding of definitions. Begin by noting that:

$$\partial\Omega_{ST} \cap K = \bigcup_{\substack{V \notin \mathcal{S}_{\geq T} \\ V \subset U \in \mathcal{S}_{\geq T}}} c_V \cap K$$

and

$$\partial\Sigma(T) \cap K = \bigcup_{U \subsetneq T} c_U \cap K.$$

To conclude the proof, observe that for W right-angled, a subset V satisfying $V \notin \mathcal{S}_{\geq T}$ and $V \subset U \in \mathcal{S}_{\geq T}$ is a disjoint union $V = A \sqcup B$, where $A \in \mathcal{S}(Lk(T))$ and $B \subsetneq T$. Conversely, any such disjoint union $A \sqcup B \notin \mathcal{S}_{\geq T}$ (by definition it cannot contain T), and satisfies $A \sqcup B \subset A \cup T \in \mathcal{S}_{\geq T}$.

∎

Theorem 4.3.2 (Compare [6, Theorem 20.6.22])**.** *Suppose that $T \in \mathcal{S}^{(k)}$ and that $\mathbf{q} \in \mathcal{R}_{St(T)}$. Then $L^2_{\mathbf{q}}H_*(\Omega_{ST}, \partial\Omega_{ST})$ is concentrated in dimension k. If (W,S) is assumed to be right-angled, then the same statement holds if we replace $\mathcal{R}_{St(T)}$ by $\mathcal{R}_{Lk(T)}$.*

Proof. For the proof we temporarily switch notation and denote the (U,T)–ruin by $\Omega(U,T)$. We first make an observation about ruins. We note that for every $V \subseteq S$, $\Omega(V,T) = \Omega(St(T,V),T)$, the point being that $\Omega(V,T)$ consists of Coxeter cells corresponding to spherical subsets of V containing T, and $St(T,V)$ is the union of all such subsets. In particular, $\Omega(U,T) = \Omega(St(T,U),T))$, and hence $\Omega(U,T)$ is a subcomplex of $\Sigma(St(T,U))$.

The proof is now by induction on k. We will show that for $U \subset S$ and $T \in \mathcal{S}(U)^{(k)}$, $L^2_{\mathbf{q}}H_*(\Omega(U,T),\partial)$ is concentrated in dimension k. For the base case $k = 0$, note that $St(\varnothing,U) = U$, $\Omega(U,\varnothing) = \Sigma(U)$, and $\partial\Omega(U,\varnothing) = \varnothing$. Hence, for $k = 0$, the theorem asserts that for $\mathbf{q} \in \mathcal{R}_U$, $L^2_{\mathbf{q}}H_*(\Sigma(U))$ is concentrated in dimension 0, which is Proposition 3.3.1.

Now, suppose the theorem is true for $k-1$ and let $T \in \mathcal{S}(U)^{(k)}$. Let $s \in T$, $V = T - s$ and consider the long exact sequence:

$$L^2_{\mathbf{q}}H_*(\Omega(St(T,U)-s,V),\partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(St(T,U),V),\partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(St(T,U),T),\partial)$$

Note that

$$\Omega(St(T,U),V) = \Omega(St(V,St(T,U)),V)$$

and

$$\Omega(St(T,U)-s,V)=\Omega(St(V,St(T,U)-s),V).$$

Since $St(V,St(T,U))\subseteq St(T,U)$ and $St(V,St(T,U)-s)\subseteq St(T,U)$, it follows that $\mathcal{R}_{St(T,U)}\subseteq\mathcal{R}_{St(V,St(T,U))}$ and $\mathcal{R}_{St(T,U)}\subseteq\mathcal{R}_{St(V,St(T,U)-s)}$. Since $\mathbf{q}\in\mathcal{R}_{St(T,U)}$, it follows by induction that the left-hand term and the middle term of the exact sequence are both concentrated in dimension $k-1$. Since $L^2_{\mathbf{q}}H_*(\Omega(St(T,U),T),\partial)$ vanishes for $*<k$ (Proposition 4.2.1), it follows that $L^2_{\mathbf{q}}H_*(\Omega(St(T,U),T),\partial)=L^2_{\mathbf{q}}H_*(\Omega(U,T),\partial)$ is concentrated in dimension k.

Now, suppose that (W,S) is right-angled. Consider the long exact sequence of the pair $(\Omega(S,T),\partial\Omega(S,T))$:

$$\cdots\longrightarrow L^2_{\mathbf{q}}H_*(\partial\Omega(S,T))\longrightarrow L^2_{\mathbf{q}}H_*(\Omega(S,T))\longrightarrow L^2_{\mathbf{q}}H_*(\Omega(S,T),\partial)\longrightarrow\cdots$$

Since W is right-angled, it follows from Lemma 4.3.1 that $\Omega(S,T)=\Sigma(Lk(T))\times\Sigma(T)$ and $\partial\Omega(S,T)=\Sigma(Lk(T))\times\partial\Sigma(T)$. By the the Künneth formula,

$$L^2_{\mathbf{q}}b_m(\Omega(S,T))=\sum_{i+j=m}L^2_{\mathbf{q}}b_i(\Sigma(Lk(T)))\cdot L^2_{\mathbf{q}}b_j(\Sigma(T)),$$
$$L^2_{\mathbf{q}}b_m(\partial\Omega(S,T))=\sum_{i+j=m}L^2_{\mathbf{q}}b_i(\Sigma(Lk(T)))\cdot L^2_{\mathbf{q}}b_j(\partial\Sigma(T)).$$

Since W_T is finite and $\mathbf{q}\in\mathcal{R}_{Lk(T)}$, Proposition 3.3.1 implies that $L^2_{\mathbf{q}}b_*(\Sigma(Lk(T)))$ and $L^2_{\mathbf{q}}b_*(\Sigma(T))$ are both concentrated in degree 0, and hence $L^2_{\mathbf{q}}H_*(\Omega(S,T))$ is concentrated in dimension 0. Similarly, $L^2_{\mathbf{q}}H_*(\partial\Omega(S,T))$ vanishes above dimension $k-1$ (this is because, since W_T is finite, $\Sigma(T)$ is topologically a disjoint collection of k–disks with boundary $\partial\Sigma(T)$). As Proposition 4.2.1 implies that $L^2_{\mathbf{q}}H_*(\Omega(St(T,U),T),\partial)$ vanishes for $*<k$, the long exact sequence for the pair implies that $L^2_{\mathbf{q}}H_*(\Omega(S,T),\partial)$ is concentrated in dimension k. ∎

Remark 4.3.3. Suppose that $T\in\mathcal{S}$ and that $\mathbf{q}\in\mathcal{R}_{St(T)}$. Then, for $U\in\mathcal{S}_{\geq T}$, Theorem 4.3.2 implies that $L^2_{\mathbf{q}}H_*(\Omega(S,U),\partial)$ is concentrated in dimension $|U|$. This is because, if $T\subset U$, then $St(U)\subset St(T)$ (similarily, $Lk(U)\subset Lk(T)$). Therefore

$\mathbf{q} \in \mathcal{R}_{St(T)} \subseteq \mathcal{R}_{St(U)}$. The analogous statements hold when W is assumed to be right-angled and $\mathbf{q} \in \mathcal{R}_{Lk(T)}$.

Given a subset $T \subset S$, define

$$W^T := \{w \in W \mid l(ws) < l(w) \text{ for } s \in T\}$$

and

$$W^T(\mathbf{t}) = \sum_{w \in W^T} t_w.$$

The following is now a consequence of Theorem 4.3.2 and [6, Corollary 20.6.6].

Corollary 4.3.4. *Suppose that $T \in \mathcal{S}^{(k)}$ and that $\mathbf{q} \in \mathcal{R}_{St(T)}$. Then $L^2_{\mathbf{q}}H_*(\Omega(S,T),\partial)$ is concentrated in dimension k and*

$$L^2_{\mathbf{q}}b_k(\Omega(S,T),\partial) = \frac{W^T(\mathbf{q})}{W(\mathbf{q})}.$$

If (W,S) is assumed to be right-angled, then the same statement holds if we replace $\mathcal{R}_{St(T)}$ by $\mathcal{R}_{Lk(T)}$.

4.4 A spectral sequence

In this section, we define a spectral sequence following the line laid down in [10, § 2].

A *poset of coefficients* is a contravariant functor $\mathcal{A}$ from a poset $\mathcal{P}$ to the category of abelian groups. In other words, it is a collection $\{\mathcal{A}\}_{a \in \mathcal{P}}$ of abelian groups together with homomorphisms $\phi_{ba} : \mathcal{A}_a \to \mathcal{A}_b$, defined whenever $a > b$, such that $\phi_{ca} = \phi_{cb}\phi_{ba}$, whenever $a > b > c$. The functor $\mathcal{A}$ gives us a system of coefficients on the cell complex $\mathrm{Flag}(\mathcal{P})$: it associates to the simplex σ the abelian group $\mathcal{A}_{\min(\sigma)}$. Hence, we get a cochain complex

$$C^j(\mathrm{Flag}(\mathcal{P});\mathcal{A}) := \bigoplus_{\sigma \in \mathrm{Flag}(\mathcal{P})^{(j)}} \mathcal{A}_{\min(\sigma)},$$

where $\mathrm{Flag}(\mathcal{P})^{(j)}$ denotes the set of j–simplices in $\mathrm{Flag}(\mathcal{P})$.

Let Y be a CW complex. A *poset of spaces* in Y over $\mathcal{P}$ is a cover $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ of Y by subcomplexes indexed by $\mathcal{P}$ so that if $N(\mathcal{V})$ denotes the nerve of the cover, then

(i) $a < b \Longrightarrow Y_a \subset Y_b$,

(ii) the vertex set $\mathrm{Vert}(\sigma)$ of each simplex in $N(\mathcal{V})$ has the greatest lower bound $\wedge\sigma$ in $\mathcal{P}$, and

(iii) $\mathcal{V}$ is closed under taking finite nonempty intersections, i.e., for any simplex σ of $N(\mathcal{V})$,
$$\bigcap_{a \in \sigma} Y_a = Y_{\wedge\sigma}.$$

Note that any cover leads to a poset of spaces by taking all nonempty intersections as elements of the new cover and removing duplicates. The resulting poset is the set of all nonempty intersections, ordered by inclusion.

The following lemmas appearing in [10] define a spectral sequence associated to a poset of spaces, and give conditions for the sequence to degenerate.

Lemma 4.4.1 ([10, Lemma 2.1])**.** *Suppose $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for Y over $\mathcal{P}$. There is a Mayer–Vietoris type spectral sequence converging to $H^*(Y)$ with E_1–term:*
$$E_1^{i,j} = C^i(\mathit{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),$$
and E_2–term:
$$E_2^{i,j} = H^i(\mathit{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),$$
where the coefficient system $\mathcal{H}^j(\mathcal{V})$ is given by $\mathcal{H}^j(\mathcal{V})(\sigma) = H^j(Y_{\min(\sigma)})$.

Lemma 4.4.2 ([10, Lemma 2.2])**.** *Suppose that $\mathcal{V} := \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for Y over $\mathcal{P}$. If for every $a \in \mathcal{P}$, the induced homomorphism $H^*(Y_a) \to H^*(Y_{<a})$ is the zero map, then the spectral sequence degenerates at E_2 and*
$$H^*(Y) = \bigoplus_{a \in \mathcal{P}} H^i(\mathit{Flag}(\mathcal{P}_{\geq a}), \mathit{Flag}(\mathcal{P}_{>a}), H^j(Y_a)).$$

4.5 $L^2_{\mathbf{q}}$–(co)homology of $(\Sigma, \Sigma^{(k-1)})$

To simplify notation, write Σ for Σ_{cc} and let $\Sigma^{(k-1)}$ denote the $(k-1)$–skeleton of Σ_{cc}. For the proofs in this section, we will also write $H^*_{\mathbf{q}}(\mathcal{U})$ for $H^*_W(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W))$ (see Section 3.2.2 for the definition of $H^*_W(\mathcal{U}, \mathcal{N}_{\mathbf{q}}(W))$).

Lemma 4.5.1. *Let Δ denote the standard n–simplex and let $\Delta^{(k)}$ be its k–skeleton, $k < n$. Then the reduced homology $\tilde{H}_*(\Delta^{(k)})$ (with coefficients in $\mathbb{R}$) is concentrated in dimension k. Furthermore, $b_k(\Delta^{(k)}) = \binom{n}{k+1}$.*

Proof. Δ is contractible, so it follows that the augmented chain complex $C_*(\Delta)$ is exact. In particular, $C_*(\Delta^{(k)})$ has the same homology as $C_*(\Delta)$ whenever $l < k$, so $\tilde{H}_l(\Delta^{(k)}) = 0$ whenever $l < k$. We now must show that $\tilde{H}_k(\Delta^{(k)})$ has the claimed dimension. First, note that $C_k(\Delta^{(k)}) = C_k(\Delta)$ has dimension $\binom{n+1}{k+1}$, as there is a k–simplex for each set of $k+1$ vertices. We now proceed by induction on k. If $k = 0$, then $\Delta^{(0)}$ is just $n+1$ vertices, so $\tilde{H}_0(\Delta^{(0)})$ has dimension $\binom{n}{1} = n$. Suppose that the claim is true for $k-1$, and consider the chain complex:

$$0 \longrightarrow C_k(\Delta) \longrightarrow C_{k-1}(\Delta) \longrightarrow \cdots$$

By induction, $\ker \partial_{k-1}$ is has dimension $\binom{n}{k}$. Since $\tilde{H}_{k-1}(\Delta) = 0$, it follows that $\mathrm{im}\partial_k$ also has dimension $\binom{n}{k}$. So, $\ker \partial_k$ has dimension $\binom{n+1}{k+1} - \binom{n}{k} = \binom{n}{k+1}$. ∎

Theorem 4.5.2. *Let $k \geq 1$. Suppose that for every $T \in \mathcal{S}^{(k)}$, $\mathbf{q} \in \mathcal{R}_{St(T)}$, and let $\Sigma^{(k-1)}$ denote the $(k-1)$–skeleton of Σ. Then $L^2_{\mathbf{q}}H_*(\Sigma, \Sigma^{(k-1)})$ is concentrated in dimension k. Furthermore,*

$$L^2_{\mathbf{q}}b_k(\Sigma, \Sigma^{(k-1)}) = \sum_{U \in \mathcal{S}^{(\geq k)}} \binom{|U|-1}{k-1} L^2_{\mathbf{q}}b_{|U|}(\Omega_U, \partial).$$

If (W, S) is assumed to be right-angled, then the same statement holds if we replace $\mathcal{R}_{St(T)}$ by $\mathcal{R}_{Lk(T)}$.

Proof. We will show that $H^*_{\mathbf{q}}(\Sigma, \Sigma^{(k-1)})$ is concentrated in dimension k.

Consider the relative cochain complex $L^2_{\mathbf{q}}C^*(\Sigma, \Sigma^{(k-1)})$. We have that

$$L^2_{\mathbf{q}}C^i(\Sigma, \Sigma^{(k-1)}) = \begin{cases} 0, & i \le k-1; \\ \bigoplus_{U \in \mathcal{S}^{(i)}} H_U, & i > k-1. \end{cases}$$

Set $C_{-i}(\Sigma, \Sigma^{(k-1)}) = L^2_{\mathbf{q}}C^i(\Sigma, \Sigma^{(k-1)})$. Then $C_*(\Sigma, \Sigma^{(k-1)})$ is a chain complex. Now, for every $T \in \mathcal{S}$, set $\tilde{\Omega}_T = \Omega_{ST}/\partial\Omega_{ST}$ and set $C_{-i}(\tilde{\Omega}_T) = L^2_{\mathbf{q}}C^i(\tilde{\Omega}_T) = L^2_{\mathbf{q}}C^i(\Omega_{ST}, \partial\Omega_{ST})$. In this way, we have made the cochain complex of every (S,T)–ruin a subcomplex of the re-indexed relative cochain complex $C_*(\Sigma, \Sigma^{(k-1)})$ (a similar trick works using ordinary cochain complexes).

Let $\mathcal{P}$ be the poset $\mathcal{S}^{(\ge k)}$ with the order reversed. By the above re-indexing of cochain complexes, $\{\tilde{\Omega}_T\}_{T \in \mathcal{P}}$ is a poset of spaces over $Y = \Sigma/\Sigma^{(k-1)}$, and hence we have the spectral sequence of Lemma 4.4.1.

We first establish the condition of Lemma 4.4.2. So, we claim that for every $U \in \mathcal{P}$, the induced map $H^*_{\mathbf{q}}(\tilde{\Omega}_U) \to H^*_{\mathbf{q}}(\tilde{\Omega}_{<U})$ is the zero map. To prove the claim we will show that $H^{-|U|}_{\mathbf{q}}(\tilde{\Omega}_{<U}) = 0$, as this implies $H^*_{\mathbf{q}}(\tilde{\Omega}_U) \to H^*_{\mathbf{q}}(\tilde{\Omega}_{<U})$ is the zero map since $H^*_{\mathbf{q}}(\tilde{\Omega}_U)$ is concentrated in dimension $-|U|$ (see Remark 4.3.3). Recall that $\tilde{\Omega}_{<U} = \bigcup_{T \in \mathcal{P}_{<U}} \tilde{\Omega}_T$. The proof is by induction on the number of elements in the union. For the base case, note that for every spherical V properly containing U, the induced map $H^*_{\mathbf{q}}(\tilde{\Omega}_U) \to H^*_{\mathbf{q}}(\tilde{\Omega}_V)$ is the zero map. This is because of Theorem 4.3.2, which states that $H^*_{\mathbf{q}}(\tilde{\Omega}_U)$ and $H^*_{\mathbf{q}}(\tilde{\Omega}_V)$ are concentrated in dimension $-|U|$ and $-|V|$, respectively, and $-|V| < -|U|$. Now, let $\mathcal{C}$ be a subcollection of elements of $\mathcal{P}_{<U}$ and let $B = \bigcup_{T \in \mathcal{C}} \tilde{\Omega}_T$. We wish to show that $H^{-|U|}_{\mathbf{q}}(B) = 0$. Write $B = A \cup \tilde{\Omega}_V$, where $V \in \mathcal{C}$ and $A = \bigcup_{\substack{T \in \mathcal{C} \\ T \neq V}} \tilde{\Omega}_T$. Then we have the Mayer–Vietoris sequence:

$$H^{-|U|-1}_{\mathbf{q}}(A \cap \tilde{\Omega}_V) \longrightarrow H^{-|U|}_{\mathbf{q}}(B) \longrightarrow H^{-|U|}_{\mathbf{q}}(A) \oplus H^{-|U|}_{\mathbf{q}}(\tilde{\Omega}_V)$$

By induction, $H^*_{\mathbf{q}}(A)$ vanishes for $* \ge -|U|-1$, and by Theorem 4.3.2, $H^*_{\mathbf{q}}(\tilde{\Omega}_V)$ is concentrated in $-|V| < -|U|$. We now claim $H^*_{\mathbf{q}}(A \cap \tilde{\Omega}_V)$ vanishes for $* \ge -|U|-1$, as this implies $H^{-|U|}_{\mathbf{q}}(B) = 0$. We observe that

$$\begin{aligned} A \cap \tilde{\Omega}_V &= \bigcup_{\substack{T \in \mathcal{C} \\ T \neq V}} \tilde{\Omega}_T \cap \tilde{\Omega}_V \\ &= \bigcup_{\substack{T \in \mathcal{C} \\ T \cup V \in \mathcal{S}}} \tilde{\Omega}_{T \cup V} \end{aligned}$$

The last inequality follows from the fact that $H_T \cap H_V = H_{T\cup V}$ whenever $T \cup V$ is spherical (see Section 4.1). Thus $A \cap \tilde{\Omega}_V$ is the union of elements corresponding to a subcollection of $\mathcal{C}$. Therefore the claim follows by induction.

We have established the condition in Lemma 4.4.2, and hence

$$H^{-n}_{\mathbf{q}}(Y) = \bigoplus_{U \in \mathcal{P}} H^{|U|-n}(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}); H^{-|U|}_{\mathbf{q}}(\tilde{\Omega}_U)) \tag{4.2}$$

The strategy of the proof now is as follows. By Theorem 4.3.2, for every $U \in \mathcal{P}$, $H^*_{\mathbf{q}}(\tilde{\Omega}_U)$ is concentrated in dimension $-|U|$. So, we are done if we show that for every $U \in \mathcal{P}$, $H^*(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}))$ is concentrated in dimension $|U| - k$. This implies $E_2^{i,j} = 0$ unless $i + j = -k$, and by (4.2), $H^*_{\mathbf{q}}(Y)$ is concentrated in dimension $-k$. Re-indexing our complexes, it follows that the cohomology of the complex $L^2_{\mathbf{q}}C^*(\Sigma, \Sigma^{(k-1)})$ is concentrated in dimension k.

We now claim that for $U \in \mathcal{P}$ with $m = |U|$, $H^*(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}))$ is concentrated in dimension $m-k$ and free of rank $\binom{m-1}{k-1}$. Since the geometric realization of $\mathrm{Flag}(\mathcal{P}_{\geq U})$ is contractible, by the long exact sequence for the pair it suffices to show that the reduced cohomology $\tilde{H}^*(\mathrm{Flag}(\mathcal{P}_{>U}))$ is concentrated in dimension $m-k-1$. Note that if $m = k$, then we are done since $\mathrm{Flag}(\mathcal{P}_{>U}) = \varnothing$. Also, note that for the special case where $m - k = 1$, the map $H^0(\mathrm{Flag}(\mathcal{P}_{\geq U})) \to H^0(\mathrm{Flag}(\mathcal{P}_{>U}))$ in the long exact sequence for the pair $(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}))$ is injective ($\mathrm{Flag}(\mathcal{P}_{\geq U})$ is the cone on $\mathrm{Flag}(\mathcal{P}_{>U})$), so showing $\tilde{H}^*(\mathrm{Flag}(\mathcal{P}_{>U}))$ is concentrated in dimension $m - k - 1 = 0$ does in fact suffice.

Consider the poset $\mathcal{S}^{op}$, which is the poset $\mathcal{S}$ of spherical subsets with order reversed. Note that $\mathrm{Flag}(\mathcal{S}^{op}_{>U}) \cong \mathrm{Flag}(\mathcal{S}_{<U})$. The geometric realization of $\mathrm{Flag}(\mathcal{S}_{<U})$ is $b\Delta$, where $b\Delta$ is the barycentric subdivision of the $(m-1)$–dimensional simplex Δ. This is because $\mathcal{S}_{<U}$ is the poset of proper subsets of U. Note that $\mathrm{Flag}(\mathcal{P}_{>U})$ is a subcomplex of $\mathrm{Flag}(\mathcal{S}^{op}_{>U})$, and more precisely, the geometric realization of $\mathrm{Flag}(\mathcal{P}_{>U})$ is the subcomplex of barycentric subdivision of $\partial\Delta$ (recall that $k \geq 1$) obtained by removing barycenters corresponding to spherical subsets of cardinality less than k.

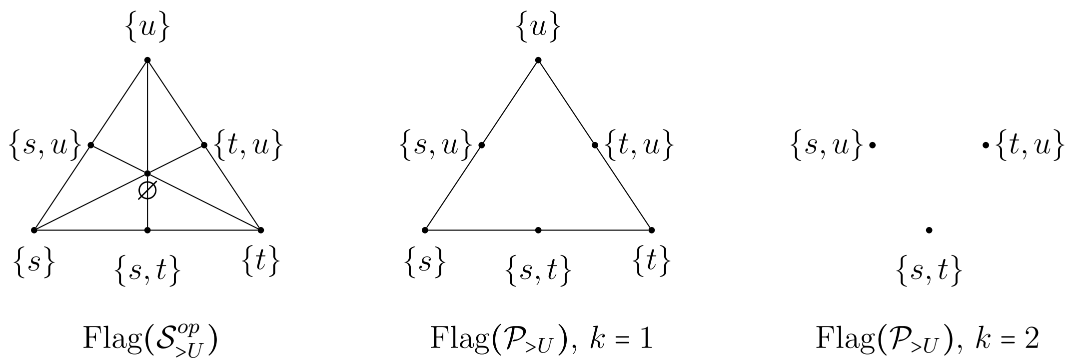


Figure 4.1: Geometric realizations when $U = \{s,t,u\}$

These barycenters correspond to faces of $\partial\Delta$ of dimension less than or equal to $k-2$. Hence

$$\tilde{H}^*(\mathrm{Flag}(\mathcal{P}_{>U})) \cong \tilde{H}^*(\partial\Delta - \Delta^{(k-2)}),$$

where $\Delta^{(k-2)}$ denotes the $(k-2)$–skeleton of Δ. Note that if $k=1$, then $\Delta^{(k-2)} = \varnothing$, and we are done as the reduced homology $\tilde{H}^*(\partial\Delta)$ is concentrated in dimension $m-2$. So, suppose $k>1$. By Alexander Duality,

$$\tilde{H}_*(\Delta^{(k-2)}) \cong \tilde{H}^{m-*-3}(\partial\Delta - \Delta^{(k-2)}).$$

By Lemma 4.5.1, the reduced homology $\tilde{H}_*(\Delta^{(k-2)})$ is concentrated in dimension $k-2$. Furthermore, $\tilde{H}_*(\Delta^{(k-2)})$ has dimension $\binom{m-1}{k-1}$. It follows that $\tilde{H}^*(\partial\Delta - \Delta^{(k-2)})$ is concentrated in dimension $m-k-1$ and of dimension $\binom{m-1}{k-1}$, and therefore the same holds for $\tilde{H}^*(\mathrm{Flag}(\mathcal{P}_{>U}))$.

Thus, we have shown that $H^*_{\mathbf{q}}(Y)$ is concentrated in dimension $-k$ and by (4.2)

$$H^{-k}_{\mathbf{q}}(Y) = \bigoplus_{U\in\mathcal{P}} H^{|U|-k}(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}); H^{-|U|}_{\mathbf{q}}(\tilde{\Omega}_U)).$$

In particular, since $H^{|U|-k}(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U}))$ is free and of dimension $\binom{|U|-1}{k-1}$,

$$L^2_{\mathbf{q}}b_{-k}(Y) = \sum_{U\in\mathcal{P}} b^{|U|-k}(\mathrm{Flag}(\mathcal{P}_{\geq U}), \mathrm{Flag}(\mathcal{P}_{>U})) \cdot L^2_{\mathbf{q}}b_{-|U|}(\tilde{\Omega}_U)$$
$$= \sum_{U\in\mathcal{P}} \binom{|U|-1}{k-1} L^2_{\mathbf{q}}b_{-|U|}(\tilde{\Omega}_U).$$

The assertion now follows after recalling that we are making computations with respect to re-indexed complexes. Specifically, $L^2_{\mathbf{q}}b_{-k}(Y) = L^2_{\mathbf{q}}b_k(\Sigma, \Sigma^{(k-1)})$ and $L^2_{\mathbf{q}}b_{-|U|}(\tilde{\Omega}_U) = L^2_{\mathbf{q}}b_{|U|}(\Omega_U, \partial)$. ∎

Using Corollary 4.3.4, we obtain the following formula for $L^2_{\mathbf{q}}$–Betti numbers.

Corollary 4.5.3. *Let $k \geq 1$. Suppose that for every $T \in \mathcal{S}^{(k)}$, $\mathbf{q} \in \mathcal{R}_{St(T)}$. Then*

$$L^2_{\mathbf{q}}b_n(\Sigma, \Sigma^{(k-1)}) = \begin{cases} \sum_{U\in\mathcal{S}^{(\geq k)}} \binom{|U|-1}{k-1} \frac{W^U(\mathbf{q})}{W(\mathbf{q})}, & n = k; \\ 0, & \textit{otherwise.} \end{cases}$$

If (W,S) is assumed to be right-angled, then the same statement holds if we replace $\mathcal{R}_{St(T)}$ by $\mathcal{R}_{Lk(T)}$.

Remark 4.5.4. Note that a formula for $L^2_{\mathbf{q}}b_k(\Sigma, \Sigma^{(k-1)})$ could also be derived from an Euler characteristic argument, and it is the same as the formula above by [6, Lemma 17.1.8].

4.6 Some Consequences

Corollary 4.6.1. *Suppose that for every $T \in \mathcal{S}^{(k)}$, $\mathbf{q} \in \mathcal{R}_{St(T)}$. Then*

$$L^2_{\mathbf{q}}H_n(\Sigma) = 0 \textit{ for } n > k.$$

If (W,S) is assumed to be right-angled, then the same statement holds if we replace $\mathcal{R}_{St(T)}$ by $\mathcal{R}_{Lk(T)}$.

Proof. For the case where $k = 0$, this is just Proposition 3.3.1, so suppose $k \geq 1$. By Theorem 4.5.2, $L^2_{\mathbf{q}}H_*(\Sigma, \Sigma^{(k-1)})$ is concentrated in dimension k. Therefore the long exact sequence for the pair $(\Sigma, \Sigma^{(k-1)})$ implies the assertion. ∎

Example 4.6.2. Suppose that (W,S) is right-angled and that the corresponding nerve L is a flag triangulation of S^2. Furthermore, suppose that $\mathbf{q} = q$, a positive real number. Let ρ and ρ_T denote the radii of convergence of the growth series $W(t)$ and $W_T(t)$, respectively. In [6, Example 17.4.3], it was computed that

$$\rho = \frac{(f_0 - 4) - \sqrt{(f_0-4)^2 - 4}}{2},$$

where f_0 is the number of vertices of L (note that since L is flag, $f_0 \geq 6$).

The link of every vertex of l is a k–gon, with $k \geq 4$ and $k < f_0$. If $W_{Lk(v)}$ denotes the special subgroup corresponding the the link of a vertex v, and $Lk(v)$ has k vertices, then by an easy computation (see [6, Example 17.1.15]) we have that

$$\rho_{Lk(v)} = \frac{(k-2) - \sqrt{k^2 - 4k}}{2}.$$

Note that $\rho_{Lk(v)}$ is a decreasing function of k when $k \geq 4$. If v_0 is the vertex of L whose link has the most vertices, then Corollary 4.6.1 implies that $L^2_q H_*(\Sigma_L)$ is concentrated in dimension 1 whenever $\rho < q < \rho_{Lk(v_0)}$. This is was already known, as $L^2_q b_2(\Sigma_L) = 0$ for $q \leq 1$ [8, Theorem 16.13].

We present some further consequences of Theorem 4.5.2. But first, some definitions. A locally finite cell complex Λ is an *n–dimensional pseudomanifold* if each maximal cell of Λ is n–dimensional and each $(n-1)$–cell is a face of precisely two n–cells. A pseudomanifold Λ is *orientable* if one can choose orientations for the top-dimensional cells so that their sum is a (possibly infinite) cycle. We now say that a Coxeter system (W,S) is *type PM^n* if its nerve L is an orientable $(n-1)$–dimensional pseudomanifold with the property that the complement of the codimension-two skeleton of L is connected.

Corollary 4.6.3. *Suppose that (W,S) is right-angled and of type PM^n. Then for $\mathbf{q} \leq \mathbf{1}$,*

$$L^2_{\mathbf{q}} b_k(\Sigma^{(n-2)}) = \begin{cases} \sum_{U \in \mathcal{S}^{(\geq n-1)}} \binom{|U|-1}{n-2} \frac{W^U(\mathbf{q})}{W(\mathbf{q})} + L^2_{\mathbf{q}} b_{n-2}(\Sigma) - L^2_{\mathbf{q}} b_{n-1}(\Sigma), & k = n-2; \\ L^2_{\mathbf{q}} b_k(\Sigma), & \textit{otherwise.} \end{cases}$$

Proof. Since (W,S) is of type PM^n, the nerve L is a $(n-1)$–pseudomanifold. Let $T \in \mathcal{S}^{n-1}$. Then the corresponding geometric simplex σ_T in L has dimension $n-2$, and since L is an $(n-1)$–pseudomanifold, it follows that σ_T is contained in precisely two $(n-1)$–simplices σ_U and σ_V, where $U = T \cup \{s\}$ and $V = T \cup \{t\}$ for some $s,t \in S$. Since L has dimension $n-1$ and (W,S) is right-angled, it follows that $m_{st} = \infty$ (otherwise, $U \cup V$ would span an n–simplex in L). It follows that $Lk(T) = \{s,t\}$, and that $W_{Lk(T)} = D_\infty$. Thus, if $\mathbf{q} \in \mathcal{R}_{Lk(T)}$, then $\mathbf{q} < \mathbf{1}$. Since $T \in \mathcal{S}^{n-1}$ was arbitrary, it follows that if for every $T \in \mathcal{S}^{n-1}$, $\mathbf{q} \in \mathcal{R}_{Lk(T)}$, then $\mathbf{q} < \mathbf{1}$.

By Theorem 4.5.2, $L^2_{\mathbf{q}}H_*(\Sigma, \Sigma^{(n-2)})$ is concentrated in dimension $n-1$. Consider the long exact sequence for the pair $(\Sigma, \Sigma^{(n-2)})$:

$$0 \longrightarrow L^2_{\mathbf{q}}H_{n-1}(\Sigma) \longrightarrow L^2_{\mathbf{q}}H_{n-1}(\Sigma, \Sigma^{(n-2)}) \longrightarrow L^2_{\mathbf{q}}H_{n-2}(\Sigma^{(n-2)}) \longrightarrow L^2_{\mathbf{q}}H_{n-2}(\Sigma) \longrightarrow 0$$

A dimension count and continuity of weighted $L^2_{\mathbf{q}}$–Betti numbers now implies the assertion. ∎

When (W,S) is of type PM^3, we have the following computation thanks to Corollary 4.6.3. Recall, that $\Sigma^{(1)}$ is the Cayley graph of W (Proposition 2.8.1), so the following corollary gives a formula for weighted L^2–Betti numbers of the Cayley graph when W is of type PM^3.

Corollary 4.6.4. *Suppose that (W,S) is right-angled and of type PM^3. Furthermore, suppose that $\mathbf{q} \le \mathbf{1}$ and $\mathbf{q} \notin \mathcal{R}$. Then*

$$L^2_{\mathbf{q}}b_1(\Sigma^{(1)}) = -\frac{1}{W(\mathbf{q})} + \sum_{U \in \mathcal{S}^{(\ge 2)}} (|U|-1)\frac{W^U(\mathbf{q})}{W(\mathbf{q})}$$

Proof. Since Σ is a pseudomanifold, it follows that $L^2_{\mathbf{q}}b_3(\Sigma) = 0$ and since $\mathbf{q} \notin \mathcal{R}$, $L^2_{\mathbf{q}}b_0(\Sigma) = 0$. The assertion follows from Corollary 4.6.3, as $\chi_{\mathbf{q}}(\Sigma) = L^2_{\mathbf{q}}b_2(\Sigma) - L^2_{\mathbf{q}}b_1(\Sigma) = \frac{1}{W(\mathbf{q})}$. ∎

Example 4.6.5. Suppose that (W,S) is right-angled and of type PM^3, and let $\mathbf{q} = q$, a positive real number. Recall the f–polynomial $f_L(t)$ of L. It is defined by

$$f_L(t) := \sum_{i=0}^{3} f_{i-1} t^i,$$

where f_m is the number of m–simplices of L and $f_{-1} = 1$. By [6, Proposition 17.4.2], we have the following formula:

$$\frac{1}{W(t)} = (1+t)^3 f_L\left(\frac{-t}{1+t}\right).$$

This simplifies to

$$\begin{aligned}\frac{1}{W(t)} &= 1-(f_0-3)t+(f_1-2f_0+3)t^2-(f_0-f_1+f_2-1)t^3\\ &= 1-(f_0-3)t+(f_0+3-3\chi(L))t^2-(\chi(L)-1)t^3.\end{aligned}$$

Here $\chi(L)$ is the Euler characteristic of L. Note that the second equality follows by using the facts that $\chi(L) = f_0 - f_1 + f_2$ and $3f_2 = 2f_1$ (this second formula holds because each edge is contained in exactly two 2–simplices, and each 2–simplex contains exactly three edges).

The radius of convergence ρ of $W(t)$ is the smallest modulus of a root of the above polynomial. Since W is of type PM^3, the link of every vertex of L is 1–pseudomanifold (in particular, just homeomorphic to S^1). Thus, just as in Example 4.6.2, the radius of convergence $\rho_{Lk(v)}$ of the special subgroup $W_{Lk(v)}$ has the formula:

$$\rho_{Lk(v)} = \frac{(k-2)-\sqrt{k^2-4k}}{2},$$

where $Lk(v)$ is the link of the vertex v and k is the number of vertices in $Lk(v)$. If v_0 is the vertex of L whose link has the maximal number of vertices, then Corollary 4.6.1 implies that $L^2_q H_*(\Sigma_L) = 0$ is concentrated in dimension 1 whenever $\rho < q < \rho_{Lk(v_0)}$.

The main point is that ρ is explicitly computable. For example, if L is a flag triangulation of a torus (or, more generally, a flag triangulation of a surface of genus $g \geq 1$), it is still an open conjecture that $L^2_q b_*(\Sigma_L) = 0$ is concentrated in degree 2 for $q = 1$ (see [9, Conjecture 11.5.1]), but on the other hand Corollary 4.6.1 allows us to conclude that $L^2_q b_2(\Sigma_L) = 0$ for $q < \rho_{Lk(v_0)}$.

Chapter 5

The Weighted Singer Conjecture

Appearing in [8], the following is the appropriate reformulation of the the Singer Conjecture for Coxeter groups [9] for weighted L^2–(co)homology:

Conjecture 5.0.1 (Weighted Singer Conjecture)**.** Suppose that the nerve L is a triangulation of S^{n-1}. Then

$$L^2_{\mathbf{q}}H_k(\Sigma_L) = 0 \text{ for } k > \frac{n}{2} \text{ and } \mathbf{q} \leq \mathbf{1}.$$

By weighted Poincaré duality, this is equivalent to the conjecture that if $\mathbf{q} \geq \mathbf{1}$ and $k < \frac{n}{2}$, then $L^2_{\mathbf{q}}H_k(\Sigma)$ vanishes. The conjecture is known for elementary reasons for $n \leq 2$, and in [8], it is proved for the case where W is right-angled and $n \leq 4$. Furthermore, it was shown in in [8] that Conjecture 5.0.1 for n odd implies Conjecture 5.0.1 for n even, under the assumption that W is right-angled.

The original Singer Conjecture for Coxeter groups was formulated for $\mathbf{q} = \mathbf{1}$ in [9] and concluded that the L^2–(co)homology is concentrated in dimension $\frac{n}{2}$. The original conjecture is known for elementary reasons for $n \leq 2$ and holds by a result of Lott and Lück [14], in conjunction with the validity of the Geometrization Conjecture for 3–manifolds [17], for $n = 3$. It was proved by Davis–Okun [9] for the case where W is right-angled and $n \leq 4$. It was later proved for the case where W is an even Coxeter group and $n \leq 4$ by Schroeder [18], under the assumption that the nerve L is a flag complex. Due to recent work of Okun–Schreve [16, Theorem 4.9], the conjecture is now known in full generality whenever $\mathbf{q} = \mathbf{1}$ and $n \leq 4$. In fact, using induction and [16, Theorem 4.5, Lemma 4.6, Corollary 4.7] proves the following theorem.

Theorem 5.0.2. *Suppose that the nerve L is an $(n-1)$–sphere or an $(n-1)$–disk. Then*

$$L^2_{\mathbf{1}}H_k(\Sigma_L) = 0 \textit{ for } k \geq n-1.$$

In this chapter, we present a proof of Conjecture 5.0.1 in dimension three that encompasses all but nine Coxeter groups. Then, under some restrictions on the nerve of the Coxeter group, we obtain partial results whenever $n = 4$ (in particular, the conjecture holds for $n = 4$ if the nerve of the corresponding Coxeter group is a flag complex). We then extend our results in dimension four to prove a general version of the conjecture for the case where the nerve of the Coxeter group is assumed to be a flag triangulation of a 3–manifold.

5.1 The case where L is a disk

Note that if L is a triangulation of the $(n-1)$–disk, then Σ_L is an n–manifold with boundary. We now obtain the following theorem, which whenever $n = 3, 4$ can be thought of as a version of Conjecture 5.0.1 for the case where Σ_L is an n–manifold with boundary.

Theorem 5.1.1. *Suppose that the nerve L is an $(n-1)$–disk. Then*

$$L^2_{\mathbf{q}}H_k(\Sigma_L) = 0 \textit{ for } k \geq n-1 \textit{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof. By Theorem 5.0.2, we have that $L^2_{\mathbf{1}}H_k(\Sigma_L) = 0$ for $k \geq n-1$. Furthermore, Proposition 2.9.1 implies that $\operatorname{vcd} W \leq n-1$, and hence we are done by Lemma 3.3.8. ∎

5.2 A cell structure on K

Suppose that L is the labeled nerve of a Coxeter system, homeomorphic to the n–sphere. For every $T \in \mathcal{S}$, define K_T to be the geometric realization of the poset

$\mathcal{S}_{\geq T} = \{U \in \mathcal{S} \mid T \subseteq U\}$. In other words, K_T is the union of all closed simplices in bL with minimum vertex T, so K_T is the cone on the barycentric subdivision of the link of T in L. If L is a triangulation of the n–sphere, then it follows that links of simplices T of L are spheres of dimension $n - |T|$. Thus it follows that each K_T is a $(n - |T| + 1)$–disk, hence $\{K_T\}_{T \in \mathcal{S}}$ yields a cellulation of K. We denote K with this cellulation by K_d. Note that this cellulation extends to Σ_L, and the simplicial structure on Σ_L coincides with the barycentric subdivision of this cell structure.

The codimension-one faces of K_d correspond to vertices of L, and we assign dihedral angles to K_d as follows. If $\{s, t\}$ is an edge of L, then we assign the dihedral angle π/m_{st} between the faces K_s and K_t.

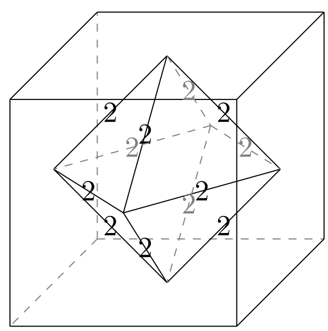


Figure 5.1: K_d when W is right-angled and the labeled nerve L is the boundary complex of an octahedron

5.3 Andreev's theorem

In [1], Andreev listed necessary and sufficient conditions for abstract three-dimensional polytopes with assigned dihedral angles $(0, \frac{\pi}{2}]$ to be realized as convex polytopes in $\mathbb{H}^3$. For these polytopes to tile $\mathbb{H}^3$, these angles must be integer submultiples of π. We now state the theorem.

Theorem 5.3.1 ([1, Theorem 2]). *Let P be an abstract three-dimensional simple polyhedron, not a simplex. The following conditions are necessary and sufficient for the existence in $\mathbb{H}^3$ of a convex polytope of finite volume of the combinatorial type*

P with the dihedral angles $\alpha_{ij} \le \frac{\pi}{2}$ (where α_{ij} is the dihedral angle between the faces F_i, F_j):

(i) If F_1, F_2 and F_3 are all the faces meeting at a vertex of P, then $\alpha_{12}+\alpha_{23}+\alpha_{31} > \pi$.

(ii) If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of the intersection satisfy $\alpha_{12} + \alpha_{23} + \alpha_{31} < \pi$.

(iii) Four faces cannot intersect cyclically with all four angles$=\frac{\pi}{2}$ unless two of the opposite faces intersect.

(iv) If P is a triangular prism, then the angles along the base and the top cannot all be $\frac{\pi}{2}$.

Our goal is to use the above theorem to formulate conditions on the labeled nerve L so that $\Sigma_L = \mathbb{H}^3$. Note that, for the rest of this thesis, we use the notation $\Sigma_L = X$ whenever Σ_L admits a W_L–invariant metric making it isometric to X.

We say that a vertex in L is a *Euclidean* 3*–vertex* if its link has three pairwise connected vertices, and if v_0, v_1, v_2 are the vertices, then the labelings on the corresponding edges satisfy:

$$\frac{\pi}{m_{v_0v_1}} + \frac{\pi}{m_{v_0v_2}} + \frac{\pi}{m_{v_1v_2}} = \pi.$$

Similarly, we say a vertex v in L is a *Euclidean* 4*–vertex* if $Lk(v)$ is a 4–gon with all edges labeled by 2.

Let C be an empty circuit in L and suppose that C is not the link of some vertex of L. If C consists of three vertices v_0, v_1, v_2, then we say that C is a *Euclidean* 3*–circuit* if the labelings on the edges of C satisfy:

$$\frac{\pi}{m_{v_0v_1}} + \frac{\pi}{m_{v_0v_2}} + \frac{\pi}{m_{v_1v_2}} = \pi.$$

Similarly, if C consists of four vertices and is not the boundary of two adjacent simplices, then we say that C is a *Euclidean* 4*–circuit* if the labels on the edges of C are all equal to 2.

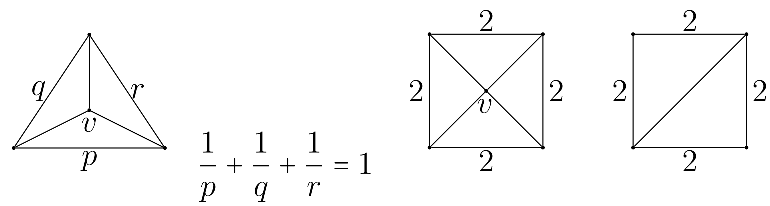


Figure 5.2: The two figures on the left show Euclidean vertices, while the far right is *not* a Euclidean circuit

The nerve L of a Coxeter system (W, S) has a natural piecewise spherical structure, and under this structure, if $s, t \in S$ are connected by an edge in L, then the edge has length $\pi - \pi/m_{st}$, where $(st)^{m_{st}} = 1$. Hence L inherits the structure of a *metric flag complex* [6, Lemma 12.3.1], meaning that any collection of pairwise connected edges of L spans a simplex if and only if there exists a spherical simplex with the corresponding edge lengths. It follows that if v is a Euclidean 3– or 4–vertex, then $Lk(v)$ is a full subcomplex of L. Similarly, Euclidean circuits are full subcomplexes. Thus the corresponding subgroups are in fact special subgroups of W.

Suppose that L is the labeled nerve of a Coxeter system, homeomorphic to S^2, and let K_d have the prescribed dihedral angles π/m_{st} as in 5.2. It follows that if K_d satisfies the conditions of Theorem 5.3.1, then $\Sigma_L = \mathbb{H}^3$. The following theorem now becomes a special case of Theorem 5.3.1.

Theorem 5.3.2. *Suppose that L is the labeled nerve of a Coxeter system, homeomorphic to S^2, but not the boundary of a 3–simplex. Furthermore, suppose that*

- *L has no Euclidean 3– or 4–circuits.*
- *L has no Euclidean vertices.*
- *L is not the right-angled suspension of a 3–gon.*

Then $\Sigma_L = \mathbb{H}^3$.

Proof. We must show that K_d satisfies the conditions of Theorem 5.3.1. First note that condition (i) is vacuous in our case. Condition (ii) on K_d is equivalent to saying that L has no Euclidean 3–vertices and no Euclidean 3–circuits. Similarly, condition (iii) on K_d is equivalent to saying that L has no Euclidean 4–vertices or no Euclidean 4–circuits. Finally, condition (iv) on K_d is equivalent to saying that L is not the right-angled suspension of a 3–gon. ∎

For convenience we restate the above theorem in terms of special subgroups.

Theorem 5.3.3. *Suppose that the nerve L is a triangulation of S^2, but not the boundary of a 3–simplex, and let (W, S) be the corresponding Coxeter system. Furthermore, suppose that*

- *For every $T \subset S$, W_T is not a Euclidean reflection group.*
- *$W \neq W_T \times D_\infty$, where $T \subset S$ spans empty triangle in L and D_∞ is the infinite dihedral group.*

Then $\Sigma_L = \mathbb{H}^3$.

5.4 Equidistant hypersurfaces

Suppose that the Coxeter group W has nerve L that is a triangulation of S^2 and that $\Sigma_L = \mathbb{H}^3$. Let D denote the Davis chamber (in $\mathbb{H}^3$) and let W_M be a special subgroup of W. We now consider the (possibly infinite) convex polytope $W_M D$ in $\mathbb{H}^3$.

For $t > 0$, let S_t denote the t–distant surface from a component S of $\partial W_M D$. Then S_t is a smooth surface (see [4, Proposition II.2.2.1]). In fact, S_t is a union of pieces of which there are three types: hyperbolic, Euclidean, and spherical, each of which are the equidistant pieces from faces, edges, and vertices of S, respectively. The Euclidean pieces look like rectangles that are each adjacent to two hyperbolic pieces and two spherical pieces, and the spherical pieces are adjacent to Euclidean pieces.

As $W_M D$ is convex, the nearest point projection $p : \mathbb{H}^3 \cup \partial\mathbb{H}^3 \to W_M D$ is defined. If we fix $t > r > 0$, then p induces a map $p_{tr} : S_t \to S_r$.

Lemma 5.4.1. *The map $p_{tr} : S_t \to S_r$ induced by nearest point projection is $\frac{\tanh(t)}{\tanh(r)}$–quasiconformal.*

Proof. It suffices to check what p_{tr} does on each of the three types of pieces. First, note that a face of S is simply the intersection of $\partial W_M D$ with a hyperbolic plane in $\mathbb{H}^3$. Thus p_{tr} simply scales the corresponding hyperbolic pieces on S_t and S_r by a constant factor. Hence p_{tr} is conformal there. Similarly, the map p_{tr} is conformal on the spherical pieces.

Second, we consider the Euclidean piece in S_t equidistant from an edge of S. A Euclidean piece looks like a rectangle adjacent to two hyperbolic pieces at two parallel edges (parallel in the intrinsic Euclidean geometry), and the the map induced by nearest point projection $S_t \to S$ scales by a factor of $1/\cosh(t)$ in the direction of those edges. The other two edges of the Euclidean piece are each adjacent to a spherical piece. An edge like this is the arc of a circle with radius t centered at a vertex in S. Thus the edge has length $\theta \sinh(t)$, where θ is the dihedral angle at the corresponding edge of S. Hence the map p_{tr} scales by a factor of $\cosh(r)/\cosh(t)$ in the direction of the edges adjacent to the hyperbolic pieces, and scales the edges adjacent to the spherical pieces by a factor of $\sinh(r)/\sinh(t)$. Therefore p_{tr} is $\frac{\tanh(t)}{\tanh(r)}$–quasiconformal on the Euclidean pieces. ∎

5.5 The conjecture in dimension three

In this section, we prove the following theorem.

Theorem 5.5.1. *Suppose that the nerve L of a Coxeter group is a triangulation of S^2 not dual to a hyperbolic 3–simplex. Then*

$$L^2_{\mathbf{q}} H_k(\Sigma_L) = 0 \text{ for } k > 1 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Suppose that M is a complete smooth Riemannian manifold. Given a a nonnegative measurable function $f : M \to [0, \infty)$, we define a new norm on the C^∞ k–forms called the L^2_f norm by

$$||\omega||^2_f = \int_M ||\omega||^2_p f(p) dV,$$

where $||\omega||_p^2$ is the pointwise norm and dV is the volume form of M. Let $L^2_f\mathcal{C}^*(M)$ denote the weighted L^2 de Rham complex defined using the L^2_f norm.

Lemma 5.5.2. *Let M and N be smooth surfaces and suppose that $\phi : M \to N$ is a K–quasiconformal diffeomorphism. Let $g : N \to [0, \infty)$ be the function defined by $g(p) = f(\phi^{-1}(p))$. Then for every $\omega \in L^2_g\mathcal{C}^1(N)$, we have that*

$$\frac{1}{K}||\omega||_g^2 \leq ||\phi^*(\omega)||_f^2 \leq K||\omega||_g^2.$$

Proof. The pointwise norm of a 1–form is $||\omega||_p = \sup_{\substack{x \in T_pM \\ ||x||=1}} \omega(x)$, where T_pM is the tangent space of M at p. Since ϕ is K–quasiconformal, its differential $d\phi$ maps the circle $\{x \in T_pM \mid ||x|| = 1\}$ to an ellipse with semi-axis $b(p) \leq a(p)$ satisfying $\frac{a(p)}{b(p)} \leq K$. Thus for any $\omega \in L^2_{\mathbf{q}}\mathcal{C}^1(N)$,

$$b(p)||\omega||_{\phi(p)} \leq ||\phi^*(\omega)||_p \leq a(p)||\omega||_{\phi(p)}.$$

Now, let dV_M and dV_N be the respective volume forms of M and N. We have that

$$(f dV_M)_p = \frac{(g(\phi)\phi^*(dV_N))_p}{a(p)b(p)},$$

so for L^2_f norms we have

$$\begin{aligned}
||\phi^*(\omega)||_f^2 &= \int_M ||\phi^*(\omega)||_p^2 f(p) dV_M \\
&\leq \int_M \frac{a(p)}{b(p)}||\omega||^2_{\phi(p)} g(\phi(p))\phi^*(dV_N) \\
&\leq K \int_M ||\omega||^2_{\phi(p)} g(\phi(p))\phi^*(dV_N) \\
&= K \int_N ||\omega||_x^2 g(x) dV_N = K||\omega||_g^2
\end{aligned}$$

The remaining inequality follows similarly. ∎

Suppose that the nerve L of W is a triangulation of S^2 and that $\Sigma_L = \mathbb{H}^3$. Define f to be the function $f(p) = q_w$, where $w \in W_L$ is a word of shortest length such that

$p \in wD$ (here D is the Davis chamber). Let $L^2_{\mathbf{q}}\mathcal{H}^*(\mathbb{H}^3)$ denote the weighted L^2 de Rham cohomology defined using this f.

Let W_M be an infinite special subgroup of W and let S be one of the components of $\partial W_M D$. Put coordinates (x,t) on $\mathbb{H}^3$ so that $t \in \mathbb{R}$ is the oriented distance from $p \in \mathbb{H}^3$ to the closest point $x \in S$. Fix $r > 0$, and for $t \geq r$ let S_t denote the hypersurface consisting of points of (oriented) distance t from S. Let $p_{tr} : S_t \to S_r$ be the map induced by nearest point projection, and let ϕ_{tr} denote the inverse of p_{tr}. By Lemma 5.4.1, p_{tr} is $K(t)$–quasiconformal, with $K(t) = \frac{\tanh(t)}{\tanh(r)}$, and hence so is its inverse $\phi_{tr} : S_r \to S_t$. Let $i_r : S_r \to \mathbb{H}^3$ and $i_t : S_t \to \mathbb{H}^3$ be the inclusions. Then i_r and $i_t \circ \phi_{tr}$ are properly homotopic.

We now adapt the argument after [8, Theorem 16.10] to prove the following lemma.

Lemma 5.5.3. *If* $\mathbf{q} \geq \mathbf{1}$, *then the map* $i_r^* : L^2_{\mathbf{q}}\mathcal{H}^1(\mathbb{H}^3) \to L^2_{\mathbf{q}}\mathcal{H}^1(S_r)$ *induced by the inclusion* i_r *is the zero map.*

Proof. Set $g(x,y) = f(x,0)$, so $f(x,y) \geq g(x,y)$, and let ω be a closed L^2_f 1–form on $\mathbb{H}^3$. We now show that the restriction $i_r^*(\omega)$ to S_r represents the zero class in in reduced L^2_f-cohomology. For the remainder of the proof, we will use the notation $\|[\alpha]\|_g$ and $\|[\alpha]\|_x$ to denote the respective L^2_g norm and pointwise norm of the harmonic representative of the cohomology class $[\alpha]$.

Suppose for a contradiction that $[i_r^*(\omega)] \neq 0$. Then $\|i_r^*(\omega)\|_g \geq \|[i_r^*(\omega)]\|_g > 0$. By Lemma 5.5.2, it follows that $\|\phi_{tr}^*(i_t^*(\omega))\|_g^2 \leq K(t)\|i_t^*(\omega)\|_g^2$, and since i_r and $i_t \circ \phi_{tr}$ are properly homotopic, $[i_r^*(\omega)] = [\phi_{tr}^*(i_t^*(\omega))]$. Therefore

$$K(t)\|i_t^*(\omega)\|_g^2 \geq \|[i_r^*(\omega)]\|_g^2 > 0.$$

Now, $i_t^*(\omega)$ is just a restriction of ω, so we have the pointwise inequality $\|\omega\|_x \geq \|i_t^*(\omega)\|_x$. Using Fubini's Theorem, we compute

$$\begin{aligned}\|\omega\|_g^2 &= \int_{\mathbb{H}^3} \|\omega\|_x^2 g(x,y)dV \geq \int_r^\infty \int_{S_t} \|\omega\|_x^2 g(x,y)dAdt \geq \int_r^\infty \int_{S_t} \|i_t^*(\omega)\|_x^2 g(x,y)dAdt \\ &= \int_r^\infty \|i_t^*(\omega)\|_g^2 dt \geq \int_r^\infty \frac{\tanh(r)}{\tanh(t)}\|[i_r^*(\omega)]\|_g^2 dt = \infty.\end{aligned}$$

Since $\|\omega\|_f \geq \|\omega\|_g$, this contradicts the assumption that the L^2_f norm of ω is finite. ∎

Suppose that L is the nerve of a Coxeter group W_L and that A is a full subcomplex of L. For the proofs that follow, note that $\dim_{\mathbf{q}} L^2_{\mathbf{q}}H_k(W_L\Sigma_A) = L^2_{\mathbf{q}}b_k(\Sigma_A)$ (see [6, pg. 352 (vi)]).

Lemma 5.5.4. *Suppose that the nerve L is a triangulation of S^2 and that there exists a full subcomplex 1–sphere M of L that separates L into two full 2–disks L_1 and L_2 with boundary M. Furthermore, suppose that one of the following holds:*

(i) $\Sigma_M = \mathbb{R}^2$.

(ii) $\Sigma_L = \mathbb{H}^3$.

Then

$$L^2_{\mathbf{q}}H_k(\Sigma_L) \textit{ for } k \geq 2 \textit{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof. Since Σ_L is a 3–manifold, it follows that $L^2_{\mathbf{q}}b_3(\Sigma_L) = 0$ [6, Proposition 20.4.1]. Hence we must show that $L^2_{\mathbf{q}}b_2(\Sigma_L) = 0$. Consider the following Mayer–Vietoris sequence applied to $L = L_1 \cup_M L_2$:

$$\cdots \to L^2_{\mathbf{q}}H_2(W_L\Sigma_{L_1}) \oplus L^2_{\mathbf{q}}H_2(W_L\Sigma_{L_2}) \to L^2_{\mathbf{q}}H_2(\Sigma_L) \to L^2_{\mathbf{q}}H_1(W_L\Sigma_M) \to \cdots$$

By Theorem 5.1.1, we have that $L^2_{\mathbf{q}}H_2(W_L\Sigma_{L_1}) = L^2_{\mathbf{q}}H_2(W_L\Sigma_{L_2}) = 0$. If (i) holds, then Theorem 3.3.4 implies that $L^2_{\mathbf{q}}H_1(\Sigma_M) = 0$, and we are done. If (ii) holds, we argue that the connecting homomorphism $\partial_* : L^2_{\mathbf{q}}H_2(\Sigma_L) \to L^2_{\mathbf{q}}H_1(W_L\Sigma_M)$ is the zero map. By [8, Lemma 16.2], we reduce the proof to showing that the map induced by inclusion $i_* : L^2_{\mathbf{q}^{-1}}H_1(W_L\Sigma_M) \to L^2_{\mathbf{q}^{-1}}H_1(\Sigma_L)$ is the zero map, and since $W_L\Sigma_M$ is a disjoint union of copies of Σ_M, it is enough to show that the restriction of i_* to one summand $L^2_{\mathbf{q}^{-1}}H_1(\Sigma_M)$ is zero.

Consider the infinite convex polytope W_MD, where D is the Davis chamber for W. We have that W_M acts properly and cocompactly on W_MD by isometries. In particular, if S is one of the components of ∂W_MD, then W_M acts properly and cocompactly on S, and therefore $L^2_{\mathbf{q}^{-1}}H^*(\Sigma_M) \cong L^2_{\mathbf{q}^{-1}}\mathcal{H}^*(S)$. Hence we are done if we show that map $i^* : L^2_{\mathbf{q}^{-1}}\mathcal{H}^1(\mathbb{H}^3) \to L^2_{\mathbf{q}^{-1}}\mathcal{H}^1(S)$ induced by the inclusion $i : S \to \mathbb{H}^3$ is the zero map.

Fix $r > 0$, and let S_r be the r–distant surface from S. S_r and S are properly homotopy equivalent, and this equivalence induces a weak isomorphism between $L^2_{\mathbf{q}^{-1}}\mathcal{H}^*(S)$ and $L^2_{\mathbf{q}^{-1}}\mathcal{H}^*(S_r)$. Thus we have reduced the proof to showing that the map $i_r^* : L^2_{\mathbf{q}^{-1}}\mathcal{H}^1(\mathbb{H}^3) \to L^2_{\mathbf{q}^{-1}}\mathcal{H}^1(S_r)$ induced by the inclusion $i_r : S_r \to \mathbb{H}^3$ is the zero map, and therefore we are done by Lemma 5.5.3. ∎

Remark 5.5.5. In [8, Section 16] W is strictly assumed to be right-angled, but the proof of [8, Lemma 16.2] does not use this, as it only uses properties of weighted L^2–(co)homology.

Proof of Theorem 5.5.1. We first suppose that $\Sigma_L = \mathbb{H}^3$. We need to find a full subcomplex M of L satisfying the hypothesis of Lemma 5.5.4. First we suppose that L is a flag complex. Let v be a vertex of L and set $M = Lk(v)$. Since L is flag, M is a full subcomplex of L, and since L is a triangulation of the 2–sphere, it follows that M is a 1–sphere, and we are done. Now suppose that L is not flag. Since L is not the boundary of a 3–simplex, there exists an empty 2–simplex in L. Let M denote this empty 2–simplex. Then M separates L into two full 2–disks, both with boundary M, and we are done. We now suppose that $\Sigma_L \neq \mathbb{H}^3$ and use Theorem 5.3.3 to perform a case-by-case analysis.

Case I: W contains a Euclidean special subgroup W_T. Let M be the full subcomplex of L corresponding to W_T. Then M separates L into two 2–disks both with boundary M and hence Lemma 5.5.4 (i) implies the assertion.

Case II: $W = W_T \times D_\infty$, where $T \subset S$ spans empty triangle in L. Either $\Sigma_L = \mathbb{R}^3$ or $\Sigma_L = \mathbb{H}^2 \times \mathbb{R}$. In both cases we are done by the weighted Künneth formula.

Case III: L is the boundary of a 3–simplex. By assumption, L is not dual to a hyperbolic simplex, so $\Sigma_L = \mathbb{R}^3$. Therefore we are done by [8, Corollary 14.5]. ∎

5.6 The conjecture in dimension four

In dimension four, we prove the following case of the Weighted Singer Conjecture:

Theorem 5.6.1. *Suppose that the nerve L of a Coxeter group is a triangulation of S^3. Furthermore, suppose that there exists a vertex of L such that its link is a full subcomplex of L and not dual to a hyperbolic 3–simplex. Then*

$$L^2_{\mathbf{q}}H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof. In this case, Σ_L is a 4–manifold, and hence $L^2_{\mathbf{q}}b_4(\Sigma_L) = 0$ [6, Proposition 20.4.1]. It remains to show that $L^2_{\mathbf{q}}b_3(\Sigma_L) = 0$. Suppose that the nerve L is a triangulation of S^3 and let $s \in L$ be a vertex. We make the following observations:

- The nerve L_{S-s} of the Coxeter system $(W_{S-s}, S-s)$ is a 3–disk.
- The nerve $St(s)$ of the Coxeter group $W_{St(s)}$ is a 3–disk.
- The nerve $Lk(s)$ of the Coxeter group $W_{Lk(s)}$ is a 2–sphere.

This is because the subcomplexes $St(s)$, $Lk(s)$, and L_{S-s} of L correspond to the closed star of the vertex s, link of the vertex s, and complement of the open star of s, respectively, which are all by assumption full subcomplexes of L.

Consider the following Mayer–Vietoris sequence:

$$\cdots \to L^2_{\mathbf{q}}H_3(W_L\Sigma_{L_{S-s}}) \oplus L^2_{\mathbf{q}}H_3(W_L\Sigma_{St(s)}) \to L^2_{\mathbf{q}}H_3(\Sigma_L) \to L^2_{\mathbf{q}}H_2(W_L\Sigma_{Lk(s)}) \to \cdots$$

By Theorem 5.1.1, $L^2_{\mathbf{q}}b_3(\Sigma_{St(s)}) = 0$ and $L^2_{\mathbf{q}}b_3(\Sigma_{L_{S-s}}) = 0$, and by Theorem 5.5.1, $L^2_{\mathbf{q}}b_2(\Sigma_{Lk(s)}) = 0$. Therefore by the above sequence, $L^2_{\mathbf{q}}b_3(\Sigma_L) = 0$. ∎

We obtain the following corollary.

Corollary 5.6.2. *Suppose that the nerve L of a Coxeter group is a flag triangulation of S^3. Then*

$$L^2_{\mathbf{q}}H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Proof. Since L is flag, it follows that the link of every vertex is a full subcomplex of L. Furthermore, the link of every vertex is not the boundary of a 3–simplex (and in particular, not dual to a 3–simplex). Theorem 5.6.1 now completes the proof. ∎

5.7 The case where L is a 3–manifold

In this section, we prove the following generalization of Corollary 5.6.2.

Theorem 5.7.1. *Suppose that L is a flag triangulation of a 3–manifold. Then*

$$L^2_{\mathbf{q}}H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } \mathbf{q} \leq \mathbf{1}.$$

Note that, in this case, Σ_L is a 4–pseudomanifold (i.e. every 3–cell of Σ_L is contained in precisely two 4–cells). We will prove the theorem using ruins.

Lemma 5.7.2. *Suppose that L is a flag triangulation of a 3–manifold. Then for every $t \in L$, $L^2_{\mathbf{q}}H_*(\Omega(S,t), \partial\Omega(S,t)) = 0$ for $* > 2$ and $\mathbf{q} \leq \mathbf{1}$.*

Proof. First, for $t \in L$, recall that the (S,t)–ruin has the property that

$$\Omega(S,t) = \Omega(St(t), t),$$

where $St(t) = \{s \in S \mid m_{st} < \infty\}$. Recall that $Lk(t) = St(t) - t$, and so we have the following weak exact sequence (see sequence (4.1)):

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(Lk(t))) \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(St(t))) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(S,t), \partial\Omega(S,t)) \longrightarrow \cdots$$

Note that

$$L^2_{\mathbf{q}}b_*(\Sigma(St(t))) = L^2_{\mathbf{q}}b_*(\Sigma_{St(t)}) \text{ and } L^2_{\mathbf{q}}b_*(\Sigma(Lk(t))) = L^2_{\mathbf{q}}b_*(\Sigma_{Lk(t)}),$$

where $\Sigma_{St(t)}$ and $\Sigma_{Lk(t)}$ are the Davis complexes corresponding to the subgroups $W_{St(t)}$ and $W_{Lk(t)}$, respectively. Since L is flag, the respective nerves of the groups $W_{St(t)}$ and $W_{Lk(t)}$ are a 3–disk and a 2–sphere. Furthermore, the nerve of $W_{Lk(t)}$ is not the boundary of a 3–simplex (again, L is flag). By Theorem 5.1.1, $L^2_{\mathbf{q}}b_k(\Sigma_{St(t)}) = 0$ for $k > 2$, and by Theorem 5.5.1, $L^2_{\mathbf{q}}b_k(\Sigma_{Lk(t)}) = 0$ for $k > 1$. Therefore weak exactness of the sequence implies that $L^2_{\mathbf{q}}H_*(\Omega(S,t), \partial\Omega(S,t)) = 0$ for $* > 2$. ∎

We now adapt the argument of [18] to complete the proof of Theorem 5.7.1. The main point is that we are able to prove the following lemma for $\mathbf{q} \le \mathbf{1}$.

Lemma 5.7.3 (Compare [18, Lemma 4.1])**.** *For every $T \in \mathcal{S}^{(2)}$ and $U \subset S$ with $T \subset U$, we have $L^2_{\mathbf{q}}H_4(\Omega(U,T), \partial\Omega(U,T)) = 0$ for $\mathbf{q} \le \mathbf{1}$.*

Proof. Once we establish the lemma for $\mathbf{q} = \mathbf{1}$, we apply the argument in Lemma 3.3.8 to obtain the result for $\mathbf{q} \le \mathbf{1}$ (see Remark 3.3.9).

Assume that $\Omega(U,T)$ contains 4–dimensional cells, otherwise we are done. Then every codimension-one face of a 4–cell in $\Omega(U,T)$ is either free (not the face of another 4–cell) or contained in precisely one other 4–cell (Σ_{cc} is a 4–pseudomanifold).

If every codimension-one face of a 4–cell is free, then this cell has faces not contained in $\partial\Omega(U,T)$. Thus a relative cycle cannot be supported on this cell.

So, we assume that cells of type $T' \in \mathcal{S}(U)^{(4)}_{>T}$ have a codimension-one face of type R that is not free. This face must be contained in another 4–cell of type $T'' \in \mathcal{S}(U)^{(4)}_{>T}$. Thus $T' = R \cup \{t\}$ and $T'' = R \cup \{s\}$ for some $s, t \in S$. Since L is flag and 3–dimensional, $m_{st} = \infty$. Hence we obtain a sequence of adjacent 4–cells $W_{T'}, W_{T''}, sW_{T'}, stW_{T''}, stsW_{T'}, \ldots$. Furthermore, a relative 4–cycle must be constant on adjacent cells of type T' and T'', and since we have an infinite sequence of such adjacent cells, this constant must be zero. ∎

The rest of the argument now follows [18] line by line. We repeat it for the sake of completeness.

Lemma 5.7.4 ([18, Proposition 4.2])**.** *For every $t \in T$ and $U \subset S$ with $t \in U$, we have $L^2_{\mathbf{q}}H_*(\Omega(U,t), \partial\Omega(U,t)) = 0$ for $* > 2$ and $\mathbf{q} \le \mathbf{1}$.*

Proof. The proof is by induction on $\mathrm{Card}(S - U)$, Lemma 5.7.2 serving as the base case. Let $s \in S$ and set $V = U \cup \{s\}$. If $m_{st} = \infty$, then $\Omega(U,t) = \Omega(V,t)$ and we are done by induction. Otherwise, consider the weak exact sequence (see sequence (4.1):

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(U,t),\partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(V,t),\partial) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(V,\{s,t\}),\partial) \longrightarrow \cdots$$

By Lemma 5.7.3, $L^2_{\mathbf{q}}H_4(\Omega(V,\{s,t\}),\partial) = 0$ and by induction, $L^2_{\mathbf{q}}H_*(\Omega(V,t),\partial) = 0$ for $* > 2$. Therefore $L^2_{\mathbf{q}}H_*(\Omega(U,t),\partial) = 0$ for $* > 2$. ∎

Proof of Theorem 5.7.1. For every $U \subset S$ and $t \in U$, we have the following weak exact sequence:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(U-t)) \longrightarrow L^2_{\mathbf{q}}H_*(\Sigma(U)) \longrightarrow L^2_{\mathbf{q}}H_*(\Omega(U,t),\partial) \longrightarrow \cdots$$

By Lemma 5.7.4, $L^2_{\mathbf{q}}H_*(\Omega(U,t),\partial) = 0$ for $* > 2$, and hence by weak exactness,

$$L^2_{\mathbf{q}}H_*(\Sigma(U-t)) \cong L^2_{\mathbf{q}}H_*(\Sigma(U)) \text{ for } * > 2.$$

It follows that $L^2_{\mathbf{q}}H_*(\Sigma(S)) \cong L^2_{\mathbf{q}}H_*(\Sigma(\varnothing))$ for $* > 2$, and hence the theorem. ∎

Chapter 6

The Fattened Davis Complex

We will now construct a complex which is a "fattened" version of the Davis complex. This new thickened complex will be a homology manifold with boundary possessing the Davis complex as a W–equivariant retract. For the remainder of this thesis we suppose that W is an infinite Coxeter group.

6.1 Construction

Given a Coxeter system (W, S), we find a compact P with mirror structure $(P_s)_{s \in S}$ as follows. Let P^* be a cell complex with vertex set S that is a GHS^{n-1}, with $n - 1 > \dim L$, such that the nerve L is a subcomplex of P^*. Take P to be the (P^*, S)–chamber.

Denote by $\mathcal{P}$ the collection of proper nonempty subsets T of S with $P_T \neq \varnothing$. We denote by $\mathcal{N}_P$ the subcollection of $\mathcal{P}$ corresponding to non-spherical subsets. For $T \in \mathcal{P}$, we denote a neighborhood of the face P_T by $N(P_T)$ and the corresponding closed neighborhood by $\bar{N}(P_T)$.

We begin by building a regular neighborhood of ∂P in P. Start by choosing neighborhoods of codimension–n faces so that for any two codimension–n faces P_U and P_V we have $\bar{N}(P_U) \cap \bar{N}(P_V) = \varnothing$. Then we choose neighborhoods of codimension–$(n-1)$ faces so that for any two codimension–$(n-1)$ faces P_U and P_V we have:

$$\bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V). \tag{6.1}$$

If $U \cup V \notin \mathcal{P}$, then we take $N(P_U \cap P_V) = \varnothing$. We proceed inductively, employing condition (6.1) at each step until we obtain the collection $\{N(P_T)\}_{T \in \mathcal{P}}$. This collection gives us a regular neighborhood of ∂P.

Finally, we realize the neighborhoods $\{N(P_T)\}_{T \in \mathcal{P}}$ in the above construction as $\{N_T \times P_T\}_{T \in \mathcal{P}}$, where N_T is a neighborhood of the cone point in Cone(σ_T) and σ_T is the geometric cell in P^* spanned by T (note that we can always do this, see the discussion in Section 2.7.1).

We now define

$$K^f := P - \bigcup_{T \in \mathcal{N}_P} N(P_T).$$

We call K^f the *fattened Davis chamber*.

Note that the mirror structure $(P_s)_{s \in S}$ on P induces a mirror structure $(K_s^f)_{s \in S}$ on K^f. Define $\Phi_L := \mathcal{U}(W, K^f)$. We call Φ_L the *fattened Davis complex*.

Given a $T \in \mathcal{N}_P$, we denote by $K^f(T)$ the fattened Davis chamber corresponding to σ_T and Coxeter system (W_T, T) (recall that the geometric cell σ_T has a natural W_T mirror structure).

Remark 6.1.1. For any Coxeter system (W, S), one can always find a P^* for the above construction: simply let P^* be the boundary of the standard $(|S| - 1)$–dimensional simplex $\Delta^{|S|-1}$. Then P is the barycentric subdivision of $\Delta^{|S|-1}$, and the Davis chamber K can then be viewed as a subcomplex of the barycentric subdivision of P spanned by the barycenters of spherical faces. One can see this using the language of posets. Note that K is the geometric realization of the poset $\mathcal{S}$ and P is the geometric realization of the poset of proper subsets of S. The natural inclusion of posets now induces the desired inclusion of K into P. The mirror structure $(K_s)_{s \in S}$ on K is now induced by the mirror structure $(P_s)_{s \in S}$ on P. In this case $\mathcal{U}(W, P)$ is the traditional Coxeter complex, and we are essentially viewing Σ_L as a subcomplex of the barycentric subdivision of the Coxeter complex.

6.2 Properties of Φ_L

W is assumed to be infinite, so via the choice of P for construction, the Davis chamber is the subcomplex of P spanned by vertices of P corresponding to spherical faces. Hence we have the following inclusions: $K \subset K^f \subset P$ (See Figure 6.1).

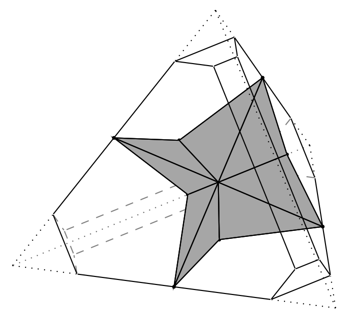


Figure 6.1: $K \subset K^f \subset P$ when $W = D_\infty \times D_\infty$ and $P = \Delta^3$

Note that there is a face preserving deformation retraction of K^f onto K, thus we have the following:

Proposition 6.2.1. *Φ_L W–equivariantly deformation retracts onto Σ_L.*

Proposition 6.2.2. *Φ_L is a locally compact contractible homology n–manifold with boundary $\partial\Phi_L$.*

Proof. Since Σ_L is contractible, it follows from Proposition 6.2.1 that Φ_L is contractible. Moreover, K^f is compact since it is closed in P (P is compact), so Φ_L is locally compact.

As before, give K^f the mirror structure $(K^f_s)_{s\in S}$ induced from P, and declare $K^f_e = \partial K^f - \bigcup_{T\in\mathcal{S}_{>\varnothing}}(K^f_T - \partial K^f_T)$, where e is the identity element of W. According to Proposition 2.5.3, it remains to show that K^f is a partially S–mirrored homology manifold with corners. Let $S' = S \cup \{e\}$ and note that by construction $K^f_T = \varnothing$ if and only if T is not spherical. So, we are done if we show that for every spherical $T \subset S'$, K^f_T has dimension $n - |T|$.

If $e \notin T$, then we are done since $(P_T, \partial P_T)$ is a $GHD^{n-|T|}$. This is because P is by definition the (P^*, S)–chamber and the nerve L was assumed to be a subcomplex of P^*. Hence, since T is spherical, σ_T, the geometric cell in P^* corresponding to T, is a simplex of dimension $|T| - 1$. Therefore the dimension of P_T is equal to $n - \dim \sigma_T - 1 = n - |T|$.

If $e \in T$, then $U = T - \{e\}$ is spherical, and by the above discussion K^f_U has dimension $n - |U| = n - |T| + 1$. Then $K^f_T = K^f_U \cap K^f_e = \partial K^f_U$ has dimension $n - |T|$. ∎

Remark 6.2.3. If $P = \Delta^{|S|-1}$, then the Coxeter complex $\mathcal{U}(W, P)$ is a PL-manifold away from faces with infinite stabilizers. This is because the links of faces corresponding to spherical subsets T are homeomorphic to the Coxeter complex of the corresponding group W_T. Since W_T is finite, this Coxeter complex is homeomorphic to a sphere of appropriate dimension. Since we obtain Φ_L by removing neighborhoods of non-spherical faces (faces with infinite stabilizers), it follows that Φ_L is a PL-manifold with boundary.

6.3 The structure of $\partial\Phi_L$

The main goal of this section is to understand the structure of $\partial\Phi_L$. The first proposition will tell us that ∂K^f can be broken up into pieces, each of which has a nice product structure. This decomposition of ∂K^f then leads us to a cover of $\partial\Phi_L$ which will be used to study the algebraic topology of $\partial\Phi_L$.

For $T \in \mathcal{N}_P$ define

$$C_T = \partial N(P_T) - \bigcup_{U \in \mathcal{N}_P} N(P_U),$$

$$\Lambda_T = P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} N(P_U).$$

Proposition 6.3.1. *(i) Suppose that $U, V \in \mathcal{N}_P$. Then $C_U \cap C_V \neq \varnothing$ if and only if $U \subset V$ or $V \subset U$.*

(ii) If $T \in \mathcal{N}_P$ then

$$C_T \approx K^f(T) \times \Lambda_T.$$

(iii) Suppose that $T_1, T_2 \in \mathcal{N}_P$ with $T_1 \subset T_2$. Then

$$C_{T_1} \cap C_{T_2} \approx K^f(T_1) \times \Lambda_{T_2}.$$

Proof. For (i), one implication is obvious. If $U \subset V$, then P_V is a face of P_U. Thus $C_U \cap C_V \neq \varnothing$. For the reverse implication, suppose that $U \not\subset V$ and $V \not\subset U$. By construction and condition (6.1), either $\bar{N}(P_U) \cap \bar{N}(P_V) = \varnothing$ or $\bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V)$. The former case immediately implies that $C_U \cap C_V = \varnothing$, and the latter case implies that the intersection $\partial N(P_U) \cap \partial N(P_V)$ is removed at some point in the construction of the fattened Davis chamber, hence $C_U \cap C_V = \varnothing$.

For (ii), recall that we have realized the collection $\{N(P_T)\}_{T \in \mathcal{N}_P}$ as neighborhoods $\{N_T \times P_T\}_{T \in \mathcal{N}_P}$, where N_T is a neighborhood of the cone point in $\mathrm{Cone}(\sigma_T)$.

Now, for each $U \subset T$, let α_U denote the face in σ_T corresponding to P_U. More precisely, σ_T has a W_T mirror structure, and α_U is the intersection of mirrors corresponding to $U \subset T$. We can express the neighborhoods in the construction of $K^f(T)$ as neighborhoods $\{\alpha_U \times N'_U\}_{\substack{U \in \mathcal{N}_P \\ U \subset T}}$, where N'_U is a neighborhood of the cone point in $\mathrm{Cone}(\mathrm{Lk}(\alpha_U, \sigma_T))$. Here $\mathrm{Lk}(\alpha_U, \sigma_T)$ denotes the link of the face α_U in σ_T. In particular,

$$K^f(T) = \sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} \alpha_U \times N'_U.$$

Now, we have that $\mathrm{Lk}(\alpha_U, \sigma_T) \approx \sigma_U$, so $N'_U \approx N_U$. Hence

$$K^f(T) \approx \sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U.$$

Moreover, we can write Λ_T and C_T as

$$\Lambda_T = P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U,$$

$$C_T = (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \neq T}} P_U \times N_U.$$

We now show that $C_T \approx K^f(T) \times \Lambda_T$. Note that $K^f(T) \times \Lambda_T = (K^f(T) \times P_T) \cap (\sigma_T \times \Lambda_T)$, so we begin unwinding definitions. We first observe that

$$K^f(T) \times P_T \approx \left(\sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U\right) \times P_T \approx (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U.$$

This is because P_T is a face of each of the P_U's. Similarly, we have

$$\sigma_T \times \Lambda_T = \sigma_T \times \left(P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U\right) \approx (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U.$$

This follows from the fact that P_U's are faces of P_T. Thus we have shown that $K^f(T) \times \Lambda_T = (K^f(T) \times P_T) \cap (\sigma_T \times \Lambda_T) \approx C_T$, therefore proving (ii).

We now prove (iii). By (ii),

$$C_{T_1} \cap C_{T_2} \approx (K^f(T_1) \cap K^f(T_2)) \times (\Lambda_{T_1} \cap \Lambda_{T_2}).$$

It now simply remains to unwind the definitions. Since $T_1 \subset T_2$, it follows that P_{T_2} is a face of P_{T_1}. In particular, $\sigma_{T_1} \cap \sigma_{T_2} = \sigma_{T_1}$ and hence

$$\begin{aligned} K^f(T_1) \cap K^f(T_2) &\approx \sigma_{T_1} \cap \sigma_{T_2} - \bigcup_{\substack{U,V \in \mathcal{N}_P \\ U \subset T_1 \\ V \subset T_2}} N(P_U) \cup N(P_V) \\ &\approx \sigma_{T_1} - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T_1}} N(P_U) \\ &\approx K^f(T_1) \end{aligned}$$

A similar computation shows that $\Lambda_{T_1} \cap \Lambda_{T_2} \approx \Lambda_{T_2}$, thus completing the proof of the proposition. ∎

Proposition 6.3.2.

$$\partial \Phi_L = \bigcup_j \bigsqcup_{T \in \mathcal{N}_P^{(j)}} \mathcal{U}(W, C_T),$$

where $\mathcal{N}_P^{(j)} = \{T \in \mathcal{N}_P \mid \mathit{Card}(T) = j\}$.

Proof. The fact that one can decompose $\partial \Phi_L$ in this way is clear by construction, and the second union is in fact a disjoint union by Proposition 6.3.1 (i). ∎

6.4 Algebraic topology of Φ_L and $\partial\Phi_L$

We now turn our attention to studying the algebraic topology of Φ_L and $\partial\Phi_L$. We first begin with a corollary of Proposition 6.2.1.

Corollary 6.4.1.

$$L^2_{\mathbf{q}}H_*(\Phi_L) \cong L^2_{\mathbf{q}}H_*(\Sigma_L).$$

Not only does Φ_L have the same weighted L^2–(co)homology as Σ_L, but by Proposition 6.2.2, Φ_L is a locally compact homology manifold with boundary. Thus we have weighted Poincaré duality for Φ_L at our disposal. With this in mind, we prove the following lemma.

Lemma 6.4.2. *Suppose that (W, S) is a Coxeter system with $\operatorname{vcd} W = m$ and that Φ_L is a homology n–manifold with boundary with $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$.*

(i) If $n - m = 1$ and $L^2_{\mathbf{q}^{-1}}b_m(\Phi_L) = 0$ then $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$.

(ii) If $n - m \geq 2$ then $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$.

Proof. Consider the long exact sequence for the pair $(\Phi_L, \partial\Phi_L)$:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_1(\partial\Phi_L) \longrightarrow L^2_{\mathbf{q}}H_1(\Phi_L) \longrightarrow L^2_{\mathbf{q}}H_1(\Phi_L, \partial\Phi_L) \longrightarrow \cdots$$

By weighted Poincaré duality

$$L^2_{\mathbf{q}}H_1(\Phi_L, \partial\Phi_L) \cong L^2_{\mathbf{q}^{-1}}H_{n-1}(\Phi_L).$$

Now, by assumption $L^2_{\mathbf{q}}H_1(\partial\Phi_L) = 0$, so by weak exactness we must show that $L^2_{\mathbf{q}^{-1}}H_{n-1}(\Phi_L) = 0$. We will then be done by Corollary 6.4.1, which says that $L^2_{\mathbf{q}}H_1(\Sigma_L) = L^2_{\mathbf{q}}H_1(\Phi_L) = 0$.

For (i), we have that $L^2_{\mathbf{q}^{-1}}b_m(\Phi_L) = 0$. Since $n - m = 1$, we have that $m = n - 1$, so it follows that $L^2_{\mathbf{q}^{-1}}H_{n-1}(\Phi_L) = 0$. For (ii), we have that $n - m \geq 2$, so $n - 1 \geq m + 1$. Since $\operatorname{vcd} W = m$, Corollary 3.3.7 implies that

$$L^2_{\mathbf{q}^{-1}}H_{n-1}(\Sigma_L) = L^2_{\mathbf{q}^{-1}}H_{n-1}(\Phi_L) = 0.$$

∎

We devote the remainder of the section to studying the algebraic topology of $\partial\Phi_L$. The following is a corollary of Proposition 6.3.1.

Corollary 6.4.3. *(i) If $T \in \mathcal{N}_P$, then for every $k \geq 0$*

$$L^2_{\mathbf{q}}b_k(\mathcal{U}(W, C_T)) = L^2_{\mathbf{q}}b_k(\Phi_{L_T}) = L^2_{\mathbf{q}}b_k(\Sigma_{L_T}),$$

where L_T is the subcomplex of L corresponding to the subgroup W_T.

(ii) Suppose that $T_1, T_2 \in \mathcal{N}_P$ with $T_1 \subset T_2$. Then for every $k \geq 0$

$$L^2_{\mathbf{q}}b_k(\mathcal{U}(W, C_{T_1}) \cap \mathcal{U}(W, C_{T_2})) = L^2_{\mathbf{q}}b_k(\Phi_{L_{T_1}}) = L^2_{\mathbf{q}}b_k(\Sigma_{L_{T_1}}),$$

where L_{T_1} is the subcomplex of L corresponding to the subgroup W_{T_1}.

Remark 6.4.4. The $L^2_{\mathbf{q}}$–Betti numbers on the center and the right of the equations in (i) and (ii) are computed with respect to the special subgroups W_T (respectively W_{T_1}) of W, while the ones on the far left side of the equations are computed with respect to W.

Proof. We prove only (i) as the proof of (ii) is similar. Proposition 6.3.1 implies that $C_T \approx K^f(T) \times \Lambda_T$ as mirrored spaces, where Λ_T is contractible and has no mirror structure. Therefore $\mathcal{U}(W, C_T)$ is W–equivariantly homotopy equivalent to $\mathcal{U}(W, K^f(T))$. Now, $L^2_{\mathbf{q}}H_*(\mathcal{U}(W, K^f(T)))$ is just the completion of

$$L^2_{\mathbf{q}}(W) \otimes_{\mathbb{R}_{\mathbf{q}}(W_T)} L^2_{\mathbf{q}}H_*\left(\mathcal{U}(W_T, K^f(T))\right),$$

so for every $k \geq 0$,

$$L^2_{\mathbf{q}}b_k\left(\mathcal{U}(W, K^f(T))\right) = L^2_{\mathbf{q}}b_k\left(\mathcal{U}(W_T, K^f(T))\right) = L^2_{\mathbf{q}}b_k(\Phi_{L_T}).$$

∎

Consider the cover $\mathcal{V} = \{\mathcal{U}(W, C_T)\}_{T \in \mathcal{N}_P}$ of $\partial\Phi_L$ in Proposition 6.3.2. The cover $\mathcal{V}$ will have intersections of variable depth, so we obtain a spectral sequence following [3, Ch. VII, §3,4]:

Proposition 6.4.5. *There is a Mayer–Vietoris type spectral sequence converging to $H_*^W(\partial\Phi_L, \mathcal{N}_{\mathbf{q}}(W))$ with E_1–term:*

$$E_1^{i,j} = \bigoplus_{\substack{\sigma\in\mathrm{Flag}(\mathcal{N}_P)\\ \dim\sigma=i}} H_j^W(\mathcal{U}(W, C_{\min\sigma}), \mathcal{N}_{\mathbf{q}}(W)).$$

Proof. Let $N(\mathcal{V})$ denote the nerve of the cover $\mathcal{V}$. It is the abstract simplicial complex whose vertex set is $\mathcal{N}_P$ and whose simplices are the non-empty subsets $\sigma \subset \mathcal{N}_P$ such that the intersection $V_\sigma = \bigcap_{T\in\sigma}\mathcal{U}(W, C_T)$ is non-empty. Following [3, Ch. VII, §3,4], there is a Mayer–Vietoris type spectral sequence converging to $H_*^W(\partial\Phi_L, \mathcal{N}_{\mathbf{q}}(W))$ with E_1–term:

$$E_1^{i,j} = \bigoplus_{\substack{\sigma\in N(\mathcal{V})\\ \dim\sigma=i}} H_j^W(V_\sigma, \mathcal{N}_{\mathbf{q}}(W)).$$

We have that $V_\sigma \neq \varnothing$ if and only if $\bigcap_{T\in\sigma} C_T \neq \varnothing$, and applying Proposition 6.3.1 inductively, this happens if and only if the vertices of σ form a chain $T_{i_1} \subset T_{i_2} \subset \cdots \subset T_{i_k}$. This observation shows that $N(\mathcal{V}) = \mathrm{Flag}(\mathcal{N}_P)$. Now, applying Proposition 6.3.1 inductively, it follows that $V_\sigma \approx \mathcal{U}(W, C_{T_{i_1}})$. Hence $H_*^W(V_\sigma, \mathcal{N}_{\mathbf{q}}(W)) = H_*^W(\mathcal{U}(W, C_{T_{i_1}}), \mathcal{N}_{\mathbf{q}}(W))$, so the terms in the spectral sequence are the ones claimed. ∎

For later computations, note that Corollary 6.4.3 implies:

$$\begin{aligned} L^2_{\mathbf{q}} b_*(\mathcal{U}(W, C_{\min\sigma})) &= \dim_{\mathcal{N}_{\mathbf{q}}} H_*^W(\mathcal{U}(W, C_{\min\sigma})) \\ &= L^2_{\mathbf{q}} b_*(\Phi_{L_{\min\sigma}}) \\ &= L^2_{\mathbf{q}} b_*(\Sigma_{L_{\min\sigma}}). \end{aligned}$$

Chapter 7

Computations

In this section we will use the fattened Davis complex to make concrete computations. We first begin by considering the case where the nerve L of the Coxeter system (W,S) is a graph. Note that for this special case Σ_L is two-dimensional. We then briefly discuss how we can use our computations to produce examples of Coxeter groups for which the Weighted Singer Conjecture holds. We then direct our attention to quasi-Lánner groups, and finish with computations for 2–spherical Coxeter groups whose corresponding nerves are no longer restricted to be graphs.

Let K_n denote the complete graph on n vertices. Recall that a Coxeter system is 2*–spherical* if the one-skeleton of its nerve is K_n for some n. For the purpose of figures and examples, we will distinguish the special case where the labeled nerve $L = K_n(3)$, where $K_n(3)$ denotes the complete graph on n vertices with every edge labeled by 3.

Unless stated otherwise, the standing assumption in this chapter is that $\mathbf{q} \geq \mathbf{1}$.

7.1 The case where L is a graph

Suppose that the labeled nerve L is the one-skeleton of an n–dimensional cell complex Λ, where $n \geq 2$. We say that a 2–cell of Λ is *Euclidean* if the corresponding special subgroup generated by the vertices of that cell is a Euclidean reflection group. Note that the only possible labels on a Euclidean cell are $m_{st} \in \{2,3,4,6\}$.

Before proving the main theorem of this section, we begin with a lemma. The special case of the lemma when $\mathbf{q} = \mathbf{1}$ is closely related to a result of Schroeder [19, Theorem 4.6]. We provide an argument which is analogous to that of Schroeder in his proof.

Lemma 7.1.1. *Suppose that the labeled nerve L is the one-skeleton of a cellulation of S^2. Then*

$$L^2_{\mathbf{q}} b_2(\Sigma_L) = 0 \textit{ for } \mathbf{q} \leq \mathbf{1}.$$

Proof. In light of Lemma 3.3.8, we must show that $L^2_{\mathbf{1}} b_2(\Sigma_L) = 0$. We begin by building L to a triangulation of S^2 by coning on empty 2–cells and labeling the new edges by 2's, at each step keeping track of the $L^2_{\mathbf{1}}$–(co)homology with a Mayer–Vietoris sequence. More precisely, start with $T_1 \subset S$ corresponding to an empty 2–cell L_{T_1} in L and denote by CL_{T_1} the right-angled cone on L_{T_1}. The corresponding special subgroup W_{T_1} is infinite, and it acts properly and cocompactly by reflections on either $\mathbb{R}^2$ or $\mathbb{H}^2$. In both cases $L^2_{\mathbf{1}} H_2(\Sigma_{L_{T_1}}) = 0$ and hence the Künneth formula implies that $L^2_{\mathbf{1}} H_2(\Sigma_{CL_{T_1}}) = 0$. We have the following Mayer–Vietoris sequence:

$$\cdots \longrightarrow L^2_{\mathbf{1}} H_2(\Sigma_{L_{T_1}}) \longrightarrow L^2_{\mathbf{1}} H_2(\Sigma_{CL_{T_1}}) \oplus L^2_{\mathbf{1}} H_2(\Sigma_L) \xrightarrow{f_1} L^2_{\mathbf{1}} H_2(\Sigma_{L \cup CL_{T_1}}) \longrightarrow \cdots$$

In particular, the map f_1 is injective. We then choose another $T_2 \subset S$ corresponding to an empty 2–cell L_{T_2} in L and denote by CL_{T_2} the right-angled cone on L_{T_2}. By a similar argument, the map f_2 in the following Mayer–Vietoris sequence is injective:

$$\cdots \longrightarrow L^2_{\mathbf{1}} H_2(\Sigma_{CL_{T_2}}) \oplus L^2_{\mathbf{1}} H_2(\Sigma_{L \cup CL_{T_2}}) \xrightarrow{f_2} L^2_{\mathbf{1}} H_2(\Sigma_{L \cup CL_{T_1} \cup CL_{T_2}}) \longrightarrow \cdots$$

Proceed inductively until all empty 2–cells have been coned off and denote the newly promoted nerve by L'. The f_i's yield a sequence of injective maps:

$$L^2_{\mathbf{1}} H_2(\Sigma_L) \hookrightarrow L^2_{\mathbf{1}} H_2(\Sigma_{L \cup CL_{T_1}}) \hookrightarrow \cdots \hookrightarrow L^2_{\mathbf{1}} H_2(\Sigma_{L'})$$

Since L' is a triangulation of S^2, it follows that $\Sigma_{L'}$ is a 3–manifold. Now, a result of Lott and Luck [14], in conjunction with the validity of the Geometrization Conjecture for 3–manifolds [17], implies that $L^2_{\mathbf{1}}H_*(\Sigma_{L'})$ vanishes in all dimensions. In particular, $L^2_{\mathbf{1}}b_2(\Sigma_L) = 0$. ∎

Remark 7.1.2. Schroeder proves a more general theorem for $\mathbf{q} = \mathbf{1}$ [19, Theorem 4.6]. A metric flag complex L is *planar* if it can be embedded as a proper subcomplex of a triangulation of the 2–sphere (see the discussion before Theorem 5.3.3 for the definition of metric flag complex). Schroeder proves that if the nerve L of a Coxeter system is planar, then $L^2_{\mathbf{1}}b_k(\Sigma_L) = 0$ for $k \geq 2$. If L is planar and W is the corresponding Coxeter group, then Proposition 2.9.1 implies that $\operatorname{vcd} W \leq 2$. Therefore we can use Lemma 3.3.8 to deduce that $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ for $k \geq 2$ and $\mathbf{q} \leq \mathbf{1}$.

Theorem 7.1.3. *Suppose that the labeled nerve L is the one-skeleton of a cell complex that is a GHS^n, $n \geq 2$, where all 2–cells are Euclidean, and let (W,S) denote the corresponding Coxeter system. Then $L^2_{\mathbf{q}}b_*(\Sigma_L)$ is concentrated in degree 2.*

Furthermore,

$$L^2_{\mathbf{q}}b_2(\Sigma_L) = 1 - \sum_{s\in S}\frac{q_s}{1+q_s} + \sum_{\substack{s,t\in S\\ m_{st}=2}}\frac{q_sq_t}{1+q_s+q_t+q_sq_t} + \sum_{\substack{s,t\in S\\ m_{st}=3}}\frac{q_s^3}{1+2q_s+2q_s^2+q_s^3}+$$
$$+\sum_{\substack{s,t\in S\\ m_{st}=4}}\frac{q_s^2q_t^2}{1+q_s+q_t+2q_sq_t+q_s^2q_t+q_sq_t^2+q_s^2q_t^2}+$$
$$+\sum_{\substack{s,t\in S\\ m_{st}=6}}\frac{q_s^3q_t^3}{1+q_s+q_t+2q_sq_t+q_s^2q_t+q_sq_t^2+2q_s^2q_t^2+q_s^2q_t^3+q_s^3q_t^2+q_s^3q_t^3}.$$

Proof. Proposition 3.3.1 implies that $L^2_{\mathbf{q}}b_0(\Sigma_L) = 0$. Proposition 3.3.2, along with Theorem 2.2.1, explicitly compute the formula for $L^2_{\mathbf{q}}b_2(\Sigma_L)$. We now turn our attention to showing $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$. For the construction of the fattened Davis complex, we will use the given cell complex as P^*.

We prove the theorem by induction on n. For the base case $n = 2$, first note that for every $T \in \mathcal{N}_P$, σ_T is Euclidean. Hence Proposition 6.3.1 implies that each C_T appearing in ∂K^f corresponds to a set $T \in \mathcal{N}_P$ where W_T is Euclidean reflection group. Thus Corollary 6.4.3 and Theorem 3.3.4 imply that $L^2_{\mathbf{q}}b_1(\mathcal{U}(W, C_T)) = 0$. This and Proposition 3.3.1 imply that the $E_1^{0,1}$ and $E_1^{1,0}$ terms in the E_1 sheet of the spectral sequence in Proposition 6.4.5 are zero, which in turn implies that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. Now, note that Φ_L is three-dimensional and $\operatorname{vcd} W = 2$. Moreover, by Lemma 7.1.1, $L^2_{\mathbf{q}^{-1}}H_2(\Sigma_L) = 0$. Therefore, via Lemma 6.4.2 (i), we reach the conclusion that $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$.

Now, suppose the theorem is true for $m < n$. Since Σ_L is two-dimensional, Lemma 6.4.2 (ii) tells us that we are done if we show that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. Let $T \in \mathcal{N}_P$. Then σ_T is the $(\partial\sigma_T, T)$–chamber, where σ_T is the geometric cell in P^* spanned by T. In particular, $\partial\sigma_T$ is a cell complex that is GHS^m, $m < n$, and since all 2–cells of P^* are Euclidean, it follows that all 2–cells of $\partial\sigma_T$ are Euclidean. Hence, by induction and Corollary 6.4.3, it follows that for every $T \in \mathcal{N}_P$, $L^2_{\mathbf{q}}b_1(\mathcal{U}(W, C_T)) = L^2_{\mathbf{q}}b_1(\Sigma_{L_T}) = 0$. This and Proposition 3.3.1 imply that the $E_1^{0,1}$ and $E_1^{1,0}$ terms in the E_1 sheet of the spectral sequence in Proposition 6.4.5 are zero, which in turn implies that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. ■

Consider the special case of Theorem 7.1.3 when $n = 2$. In this case, Theorem 7.1.3, along with Lemma 7.1.1, explicitly compute the $L^2_{\mathbf{q}}$–Betti numbers for all $\mathbf{q}$: they are always concentrated in a single dimension. We emphasize this in the following corollary.

Corollary 7.1.4. *Suppose that the labeled nerve L is the one-skeleton of a cell complex that is a GHS^2, where all 2–cells are Euclidean.*

- *If $\mathbf{q} \in \bar{\mathcal{R}}$, then $L^2_{\mathbf{q}}H_*(\Sigma_L)$ is concentrated in dimension 0.*
- *If $\mathbf{q} \notin \mathcal{R}$ and $\mathbf{q} \le \mathbf{1}$, then $L^2_{\mathbf{q}}H_*(\Sigma_L)$ is concentrated in dimension 1.*
- *If $\mathbf{q} \ge \mathbf{1}$, then $L^2_{\mathbf{q}}H_*(\Sigma_L)$ is concentrated in dimension 2.*

Furthermore,

$$\chi_{\mathbf{q}}(\Sigma_L) = 1 - \sum_{s\in S} \frac{q_s}{1+q_s} + \sum_{\substack{s,t\in S\\ m_{st}=2}} \frac{q_s q_t}{1+q_s+q_t+q_s q_t} + \sum_{\substack{s,t\in S\\ m_{st}=3}} \frac{q_s^3}{1+2q_s+2q_s^2+q_s^3} +$$
$$+ \sum_{\substack{s,t\in S\\ m_{st}=4}} \frac{q_s^2 q_t^2}{1+q_s+q_t+2q_s q_t+q_s^2 q_t+q_s q_t^2+q_s^2 q_t^2} +$$
$$+ \sum_{\substack{s,t\in S\\ m_{st}=6}} \frac{q_s^3 q_t^3}{1+q_s+q_t+2q_s q_t+q_s^2 q_t+q_s q_t^2+2q_s^2 q_t^2+q_s^2 q_t^3+q_s^3 q_t^2+q_s^3 q_t^3}.$$

If we place some restrictions on either our labels or the cell complex, then the formulas in Theorem 7.1.3 become relatively simple, as illustrated by the following corollaries.

Corollary 7.1.5. *Suppose that L is the one-skeleton of a cell complex that is a GHS^n, $n \geq 2$, where all 2–cells are 2–simplices. Give L the labels $m_{st} = 3$. Then $L^2_q b_*(\Sigma_L)$ is concentrated in degree 2.*

Furthermore,

$$L^2_q b_2(\Sigma_L) = 1 - \frac{Vq}{1+q} + \frac{Eq^3}{1+2q+2q^2+q^3},$$

where V and E are the number of vertices and edges of L, respectively.

Recall that an n–dimensional octahedron has $2n$ vertices and $2n(n-1)$ edges.

Corollary 7.1.6. *Suppose that L the one skeleton of an n–dimensional octahedron with $n \geq 3$ and the labels $m_{st} = 3$. Then $L^2_{\mathbf{q}} b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_q b_2(\Sigma_L) = 1 - \frac{2nq}{1+q} + \frac{2n(n-1)q^3}{(1+2q+2q^2+q^3)}.$$

Corollary 7.1.7. *Let $L = K_n(3)$ with $n \geq 3$. Then $L^2_{\mathbf{q}} b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_q b_2(\Sigma_L) = 1 - \frac{nq}{1+q} + \frac{n(n-1)q^3}{2(1+2q+2q^2+q^3)}.$$

Remark 7.1.8. Note that under the hypothesis of the above corollaries, all generators in S are conjugate, so in this case $\mathbf{q} = q$, where $q \geq 1$ is a positive real number.

If we assume that W is right-angled, we have the following consequences of Theorem 7.1.3.

Corollary 7.1.9. *Suppose that L is the one-skeleton of a cell complex that is a GHS^n, $n \geq 2$, where all 2–cells are 2–cubes. Give L the labels $m_{st} = 2$. Then $L^2_{\mathbf{q}}b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_{\mathbf{q}}b_2(\Sigma_L) = 1 - \sum_{s\in S} \frac{q_s}{1+q_s} + \sum_{\{s,t\}\in\mathcal{S}} \frac{q_s q_t}{1+q_s+q_t+q_s q_t}.$$

Analogous to the case where $L = K_n(3)$, let $C_n(2)$ denote the one-skeleton of an n–cube with edges labeled by 2. If we assume that $L = C_n(2)$ and that $\mathbf{q} = q$, where q is a positive real number, then we obtain simple formulas for the $L^2_{\mathbf{q}}$–Betti numbers. Recall that an n–cube has 2^n vertices and $n2^{n-1}$ edges.

Corollary 7.1.10. *Let $L = C_n(2)$ with $n \geq 2$. Then $L^2_q b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_q b_2(\Sigma_L) = 1 - \frac{2^n q}{1+q} + \frac{n2^{n-1}q^2}{1+2q+q^2}.$$

We can also allow ourselves to remove some edges from $L = K_n(3)$. We denote by $K^l_n(3)$ the complete graph on n vertices, labeled by 3's and with l edges removed. We have the following consequence of Corollary 7.1.7.

Corollary 7.1.11. *Suppose that $L = K^l_n(3)$, where $n \geq 5$ and $l \leq n-4$. Then $L^2_q b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_q b_2(\Sigma_L) = 1 - \frac{nq}{1+q} + \frac{n(n-1)q^3}{2(1+2q+2q^2+q^3)} - \frac{lq^3}{1+2q+2q^2+q^3}.$$

Proof. We first note that removing an edge from $K_n(3)$ splits the graph into two copies of $K_{n-1}(3)$ intersecting at $K_{n-2}(3)$. Since $n \geq 5$ and $q \geq 1$ we have the following Mayer–Vietoris sequence:

$$\cdots \longrightarrow L^2_q H_1(\Sigma_{K_{n-2}}) \longrightarrow L^2_q H_1(\Sigma_{K_{n-1}}) \oplus L^2_q H_1(\Sigma_{K_{n-1}}) \longrightarrow L^2_q H_1(\Sigma_{K^1_n}) \longrightarrow 0 \qquad (\star)$$

We first handle the case where $L = K_5^1(3)$. Removing an edge from $K_5(3)$ splits the graph into two copies of $K_4(3)$ intersecting at $K_3(3)$. Corollary 7.1.7 computes the L_q^2–(co)homology of each of the pieces in this decomposition and applying the sequence $(\star)$ now proves the assertion for the case $L = K_5^1(3)$.

The proof for $L = K_n^l(3)$ is now by induction, the above computation serving as the base case. Suppose that the theorem is true for $m < n$. Begin by removing an edge from $K_n(3)$, splitting it as two copies of $K_{n-1}(3)$ intersecting at $K_{n-2}(3)$. Now, we remove the remaining $l-1 \leq n-5$ edges from each of the graphs in the splitting, the worst case scenario being that we remove $l-1$ edges from $K_{n-2}(3)$ (which in turn removes $l-1$ edges from each copy of $K_{n-1}(3)$). Nevertheless, the inductive hypothesis is satisfied for each of the K_{n-1}'s in the splitting no matter how the remaining edges are removed. Applying a Mayer–Vietoris sequence analogous to $(\star)$ now shows that the theorem holds for $L = K_n^l(3)$. ∎

With the help of ruins (see Section 4.2), we are also able to make computations when we change some labels on $L = K_n(3)$.

Theorem 7.1.12. *Let $L = K_n$, the complete graph on n vertices with $n \geq 5$. Let $k \leq n-4$, and suppose that we label k edges of L with $m_{st} \in \mathbb{N} - \{1,3\}$ and label the remaining edges by 3. Then $L^2_{\mathbf{q}}b_*(\Sigma_L)$ is concentrated in degree 2.*

Proof. The proof is by induction on n. First consider the case where $L = K_5$ with one label $m_{st} \in \mathbb{N} - \{1,3\}$. Then by Corollary 7.1.7, $L^2b_1(\Sigma(S-s)) = L^2b_1(\Sigma_{K_4(3)}) = 0$. According to sequence (4.1), it remains to show that $L^2_{\mathbf{q}}H_1(\Omega_{S\{s\}}, \partial) = 0$. We turn our attention to sequence (4.1) with $U = S$, $T = \{s,t\}$, $U' = S - t$, and $T' = \{s\}$. By Proposition 4.2.1, $L^2_{\mathbf{q}}H_1(\Omega_{ST}, \partial) = 0$, the point being that the relative chain complex of $(\Omega_{ST}, \partial\Omega_{ST})$ has no one-dimensional cells. So, by weak exactness, it remains to show that $L^2_{\mathbf{q}}H_1(\Omega_{U'T'}, \partial) = 0$. We consider the following weak exact sequence:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_1(\Sigma(S-\{s,t\})) \longrightarrow L^2_{\mathbf{q}}H_1(\Sigma(S-t)) \longrightarrow L^2_{\mathbf{q}}H_1(\Omega_{U'T'}, \partial) \longrightarrow \cdots$$

Note that

$$L^2_{\mathbf{q}}b_0(\Sigma(S-\{s,t\})) = L^2_{\mathbf{q}}b_0(\Sigma_{K_3(3)}) = 0$$

and

$$L^2_{\mathbf{q}} b_1(\Sigma(S - t)) = L^2_{\mathbf{q}} b_1(\Sigma_{K_4(3)}) = 0$$

by Theorem 3.3.4 and Corollary 7.1.7, respectively. By weak exactness, $L^2_{\mathbf{q}} H_1(\Omega_{U'T'}, \partial) = 0$, and hence $L^2_{\mathbf{q}} H_1(\Omega_{S\{s\}}, \partial) = 0$, thus proving the assertion for $L = K_5$.

Now, suppose that the theorem is true for $L = K_m$, $m < n$. We wish to show the theorem is true for $L = K_n$. Begin by choosing an edge e with vertices s and t and label different from 3. We now observe that $L^2 b_1(\Sigma(S - s)) = L^2 b_1(\Sigma_{K_{n-1}}) = 0$ by the inductive hypothesis, since K_{n-1} now has at most $n - 5$ edges with a label different from 3. Similarly, the inductive hypothesis implies $L^2_{\mathbf{q}} b_1(\Sigma(S - t)) = 0$ and $L^2_{\mathbf{q}} b_0(\Sigma(S - \{s, t\})) = 0$. Hence the weak exact sequences used in the proof for the case $L = K_5$ allow us to conclude that $L^2_{\mathbf{q}} b_1(\Sigma_L) = L^2_{\mathbf{q}} b_1(\Sigma(S)) = 0$. ∎

Remark 7.1.13. Note that in conjunction with Theorem 3.3.4 and Corollary 7.1.4, the above argument gives an alternate proof of Corollary 7.1.7.

7.2 Connection to the Weighted Singer Conjecture

We note that Theorem 7.1.3 provides convincing evidence for the validity of a weighted version of Theorem 5.0.2 when L is a triangulation of the $(n-1)$–sphere. Suppose that the labeled nerve L' is the one-skeleton of a cellulation of a GHS^{n-1}, $n \geq 3$, where all 2–cells are Euclidean. Build L' to a triangulation that is a GHS^{n-1} by coning on each empty cell and labeling new edges by 2. In other words, perform the following sequence of right-angled cones. First begin by coning on each empty 2–cell, then on each empty 3–cell, and so on, until each empty cell has been coned off (if $n = 3$, this process stops when each empty 2–cell has been coned off).

Theorem 7.2.1. *Suppose that the nerve L a GHS^{n-1}, $n \geq 3$, obtained via the above construction and suppose that $\mathbf{q} \geq \mathbf{1}$. Then*

$$L^2_{\mathbf{q}} b_k(\Sigma_L) = 0 \textit{ for } k \leq 1.$$

Proof. The proof of the theorem follows the strategy of Lemma 7.1.1: one performs careful book-keeping using Mayer–Vietoris sequences when constructing L from L'. Theorem 7.1.3 tells us that L' originally satisfies $L^2_{\mathbf{q}}b_1(\Sigma_{L'}) = 0$. To construct L from L', we first began by coning empty 2–cells, then successively coning higher dimensional cells, labeling new edges by 2. If at each step of this process we employ a Mayer–Vietoris sequence, then Theorem 7.1.3, in conjunction with the fact that right-angled cones will not develop new homology below dimension 2, implies that $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$. ∎

7.3 Quasi-Lánner groups

A 2–spherical Coxeter group W is *quasi-Lánner* if it acts properly (but not co-compactly) on hyperbolic space $\mathbb{H}^n$ by reflections with fundamental chamber an n–simplex of finite volume. For brevity, we say that W is of type QL_n. Quasi-Lánner groups have been classified and only exist in dimensions 3 through 10. For a complete list, see [13, §6.9]. We note that the Coxeter group with corresponding nerve $L = K_4(3)$ is on the list.

All non-spherical proper special subgroups of a quasi-Lánner group are Euclidean and on the list appearing in [13, pg. 34]. Moreover, if W is of type QL_n, then the only proper infinite special subgroups are those W_T with $|T| = n-1$. Hence, by Proposition 2.9.1, if W is of type QL_n, then $\operatorname{vcd} W = n-1$. With this observation, we prove the following theorem.

Theorem 7.3.1. *Suppose that W is of type QL_n. Then $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ whenever $k \geq n-1$ and $\mathbf{q} \leq \mathbf{1}$, or $k \leq 1$ and $\mathbf{q} \geq \mathbf{1}$.*

Proof. We first suppose that $\mathbf{q} = \mathbf{1}$. Since W is of type QL_n, we can realize a finite volume n–simplex in hyperbolic space $\mathbb{H}^n$, with W acting by reflections along codimension-one faces (note that this simplex has some ideal vertices). By a theorem of Cheeger–Gromov [5], $L^2_{\mathbf{1}}H_k(\Sigma_L) \cong L^2_{\mathbf{1}}\mathcal{H}^k(\mathbb{H}^n)$, where $L^2_{\mathbf{1}}\mathcal{H}^k$ denotes the L^2 de Rham cohomology. By a theorem of Dodziuk [11], $L^2_{\mathbf{1}}\mathcal{H}^k(\mathbb{H}^n) = 0$ for all $k \geq 0$ if n is odd, and is concentrated in dimension $\frac{n}{2}$ if n is even. In particular, $L^2_{\mathbf{1}} b_{n-1}(\Sigma_L) = 0$. The result for $\mathbf{q} \leq \mathbf{1}$ now follows by Lemma 3.3.8 and the fact that $\operatorname{vcd} W = n - 1$.

Now, suppose that $\mathbf{q} \geq \mathbf{1}$. Consider the fattened Davis complex Φ_L with respect to $P = \Delta^n$, the standard n–simplex (see Remark 6.1.1 and Figure 7.1).

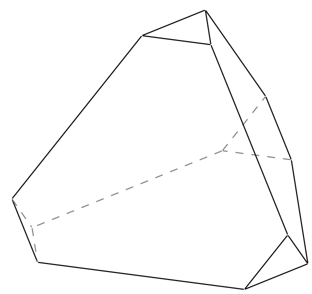


Figure 7.1: K^f when $L = K_4(3)$

Weighted Poincaré duality implies that

$$L^2_{\mathbf{q}} H_1(\Phi_L, \partial\Phi_L) \cong L^2_{\mathbf{q}^{-1}} H_{n-1}(\Phi_L) \cong L^2_{\mathbf{q}^{-1}} H_{n-1}(\Sigma_L) = 0,$$

so by the long exact sequence for the pair $(\Phi_L, \partial\Phi_L)$ it remains to show $L^2_{\mathbf{q}} H_1(\partial\Phi_L) = 0$. Proposition 6.3.2 implies that each C_T appearing in ∂K^f corresponds to a set $T \in \mathcal{N}_P$ with W_T a Euclidean reflection group. In particular, Corollary 6.4.3 and Theorem 3.3.4 imply that $L^2_{\mathbf{q}} b_1(\mathcal{U}(W, C_T)) = 0$. Hence the $E_1^{0,1}$ term in the E_1 sheet of the spectral sequence of Proposition 6.4.5 is zero. By Proposition 3.3.1, the first row of the E_1 sheet is also zero, and in particular $E_1^{1,0}$ is zero. Therefore $L^2_{\mathbf{q}} b_1(\partial\Phi_L) = 0$. ■

Of important note is the case when W is QL_3. In this special case, Theorem 7.3.1 explicitly computes the $L^2_{\mathbf{q}}$–Betti numbers for all $\mathbf{q}$: they are always concentrated in a single dimension.

Corollary 7.3.2. *Suppose that W is of type QL_3. Then*

- *If $\mathbf{q} \in \bar{\mathcal{R}}$, then $L^2_{\mathbf{q}} H_*(\Sigma_L)$ is concentrated in dimension* 0.

- *If* $\mathbf{q} \notin \mathcal{R}$ *and* $\mathbf{q} \leq \mathbf{1}$*, then* $L^2_{\mathbf{q}}H_*(\Sigma_L)$ *is concentrated in dimension* 1*.*
- *If* $\mathbf{q} \geq \mathbf{1}$*, then* $L^2_{\mathbf{q}}H_*(\Sigma_L)$ *is concentrated in dimension* 2*.*

Since the $L^2_{\mathbf{q}}$–(co)homology is always concentrated in a single dimension, one can use Proposition 3.3.2, along with Theorem 2.2.1, to obtain explicit formulas for the $L^2_{\mathbf{q}}$–Betti numbers.

7.4 Other 2–spherical groups

We now perform computations for other 2–spherical groups, removing the restriction that the nerve L is a graph. Given a Coxeter system (W, S), we make a particular choice of P for the construction of Φ_L, namely $P = \Delta^{|S|-1}$, the standard $(|S|-1)$–simplex (see Remark 6.1.1).

While one could argue the following lemma using the spectral sequence, we use a simple Mayer–Vietoris sequence argument to illustrate the technique behind the machinery.

Lemma 7.4.1. *Suppose that* (W, S) *is infinite* 2*–spherical with* $|S| = 5$ *and* $\operatorname{vcd} W \leq 3$*. Furthermore, suppose that every infinite special subgroup* W_T*, with* $|T| = 3, 4$*, is Euclidean or* QL_3*, and that* $L^2_{\mathbf{1}}b_3(\Sigma_L) = 0$*. Then* $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ *for* $k < 2$*.*

Proof. We wish to reduce the proof to showing that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. If $\operatorname{vcd} W = 2$, then this is accomplished by Lemma 6.4.2 (ii). If $\operatorname{vcd} W = 3$, then according to Lemma 6.4.2 (i), we reduce the proof to showing $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$ if we show that $L^2_{\mathbf{q}^{-1}}b_3(\Sigma_L) = 0$. By Lemma 3.3.8, we reach this conclusion since by assumption $L^2_{\mathbf{1}}b_3(\Sigma_L) = 0$. So, to complete the proof, we must show that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$.

Let $\mathcal{N}_P^{(j)} = \{T \in \mathcal{N}_P \mid \operatorname{Card}(T) = j\}$ and set

$$A_j = \bigsqcup_{T \in \mathcal{N}_P^{(j)}} \mathcal{U}(W, C_T).$$

Note that $|S| = 5$ and all proper non-spherical subsets T have order 3 or 4, so by Proposition 6.3.2, $\partial\Phi_L = A_3 \cup A_4$. Figure 7.2 illustrates the chamber for $\partial\Phi_L$ for the case where $L = K_5(3)$.

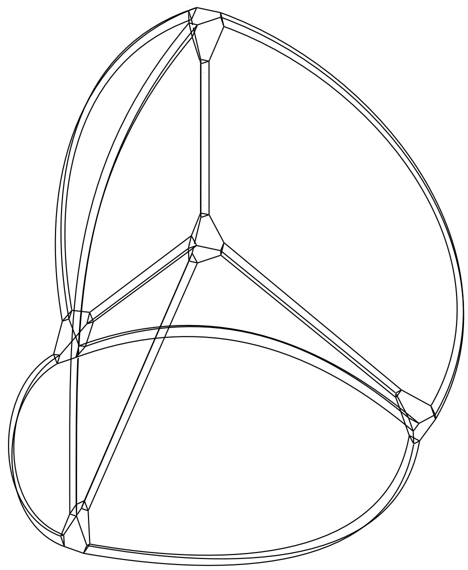


Figure 7.2: Fundamental chamber for $\partial\Phi_L$ when $L = K_5(3)$

By Proposition 6.3.1 (i),

$$A_3 \cap A_4 = \bigsqcup_{\substack{U \in \mathcal{N}_P^{(3)} \\ V \in \mathcal{N}_P^{(4)} \\ U \subset V}} \mathcal{U}(W, C_U) \cap \mathcal{U}(W, C_V)$$

By Corollary 6.4.3,

$$\begin{aligned} L^2_{\mathbf{q}} b_k(A_j) &= \sum L^2_{\mathbf{q}} b_k(\mathcal{U}(W, C_T)) \\ &= \sum L^2_{\mathbf{q}} b_k(\Phi_{L_T}) \end{aligned}$$

and

$$\begin{aligned} L^2_{\mathbf{q}} b_k(A_3 \cap A_4) &= \sum L^2_{\mathbf{q}} b_k(\mathcal{U}(W, C_T)) \\ &= \sum L^2_{\mathbf{q}} b_k(\Phi_{L_T}) \end{aligned}$$

Here L_T is the subcomplex of L corresponding to the infinite subgroup W_T, which is either Euclidean or of type QL_3. By Theorem 3.3.4 and Theorem 7.3.1,

$L^2_{\mathbf{q}}b_k(\Phi_{L_T}) = 0$ for $k < 2$. Hence

$$L^2_{\mathbf{q}}H_k(A_j) = 0 \text{ for } j = 3,4 \text{ and } L^2_{\mathbf{q}}H_k(A_3 \cap A_4) = 0 \text{ for } k < 2. \qquad (\diamond)$$

Now, consider the Mayer–Vietoris sequence:

$$\cdots \longrightarrow L^2_{\mathbf{q}}H_k(A_3 \cap A_4) \longrightarrow L^2_{\mathbf{q}}H_k(A_3) \oplus L^2_{\mathbf{q}}H_k(A_4) \longrightarrow L^2_{\mathbf{q}}H_k(\partial\Phi_L) \longrightarrow \cdots$$

Inputting $(\diamond)$ into this sequence yields

$$L^2_{\mathbf{q}}H_k(\partial\Phi_L) = 0 \text{ for } k < 2.$$

Lemma 6.4.2 now concludes that $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$. ∎

Theorem 7.4.2. *Suppose that (W,S) is infinite 2–spherical with $|S| \geq 5$. Suppose furthermore that:*

1. *For every $T \subseteq S$ with $|T| \geq 5$, $\operatorname{vcd} W_T \leq |T| - 2$.*
2. *$L^2_{\mathbf{1}}b_{|S|-2}(\Sigma_L) = 0$.*
3. *Every infinite subgroup W_T, with $|T| = 3,4$, is Euclidean or QL_3.*

Then $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ for $k < 2$.

Proof. The statement for $L^2_{\mathbf{q}}b_0(\Sigma_L)$ follows from Proposition 3.3.1. So, we turn our attention to showing $L^2_{\mathbf{q}}b_1(\Sigma_L) = 0$. The proof of the theorem is now by induction on $|S|$, Lemma 7.4.1 serving as the base case. By Lemma 3.3.8, since $\operatorname{vcd} W \leq |S|-2$, it follows that $L^2_{\mathbf{q}^{-1}}b_{|S|-2}(\Sigma_L) = 0$. Furthermore, Φ_L has dimension $|S| - 1$, so by Lemma 6.4.2 it now suffices to show that $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. By assumption, every non-spherical special subgroup W_U with $|U| = 3,4$ is Euclidean or QL_3. Thus every non-spherical special subgroup W_U, with $4 < |U| < |S|$ satisfies the inductive hypothesis. Therefore by induction, Theorem 3.3.4, and Theorem 7.3.1, for any $T \in \mathcal{N}_P$ we have that $L^2_{\mathbf{q}}b_1(\Sigma_{L_T}) = 0$ (Here L_T is the subcomplex of L corresponding to the special subgroup W_T).

Hence by Corollary 6.4.3 (i), for every $T \in \mathcal{N}_P$

$$L^2_{\mathbf{q}}b_1(\mathcal{U}(W, C_T)) = L^2_{\mathbf{q}}b_1(\Sigma_{L_T}) = 0.$$

This implies that the $E_1^{0,1}$ term in the E_1 sheet of the spectral sequence of Proposition 6.4.5 is zero. By Proposition 3.3.1, the first row of the E_1 sheet is also zero, and in particular $E_1^{1,0}$ is zero. Therefore $L^2_{\mathbf{q}}b_1(\partial\Phi_L) = 0$. ∎

With the help of Theorem 5.0.2, we drop condition 2 in Theorem 7.4.2.

Corollary 7.4.3. *Suppose that (W, S) is infinite 2–spherical with $|S| \geq 5$. Suppose furthermore that:*

1. *For every $T \subseteq S$ with $|T| \geq 5$, $\operatorname{vcd} W_T \leq |T| - 2$.*

2. *Every infinite subgroup W_T, with $|T| = 3, 4$, is Euclidean or QL_3.*

Then $L^2_{\mathbf{q}}b_k(\Sigma_L) = 0$ for $k < 2$.

Proof. Note that condition 2 in Theorem 7.4.2 is vacuously satisfied if $\operatorname{vcd} W \leq |S| - 3$, so we suppose that $\operatorname{vcd} W = |S| - 2$. We must show that $L^2_{\mathbf{1}}b_{|S|-2}(\Sigma_L) = 0$, and to do this, we use an argument analogous to the one in Lemma 7.1.1. We first begin by coning empty 2–simplices of L, and then empty 3–simplices, and so on, until all empty simplices have been coned off. We then label all new edges by 2. In this way we obtain a newly promoted nerve L' which is a triangulation of $S^{|S|-2}$, and in particular, $\Sigma_{L'}$ is an $(|S| - 1)$–manifold. By Theorem 5.0.2, $L^2_{\mathbf{1}}b_{|S|-2}(\Sigma_{L'}) = 0$, and using the arguments of Lemma 7.1.1, we can conclude that $L^2_{\mathbf{1}}b_{|S|-2}(\Sigma_L) = 0$. ∎

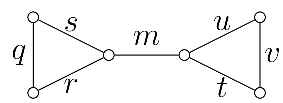
As a corollary to Theorem 7.4.2, we also obtain a specialized version of Conjecture 5.0.1 where $n = 4$ and W is 2–spherical.

Corollary 7.4.4. *Suppose that (W, S) is 2–spherical with $|S| \geq 6$ and that the nerve L is a triangulation of S^3. Furthermore, suppose that every infinite special subgroup W_T, with $|T| = 3, 4$, is Euclidean or QL_3. Then*

$$L^2_{\mathbf{q}}b_k(\Sigma_L) = 0 \textit{ for } k < 2.$$

Proof. Since L is a triangulation of S^3, it follows that $\operatorname{vcd} W = 4$. In particular, W satisfies the hypothesis of Theorem 7.4.2. ■

Remark 7.4.5. Figure 7.3 gives examples of Coxeter diagrams whose corresponding Coxeter system (W, S) has $|S| = 6$ and satisfies the hypothesis of Corollary 7.4.4 (if two vertices are not connected, then the implied label between them is 2). The author does not know whether there exist examples whenever $|S| \geq 7$.



$\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1, \quad \frac{1}{t} + \frac{1}{u} + \frac{1}{v} = 1, \quad m = 2, 3, 4$

- If $m = 3$, then $s, r, u, t \neq 6$ and either $s, r \neq 4$ or $u, t \neq 4$.
- If $m = 4$, then $s, r, u, t \neq 4, 6$.

Figure 7.3: 2–spherical Coxeter diagrams satisfying the hypothesis of Corollary 7.4.4

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