May 2015

Infinitely Generated Clifford Algebras and Wedge Representations of gl∞|∞

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A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

at

The University of Wisconsin–Milwaukee

May 2015

ABSTRACT

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The University of Wisconsin-Milwaukee, 2015
Under the Supervision of Dr. Yi Ming Zou

The goal of this dissertation is to explore representations of $\mathfrak{gl}_{\infty|\infty}$ and associated Clifford superalgebras. The machinery utilized is motivated by developing an alternate superalgebra analogue to the Lie algebra theory developed by Kac [17]. Kac and van de Leur first developed a super analogue, but it had various departures from a natural extension of their Lie algebra approach, made most certainly for the physics consequences. In an effort to establish a natural mathematical analogue, we construct a theory distinct from the super analogue developed by Kac and van de Leur [16]. We first construct an irreducible representation of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ on an infinite-dimensional wedge space $\mathfrak{F}$ that permits the presence of infinitely many odd parity vectors. We then develop corresponding operators on $\mathfrak{F}$ which serve as generators for a new Clifford superalgebra, whose structure is also examined. From here, we extend our representation to $\widehat{\mathfrak{gl}}_{\infty|\infty}$, the central extension of $\mathfrak{gl}_{\infty|\infty}$, and develop a correspondence between a subsuperalgebra of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and the Clifford superalgebra previously constructed. Finally, we begin to provide a context to study all Clifford algebras of an infinite-dimensional non-degenerate real quadratic space X. We focus mainly on developing the Clifford group and examining its connection to the group of orthogonal automorphisms on X.

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ACKNOWLEDGMENTS

First, I want to express my deepest gratitude to my advisor, Dr. Yi Ming Zou, for his exceptional patience and encouragement. His guidance and assistance have made this work possible, and I am indebted to his countless contributions to my development as a mathematician.

I am also grateful to the rest of my committee, Drs. Allen Bell, Craig Guilbault, Ian Musson, and Jeb Willenbring for supporting my development and providing models of excellence that I hope to emulate throughout my career. In addition, I want to thank Drs. Kyle Swanson and Xianwei Van Harpen for helping me balance my passion for teaching with this work. There is a long list of mathematicians not mentioned here whose work has been an inspiration, each deserving of thanks.

In my daily work, I have been fortunate enough to develop close relationships with many other people. First, I would like to thank Dr. Jason Gaddis, whose willingness to share advice and resources has benefitted me greatly. I want to record my sincere thanks to my cousin, Dr. Justin Z. Schroeder, whose collaboration continually proved tempered and strong. I am also grateful to my close graduate school companions - namely Molly Moran, Tom Goeppinger, and Justin Trulen - for helping me maintain my sanity. Finally, I count many of the staff, faculty, and other graduate students at UWM as friends, and although I have benefitted from too many people to mention them all by name, I count myself lucky to have known them so well.

Beyond mathematics, I would like to thank my sisters. They have often been seemingly marginalized due to my conflicting commitments, yet their understanding has been unwavering and their presence has always lifted me. This journey reaffirmed what I have always known - they are both genuinely beautiful human beings. Finally, I must name my parents, Jim and Dev. They have always been a source of, and facilitated, great joy in my life. Their support and understanding enabled me to pursue my passions, challenge myself, and find new purpose. No man has received more from his parents. Thank you.

Chapter 1

Background

1.1 Introduction

Infinite dimensional Lie theory has received great interest from both the mathematics and theoretical physics communities since the 1960s. The progenitors of Lie theory saw these structures as groups of symmetries of an object, and the corresponding set of infinitesimal transformations. The issue of classifying simple finite-dimensional Lie algebras of vector fields was solved by Wilhelm Killing and Elie Cartan over one-hundred years ago. Since then, the theory of finite-dimensional Lie groups and Lie algebras continually developed, but a resurgence in the study of infinite-dimensional Lie algebras did not occur until the mid-1960's with the work of I.M. Singer and Shlomo Sternberg [28], which outlined the machinery of filtered and graded Lie algebras.

At the present, infinite-dimensional Lie groups and algebras, along with their representations, is a mature field. There are four classes of infinite-dimensional Lie groups and algebras that have been studied fairly extensively. First, the Lie algebras of vector fields, along with the corresponding groups of diffeomorphisms of a manifold, especially the cohomology theory of infinite-dimensional Lie algebras of vector fields on a finite-dimensional manifold. This area, along with classifying representations of the groups of diffeomorphisms of a manifold, yielded many geometric applications. Next, the class of Lie groups or algebras of smooth mappings of a given manifold on to a finite-dimensional Lie group or algebra, has received study from mathematicians and theoretical physicists. Most of the study here has been limited to specific families of representations [18]. Third is the class consisting of classical Lie groups and algebras of operators in a Hilbert or Banach space, which focuses again on the structure of these Lie groups and algebras, as well as their representations. Finally, the fourth class is the class of Kac-Moody algebras. This area has seen the most growth in recent years, and has been the most fruitful topic in infinite dimensional Lie theory.

To motivate our study, we now give a very brief account of Boson-Fermion correspondence, culminating in an overview of the super-analogue developed by [16]. In particle physics, the state space for a system of a variable number of elementary particles is often called the Fock space. Here there are two distinct types of elementary particles, bosons and fermions, and each have different Fock spaces. In the case of fermionic Fock spaces, they can be viewed naturally as representations of a Clifford algebra, whose generators can be identified with the adding or removing a particle in a given pure energy state. Similarly, a bosonic Fock space is naturally realized as a representation of a Weyl algebra. To study the symmetries of Fock spaces, we are interested in the various algebras that naturally act on Fock spaces, and how these actions are related to each other and our Clifford algebra generators.

If we have a fermionic Fock space F over $\mathbb{C}$, it is then an infinite-dimensional vector space. We can find a standard basis, which can be indexed in various ways. In Chapter 2 we discuss an indexing by Maya diagrams, the related charged partitions, and ordered wedge products. A bosonic Fock B space can be viewed as a space of polynomials in infinitely many variables over $\mathbb{C}$. That way, we can find a standard basis for B using Schur functions. Then, using the fact that Schur functions are indexed by partitions, we can define a bijection between the standard bases of F and B, which will then be extended to a vector space isomorphism. It is important to note that there are many isomorphisms of vector spaces B and F. The chosen construction outlined in [17, 18] and Chapter 2 is adopted to study representation theory of the algebra acting on the space. A natural question centers on how these constructions may be extended to the Lie superalgebra theory. Lie superalgebras are a generalization of Lie algebras to a $\mathbb{Z}_2$-grading, and these superalgebras are important in describing supersymmetry in theoretical physics. This area was first explored by Kac and van de Leur in [16] with a particular super Fock space. Our motivation for delving into the super-analogue was to find a construction that felt more "natural" given the action of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ on a different super Fock space, and the potential for representation theoretic results.

We now look at the super analogue available in [16]. Kac and van de Leur constructed a representation of $\mathfrak{gl}_{\infty|\infty}$ by first defining a Clifford superalgebra $\mathcal{C}l$ with generators ψ_i, ψ_i^*. Next they defined the associated spin module V, that served as the super Fock space, with a non-zero *even* monomial $|0\rangle$ as a generator. This is done by taking an infinite-dimensional complex vector superspace Ψ, which is simply a $\mathbb{Z}_2$-graded vector space, and then identifying Ψ with the space of column vectors whose coordinates are indexed by $\frac{1}{2}\mathbb{Z}$. Then V is the infinite wedge space generated as a $\mathcal{C}l$-module by the infinite monomial $|0\rangle = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ made of infinitely many nonzero *even* vectors. The so-called standard representation of $\mathfrak{gl}_{\infty|\infty}$ on Ψ, which is the infinite dimensional analogue of the standard representation of the Lie super algebra $\mathfrak{gl}(m,n)$, can be realized using the Clifford superalgebra via the map

$$E_{ij} \to (-1)^{|j|}\psi_i\psi_j^*$$

Hence a representation of π of $\mathfrak{gl}_{\infty|\infty}$ can be defined on the spin module V. This then admits two separate decompositions of V,

$$V = \bigoplus_{m\in\mathbb{Z}} V_m \quad \text{and} \quad V = \bigoplus_{m\in\mathbb{Z}} V_{(f);m}$$

via *charge number* and *fermionic charge number*, respectively.

The former decomposition yields spaces that are invariant and irreducible with respect to $\mathfrak{gl}_{\infty|\infty}$, and therefore lends itself to study of the highest weight theory of $\mathfrak{gl}_{\infty|\infty}$. In fact, by utilizing the relations of the Clifford superalgebra this study reveals that no non-trivial highest weight representation of $\mathfrak{gl}_{\infty|\infty}$ is unitary. Interestingly, each of these decompositions comes into play in developing the super analogue of the boson-fermion correspondence.

In order to see this, we extend the previous representation to the Lie superalgebra $\mathfrak{a}_{\infty|\infty}$ and its central extension $\bar{\mathfrak{a}}_{\infty|\infty}$, thus allowing us to introduce the principal subsuperalgebra $\widetilde{\mathfrak{g}}$ of $\bar{\mathfrak{a}}_{\infty|\infty}$. The basis elements $e(n)$, $f(n)$, $\lambda(n)$, and $\mu(n)$ of this subsuperalgebra $\widetilde{\mathfrak{g}}$ play an important role in the super boson-fermion correspondence, and can be described in terms of the generators of $\mathcal{C}l$.

View ψ_i and ψ_i^* as operators mapping V into its formal completion $\widehat{V}$, and introduce generating series of ψ_i and ψ_i^*:

$$\begin{aligned} \psi_{\bar{0}}(z) &= \sum_{i\in\mathbb{Z}} \psi_i z^i, & \psi_{\bar{0}}^*(z) &= \sum_{i\in\mathbb{Z}} \psi_i z^{-i} \\ \psi_{\bar{1}}(z) &= \sum_{i\in\frac{1}{2}+\mathbb{Z}} \psi_i z^i, & \psi_{\bar{1}}^*(z) &= -\sum_{i\in\frac{1}{2}+\mathbb{Z}} \psi_i z^{-i} \end{aligned}$$

Now define the projection P to be the generating series for the sum of projections of P_m of V onto $V_{f(m)}$. Also, let Q be an appropriate operator mapping a fermionic charge space $V_{(f),m}$ to $V_{(f),m+1}$. One then has a way of explicitly describing the generating series of the ψ_i and ψ_i^* in terms of the well-known vertex operator

$$\Gamma(z) = \Gamma_-(z)\Gamma_+(z) : V \to \widehat{V},$$

where

$$\Gamma_-(z) = \exp\left(\sum_{n>0} \frac{\lambda(-n)}{n} z^n\right), \qquad \Gamma_+(z) = \exp\left(-\sum_{n>0} \frac{\lambda(n)}{n} z^{-n}\right).$$

The following "nice" description of these operators is known as the super boson-fermion correspondence:

Theorem (Super boson - fermion correspondence [16]).

$$\begin{aligned} \psi_{\bar{0}}(z) &= P(z)\; Q\; \Gamma_-(z)\; \Gamma_+(z) \\ \psi_{\bar{0}}^*(z) &= Q^{-1}\; P(z)^{-1}\; \Gamma_-(z)^{-1}\; \Gamma_+^{-1}(z) \\ \psi_{\bar{1}}(z) &= -P(z)\; Q\; \Gamma_-(z)\; e(z)\; \Gamma_+(z) \\ \psi_{\bar{1}}^*(z) &= Q^{-1}\; P(z)^{-1}\; \Gamma_-(z)^{-1}\; f(z)\; \Gamma_+^{-1}(z) \end{aligned}$$

There are various immediate applications of this correspondence. First, this provides an explicit way of comparing expressions for q-dimensions of representations of $\widetilde{\mathfrak{g}}_{1|1}$, through which new combinatorial identities were derived by computing characters of representations of $\mathfrak{gl}_{\infty|\infty}$ in two different ways. In addition, a series of irreducible highest weigh modules of the affine superalgebra $\widetilde{\mathfrak{g}}_{n|n}$ are constructed, and an explicit formula for the q-dimension is found [16].

We will begin by introducing basic definitions and results that will be utilized throughout this paper. The goal here is to provide a simple introduction and reference, as these structures will be discussed in greater detail throughout the remaining chapters.

1.2 Partitions

In this section, we will briefly review and introduce some basic structures and properties that will prove helpful in connecting and visualizing abstract structures in the following chapters.

A *partition* can be thought of as a finite (or infinite where only finitely many parts are non-zero) tuple

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

of weakly decreasing non-negative integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$$

where at most finitely many λ_i are non-zero. We say that the *length* of the partition, $\ell(\lambda)$, is equal to the number of non-zero parts λ_i, and for any partition, the *size* of the partition is the sum of all its parts,

$$n = \sum_{i=1}^{\ell} \lambda_i.$$

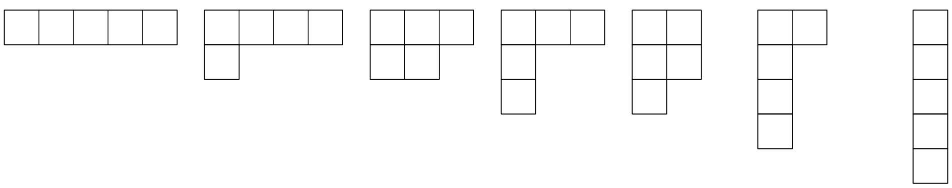
We say that λ is a partition of n, denoted $\lambda \vdash n$. A common notation for a partition is simply the sequence of values of the λ_i's, except that if a particular value of a

λ_i is repeated one shows this by putting an appropriate power on that value. For example, the partition $\lambda = (3, 3, 2, 1, 1)$ of $n = 10$ would be written as

$$(3^2, 2, 1^2).$$

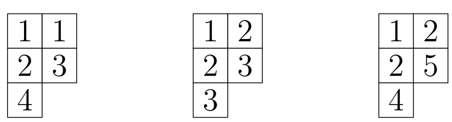
It is often useful to represent a partition in a particular way, that is as a collection of unit squares on then integer lattice. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n, we say that a *Young diagram of shape* λ is an array of n boxes in ℓ rows, left aligned such that row i contains λ_i boxes.

Example 1.2.1. The seven partitions of the integer 5 are given below with their corresponding Young diagrams

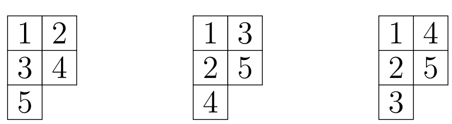


(5) (4, 1) (3, 2) (3, 1, 1) (2, 2, 1) (2, 1, 1, 1) (1, 1, 1, 1, 1)

We can give further structure to these shapes by assigning a labeling of cells of a Young diagram. A *semistandard Young tableau* is a labeling of a Young diagram T which is weakly increasing along rows and strictly increasing down columns. For example,



If we require labeling to be strictly increasing along rows and down columns, we then call the labeling a *standard Young tableau* .



These are all different tableaux of the same shape given by the partition $\lambda = (2, 2, 1)$ of 5. We will mainly see these structures in Chapter 2 and Appendix A.

There is an ever-expanding amount of literature on the role that Young diagrams, Young tableaux, and their variants, play in representation theory. It is worth mentioning that every partition of n determines a Young diagram which determines an irreducible representation of the symmetric group S_n. The symmetric group on a finite set of n symbols is the group whose elements are permutations of the n symbols, and the group operation is composition. In fact, all inequivalent irreducible representations of S_n over the complex numbers are determined by partitions of n (and thus Young diagrams). The dimension of an irreducible representation associated with a given Young diagram is then determined via the product of the what is known as the *hook-lengths* of all its elements. Further, the Robinson-Schensted Correspondence and its generalizations yield a plethora of applications of these structures in relation to representation theory ideas. For an introduction to these connections, see [25].

The last thing we need to define in this section are *Schur polynomials.* These polynomials form a basis for the space of all symmetric polynomials, and there are various ways to go about defining them. A Schur polynomial depends on a partition λ of a positive integer n, and we will use that relationship to define it here.

Definition 1.2.2. Fix λ and a bound N on the size of the entries in each semistandard tableau T. Then let

$$x^T = \prod_{i=1}^{N} x_i^j$$

where j is the number of i's in T. Then the **Schur polynomial** is

$$s_\lambda(x_1, \ldots, x_N) := \sum_{semistandard\ T} x^T$$

Example 1.2.3. We will construct a basic example using $\lambda = (2, 1)$. First we will give all the possible semistandard tableaux of shape λ,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Then the Schur polynomial corresponding to λ is

$$s_{(2,1)}(x_1, x_2, x_3) = {x_1}^2 x_2 + {x_1}^2 x_3 + x_1 {x_2}^2 + x_1 x_2 x_3 + x_1 x_3 x_2 + x_1 {x_3}^2 + {x_2}^2 x_3 + x_2 {x_3}^2$$

These polynomials are certain homogeneous symmetric polynomials in n indeterminates with integer coefficients and correspond to the irreducible representations of S_n. One of the main problems in the field of representation theory is the decomposition of a representation into irreducible components realized as irreducible modules. For example, the Littlewood-Richardson rule can be used in a general linear group to find the decomposition of a tensor product into irreducibles by looking at the corresponding Schur functions. We revisit Schur polynomials in Chapter 2 and give them a unique context there.

1.3 Lie algebras

In this section we will present some elementary definitions and properties of the theory of Lie algebras and Lie superalgebras. The goal is to provide several fundamental definitions that are needed through the following chapters. For more thorough treatments, one should see [15, 6]. More advanced readers may want to skip to Chapter 2.

Until Chapter 4, we will work over the field of complex numbers $\mathbb{C}$. We begin with the definition of a Lie algebra.

Definition 1.3.1. A complex **Lie algebra** is a vector space $\mathfrak{g}$ over $\mathbb{C}$ with an bilinear operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad x, y \in \mathfrak{g}$$

called the Lie bracket (or commutator), such that the following axioms are satisfied:

- It is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
- It satisfies the Jacobi Identitiy: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

As Chapter 2 will be dealing with representations of Lie algebras, we want to define the following:

Definition 1.3.2. A **Lie algebra homomorphism** is a linear map $\varphi \in Hom(\mathfrak{g}, \mathfrak{h})$ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that it is compatible with the Lie bracket:

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

Note that the bracket on the left-hand side is taken in $\mathfrak{g}$, while on the right-hand side is taken in $\mathfrak{h}$.

Example 1.3.3. We will now look at a very important and basic Lie algebra. Let V be a vector space and let $\mathrm{End}(V)$ be the set of endomorphisms of V, which is an associative algebra. Then $\mathrm{End}(V)$ equipped with the bracket $[X, Y] = XY - YX$ forms a Lie algebra called the **general linear Lie algebra**, denoted $\mathfrak{gl}(V)$. When $V = \mathbb{C}^n$, we also write $\mathfrak{gl}(n)$ for $\mathfrak{gl}(V)$. Now a **representation** of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra homomorphism:

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V).$$

If $\dim V = n$, we choose an ordered basis $\{v_i\}_{i=1}^n$ for V, with the elementary matrices accordingly denoted E_{ij}, then $\mathfrak{gl}(V)$ can be realized as $n \times n$ complex matrices.

We now give a short discussion of the exterior algebra in Lie algebra case, as we will see it again in Chapter 2. Given a vector space V, the exterior algebra ΛV is generated by the elements of V using the operations of addition and scalar multiplication, an associative binary operation $\wedge$ called the exterior or wedge product. These operations are subject to the identities necessary for ΛV to be an associative algebra, as well as the identity

$$v \wedge v = 0 \text{ for all } v \text{ in } V.$$

As we are working over $\mathbb{C}$, we may replace the relation $v \wedge v = 0$ by the relation

$$v \wedge w = -w \wedge v$$

for all $v, w \in V$. If we are not in characteristic 2, then the second relation implies the first, while the converse holds in any characteristic. The exterior algebra of a vector space V is also called the *Grassmann algebra associated with V.*

1.4 Lie superalgebras

In order to establish the necessary background for Chapter 3, we will begin with the superalgebra analogues to many of the definitions found in the previous section. We begin with a vector superspace V, which is a vector space that is endowed with a $\mathbb{Z}_2$-grading

$$V = V_{\bar{0}} \oplus V_{\bar{1}}.$$

For a finite-dimensional superspace, its dimension is often given as the sum $\dim V = \dim V_0 + \dim V_1$. For a homogeneous element $v \in V_i$, the *parity* of v is denoted $|v| = \bar{i}$, $\bar{i} \in \mathbb{Z}_2$. An element in $V_{\bar{0}}$ is called *even* , and an element in $V_{\bar{1}}$ is called *odd.* Given two superspaces, V and U, the space of linear transformations from V to U is also a superspace. In particular, the space of endomoporphisms of V is a vector superspace, and will be denoted $\mathrm{End}(V)$.

Definition 1.4.1. A **superalgebra** $\mathcal{A}$ is a vector superspace equipped with a bilinear multiplication satisfying $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$, for $i, j \in \mathbb{Z}_2$

One understands modules over $\mathcal{A}$, and subalgebras and ideals of $\mathcal{A}$ in a $\mathbb{Z}_2$-graded sense. As expected, a superalgebra containing no nontrivial ideal is called *simple.*

Definition 1.4.2. A **homomorphism** between modules M and N of a superalgebra $\mathcal{A}$ is a linear map $f : M \to N$ satisfying

$$f(am) = af(m), \quad a \in \mathcal{A}, m \in M.$$

Definition 1.4.3. A **Lie superalgebra** is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bilinear operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie superbracket (or supercommutator), such that the following two axioms are satisfied for homogeneous elements

- Skew-supersymmetry: $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$.

- Super Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the even part $\mathfrak{g}_0$ is a Lie algebra. Hence, if $\mathfrak{g}_1 = 0$, then $\mathfrak{g}$ is just a usual Lie algebra.

Definition 1.4.4. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie superalgebras. A **Lie superalgebra homomorphism** is an even linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ satisfying

$$\varphi([a,b]) = [\varphi(a), \varphi(b)], \text{ for all } a, b \in \mathfrak{g}.$$

Example 1.4.5. Let $\mathrm{End}(V)$ denote the endomorphisms of a super vector space V. $\mathrm{End}(V)$ is a super vector space itself, where $\mathrm{End}(V)_0$ are the endomorphisms preserving parity, and $\mathrm{End}(V)_1$ are those reversing it.

Analogous to the Lie algebra case, equip $\mathrm{End}(V)$ with the supercommutator, and it forms a Lie superalgebra called the **general linear Lie superalgebra** and denoted by $\mathfrak{gl}(V)$. Further, when $V = \mathbb{C}^{m|n}$, we write $\mathfrak{gl}(m|n)$ for $\mathfrak{gl}(V)$. Next, assign an ordered basis for V, where $\{v_i\}_{i=1}^{m}$ are the even basis vectors belonging to V_0, and $\{v_i\}_{i=m+1}^{m+n}$ are the odd basis vectors of V_1. Let E_{ij} be the corresponding elementary matrices, then $\mathfrak{gl}(m|n)$ can be realized as $(m+n) \times (m+n)$ complex matrices of the block form

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, and d are respectively $m \times m$, $m \times n$, $n \times m$, and $n \times n$ matrices.

Remark 1.4.6. It is worth noting that we can make any associative superalgebra $\mathcal{A}$ into a Lie superalgebra by taking assigning it the super-commutator,

$$[a,b] = ab - (-1)^{|a||b|}ba$$

We will need to consider the tensor product of two superalgebras V and W, which is a superalgebra $V \otimes W$ with a multiplication given by,

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = (-1)^{|w_1||v_2|}(v_1 v_2 \otimes w_1 w_2).$$

Note that if either V or W is even, this becomes the ordinary ungraded tensor product, though it still yields a graded tensor. However, in general, the super tensor product is distinct from the tensor product of V and W regarded as ungraded algebras. So the category of vector superspaces admits tensor products that have a natural $\mathbb{Z}_2$-grading. If V and W are super vector spaces, then we have for $V \otimes W$,

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

Then $V \otimes W \cong W \otimes V$ by the *commutativity map*

$$\alpha_{V,W} : V \otimes W \to W \otimes V$$

where $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$. This is a special case of what is commonly called the "sign rule". This principle appears frequently when dealing with Lie superalgebras.

For a vector superspace V, the exterior algebra ΛV is often called the *Grassmann algebra* over V. This ΛV, or $\Lambda^{\cdot} V$, is the free graded superalgebra on V. Explicitly, this is the quotient of the tensor algebra $T(V)$ by the ideal generated by elements of the form

$$v \otimes w + (-1)^{|v|\cdot|w|} w \otimes v.$$

The product in this algebra is denoted with a wedge, and called the wedge product. It obeys the relation

$$v \wedge w = -(-1)^{|v|\cdot|w|} w \wedge v.$$

This will be of vital importance to us in Chapter 3.

The final introductory definition begins with a *perfect* Lie superalgebra $\mathfrak{g}$, meaning if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then we have the following definition:

Definition 1.4.7. Let $\mathfrak{a}$ be a Lie superalgebra over a commutative ring k. A short exact sequence of Lie superalgebras

$$0 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{u} \xrightarrow{\alpha} \mathfrak{a} \longrightarrow 0$$

is called a **central extension** of $\mathfrak{a}$ by $\mathfrak{c}$ if the image of $\mathfrak{c}$ is central in $\mathfrak{u}$. Thus, $\mathfrak{c}_{\bar{1}} = \{0\}$ and $[\mathfrak{c}, \mathfrak{u}] = 0$. Here, $\mathfrak{c}$ is called the kernel of the central extension. We denote the above central extension by $\alpha : \mathfrak{u} \to \mathfrak{a}$.

1.5 Thesis outline

The remainder of this thesis is organized as follows: in Chapter 2 we follow the work of Kac in [17, 18] to construct the fermionic Fock space F. We will utilize a basis of semi-infinite monomials in F and identify these elements with both Maya diagrams and Young diagrams. We then introduce the Lie algebra $\mathfrak{gl}_\infty$ of linear operators, which will be realized as

$$\mathfrak{gl}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid \text{ only a finite number of } a_{ij} \text{ are non-zero } \}.$$

We will relate the action of $\mathfrak{gl}_\infty$ on F with wedging and contracting operators on F, which we will show generate a Clifford algebra.

We then gear up for introduction of the Boson-Fermion correspondence by extending our representation to the larger Lie algebra $\overline{\mathfrak{a}}_\infty$ and its central extension $\mathfrak{a}_\infty$. We then construct an important subalgebra of $\mathfrak{a}_\infty$ called a Heisenberg Lie algebra. Chapter 2 concludes by examining the machinery needed for relating the Clifford algebra and Heisenberg Lie algebra in the context of what is called the boson-fermion correspondence.

In Chapter 3, we will provide our approach in developing an analogue of Chapter 2 in the super case. Here we construct a new irreducible representation for the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ and its central extension, $\widehat{\mathfrak{gl}}_{\infty|\infty}$. We then develop corresponding operators used first to generate a **new** Clifford superalgebra $\widehat{\mathcal{C}l}$, then embed $\mathfrak{gl}_{\infty|\infty}$ into $\widehat{\mathcal{C}l}$. We then develop a new subsuperalgebra $\mathfrak{s}$ of $\widehat{\mathfrak{gl}}_{\infty|\infty}$, and establish correspondence between elements of $\mathfrak{s}$ and elements from our new Clifford superalgebra. Further analysis of representation theoretical consequences of our construction is done, followed by an outline of the issues with our approach, and current open questions.

In Chapter 4, we briefly review the well-known classification of finite-dimensional Clifford algebras of a real (resp. complex) quadratic space. As a foundation of our interest in the infinite-dimensional representation theory of Lie algebras and superalgebras, we began to examine infinitely-generated Clifford algebras via the approach of quadratic forms. This chapter serves as a guideline and motivation for some of the extended results that close out the main body of this dissertation. The appendices will consist of a connection between the representation theory in Chapter 3 and Young diagrams that should serve useful in dimensional computations, and a series of important computational proofs from Chapter 3.

Chapter 2

Wedge representations of affine Lie algebras

We will recall the fermionic Fock space F and the corresponding bosonic space B. We examine the a natural identification between these two spaces, which is part of the well-known boson-fermion correspondence. This connection is important to this dissertation as it gives a non-trivial relationship between a Clifford algebra and a Heisenberg Lie algebra (realized as a subalgebra of the Weyl algebra) that act on F and B. To accomplish this we introduce a series of Lie algebras. Originally, the following served, in part, as a representation theoretical interpretation of the Dirac theory of the positron. We will follow [17] fairly closely and a complete description of the results which follow, along with many others, can be found there and in [18]

2.1 The Fermionic Fock Space

Take an infinite dimensional vector space

$$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j$$

with a fixed basis $\{v_i\}_{i \in \mathbb{Z}}$. We will mainly concern ourselves with a space F constructed from V called the *fermionic Fock* space. To construct F, we first need the exterior (wedge) product on V, which is associative and anti-symmetric so that

$$v \wedge w = -w \wedge v.$$

We then introduce F as the complex vector space with a bsis consisting of *semi-infinite monomials* $v_{i_0} \wedge v_{i_1} \wedge \dots$, with $i_j \in \mathbb{Z}$, that satisfy

1. $i_0 > i_{-1} > i_{-2} \dots$
2. $i_k = i_{k-1} - 1$ for $k \gg 0$

According to Dirac's positron theory, the first condition reflects what is often referred to as the Pauli exclusion principle. Also, in the context of that theory, the latter condition relates that all but a finite number of negative energy states are occupied [17]. We will refer to semi-infinite monomials simply as *monomials* for the remainder of this chapter. We call the monomial

$$v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$$

the *vacuum vector of charge* 0. Further, we will define the *charge* decomposition

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$$

by letting

$$|m\rangle = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots$$

denote the *vacuum vector of charge* m. Then each $F^{(m)}$ is the linear span of all monomials which differ from $|m\rangle$ in finitely many positions, that is, those of charge m.

Example 2.1.1. According to this definition, both the monomial

$$v_9 \wedge v_7 \wedge v_5 \wedge v_3 \wedge v_0 \wedge v_{-3} \wedge v_{-4} \wedge \dots$$

and the monomial

$$v_{13} \wedge v_5 \wedge v_4 \wedge v_0 \wedge v_{-1} \wedge v_{-3} \wedge v_{-4} \wedge \dots$$

are elements of $F^{(2)}$.

It is convenient to define the energy of a monomial as in [18], as opposed to Dirac's equivalent definition which appeared years earlier. Given a monomial φ of charge m, we can then associate to it a partition, as defined in Chapter 1,

$$\lambda^{\varphi} = (\lambda_0, \lambda_1, \dots)$$

by assigning $\lambda_j = i_j - (m - j)$. This establishes a bijective correspondence between the set of all monomials of a given charge m and the set of all partitions. Next, define the *energy* of φ to be equal to the size $|\lambda^{\varphi}|$ of the associated partition, namely

$$|\lambda^{\varphi}| = \sum_i \lambda_i.$$

Now if we let $F_k^{(m)}$ denote the linear span of all semi-infinite monomials of charge m and energy k, we have the vector space decomposition:

$$F^{(m)} = \bigoplus F_k^{(m)}.$$

By the above construction, we have a corresponding q-dimension

$$\dim_q F^{(m)} := \sum_{k \in \mathbb{Z}} \dim F_k^{(m)} q^k = \frac{1}{(q)_\infty} \tag{†}$$

where $\dim F_k^{(m)} = p(k)$, or the number of partitions of k, and (†) is the corresponding generating function.

There are multiple ways to view monomials. One is using *Maya diagrams,* or a two-coloring of integers. Each colored integer is usually called a *stone.* We typically view vectors of V present in a monomial as black stones in the corresponding Maya diagram, and absent vectors as white stones.

Example 2.1.2. Given the monomial

$$\varphi = v_6 \wedge v_5 \wedge v_2 \wedge v_0 \wedge v_{-1} \wedge v_{-2} \wedge v_{-3} \wedge v_{-4} \wedge \dots \quad ,$$

the corresponding Maya diagram is shown in the below figure.

●	●	○	○	●	○	|	●	●	●	●	●	⋯
6	5	4	3	2	1		0	−1	−2	−3	−4	

It is well-known that there is a one-to-one correspondence between the set of Maya diagrams and the set of Young diagrams. We illustrate the connection with φ from above.

Begin with the Maya diagram for φ and determine the charge of φ. In our example, φ differs from $|3\rangle$ in only a finite number of places, so the charge is 3. Then fix the corresponding position of the Maya diagram as the origin. We then arrange the positions i in the Maya diagram with $i > 3$ down the vertical axis perpendicular to the tail end of the monomial, creating a portion of an integer lattice. Now, to construct the corresponding Young diagram, begin with the first filled position of φ, so 6, and create a ray moving right one unit. Now, we move onto the next possible position, 5, and continue the ray in the following manner:

- If 5 is a filled position, continue with a horizontal ray one unit to the right;
- If 5 is an unfilled position, continue with a vertical unit ray one unit up.

We continue this pattern until our series of rays intersects with the horizontal axis. By filling in the unit grid, we see we have created a Young diagram corresponding to a particular monomial of charge 3. In fact, we know the energy of this monomial as well, which is the integer that the Young diagram is partitioning. Figure 2.1 is the corresponding construction of the Young diagram for φ.

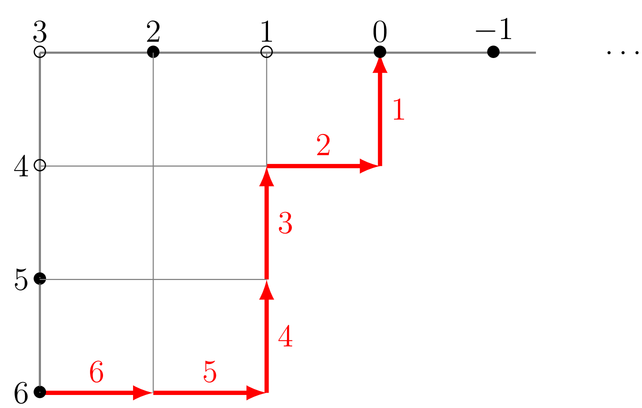


Figure 2.1: Connection between Maya and Young Diagrams

2.2 The basic representation of $\mathfrak{gl}_\infty$

The following construction of the basic representation of $\mathfrak{gl}_\infty$, as well as the corresponding highest weight theory of GL_∞ is well-documented since [17]. The representation theoretical results stemming from these concepts are applicable to soliton equations in the MKP hierarchies, many of which have been realized by the Kyoto school, beginning with Jimbo and Miwa [14], and continuing later with the addition of Date [22].

We begin by identifying V with the space of column vectors whose coordinates are indexed by $\mathbb{Z}$, and having all but a finite number of them equal to 0, via the map

$$\sum_j c_j v_j \to (c_j)_{j\in\mathbb{Z}}.$$

We introduce the Lie algebra

$$\mathfrak{gl}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid \text{ all but finitely many } a_{ij} \text{ are } 0\ \}$$

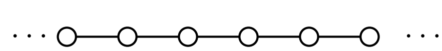
with the usual Lie bracket and the usual action on V:

$$E_{ij}v_j = v_i.$$

Remark 2.2.1. We can view $\mathfrak{gl}_\infty$ as a Kac-Moody Algebra on the Chevalley generators

$$e_i = E_{i,i+1},\ f_i = E_{i+1,i},\ h_i = E_{i,i} - E_{i+1,i+1}, \quad i \in \mathbb{Z},$$

where its Dynkin diagram is the infinite chain extending in both directions [17].



We then want to define a representation π_m of $\mathfrak{gl}_\infty$ on $F^{(m)}$ inspired by the action of $\mathfrak{gl}_\infty$ on $F^{(m)}$. Given a monomial $v_{i_0} \wedge v_{i_1} \wedge \cdots \in F^{(m)}$, our representation π_m will be defined by

$\pi_m(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots)$

$$= E_{ij}v_{i_0} \wedge v_{i_1} \wedge \dots + v_{i_0} \wedge E_{ij}v_{i_1} \wedge v_{i_2} \wedge \dots + \dots$$
$$= \sum_{i_k \in \mathbb{Z}} \dots \wedge v_{i_{k-1}} \wedge E_{ij}v_{i_k} \wedge v_{i_{k+1}}$$

It is useful to reiterate that multilinearity and anticommutativity are satisfied here, which is often used to highlight the *Pauli exclusion principle*, ensuring that a monomial of the form,

$$\dots \wedge v \wedge \dots \wedge v \wedge \dots$$

will be 0. This fact, paired with the corresponding action of $\pi_m(E_{ij})$ on an ensemble, corresponds to the effect of electromagnetic radiation. For more on this connection, see [17].

Proposition 2.2.2 ([17]). *The representation π_m is an irreducible representation of $\mathfrak{gl}_\infty$ on $F^{(m)}$.*

We now define a representation on all F

$$\pi = \bigoplus_{m \in \mathbb{Z}} \pi_m,$$

and consider the *principal gradation* of $\mathfrak{gl}_\infty$,

$$\mathfrak{gl}_\infty = \bigoplus_{j \in \mathbb{Z}} g_j,$$

by letting $\deg(E_{ij}) = j - i$. Here $g_k = \mathrm{span}(E_{ij})$ for E_{ij} that has degree j. Then we have

$$[g_i, g_j] = g_{i+j} \tag{2.1}$$
$$\pi_0(g_i)F_j \subseteq F_{j-i}, \qquad F_0 = \mathbb{C}|0\rangle \tag{2.2}$$

This gradation will be useful when extending the representation π to a larger Lie algebra.

2.3 Wedging and contracting operators

For $j \in \mathbb{Z}$, we introduce the *wedging* and *contracting* operators ψ_j and ψ_j^* on F as the following:

$$\psi_j(v_{i_0} \wedge v_{i_1} \wedge \dots) \quad = \begin{cases} 0 \quad \text{if } j = i_s \text{ for some } s, \\ (-1)^s v_{i_1} \wedge \dots v_{i_s} \wedge v_j \wedge v_{i_{s+1}} \wedge \dots \text{ if } i_s > j > i_{s+1}. \end{cases}$$

$$\psi_j^*(v_{i_0} \wedge v_{i_1} \wedge \dots) = \begin{cases} 0 \quad \text{if } j \neq i_s \text{ for all } s, \\ (-1)^{s+1} v_{i_1} \wedge \dots v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \dots \text{ if } j = i_s. \end{cases}$$

We then have the following result via straight-forward calculations.

Proposition 2.3.1 ([18])**.** *Given the operators ψ_j and ψ_j^* on F as defined above, the following relations hold:*

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$$
$$\psi_i \psi_j + \psi_j \psi_i = 0$$
$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

Corollary 2.3.2 ([18])**.** *The operators ψ_j and ψ_j^* generate a Clifford algebra, denoted $\mathcal{C}l$.*

Further, the $\mathcal{C}l$-module F is irreducible and

$$\psi_j(|0\rangle) = 0 \text{ for } j \leq 0 \quad ; \quad \psi_j^*(|0\rangle) = 0 \text{ for } j \geq 0$$

Here $\mathcal{C}l$ is the Clifford algebra associated with the space

$$V = \sum_i \mathbb{C}\psi_i + \sum_i \mathbb{C}\psi_i^*$$

with symmetric bilinear form $(\psi_i|\psi_j^*) = \delta_{ij}$.

Proposition 2.3.3 ([18])**.** *The Fock space F is the spin module of $\mathcal{C}l$ associated with the subspace*

$$U = \sum_{i \leq 0} \mathbb{C}\psi_i + \sum_{i > 0} \mathbb{C}\psi_i^*$$

One next sees that the embedding $\pi : \mathfrak{gl}_\infty \to \mathcal{C}l$ given by

$$\pi(E_{ij}) = \psi_i \psi_j^*$$

defines a representation π of $\mathfrak{gl}_\infty$ on F, thus also yielding a representation π_m of $\mathfrak{gl}_\infty$ on $F^{(m)}$ for each $m \in \mathbb{Z}$. The representation π_0 of $\mathfrak{gl}_\infty$ on $F^{(0)}$ is called the *basic representation.*

Now, we want to introduce a few tools that will serve us later when setting up part of the boson-fermion correspondence. If we have a collection of numbers $\lambda = \{\lambda_i\}_{i\in\mathbb{Z}}$, we say the *highest weight representation* π_λ of $\mathfrak{gl}_\infty$ is an irreducible representation on a vector space $L(\lambda)$ which admits a *highest weight vector* v_λ, such that

$$v_\lambda \neq 0, \quad \pi_\lambda(E_{ij})v_\lambda = 0 \quad \text{for } i < j; \quad \pi_\lambda(E_{ii})v_\lambda = \lambda_i v_\lambda.$$

Much can be said about $L(\lambda)$ as a result of, in part, the PBW theorem (see [17]), but we want to note that, provided the λ_i are real, $L(\lambda)$ carries a unique Hermitian form $\langle , \rangle$, called the *contravariant Hermitian form.* This form satisfies

$$\langle v_\lambda, v_\lambda \rangle = 1, \quad \text{and} \quad (\pi_\lambda(A))^* = \pi_\lambda({}^t\bar{A}), \quad \text{for } A \in \mathfrak{gl}_\infty. \tag{2.3}$$

We next introduce the infinite complex matrix group

$$\mathrm{GL}_\infty = \{A = (a_{ij})_{ij} \mid \text{A is invertible and all but a finite number of the } a_{ij} - \delta_{ij} \text{ are } 0\},$$

whose Lie algebra is $\mathfrak{gl}_\infty$. Further, we can define a representation R_0 of GL_∞ on $F^{(0)}$ by

$$R_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) = E_{ij}v_{i_0} \wedge E_{ij}v_{i_1} \wedge E_{ij}v_{i_2} \wedge \dots$$

where $E_{ij} \in \mathrm{GL}_\infty$.

Then the irreducible representation R_0 corresponds to the irreducible π_0 via

$$\exp \pi_0(A) = R_0(\exp A)$$

for $A \in \mathfrak{gl}_\infty$.

We introduce this because, not only are the interesting highest weight representations of $\mathfrak{gl}_\infty$ exponentiated to GL_∞, but GL_∞ will also be used in providing the machinery for the boson-fermion correspondence. For more on the highest weight representations of $\mathfrak{gl}_\infty$, and their corresponding representations of GL_∞ (when possible), see [17].

2.4 Vertex Operator Realization

Let us introduce two new Lie algebras: $\overline{\mathfrak{gl}}_\infty$ and $\bar{\mathfrak{a}}_\infty$, where

$$\overline{\mathfrak{gl}}_\infty = \left\{ (a_{ij})_{i,j\in\mathbb{Z}} | \begin{array}{c} \text{there are finitely many non-zero } a_{ij} \\ \text{on and below the diagonal} \end{array} \right\}$$

and

$$\bar{\mathfrak{a}}_\infty = \left\{ (a_{ij})_{i,j\in\mathbb{Z}} \mid \begin{array}{c} \text{for each } k, \text{ the number of non-zero } a_{ij} \\ \text{with } j \le k \text{ and } i \ge k \text{ is finite} \end{array} \right\}$$

Both of these Lie algebras act on a completion of V,

$$\overline{V} = \{\sum_{j\in\mathbb{Z}} c_j v_j \mid c_j = 0 \text{ for } j \gg 0\}.$$

There is an obvious extension of π_m to $\overline{\mathfrak{gl}}_\infty$. However, if we try to extend the representation π_0 (respectively, π_m) to the Lie algebra $\bar{\mathfrak{a}}_\infty$, we encounter a problem. Consider

$$\pi_0(\sum_{i\in\mathbb{Z}} \lambda_i E_{ii})|0\rangle = (\lambda_0 + \lambda_{-1} + \dots)|0\rangle$$

which is a potentially divergent series. This anomaly is removed by changing the representation π as follows:

$$\hat{\pi}(E_{ij}) = \begin{cases} \pi(E_{ij}) & \text{if } i \neq j \text{ or } i = j > 0 \\ \pi(E_{ii}) - \mathrm{I} & \text{for } i \le 0 \end{cases}$$

Note that this simply kills the vacuum vector $|0\rangle$.

Thus we get a projective representation of $\bar{\mathfrak{a}}_\infty$. We now introduce the central extension $\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}c$ with center $\mathbb{C}c$ and bracket

$$[x, y] = xy - yx + \alpha(x, y)c,$$

where the cocycle α is defined by

$$\alpha(E_{ij}, E_{ji}) = -\,\alpha(E_{ji}, E_{ij}) = I \qquad \text{if } i \leq 0 < j$$
$$\alpha(E_{ij}, E_{ji}) = 0 \qquad \text{for all other cases}$$

The usefulness of introducing $\mathfrak{a}_\infty$ is that we can consider the shift matrix:

$$\Lambda_k = \sum_{i \in \mathbb{Z}} E_{i,i+k}$$

A simple calculation in $\mathfrak{a}_\infty$ yields

$$[\Lambda_k, \Lambda_n] = k\delta_{k,-n}c. \tag{2.4}$$

By making a note of the distinct sets of elements in $\mathfrak{a}_\infty$,

$$\{\Lambda_k \mid k > 0\} \text{ and } \{\Lambda_k \mid k < 0\},$$

along with central element c, it is clear we have an infinite-dimensional Heisenberg Lie algebra .

Definition 2.4.1. The subalgebra

$$\mathfrak{s} = \sum_{k \neq 0} \mathbb{C}\Lambda_k + \mathbb{C}c,$$

will be called the *principal subalgebra* of $\mathfrak{a}_\infty$, and is often also known as the *oscillator algebra.*

We then restrict π_m of $\mathfrak{a}_\infty$ on $F^{(m)}$ to $\mathfrak{s}$ and have the following result.

Proposition 2.4.2 ([18])**.** *As $\mathfrak{s}$-modules, the $\mathfrak{a}_\infty$-modules $F^{(m)}$ are irreducible.*

2.5 Boson - Fermion Correspondence

We will now, as a source of motivation, outline the remaining machinery necessary for the boson-fermion correspondence. More details can be found in [18, 17]. Our purposes for including this here are to outline ideas for which we hope to cultivate analogues in the super case.

The operators ψ_j and ψ_j^* introduced in the last section are called *free fermions*. The **bosonization** process involves introducing *free bosons*:

$$\alpha_n = \hat{\pi}(\Lambda_n),$$

and writing them explicitly by

$$\alpha_n = \sum_{j\in\mathbb{Z}} \psi_j \psi_{j+n}^* \qquad \text{if } n \in \mathbb{Z}\backslash\{0\}$$
$$\alpha_0 = \sum_{j>0} \psi_j \psi_j^* - \sum_{j\leq 0} \psi_j^* \psi_j.$$

We now introduce the bosonic Fock space

$$B = \mathbb{C}[x_1, x_2, \ldots, q, q^{-1}],$$

which is a polynomial algebra on indeterminates x_1, x_2, $\ldots$, q, q^{-1}. Further, we take the decomposition of B via

$$B = \bigoplus_{m\in\mathbb{Z}} B^{(m)}$$

where $B^{(m)} = q^m\mathbb{C}[x_1, x_2, \ldots]$.

We then want to define a representation of the oscillator algebra $\mathfrak{s}$ on B by:

$$\pi^B(\Lambda_n) = \frac{\partial}{\partial x_n}, \qquad \pi^B(\Lambda_{-n}) = nx_n,$$
$$\pi^B(\Lambda_0) = q\frac{\partial}{\partial q}, \qquad \pi^B(c) = I$$

where $n > 0$. Then as a result of Proposition 2.4.2, and the fact that the canonical commutation relations are unique, there is a unique isomorphism of $\mathfrak{s}$-modules

$$\sigma : F \to B$$

such that $\sigma(|m\rangle) = q^m$.

The map σ transports the Hermitian form on F to a Hermitian form on B. The principal gradation

$$B^{(m)} = \oplus_{k\geq 0} B_k^{(m)}$$

is defined by

$$\deg x_j = j,$$

where the degree of a product is clearly the sum of the degrees. The transported irreducible representation $\hat{\pi}_m^B$, along with the contravariant Hermitian form $\langle , \rangle$, must satisfy

$$\langle 1, 1\rangle = 1, \qquad \text{and} \qquad (\hat{r}_m^B(\Lambda_k))^* = \hat{r}_m^B(\Lambda_{-k}). \tag{2.5}$$

A straightforward calculation yields that the Hermitian form given by

$$\langle P, Q\rangle = P(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \dots)\overline{Q}(x)|_{x=0}$$

satisfies (2.5).

Through σ, the operator of multiplication by q can be transported from B to F, obtaining the operator on F, likewise denoted q, that yields

$$q|m\rangle = |m+1\rangle, \quad q\psi_i = \psi_{i+1} q$$

for $m, i \in \mathbb{Z}$. Now, the **fermionization** process consists of constructing fermions ψ_j and ψ_j^* in terms of bosons α_i. We thus introduce the generating series

$$\psi(z) = \sum_{j\in\mathbb{Z}} z^j \psi_j, \qquad \psi^*(z) = \sum_{j\in\mathbb{Z}} z^{-j}\psi_j^*$$

which map F into its formal completion $\widehat{F}$. Note that we have two transported operators

$$\sigma\psi(z)\sigma^{-1},\ \sigma\psi^*(z)\sigma^{-1} : B \to \widehat{B},$$

where $\widehat{B}$ is the formal completion of B. The relations of operators on F,

$$[\alpha_j, \psi(z)] = z^j\psi(z)], \qquad [\alpha_j, \psi^*(z)] = -z^j\psi(z)],$$

transport to B as follows:

$$[\frac{\partial}{\partial x_j}, \sigma\psi(z)\sigma^{-1}] = z^j(\sigma\psi(z)\sigma^{-1}), \quad [x_j, \sigma\psi(z)\sigma^{-1}] = \frac{z^{-j}}{j}(\sigma\psi(z)\sigma^{-1}).$$

This is important, because up to a constant factor (depending on z) there is only one operator which maps $B^{(m)}$ into $B^{(m+1)}$ that satisfies these relations [17]. We can then define the operators

$$\Gamma_+(z) = \exp\sum_{n\geq 1}\frac{z^{-n}}{n}\alpha_n, \qquad \Gamma_-(z) = \exp\sum_{n\geq 1}\frac{z^n}{n}\alpha_{-n}.$$

If we view z as a formal parameter, then $\Gamma_\pm$ can be viewed as generating series of operators on F,

$$\Gamma_\pm(z) = \sum_{n\in\mathbb{Z}_+}\Gamma_\pm z^{\mp n}.$$

We can see these operators in the bosonic picture as

$$\Gamma_+(z) = \exp\sum_{n\geq 1}\frac{z^{-n}}{n}\frac{\partial}{\partial x_n}, \qquad \Gamma_-(z) = \exp\sum_{n\geq 1}z^n x_n.$$

The last machinery we provide here is that with any partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we associate the Schur polynomial $S_\lambda(x)$. In order to see what this polynomial is, we introduce the "elementary" Schur polynomials in a slightly different manner than Chapter 1 via the following generating series:

$$\sum_{m\in\mathbb{Z}} z^m \, \mathrm{S}_m(x) = \exp\sum_{n\geq 1} z^n x_n$$

Then we have:

$$S_m(x) = 0 \text{ for } m < 0, \quad S_0(x) = 1, \quad \text{and}$$
$$S_m(x) = \sum_{m_1+2m_2+\cdots=m}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\dots \text{ for } m > 0.$$

To clarify, consider the first few *elementary* Schur polynomials

$$\begin{aligned}S_1(x) =&x_1,\\ S_2(x) =&\frac{1}{2}x_1^2 + x_2\\ S_3(x) =&\frac{1}{6}x_1^3 + x_1x_2 + x_3,\end{aligned}$$

Now, if we have a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, the associated *Schur polynomial* will be

$$S_\lambda(\boldsymbol{x} = \det(S_{\lambda_i+j-i}(\boldsymbol{x}))_{1\leq i,j\leq|\lambda|}$$

where $\boldsymbol{x} = (x_1, x_2, \dots)$. For example, the some basic examples include:

$$\begin{aligned}S_{(1,1)} =& \det\begin{bmatrix}S_1 & S_2\\ 1 & S_1\end{bmatrix} = \frac{1}{2}x_1^2 - x_2 \ ,\\ S_{(2,1)} =& \det\begin{bmatrix}S_2 & S_3\\ 1 & S_1\end{bmatrix} = \frac{1}{3}x_1^3 - x_3 \ ,\\ S_{(2,2)} =& \det\begin{bmatrix}S_2 & S_3\\ S_1 & S_2\end{bmatrix} = \frac{1}{12}x_1^4 - x_1x_3 + x_2^2 \ ,\end{aligned}$$

Finally, recall that given a semi-infinite monomial $\varphi \in F^{(m)}$ there is an associated partition λ^φ - defined in Section 2.1 - and thus an associated Schur polynomial. Explicitly, each monomial will correspond to $S_{\lambda^\varphi}(\boldsymbol{x})$.

There are a few more tools needed to *prove* the following boson-fermion correspondence. This is due to a few calculations relying on extending the representation R of the group GL_∞ to a new representation of the group

$$\widetilde{\mathrm{GL}}_\infty = \left\{ a = (a_{ij})_{i,j\in\mathbb{Z}} \mid \begin{array}{c} a \text{ is invertible and all but a finite number of} \\ a_{ij} - \delta_{ij} \text{ with } i \geq j \text{ are } 0 \end{array} \right\}$$

See [18] for the complete proof, however we have provided the machinery for which we aim to derive analogues in Chapter 3, so we simply state the theorem here.

Theorem 2.5.1 (Boson - Fermion Correspondence). *(a) As defined above, for $z \in \mathbb{C}^\times$:*

$$\psi(z) = z^{\alpha_0}\Gamma_-(z)\Gamma_+(z)^{-1}$$
$$\psi^*(z) = q^{-1}z^{-\alpha_0}\Gamma_-^{-1}\Gamma_+(z)$$

(b) If $\varphi \in F^{(m)}$ is a semi-infinite monomial, then

$$\sigma(\varphi) = q^m S_{\lambda^\varphi}(\boldsymbol{x})$$

Chapter 3

Wedge representations of affine Lie superalgebras

3.1 The result of Kac and van de Leur

As mentioned in Chapter 1, Kac and van de Leur constructed a representation of $\mathfrak{gl}_{\infty|\infty}$ by first defining a Clifford superalgebra $\mathcal{C}l$ with generators ψ_i, ψ_i^*, and then the associated spin module V with a non-zero *even* vector $|0\rangle$ as a generator. Given an infinite-dimensional complex vector superspace Ψ, which one identifies with the space of column vectors whose coordinates are indexed by $\frac{1}{2}\mathbb{Z}$, the standard representation of $\mathfrak{gl}_{\infty|\infty}$ on Ψ associated to the Clifford superalgebra. From here, the spin module is decomposed and used in developing a super analogue of the boson-fermion correspondence [16]. Our interest is in developing the machinery of a new analogue to Chapter 2. The motivation for revisiting something similar but distinct from Kac and van de Leur originated while studying their superalgebra analogue to the Lie algebra case, and some key differences in their approach. First, they began by defining generators of a Clifford superalgebra. This dictated the infinite-dimensional wedge space they used, which was restricted in construction (thus so was the spin module). Also, unlike the Lie algebra case, this construction did not follow from a natural infinite wedge representation of $\mathfrak{gl}_{\infty|\infty}$. Due to the limitations of its infinite wedge space, some interesting structural questions that the super analogue should present were not addressed. As their approach lost some of the expected structural analogues from the Lie algebra case, and the Clifford superalgebra was no longer directly motivated by the representation of $\mathfrak{gl}_{\infty|\infty}$, we decided to construct an analogue that may address these issues. The two main distinctions in our approach are, first, we wanted a super analogue of the Fock space that allowed for infinitely many odd vectors. Second, we wanted to construct a representation of this new space that would naturally yield generators for a Clifford superalgebra, which is decidedly different from the approach in [16]. From here, we would look to establish a correspondence similar to the boson-fermion correspondence. This explicit link would serve as a launch pad to representation theoretical results. Although we are examining distinct structures (other than $\mathfrak{gl}_{\infty|\infty}$) from Kac and van del Leur, our results are clearly motivated by their work, and we will utilize their notation when our analogue mirrors theirs.

3.2 A $\mathfrak{gl}_{\infty|\infty}$ representation

Take an infinite dimensional superspace

$$V = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathbb{C}v_j$$

with even basis vectors $\{v_i \mid i \in \mathbb{Z}\}$ and odd basis vectors $\{v_k \mid k \in \frac{1}{2} + \mathbb{Z}\}$, yielding the $\mathbb{Z}_2$-gradation

$$V = V_{\bar{0}} \bigoplus V_{\bar{1}}$$

where we designate the parity of v_i as $p(i)$ or $|i| \in \mathbb{Z}_2$, with even basis vectors in $V_{\bar{0}}$ and odd basis vectors in $V_{\bar{1}}$. Now consider that the superalgebra

$$\mathfrak{gl}_{\infty|\infty} = \{(a_{ij})_{i,j \in \frac{1}{2}\mathbb{Z}} \mid \text{ all but finitely many } a_{ij} \text{ are zero}\},$$

together with the supercommutator $[\cdot,\cdot]$ forms a Lie superalgebra, as defined in Chapter 1. One may also see $\mathfrak{gl}_{\infty|\infty}$ as in [15], that is as a contragradient Lie superalgebra of infinite rank on Chevalley generators

$$e_i = E_{i,i+1}, \qquad f_i = E_{i+1,i},$$
$$h_i = E_{i,i} - E_{i+1,i+1},$$

where $i \in \frac{1}{2}\mathbb{Z}$ and E_{ij} is the usual basic matrix units with (i, j) entry 1 and the rest are 0.

It is clear that $\mathfrak{gl}_{\infty|\infty}$ operates on V via the multiplication of a matrix and a column vector,

$$E_{ij}v_j = v_i \ .$$

We define the parity of E_{ij} as $|E_{ij}| = |i + j| = |i| + |j| \in \mathbb{Z}_2$.

We now introduce $\mathfrak{F}$ as the vector space with basis consisting of monomials $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$ where $i_k \in \frac{1}{2}\mathbb{Z}$ such that

- $i_0 \geq i_1 \geq i_2 \geq \dots$
- $i_n = i_{n-1} - \frac{1}{2}$ for $n \gg 0$.

Here the possibility for duplicate vectors is realized with odd vectors, given that in the superalgebra case, the relation

$$v \wedge w = -(-1)^{|v||w|} w \wedge v \ , \tag{3.1}$$

defines the wedge product.

In essence, while in these formulas we assume multilinearity (i.e., $\cdots \wedge (\alpha u + \beta v) \wedge \cdots = \alpha(\cdots \wedge u \wedge \dots) + \beta(\cdots \wedge v \wedge \dots)$), the Pauli exclusion principle ($\cdots \wedge u \wedge \cdots \wedge u \wedge \cdots = 0$) no longer applies in general due to the presence of odd vectors. Let $m(i)$ denote the multiplicity of an odd vector v_i in a monomial in $\mathfrak{F}$. What is unique now is that $\mathfrak{F}$ contains monomials with infinitely many odd vectors, which will create a new smattering of issues when studying the structure of $\mathfrak{F}$.

Definition 3.2.1. Define the *charge* decomposition

$$\mathfrak{F} = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{F}^{(m)}$$

by letting $|m\rangle = v_m \wedge v_{m-\frac{1}{2}} \wedge v_{m-1} \wedge v_{m-\frac{3}{2}} \wedge \dots$ denote the *vacuum vector of charge m*. Then $\mathfrak{F}^{(m)}$ is the linear span of the monomials which differ from $|m\rangle$ only at a finite number of places.

The space $\mathfrak{F}^{(0)}$ is the linear span of monomials that differ from $|0\rangle = v_0 \wedge v_{-\frac{1}{2}} \wedge v_{-1} \wedge v_{-\frac{3}{2}} \wedge \dots$ in a finite number of spaces. The difference in positions between monomials in $\mathfrak{F}^{(m)}$ may occur via the presence of some v_i with $i > 0$, or via odd vectors v_i with $m(i) > 1$.

Remark 3.2.2. In the Lie algebra case discussed in Chapter 2, we had a clear bijection between the semi-infinite monomials of a given charge and the set of all partitions, Par. Our monomials have changed, and although the difference between indexing by $\frac{1}{2}\mathbb{Z}$ or $\mathbb{Z}$ can be handled via a simple doubling map, a possible issue comes from the allowing of multiple non-distinct odd vectors in φ.

Given a monomial $\varphi \in \mathfrak{F}^{(m)}$, consider what happens when we associate a partition $\lambda = (\lambda_0, \lambda_1, \dots)$ with $\varphi = v_{i_0} \wedge v_{i_1} \wedge \dots$ by comparing it to the vacuum vector $|m\rangle$ via

$$\lambda = (i_0 - m, i_1 - (m - \frac{1}{2}), i_2 - (m-1), \dots).$$

Clearly $\lambda_i = 0$ for $i \gg 0$, since φ has charge m. However, say $v_{i_{k-1}}$ and v_{i_k} are the same odd vector v_j, then $\lambda_{k-1} = j - (m - \frac{k-1}{2}) < j - (m - \frac{k}{2}) = \lambda_k$, which means λ does not belong to Par because it is not a finite *non-increasing* sequence. We note that each added multiple of an odd vector present in $\varphi \in \mathfrak{F}^{(m)}$ must be accompanied by the removal of a v_i with $i > m$ in order for φ to remain with charge m. We will revisit this in the next section when we have more tools with which to study $\mathfrak{F}^{(m)}$.

We return to $\mathfrak{F}$ and note that that action of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}^{(0)}$ is

$$E_{ij} \cdot (v_{i_0} \wedge v_{i_1} \wedge \dots) = \sum_{k \geq 0} (-1)^{|E_{ij}| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \dots \wedge E_{ij} v_{i_k} \wedge \dots$$

Define the representation r_0 of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}^{(0)}$ by:

$$r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) = \sum_{k \geq 0} \delta_{j,i_k} (-1)^k (-1)^{|j| \sum_{l=0}^{k-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_k}} \wedge \dots \tag{3.2}$$

By construction, beginning with the Lie superalgebra action and utilizing multilinearity, we have that

$$r_0([E_{ij}, E_{kl}]) = [r_0(E_{ij}), r_0(E_{kl})].$$

We ensured that our resulting monomial remains in $\mathfrak{F}^{(0)}$ (meaning the position of v_i is not an issue) due to properties of the wedge product. Thus, our resulting monomial is truly in the span of the basis elements of $\mathfrak{F}$ as defined earlier. We will further examine the reasoning behind this choice in the next section.

Also, this is always a finite sum as each monomial contains finitely many positions filled with v_j's. Refer to (3.2) and note when $|j| = \bar{0}$ or $|i| = \bar{0}$ there can be at most one term. Upon further examination, we need to consider the possibility of repeated odd vectors in our monomial. Here we will let $\Phi = \{k \mid i_k = j\}$ and examine when $|j| = \bar{1}$.

$$\begin{aligned} r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) &= \sum_{k \geq 0} \delta_{j,i_k} (-1)^k (-1)^{|j| \sum_{l=0}^{k-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_k}} \wedge \dots \\ &= \sum_{k \in \Phi} \delta_{j,i_k} (-1)^k (-1)^{|j| \sum_{l=0}^{k-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_k}} \wedge \dots \end{aligned}$$

Note that all duplicates of v_j are in consecutive positions. Thus, when applying r_0 the resulting monomials in each term of the summand will be the same, with simply a different sign depending on position. Now write $\Phi = \{q, q+1, \dots, q+(t-1), q+t\}$, where $q = \min\{\Phi\}$, and note $m(j) = |\Phi|$. We can thus simplify the sum utilizing the fact that the resulting monomial is equivalent to having moved v_{i_q}, simply with the additional sign for moving v_i past each v_j that precedes it in the monomial.

Proposition 3.2.3. *Let r_0, Φ, q, and t be defined as above. Then*

$$r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) = m(j)(-1)^q (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \tag{3.3}$$

Proof. Case 1: $|j| = \bar{0} = |i|$,
If $m(j) \neq 1$ or $m(i) \neq 0$, then $r_0(E_{ij}) = 0$.
So let $m(j) = 1$ and $m(i) = 0$, then

$$\begin{aligned} r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) &= (-1)^q v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\ &= m(j)(-1)^q (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \end{aligned}$$

Case 2: $|j| = \bar{0}$ and $|i| = \bar{1}$,

If $m(j) \neq 1$, then $r_0(E_{ij}) = 0$.

So let $m(j) = 1$, then for any multiplicity of v_i, we have

$$r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots)$$
$$= (-1)^q v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$
$$= m(j)(-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \dots$$

Case 3: $|j| = \bar{1}$, $|i| = \bar{0}$,

If $m(j) \neq 1$ or $m(i) \neq 0$, then $r_0(E_{ij}) = 0$.

So let $m(j) = 1$ and $m(i) = 0$, then

$$r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots)$$
$$= (-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$
$$= m(j)(-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$

Case 4: $|i| = \bar{1}$, $|j| = \bar{1}$,

Then for any multiplicity of both v_i and v_j, we have

$$r_0(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots)$$
$$= \sum_{r=0}^{t-1}(-1)^{q+r}(-1)^{r|i||j|}(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$
$$= (-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$
$$+ \ (-1)^{q+1}(-1)^{|i||j|}(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots +$$
$$\vdots$$
$$+ (-1)^{q+t-1}(-1)^{(t-1)|i||j|}(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots$$
$$= m(j)(-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \qquad \square$$

We can introduce the principal gradation $\mathfrak{gl}_{\infty|\infty} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j$ by putting deg $E_{ij} = j - i$, so that we have $[g_i, g_j] = g_{i+j}$. Now, we induce a *principal gradation*

$$\mathfrak{F}^{(0)} = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathfrak{F}_k^{(0)}$$

which is consistent with the gradation of $\mathfrak{gl}_{\infty|\infty}$, meaning for $a \in g_i$,

$$r_0(a)\mathfrak{F}_j^{(0)} \subseteq \mathfrak{F}_{j-i}^{(0)}. \tag{3.4}$$

Explicitly, $\mathfrak{F}_k^{(0)}$ is the linear span of all elements of $\mathfrak{F}^{(0)}$ of the form

$$r_0(E_{i_1 j_1}) r_0(E_{i_2 j_2}) \ldots r_0(E_{i_n j_n})|0\rangle \tag{3.5}$$

where

$$\sum_{\ell=1}^{n} i_\ell - \sum_{\ell=1}^{n} j_\ell = k.$$

Here we will call k the *energy* of a monomial in $\mathfrak{F}^{(m)}$ and at times may refer to the corresponding decomposition as the energy decomposition. Note, that although we have not calculated $\dim_q \mathfrak{F}^{(m)}$, it is evident (and further addressed in Section 3.3) that $\dim \mathfrak{F}^{(m)} < \infty$. This is useful in understanding the application of the PBW theorem for Lie superalgebras. For more on the superalgebra version of the PBW Theorem, see [26, 23, 15].

We defined r_0 with the hope that we would be able to use it to study $\mathfrak{F}^{(0)}$ further. In the following theorem, we see that $\mathfrak{F}^{(0)}$ has no non-trivial proper invariant subspaces under r_0.

Theorem 3.2.4. *The representation r_0 of $\mathfrak{gl}_{\infty|\infty}$ is irreducible on $\mathfrak{F}^{(0)}$.*

Proof. To show the r_0 is irreducible, first we will use the orthogonal energy decomposition constructed earlier

$$\mathfrak{F}^{(0)} = \bigoplus_{k \geq 0} \mathfrak{F}_k^{(0)}.$$

Suppose $W \subseteq \mathfrak{F}^{(0)}$ is an invariant subspace with respect to r_0. Then W and $W^\perp$ must respect the the energy decomposition of $\mathfrak{F}^{(0)}$. We can view $\mathfrak{F}^{(0)}$ as being generated by $|0\rangle$, as any monomial of charge m can be described as an element of the $\mathfrak{gl}_{\infty|\infty}$ orbit of $|0\rangle$. This can be seen clearly considering, given any monomial $\varphi \in \mathfrak{F}^{(0)}$ can be written as

$$v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge v_{i_{2k}} \wedge v_{-k-\frac{1}{2}} \wedge v_{-k-1} \wedge \dots$$

Then we can describe φ as an element of the $\mathfrak{gl}_{\infty|\infty}$ orbit of $|0\rangle$ via

$$r_0(E_{i_0,0})r_0(E_{i_1,\frac{-1}{2}})\dots r_0(E_{i_{2k},-k})|0\rangle.$$

Not only is $\mathfrak{F}^{(0)}$ generated by $|0\rangle$, but then $\mathfrak{F}_0^{(0)}$ is *spanned by* $|0\rangle$. *As both W and $W^\perp$ respect the energy decomposition, $|0\rangle$ must be contained in only one of W or $W^\perp$. If $|0\rangle \in W$, clearly each $\mathfrak{F}_k^{(0)}$ belongs to W, thus $W = \mathfrak{F}^{(0)}$ and $W^\perp = 0$. A similar argument is made if $|0\rangle \in W^\perp$.* □

Further, we can similarly define the representation r_m of $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}^{(m)}$ for every $m \in \frac{1}{2}\mathbb{Z}$. As each space $\mathfrak{F}^{(m)}$ can be generated by its corresponding vacuum vector $|m\rangle$, a similar argument to the above can be made for each $m \in \frac{1}{2}\mathbb{Z}$.

Corollary 3.2.5. *For each $m \in \frac{1}{2}\mathbb{Z}$, the representation r_m of $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}^{(m)}$ is irreducible.*

We now have a representation

$$r = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} r_m.$$

of $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}$. We will need to revisit this representation and its properties in greater detail when we consider the central extension of $\mathfrak{gl}_{\infty|\infty}$ in Section 3.5.

3.3 Creation and annihilation operators

We next look to develop creation (wedging) and annihilation (contracting) operators that will allow us to connect our representation of $\mathfrak{gl}_{\infty|\infty}$ to an algebraic structure that will further enable our study of r and $\mathfrak{F}$. That structure will be a Clifford superalgebra.

Now for $j \in \frac{1}{2}\mathbb{Z}$, we introduce two operators $\tilde{\psi}_j$ and $\tilde{\psi}_j^*$ on $\mathfrak{F}$ by the following:

$$\tilde{\psi}_j(v_{i_0} \wedge v_{i_1} \wedge \dots) = \begin{cases} 0 & \text{if } |j| = \bar{0}, \text{ and } j = i_k \text{ for some } k \\ v_j \wedge v_{i_0} \wedge v_{i_1} \wedge \dots & \text{if } j \neq i_s, \forall s, \text{ or } |j| = \bar{1} \end{cases}$$

$$\tilde{\psi}_j^*(v_{i_0} \wedge v_{i_1} \wedge \dots) = \begin{cases} 0 & \text{if } j \neq i_k \text{ for all } k, \\ m(j)(-1)^k(-1)^{|j|\sum_{l=0}^{k-1}|i_l|} v_{i_0} \wedge \dots \wedge \widehat{v_{i_k}} \wedge \dots & \text{if } j = i_k\ \exists\ k, \\ & \text{where } k = \min(\Phi) \end{cases}$$

Note that if $|j| = \bar{0}$ and $m(j) > 1$ then the monomial $v_{i_0} \wedge v_{i_1} \wedge \dots$ is simply 0.

Proposition 3.3.1. *Given the representation r of $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}$,*

$$r(E_{ij}) = \tilde{\psi}_i \tilde{\psi}_j^*$$

Proof. The proof is simply computational due to our construction. We can look at r_0, as each r_m is similarly defined on $\mathfrak{F}^{(m)}$.

$$\begin{aligned} \tilde{\psi}_i \tilde{\psi}_j^*(v_{i_0} \wedge v_{i_1} \wedge \dots) &= \delta_{j,i_k} m(j)(-1)^k(-1)^{|j|\sum_{l=0}^{k-1}|i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_k}} \wedge \dots \\ &= \delta_{j,i_k} m(j)(-1)^q(-1)^{|j|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\ &= r(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \dots) \end{aligned}$$

□

Combining the previous two results with (3.5), we have the following result which will eventually be extended to a larger Lie superalgebra.

Corollary 3.3.2. *The space $\mathfrak{F}_k^{(m)}$ is the linear span the elements of the form*

$$\tilde{\psi}_{i_n} \dots \tilde{\psi}_{i_1} \tilde{\psi}_{j_n}^* \dots \tilde{\psi}_{j_1}^* |m\rangle$$

where

$$\sum_{\ell=1}^{n} i_\ell - \sum_{\ell=1}^{n} j_\ell = k.$$

Note that the $\tilde{\psi}_{i_k}$ need not be distinct in the case that i_k is odd. Although $\mathfrak{F}_k^{(m)}$ is the linear span of these elements, we do not have a nice way to compute $\dim F_k^{(m)}$ yet, as we did in the Lie algebra case (read: partition, Young diagram). To understand $\mathfrak{F}^{(m)}$, we want to examine this further. We will outline the process here, but rigorously develop it in Appendix A.

Consider that even operators still behave nicely - meaning we can only contract even vector v_i if that vector is present in $|m\rangle$ and can only wedge in v_i if it is not present, else we annihilate the monomial. However, odd vectors may be wedged in anywhere, and in any finite multiplicity. This creates a problem trying to establish a correspondence between monomials and nice combinatorial tools, such as Young diagrams.

Then another way to realize $\mathfrak{F}^{(m)}$ might be to write any element in $\mathfrak{F}_k^{(m)}$ in the form

$$\tilde{\psi}_{i_s}^{t_s} \dots \tilde{\psi}_{i_1}^{t_1} \tilde{\psi}_{j_p}^{*} \dots \tilde{\psi}_{j_1}^{*} |m\rangle \tag{3.6}$$

where

$$\sum_{\ell=1}^{s} i_\ell t_\ell - \sum_{\gamma=1}^{p} j_\gamma = k.$$

Here $t_\alpha = 1$ if $|\alpha| = 0$, and (3.6) can be ordered such that

$$i_s < i_{s-1} \text{ and } j_p > j_{p-1}. \tag{3.7}$$

We can even further require $j_p \leq m$ to cut out the superfluous contracting of non-present vectors.

One should suspect that there is a combinatorial method for computing $\dim_q \mathfrak{F}^{(m)}$ by using a configuration similar to the above to compute $\dim \mathfrak{F}_k^{(m)}$. We begin this process in Appendix 2. In summary, by assigning a new form for each monomial via the PBW theorem, we can construct four Young diagrams that correspond to four types of operators present in the new form:

- odd vectors created with index less than the charge,
- odd vectors created with index greater than the charge,

- even vectors created with index greater than the charge,
- vectors contracted from $|m\rangle$.

This is done via a natural arrangement that is expressed with our operators. Then we assign the distance between each the index of a vector and the charge m to be half of the length of a row in a corresponding Young diagram. We can then uniquely determine a Young diagram for each of the above for types of operators, creating a unique 4-tuple of Young diagrams for each monomial. This is useful because we then know exactly which vectors are present in our monomial, and should be able to use this to study the $\dim \mathfrak{F}_k^{(m)}$. As mentioned, this area requires more attention and is thus relegated to Appendix A.

3.4 Generating a Clifford Superalgebra

A central aspect of the Boson-Fermion correspondence was creating a connection between the language of Clifford algebras and that of Heisenberg Lie algebras. Clifford algebras have an impressive and far-reaching role in various mathematical studies and applications. Studying these algebras, as well as constructing them, is what led us to the Kac papers that motivate this dissertation.

Previous results in constructing infinitely generated Clifford algebras are included in Chapter 4. Our original goal stemming from [17] was to construct a new representation of $\mathfrak{gl}_{\infty|\infty}$ (now laid out in the previous sections), corresponding operators that would generate a Clifford superalgebra for which we could study the structure and related representation theory.

Here we view the operators $\tilde{\psi}_i$, and $\tilde{\psi}_i^*$ in the context of generators of a Clifford superalgebra. We begin with the following lemma.

Lemma 3.4.1. *Given $\tilde{\psi}_i$, $\tilde{\psi}_i^*$ as defined above, they satisfy the following relations*

(1) $\tilde{\psi}_i\tilde{\psi}_j^* + (-1)^{|i||j|}\tilde{\psi}_j^*\tilde{\psi}_i = (-1)^{|i|}\delta_{i,j}$,

(2) $\tilde{\psi}_i\tilde{\psi}_j + (-1)^{|i||j|}\tilde{\psi}_j\tilde{\psi}_i = 0$,

(3) $\tilde{\psi}_i^*\tilde{\psi}_j^* + (-1)^{|i||j|}\tilde{\psi}_j^*\tilde{\psi}_i^* = 0$.

Proof. In each of the following, let

$$\varphi = v_{i_0} \wedge v_{i_1} \wedge \dots \quad .$$

For (1), let $\Theta_1 = \tilde{\psi}_i \tilde{\psi}_j^* + (-1)^{|i||j|} \tilde{\psi}_j^* \tilde{\psi}_i$. Then if $i \neq j$, we have

$$\begin{aligned}
\Theta_1(\varphi) &= \delta_{j,i_q} m(j)(-1)^q (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&\quad + (-1)^{|i||j|} \delta_{j,i_q} m(j)(-1)^{q+1} (-1)^{|i||j|} (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&= \delta_{j,i_q} m(j)(-1)^q (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&\quad + \delta_{j,i_q} m(j)(-1)^{q+1} (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&= 0
\end{aligned}$$

Now for when $i = j$, it is clear that when $|i| = \bar{0}$,

$$(\tilde{\psi}_i \tilde{\psi}_i^* + (-1)^{|i|} \tilde{\psi}_i^* \tilde{\psi}_i)(\varphi) = (\tilde{\psi}_i \tilde{\psi}_i^* + \tilde{\psi}_i^* \tilde{\psi}_i)(\varphi) = \varphi.$$

However, when $|i| = \bar{1}$, we need to be careful, as

$$(\tilde{\psi}_i \tilde{\psi}_i^* + (-1)^{|i|} \tilde{\psi}_i^* \tilde{\psi}_i)(\varphi) = (\tilde{\psi}_i \tilde{\psi}_i^* - \tilde{\psi}_i^* \tilde{\psi}_i)(\varphi).$$

Now, if v_i is present in the monomial φ, then

$$\begin{aligned}
\tilde{\psi}_i \tilde{\psi}_i^*(\varphi) - \tilde{\psi}_i^* \tilde{\psi}_i(\varphi) &= m(i)(-1)^q (-1)^{|i| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&\quad - (m(i)+1)(-1)^q (-1)^{|i| \sum_{l=0}^{q-1} |i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \dots \wedge \widehat{v_{i_q}} \wedge \dots \\
&= -\varphi
\end{aligned}$$

as the second term will have one more v_i present when contracting, thus having a coefficient of $(m(i) + 1)$ while the other signs match by construction (this is easily seen if the first monomial is re-ordered after applying the contracting operator).

If v_i is not present, then the first term annihilates the monomial, and we have

$$\begin{aligned}(\tilde{\psi}_i\tilde{\psi}_i^* - \tilde{\psi}_i^*\tilde{\psi}_i)(\varphi) &= -\tilde{\psi}_i^*\tilde{\psi}_i(\varphi) \\ &= -(-1)^q(-1)^{|i|\sum_{l=0}^{q-1}|i_l|} v_i \wedge v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \ldots\end{aligned}$$

which we can reorder, returning v_i to the position vacated by v_{i_q}, yielding the desired result of

$$(\tilde{\psi}_i\tilde{\psi}_i^* - \tilde{\psi}_i^*\tilde{\psi}_i)(\varphi) = -\varphi.$$

Thus, for $|i| \in \mathbb{Z}_2$,

$$\tilde{\psi}_i\tilde{\psi}_i^* + (-1)^{|i|}\tilde{\psi}_i^*\tilde{\psi}_i = (-1)^{|i|}\mathrm{I}.$$

Now for (2), see that if $i \neq j$ then

$$\begin{aligned}(\tilde{\psi}_i\tilde{\psi}_j + (-1)^{|i||j|}\tilde{\psi}_j\tilde{\psi}_i)(\varphi) &= v_i \wedge v_j \wedge v_{i_0} \wedge \ldots + (-1)^{|i||j|}v_j \wedge v_i \wedge v_{i_0} \wedge \ldots \\ &= v_i \wedge v_j \wedge v_{i_0} \wedge \ldots - (-1)^{2|i||j|}v_i \wedge v_j \wedge v_{i_0} \wedge \ldots \\ &= 0\end{aligned}$$

However, if $i = j$, consider first if $|i| = \bar{0}$, then

$$\begin{aligned}(\tilde{\psi}_i\tilde{\psi}_j + (-1)^{|i||j|}\tilde{\psi}_j\tilde{\psi}_i)(\varphi) &= (\tilde{\psi}_i\tilde{\psi}_i)(\varphi) + (\tilde{\psi}_i\tilde{\psi}_i)(\varphi) \\ &= 0\end{aligned}$$

as applying the same even wedging operator twice annihilates the monomial. If $|i| = \bar{1}$, then

$$\begin{aligned}(\tilde{\psi}_i\tilde{\psi}_j + (-1)^{|i||j|}\tilde{\psi}_j\tilde{\psi}_i)(\varphi) &= (\tilde{\psi}_i\tilde{\psi}_i)(\varphi) - (\tilde{\psi}_i\tilde{\psi}_i)(\varphi) \\ &= 0\end{aligned}$$

For (3), without loss of generality take $q > k$. Then we see that

$$\begin{aligned}
&\tilde{\psi}_i^* \tilde{\psi}_j^* + (-1)^{|i||j|} \tilde{\psi}_j^* \tilde{\psi}_i^* \\
&= \tilde{\psi}_i^* \left(\delta_{j,i_q} m(j)(-1)^q (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \dots \right) \\
&\quad + (-1)^{|i||j|} \tilde{\psi}_j^* \left(\delta_{i,i_k} m(i)(-1)^k (-1)^{|i| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \dots \right)
\end{aligned}$$

$$\begin{aligned}
&= \delta_{j,i_q} \delta_{i,i_k} m(i) m(j) (-1)^{q+k} (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} (-1)^{|i| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \dots \\
&\quad + (-1)^{|i||j|} \delta_{j,i_q} \delta_{i,i_k} m(j) m(i) (-1)^{q+k-1} (-1)^{|j| \left((\sum_{l=0}^{1-1} |i_l|) - |i| \right)} \\
&\qquad \cdot (-1)^{|i| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \dots
\end{aligned}$$

$$\begin{aligned}
&= \delta_{j,i_q} \delta_{i,i_k} m(i) m(j) (-1)^{q+k} (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} (-1)^{|i| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \dots \\
&\quad + \delta_{j,i_q} \delta_{i,i_k} m(j) m(i) (-1)^{q+k-1} (-1)^{|j| \sum_{l=0}^{q-1} |i_l|} (-1)^{|i| \sum_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots \wedge \widehat{v_{i_q}} \wedge \dots \\
&= 0
\end{aligned}$$

Finally, when $i = j$, consider first if $|i| = \bar{0}$, then

$$\begin{aligned}
(\tilde{\psi}_i^* \tilde{\psi}_j^* + (-1)^{|i||j|} \tilde{\psi}_j^* \tilde{\psi}_i^*)(\varphi) &= (\tilde{\psi}_i^* \tilde{\psi}_i^*)(\varphi) + (\tilde{\psi}_i^* \tilde{\psi}_i^*)(\varphi) \\
&= 0 + 0
\end{aligned}$$

as applying the same even contracting operator twice annihilates the monomial. If $|i| = \bar{1}$, then

$$\begin{aligned}
(\tilde{\psi}_i^* \tilde{\psi}_j^* + (-1)^{|i||j|} \tilde{\psi}_j^* \tilde{\psi}_i^*)(\varphi) &= (\tilde{\psi}_i^* \tilde{\psi}_i^*)(\varphi) - (\tilde{\psi}_i^* \tilde{\psi}_i^*)(\varphi) \\
&= 0
\end{aligned}$$

□

Citing the lemma and the well-known definition of Clifford superalgebra, we have our result:

Theorem 3.4.2. *The operators* $\tilde{\psi}_j$ *and* $\tilde{\psi}^*_j$ *generate a Clifford superalgebra* $\widehat{\mathcal{C}l}$.

We can then appeal to Corollary 3.3.1 for the following:

Corollary 3.4.3. *The Lie superalgebra* $\mathfrak{gl}_{\infty|\infty}$ *embeds into the Clifford superalgebra* $\mathcal{C}l$ *via*

$$r(E_{ij}) = \tilde{\psi}_i \tilde{\psi}^*_j.$$

Note that Lemma 3.4.1 paired with the previous two results gives us a completely new Clifford superalgebra, distinct from Kac and van de Leur's. The Clifford superalgebra $\widehat{\mathcal{C}l}$ does not have a positive form, and this raises interesting structural questions.

We now take a moment to highlight our reasoning for the particular positioning of v_i when defining the representation r_0 of $\mathfrak{gl}_{\infty|\infty}$ on $\mathfrak{F}$.

Remark 3.4.4 (Explanation for bringing v_i to front of monomial)**.** Recall that our original goal was to construct an infinitely generated Clifford algebra from a representation of $\mathfrak{gl}_{\infty|\infty}$. If we simply took the Lie superalgebra action, there would be a problem with explicitly defining our Clifford superalgebra generators as operators on $\mathfrak{gl}_{\infty|\infty}$. That is, if $\varphi = v_{i_0} \wedge v_{i_1} \wedge \dots$, consider r would be

$$\begin{aligned} r(E_{ij})(\varphi) &= \sum_{k \geq 0} \delta_{j,i_k} (-1)^{|E_{ij}| \sum\limits_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \dots \wedge v_{i_{k-1}} \wedge E_{ij} v_{i_k} \wedge \dots \\ &= \sum_{k \geq 0} \delta_{j,i_k} (-1)^{|E_{ij}| \sum\limits_{l=0}^{k-1} |i_l|} v_{i_0} \wedge \dots \wedge v_{i_{k-1}} \wedge v_i \wedge \dots \end{aligned} \tag{3.8}$$

The operators would *both* need to take into account the parity of i due to the possible sign coming from taking $|E_{ij}| = |i| + |j|$ involved in each operator. This issue precludes the possibility of **independent** operators $\tilde{\psi}_j$ and $\tilde{\psi}^*_j$ composing in a way that allows us construct $\widehat{\mathcal{C}l}$.

A natural instinct would be to then place v_i in its proper place considering the index ordering. Assume that $k > p$, then doing so would result in an additional sign factor

$$(-1)^{k-p}(-1)^{|i|\sum\limits_{l=p}^{k-1}|i_l|},$$

which relies on the position of both i_k and i_p. This seems to cause a similar problem. However, after developing much of the theory found in this chapter, we revisited this representation and were able to construct corresponding operators that served as Clifford superalgebra operators using cases.

For computational purposes and time constraints, we maintain our reordering of the monomial, placing v_i in the first position. We can thus express the representation in terms of independent operators. We intend to revisit the reordering where v_i is placed in its proper position considering the index ordering.

We return to the Clifford superalgebra $\widehat{\mathcal{C}l}$, and we wish to develop a corresponding spin module. In order to do so, we first highlight a few necessary results.

Proposition 3.4.5. *As a $\widehat{\mathcal{C}l}$-module, $\mathfrak{F}$ is irreducible.*

Proof. As $\mathfrak{F}$ is non-zero, it will be irreducible if $\mathfrak{F} = \widehat{\mathcal{C}l}\varphi$ for all non-zero $\varphi \in \mathfrak{F}$. This is clear, as $\widehat{\mathcal{C}l}\varphi \subseteq \mathfrak{F}$ since the operators generating $\widehat{\mathcal{C}l}$ are operators on $\mathfrak{F}$, the orbit by definition is in $\mathfrak{F}$. Further, since monomials form a basis for $\mathfrak{F}$, if we take a monomial $\varphi \in \mathfrak{F}$ it belongs to some $\mathfrak{F}^{(m)}$ by our charge decomposition. Now, by Corollary 3.3.2, that element can be written in terms of elements of $\widehat{\mathcal{C}l}$ acting on the vacuum vector $|m\rangle$. □

It is straightforward to check that in addition to relations in Lemma 3.4.1, we have

$$\tilde{\psi}_j|0\rangle = 0, \quad \text{for } j \leq 0, \quad |j| = \bar{0}; \qquad \tilde{\psi}_j^*|0\rangle = 0 \quad \text{for } j \geq 0.$$

One can associate $\widehat{\mathcal{C}l}$ with the space

$$U = \sum_i \mathbb{C}\tilde{\psi}_i + \sum_i \mathbb{C}\tilde{\psi}_i^*.$$

with bilinear form $(\tilde{\psi}_i|\tilde{\psi}^*_j) = (-1)^{|i|}\delta_{ij}$. The reason for the signage here is easily checked with Lemma 3.4.1.

Now given a maximal isotropic subspace W of U, the Clifford superalgebra has a unique irreducible module called the *spin module* which admits a non-zero vector which W annihilates. We consider the subspace of U given by

$$W = \sum_{\substack{i \leq 0 \\ |i| = \bar{0}}} \mathbb{C}\tilde{\psi}_i + \sum_{i>0} \mathbb{C}\tilde{\psi}^*_i.$$

Proposition 3.4.6. *The subspace W of U is maximally isotropic.*

Proof. Consider the nonzero vector $|0\rangle = v_0 \wedge v_{-\frac{1}{2}} \wedge v_{-1} \dots$ for which we see $W|0\rangle = 0$. This is clear, as any element in W can only create an even vector with index $i \leq 0$, and that even vector is already present in $|0\rangle$. Also, an element in W can only contract a vector with index $i \geq 0$, which will not be present in $|0\rangle$. Thus, by the definitions of both ψ_i and ψ^*_i, W is isotropic.

Seeing that W is maximally isotropic is fairly straightforward by the construction of our operators. Let $X \supsetneq W$ be maximally isotropic. Then X must contain one of the following:

(1) a generator ψ_i with $i > 0$ or $|i| = \bar{1}$, or

(2) a generator ψ^*_i with $i \leq 0$.

We immediately eliminate the possibility of X containing an element ψ_i with $|i| = \bar{1}$, as this operator will not kill any monomial. If X contains ψ_i with $i > 0$, then for any monomial $\varphi \in \mathfrak{F}$, either $\psi_i(\varphi) \neq 0$ or $\psi^*_i(\varphi) \neq 0$, depending on whether v_i is present. Yet, both ψ_i and ψ^*_i are in X. A similar argument can be made if X contains ψ_i with $i \geq 0$ to show that $X\varphi \neq$ for any $\varphi \in \mathfrak{F}$. □

Lastly, it is evident that $|0\rangle$ generates $\mathfrak{F}$ as a $\widehat{\mathcal{C}l}$−module as we can take $|0\rangle$ to any other vacuum vector $|m\rangle$ by applying necessary operators ψ_i or ψ^*_i (see Corollary 3.3.2). We say $\mathfrak{F}$ is the **spin module** associated with the subspace W. It is worth noting that $|0\rangle$ is not the only monomial that W annihilates, due to the possible presence of multiple odd vectors. However, these monomials all generate $\mathfrak{F}$ as a $\widehat{\mathcal{C}l}$-module as well.

The Clifford superalgebra $\widehat{\mathcal{C}l}$ has been constructed, with $\mathfrak{F}$ as its spin module, and we have embedded $\mathfrak{gl}_{\infty|\infty}$ into $\widehat{\mathcal{C}l}$. We now turn our focus to larger Lie superalgebras.

3.5 The central extension $\widehat{\mathfrak{gl}}_{\infty|\infty}$

For this section, we will be looking at a central extension of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$. The definition of a central extension of a Lie superalgebra can be found in Definition 1.4.7, and more details concerning this particular central extension can be found in [5]. Let Φ be as defined before Proposition 3.2.3.

Now let

$$\widehat{\mathfrak{gl}}_{\infty|\infty} = \mathfrak{gl}_{\infty|\infty} \oplus \mathbb{C}c$$

be the central extension of $\mathfrak{gl}_{\infty|\infty}$ by a one-dimensional center $\mathbb{C}c$ given by the 2-cocycle

$$\alpha(A, B) := Str([J, A]B)$$

where $J = \sum_{r \leq 0} E_{rr}$. Also, we define the *supertrace* of a matrix C by

$$\text{Str } C = \sum_{r \in \frac{1}{2}\mathbb{Z}} (-1)^{2r} c_{rr}.$$

If we assign the Cartan subalgebra $\bar{\mathfrak{h}} = \sum_{r \in \frac{1}{2}\mathbb{Z}} \mathbb{C}E_{rr} \oplus \mathbb{C}c$ degree 0, we maintain a $\frac{1}{2}\mathbb{Z}$-gradation

$$\widehat{\mathfrak{gl}}_{\infty|\infty} = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} (\widehat{\mathfrak{gl}}_{\infty|\infty})_k,$$

where $\deg E_{ij} = j - i$.

However, if we try to extend the representation r_0 (respectively, r_m) to the Lie superalgebra $\widehat{\mathfrak{gl}}_{\infty|\infty}$, we encounter a problem when we consider

$$r_0(\sum_{i \in \frac{1}{2}\mathbb{Z}} \lambda_i E_{ii})|0\rangle.$$

Consider that for every $i \in \Phi$, $r_0(E_{ii})$ sends v_i in a given position to v_i in the first position of the monomial. That is we have

$$\lambda_0 v_0 \wedge v_{-\frac{1}{2}} \wedge \cdots + \sigma(-\frac{1}{2})\lambda_{-\frac{1}{2}} v_{-\frac{1}{2}} \wedge v_0 \wedge v_{-1} \wedge \cdots + \ldots$$

where $\sigma(i)$ is the sign gained from applying $r_0(E_{ii})$. Now, simple rearranging by moving v_i to its original position yields

$$\sum_{i \leq 0} \sigma(i)\lambda_i|0\rangle$$

where $i \in \frac{1}{2}\mathbb{Z}$. The above series may very well be divergent in this context. There is a similar issue in the Lie algebra case and in Kac and van de Leur's existing super analogue, albeit with a bigger algebra. However, given we have a new representation on a *different* space, we need an new analogue.

We can remove this one with ease by changing the representation r_0 as follows:

$$\begin{aligned} \hat{r}_0(E_{ij}) &= r_0(E_{ij}), \quad \text{if } i \neq j \text{ or } i = j > 0 \\ \hat{r}_0(E_{ii}) &= r_0(E_{ii}) - \mathrm{I}, \quad \text{for } i \leq 0 \\ \hat{r}_0(c) &= \mathrm{I} \end{aligned} \tag{3.9}$$

It is important to see that this kills the vacuum vector $|0\rangle$. First, the sign obtained from taking E_{ii} to the appropriate position in $|0\rangle$ will be positive, as $|E_{ii}| = |i| + |i| = \bar{0}$. Then acting on v_i with E_{ii} results in $|0\rangle$. Now, although wedging v_i back to the front position will acquire a sign dependent on the parity of i and all the vectors it passes, the result is clearly equivalent to having left v_i in its position, which is the vacuum vector itself.

We define $\hat{r}_m$ using r_m, for all $m \in \frac{1}{2}\mathbb{Z}$, in a similar way. Then r_0 extends to obtain a linear representation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ on the space $\mathfrak{F}^{(0)}$ (resp, $\hat{r}_m$ to $\mathfrak{F}^{(m)}$). These are irreducible by the construction of our extension. Define the representation

$$\hat{r} = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \hat{r}_m$$

of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ on $\mathfrak{F}$.

If we recall our principal gradation from earlier,

$$\widehat{\mathfrak{gl}}_{\infty|\infty} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} (\widehat{\mathfrak{gl}}_{\infty|\infty})_k,$$

this then induces a principal gradation of $\mathfrak{F}^{(m)}$ consistent with the gradation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$, namely

$$\mathfrak{F}^{(m)} = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} \mathfrak{F}^{(m)}_k.$$

So now for $a \in (\widehat{\mathfrak{gl}}_{\infty|\infty})_i$,

$$r_m(a)\mathfrak{F}^{(m)}_j \subseteq \mathfrak{F}^{(m)}_{j-i}.$$

Again, we can describe the space $\mathfrak{F}^{(m)}_k$ explicitly in terms of the linear span of elements in $\mathfrak{F}^{(m)}$ of the form

$$r_m(E_{i_1j_1})r_m(E_{i_2j_2})\dots r_m(E_{i_nj_n})|m\rangle \tag{3.10}$$

where

$$\sum_{\ell=1}^{n} i_\ell - \sum_{\ell=1}^{n} j_\ell = k.$$

A result similar to Corollary 3.3.2 follows, allowing us to view these elements in terms of the Clifford superalgebra generators. These elements form a basis for $\mathfrak{F}^{(m)}_k$. Using the PBW Theorem for superalgebras together with the above, we could assume that $j_s < i_s$ for all $1 \le s \le n$, and it would then follow that $\mathfrak{F}^{(m)}_k = 0$ for $k < 0$ and $\mathfrak{F}^{(m)}_0 = \mathbb{C}|m\rangle$.

We now look to construct a subsuperalgebra, $\mathfrak{s}$, of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ for which the $\mathfrak{F}^{(m)}$ modules considered as modules of $\mathfrak{s}$ will remain irreducible. In this effort, we hope to establish a clear correspondence between our constructed Clifford superalgebra and this new subalgebra.

We begin by defining the following elements,

$$\lambda_\Psi(n) = \sum_{k\in\Psi} E_{k,k+n}, \qquad \mu_\Psi(n) = \sum_{k\in\Psi} E_{k-\frac{1}{2},k+n-\frac{1}{2}},$$
$$e_\Psi(n) = \sum_{k\in\Psi} E_{k-\frac{1}{2},k+n}, \qquad f_\Psi(n) = \sum_{k\in\Psi} E_{k,k+n-\frac{1}{2}},$$

where Ψ is a finite subset of $\mathbb{Z}$. Our next goal is to show that these elements, together with the central element c from $\widehat{\mathfrak{gl}}_{\infty|\infty}$, actually form a basis of a subsuperalgebra of $\widehat{\mathfrak{gl}}_{\infty|\infty}$, which will be $\mathfrak{s}$. To do so, we determine the following commutation relations for $\mathfrak{s}$ (the proofs to these are in Appendix B).

Note that $\gamma(n)$ is the number of elements in $(\Phi_1 \cap \Phi_2) \cap \{1, 2, \ldots, n\}$ (an explanation for its appearance is in the appendices).

Proposition 3.5.1. *The following commutation relations hold in* $\widehat{\mathfrak{gl}}_{\infty|\infty}$*,*

$$[\lambda_{\Psi_1}(n), e_{\Psi_2}(m)] = -e_{\Psi_1\cap\Psi_2}(m+n), \qquad [\lambda_{\Psi_1}(n), f_{\Psi_2}(m)] = f_{\Psi_1\cap\Psi_2}(m+n),$$
$$[\mu_{\Psi_1}(n), e_{\Psi_2}(m)] = e_{\Psi_1\cap\Psi_2}(m+n), \qquad [\mu_{\Psi_1}(n), f_{\Psi_2}(m)] = -f_{\Psi_1\cap\Psi_2}(m+n),$$
$$[\lambda_{\Psi_1}(n), \lambda_{\Psi_2}(m)] = \gamma(n)\delta_{m,-n}c, \qquad [\mu_{\Psi_1}(n), \mu_{\Psi_2}(m)] = -\gamma(n)\delta_{m,-n}c,$$
$$[\lambda_{\Psi_1}(n), \mu_{\Psi_2}(m)] = 0,$$
$$[e_{\Psi_1}(n), f_{\Psi_2}(m)] = \lambda_{\Psi_1\cap\Psi_2}(m+n) + \mu_{\Psi_1\cap\Psi_2}(m+n) - n\delta_{m,-c}c$$

This issue is revisited in the next section, and warrants further study beyond this dissertation.

Proposition 3.5.2.

$$\hat{r}_m(f_\Psi(n))|m\rangle = 0, \quad \textit{for } n \geq 1 \qquad \hat{r}_m(\lambda_\Psi(n))|m\rangle = 0 \quad \textit{for } n > 0,$$
$$\hat{r}_m(c) = I, \qquad \hat{r}_m(\lambda_\Psi(0))|m\rangle = 0,$$
$$\hat{r}_m(\mu_\Psi(0))|m\rangle = (m(k - \frac{1}{2}) - 1)|m\rangle$$

Proof. We will prove the first relation here, while the remaining computations are similar and can be found in Appendix B. We examine the first relation,

$$\begin{aligned}\hat{r}_m(f(n))|m\rangle &= \sum_{k\in\Psi}(E_{k,k+n-\frac{1}{2}})|m\rangle \\ &= \sum_{k\in\Psi}\sigma(k)v_k \wedge v_m \wedge \cdots \wedge v_{\widehat{k+n-\frac{1}{2}}} \wedge \end{aligned} \tag{3.11}$$

where $\sigma(k) = m(k)(-1)^q(-1)^{|k|\sum_{\ell=0}^{q-1}|i_\ell|}$ from Proposition (3.3.2). Consider that the sum in (3.11) will be 0 when

$$k + n - \frac{1}{2} > m$$

due to $v_{k+n-\frac{1}{2}}$ not being in $|m\rangle$. Also, the sum in (3.11) is 0 when

$$k < m$$

due to v_k being present in $|m\rangle$. Combining these two cases we see (3.11) = 0 when $n - \frac{1}{2} > 0 \implies n \geq 1$, as $n \in \mathbb{Z}$. □

We have formed $\mathfrak{s}$ using the elements $\lambda_\Psi(n)$, $\mu_\Psi(n)$, $e_\Psi(n)$, and $f_\Psi(n)$ and the central element c. Our goal now is to describe these elements in terms of the Clifford superalgebra generators, $\tilde{\psi}_i$, $\tilde{\psi}_i^*$. First we must revisit Lemma 3.4.1 to account for some issues when examining $\psi_i \psi_i^*$. Recall that, for $|i| \in \mathbb{Z}_2$,

$$\tilde{\psi}_i \tilde{\psi}_i^* + (-1)^{|i|} \tilde{\psi}_i^* \tilde{\psi}_i = (-1)^{|i|} \mathrm{I}. \tag{3.12}$$

Now we look to write the elements of $\mathfrak{s}$ in terms of $\tilde{\psi}_i$ and $\tilde{\psi}_i^*$, making note of the above equality. We introduce a method of composing the expressions $\tilde{\psi}_i$, $\tilde{\psi}_i^*$ to account for (3.12).

Definition 3.5.3. The **composition extension** of expressions $\tilde{\psi}_i \tilde{\psi}_j^*$ will be given by

$$:: \tilde{\psi}_i \tilde{\psi}_j^* :: \quad = \begin{cases} \tilde{\psi}_i \tilde{\psi}_j^* & \text{if } \; i \neq j, \text{ or if } \; i = j > 0, \\ \tilde{\psi}_i \tilde{\psi}_j^* - \mathrm{I} & \text{if } \; i = j, \; i \leq 0, \end{cases}$$

By this composition, if $i \neq j$ then by (3.9) there is no need for adjustment. For if $i = j > 0$, we appeal to both (3.9) and Lemma 3.4.1. However, if $i = j \leq 0$, according to (3.9), our representation annihilates the appropriate vacuum vector. Note that if $|i| = 0$, then we can appeal to Lemma 3.4.1 to see

$$\begin{aligned} \hat{r}(E_{ii})(\varphi) &= (\tilde{\psi}_i \tilde{\psi}_i^* - \mathrm{I})(\varphi) \\ &= -\tilde{\psi}_i^* \tilde{\psi}_i(\varphi) \\ &= 0, \end{aligned}$$

as desired. However, if $|i| = 1$, and we appeal to Lemma 3.4.1, we have the relation

$$(\tilde{\psi}_i \tilde{\psi}_i^* + \mathrm{I})(\varphi) = -\tilde{\psi}_i^* \tilde{\psi}_i(\varphi),$$

which does immediately apply to our definition of $\hat{r}$. The issue is that $\tilde{\psi}_i$ does not annihilate the monomial for odd i. It is important to notice that altering (3.9) to allow the above definition to utilize Lemma 3.4.1 in this last case where $i = j \leq 0$, and $|i| = 1$ would cause a problem with well-definedness in our representation. Hence, the adjustment is made here instead, forcing us to associate $:: \tilde{\psi}_i \tilde{\psi}_j^* ::$ with $\tilde{\psi}_i \tilde{\psi}_j^* - \mathrm{I} = 0$ according to (3.9). We now give a correspondence that results from Proposition 3.3.1, our definition of $\hat{r}$ in (3.9), and the above composition extension. It should be noted that for $i \neq j$, this is simple by design.

Theorem 3.5.4. *Given the composition extension of* $\tilde{\psi}_i \tilde{\psi}_j^*$,

$$\lambda_\Psi(n) = \sum_{k \in \Psi} :: \tilde{\psi}_k \tilde{\psi}^*{}_{k+n} ::, \qquad \mu_\Psi(n) = \sum_{k \in \Psi} :: \tilde{\psi}_{k-\frac{1}{2}} \tilde{\psi}^*{}_{k+n-\frac{1}{2}} ::,$$

$$e_\Psi(n) = \sum_{k \in \Psi} :: \tilde{\psi}_{k-\frac{1}{2}} \tilde{\psi}^*{}_{k+n} ::, \qquad f_\Psi(n) = \sum_{k \in \Psi} :: \tilde{\psi}_k \tilde{\psi}^*{}_{k+n-\frac{1}{2}} :: .$$

As a result of our new construction, there are some lingering questions that need to be addressed. The first is how to develop the last theorem starting first from the Clifford superalgebra. Here the first step would seem to be defining the generating series of the Clifford superalgebra operators:

$$\tilde{\psi}_{\bar{0}}(z) = \sum_{i \in \mathbb{Z}} \tilde{\psi}_i z^i, \qquad \tilde{\psi}^*_{\bar{0}}(z) = \sum_{i \in \mathbb{Z}} \tilde{\psi}_i z^{-i}$$

$$\tilde{\psi}_{\bar{1}}(z) = \sum_{i \in \frac{1}{2}+\mathbb{Z}} \tilde{\psi}_i z^i \qquad \tilde{\psi}^*_{\bar{1}}(z) = -\sum_{i \in \frac{1}{2}+\mathbb{Z}} \tilde{\psi}_i z^{-i}$$

where $z \in \mathbb{C}^\times$. These operators would map $\mathfrak{F}$ into its formal completion $\overline{\mathfrak{F}}$. In order to construct the formal completion of $\mathfrak{F}$, we first need to let $\overline{\mathfrak{F}^{(m)}}$ denote the formal completion of $\mathfrak{F}^{(m)}$, and then put

$$\widehat{\mathfrak{F}} = \prod_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{F}^{(m)}.$$

Another lingering question that is mentioned in Section 3.2 and Appendix A involves what $\dim_q \mathfrak{F}^{(m)}$ is precisely. However, as that question may almost be understood, the first question noted is how to explicitly formulate the rest of our correspondence.

Question 3.5.5. Can a result similar to Theorem 3.5.4 be found that describes the ψ_i and ψ_i^* in terms of the $\lambda_\Phi(n)$, $e_\Phi(n)$, $f_\Phi(n)$, and $\mu_\Phi(n)$?

Many questions involving the generating series of $\psi(z)$ and $\psi^*(z)$ remain, and hinge on the issue of divergence. An answer to this question would involve the creation of an analogue to the vertex operators Γ_- and Γ^+ from Chapter 2, and could also address what the corresponding construction of $\mathfrak{s}$ using the algebra of Laurent polynomials may be. Fully understanding this correspondence would also yield more information about our representation $\hat{r}$, including answer to the following question:

Question 3.5.6. Does the representation $\hat{r}$ of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ on $\mathfrak{F}$ remain irreducible when restricted to $\mathfrak{s}$?

Addressing the previous question would involve showing an $\mathfrak{s}$-invariant subspace of $\mathfrak{F}$ is $\widehat{\mathfrak{gl}}_{\infty|\infty}$ invariant, and should follow quickly once Question 3.5.5 is answered.

Question 3.5.7. What is the formulation of sections 3.2 through 3.5 when taken from the Weyl superalgebra instead of the Clifford superalgebra approach?

The boson-fermion correspondence helped bridge the language of Weyl algebras and Clifford algebras, and [16] has done that in a different superalgebra setting. It would be interesting to see the distinctions created by the formulation here when approaching this theory from the Weyl superalgebra viewpoint.

Finally, our main motivation for this study was constructing the Clifford superalgebra in Section 3.4. There are many open questions about the structure of infinite-dimensional Clifford algebras and superalgebras, and I would like to study the structure of this Clifford superalgebra, see [31, 29].

3.6 A note on $a_{\infty|\infty}$

Let's return again to the wedge representation r_0 of $\mathfrak{gl}_{\infty|\infty}$ on the space $\mathfrak{F}^{(0)}$. Now, we introduce the Lie superalgebra $a_{\infty|\infty}$ containing $\mathfrak{gl}_{\infty|\infty}$, defined as

$$\bar{a}_{\infty|\infty} = \left\{ (a_{ij})_{i,j\in\frac{1}{2}\mathbb{Z}} \mid \begin{array}{l} \text{for each } k \text{ the number of non-zero } a_{ij} \\ \text{with } j \leq k \text{ and } i \geq k \text{ is finite} \end{array} \right\}$$

Here $\bar{a}_{\infty|\infty}$ acts on a completion of V, namely

$$\bar{V} = \{ \sum_{j\in\frac{1}{2}\mathbb{Z}} c_j v_j \mid c_j = 0 \text{ for } j \gg 0 \}.$$

However, if we try to extend the representation r_0 (respectively, r_m) to the Lie superalgebra $\bar{a}_{\infty|\infty}$, we encounter a more serious problem than before. We still see that

$$r_0(\sum_{i\in\frac{1}{2}\mathbb{Z}} \lambda_i E_{ii})|0\rangle = \sum_{i\leq 0} \sigma(i)\lambda_i|0\rangle.$$

If this were the only issue, we could simply solve this problem as we did in the previous section. To better visualize the problem, introduce the shift matrix

$$\Lambda_k = \sum_{i\in\frac{1}{2}\mathbb{Z}} E_{i,i+k}.$$

Then for any k,

$$\begin{aligned} r_0(\lambda_k)v_{i_0} \wedge v_{i_1} \wedge \ldots &= r_0(\sum_{i\in\frac{1}{2}\mathbb{Z}} E_{i,i+k})v_{i_0} \wedge v_{i_1} \wedge \ldots \\ &= \sum_{i\in\frac{1}{2}\mathbb{Z}} r_0(E_{i,i+k})v_{i_0} \wedge v_{i_1} \wedge \ldots . \end{aligned}$$

This last sum will now have infinitely many terms.

If this issue of well-definedness were resolved, we could consider the central extension $a_{\infty|\infty} = \bar{a}_{\infty|\infty} \oplus \mathbb{C}c$ with center $\mathbb{C}c$ and bracket

$$[x, y] = xy - (-1)^{|x||y|}yx + \alpha(x, y)c,$$

where the cocycle α is defined by

$$\alpha(E_{ij}, E_{ji}) = -(-1)^{|i||j|}\alpha(E_{ji}, E_{ij}) = 1 \qquad \text{if } i \leq 0 < j$$
$$\alpha(E_{ij}, E_{ji}) = 0 \qquad \text{for all other cases .}$$

Further, compute

$$[\Lambda_k, \Lambda_n] = \begin{cases} 2\delta_{k,-n}c & \text{if } |i| = |j| = \bar{1} \\ \delta_{k,-n}c & \text{else.} \end{cases}$$

Looking at these elements as generators of a subalgebra along with central element c, one can view them in a similar way as the previous section, i.e.

$$\lambda(n) = \sum_{k \in \mathbb{Z}} E_{k,k+n}, \qquad \mu(n) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} E_{k,k+n},$$
$$e(n) = \sum_{k \in \mathbb{Z}} E_{k-\frac{1}{2},k+n}, \qquad f(n) = \sum_{k \in \mathbb{Z}} E_{k,k+n-\frac{1}{2}},$$

where the $\lambda(n)$ and $\mu(n)$ are the even generators. These two results permit us to think of the subalgebra they generate as a Heisenberg Lie superalgebra. For more, see [8].

Chapter 4

Clifford Algebra Theory

The study of Clifford algebras has intrigued scores of brilliant mathematical theorists, geometers, physicists, and engineers over the past century. Many of these well-known mathematicians - the likes of Chevellay, Bott, Weyl, and Atiyah - have become well known for other theoretical and applied work as well, but produced essential work in the theory of Clifford (or geometric) algebras [1, 7, 9, 30]. Much of the theoretical work in this area set the stage for many geometrical and physical applications, which have been unearthed by the likes of Shale and Stinespring [27] and Hestenes [13]. Work on the Dirac theory, for example, where the main success was a real formulation of the theory within the real Clifford algebra $\mathbb{R}_{1,3} = M_2(\mathbb{H})$ has been spurred by Hestenes, Lounesto, and others [12, 21, 20]. Recent applications have reached from cosmology and quantum theory to gravitation and computational geometry [4, 2]. The particular theory of infinitely generated Clifford algebra has also been expanding recently, with the focus varying from orthogonal groups [3], to ideal structures [31], to more general theory [29]. All of these avenues are of interest to the author and help illustrate the ever-expanding reach of the theory of Clifford algebras. The classification of finite-dimensional real Clifford algebras is well-known, as is the complex case. Our approach to infinitely generated Clifford algebras of real quadratic forms runs parallel to the existing finite dimensional theory, with the goal of better understanding the structural parallels to the finite dimensional algebras. Thus, we will look at the finite-dimensional case first.

4.1 Classification of finite-dimensional Clifford algebras

Ian Porteous' "Clifford Algebras and the Classical Groups" [24] provides a study of the Clifford algebras of real quadratic forms and their complexifications. The central result of his book, the classification of the conjugation anti-involution of the Clifford algebras $\mathbb{R}_{p,q}$, lends to an exhaustive treatment of the generalizations of the orthogonal and unitary groups known as the classical Lie groups. Porteous goes to great lengths to show how well-adapted various Clifford algebras are to the study of the classical groups. Here we will begin with the classification of finite-dimensional Clifford algebras, which led to our study of the infinite-dimensional cases.

The following overview of the finite dimensional approach gives a frame parallel to what we want to establish in the infinite dimensional case. We will slightly reform our definition to the case of a *real* quadratic space V. The complex case is straightforward and well-known. [24]

First, we will introduce the quadratic form on a $\mathbb{K}$-linear space V, which we will use to construct a Clifford algebra. Here we take the characteristic of $\mathbb{K}$ to be not 2, and mention that the reason for notating the field $\mathbb{K}$ will become clear at the outset of our classification of the real Clifford algebras of finite-dimension.

Definition 4.1.1. A **quadratic form** is a pair (V, q), where $q : V \to \mathbb{K}$ such that

1. $q(\mu x) = \lambda^2 q(x), \quad x \in V, \quad \lambda \in \mathbb{K};$
2. the map $b : V \times V \to \mathbb{K}$ defined by $b(x, y) = (q(x + y) - q(x) - q(y))/2$ is bilinear.

The map b is often called the **polar** of q.

Another way to realize this is given a basis $\{e_i\}_{i=1}^n$ for V, and $x = \sum_{i=1}^n x_i e_i$, we have

$$q(x) = X^t Q X,$$

where X is the column vector of the x_i and Q is a symmetric matrix having $b(e_i, e_j)$ as ij-th entry. Note that we may assign a canonical basis of V. Further, if $V = \mathbb{K}^n$

and $\{e_i\}_{i=1}^n$ is the canonical basis of V, we can identify any quadratic form on V with a homogeneous polynomial in n variables of degree 2 over $\mathbb{K}$ by

$$q(x_1, \ldots, x_n) = \sum_{1 \le i,j \le n} q_{ij} x_i x_j$$

where the q_{ij} are the entries of the symmetric matrix Q, called the coefficient matrix of q.

A real linear space with quadratic form will be called a *(real) quadratic space,* and such a space is said to be *positive-definite,* if for all nonzero $x \in V$, we have $q(x) > 0$. Also, if X and Y are real quadratic spaces with a symmetric scalar product $\cdot$, then a map $f : X \to Y$ is said to be an *orthogonal map* if it is linear and, for all $a, b \in X$,

$$f(a) \cdot f(b) = a \cdot b.$$

The possible non-degenerate quadratic forms are classified by pairs (p, q) called the *signature* of the form, where $p + q = n$ and in some basis of $\mathbb{R}^n$ we can write

$$q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n} x_i^2.$$

The real linear space with this form is often denoted $\mathbb{R}^{p,q}$.

Example 4.1.2. Euclidean space is an example of a quadratic space with signature $(n, 0)$.

Example 4.1.3. The linear space $\mathbb{R}^{2n}$ with the scalar product

$$(x, y) \mapsto \sum_{1 \le i \le n} (x_i y_{n+i} + x_{n+i} y_i)$$

is typically denoted $\mathbb{R}^{2n}_{hb}$, with $\mathbb{R}^2_{hb}$ being known as the standard hyperbolic plane.

Example 4.1.4. An important example is Minkowski space. It is $\mathbb{R}^4$ with

$$q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 - x_3^2 - x_4^2$$

Typically, the $x_1 = ct-$coordinate comes from physics, where c is the speed of the light (often set to 1 for theoretical reasoning), and t is time.

If we have a quadratic space (V, q) with polar b, and subspace $W \subseteq V$. Define the orthogonal complement of W, which we will need later on, as

$$W_{\perp} = \{v \in V |\ b(v, w) = 0 \quad \text{for all} \quad w \in W\}.$$

In other words, $W_{\perp}$ is the maximal subspace of V which is orthogonal to W.

It is useful to understand there are multiple ways of defining quadratic forms. Below is a well-known result connecting four of these avenues:

Theorem 4.1.5 ([19]). *For $n \in \mathbb{Z}$, there are canonical bijections between the following sets:*

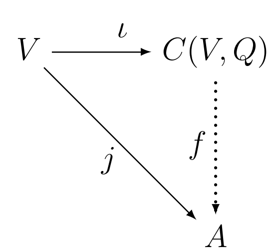
1. *The set of homogeneous quadratic polynomials $q(t) = q(t_1, \dots, t_n)$.*
2. *The set of homogeneous quadratic functions on $\mathbb{K}^n$.*
3. *The set of symmetric bilinear forms on $\mathbb{K}^n$.*
4. *The set of symmetric $n \times n$ matrices on $\mathbb{K}^n$.*

We now wish to draw out the idea of a Clifford algebra for a non-denegerate real quadratic space. Thus, we begin with a formation of Clifford algebras using a quadratic form.

Definition 4.1.6. A Clifford algebra, $C(V, q)$, is a unital associative algebra over $\mathbb{K}$ together with a linear map $\iota : V \to C(V, q)$ satisfying

$$\iota(v)^2 = -q(v)1, \quad \text{for all } v \in V,$$

defined by the following universal property that given any associative algebra A over $\mathbb{K}$ and any linear map $j : V \to A$ such that $j(v)^2 = -q(v)1_A$ for all $v \in V$, there is a unique homomorphism $f : C(V, q) \to A$ such that the diagram commutes:



It should be noted that the minus sign in the above definition can be removed by replacing the quadratic space V with its negative. However, it often arises in applications, and we will keep it for the outset. From here out, we will assume that our real quadratic spaces are non-degenerate.

A Clifford algebra for an n-dimensional space V may always be constructed as follows [10]: Begin with the tensor algebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots,$$

and then enforce the fundamental identity by taking a suitable quotient. In our case we want to set $C(V, q) = T(V)/I(q)$ where $I(q)$ is the two-sided ideal in $T(V)$ generated by all elements of the form

$$\{v \otimes v + q(v) \cdot 1 \text{ for all } v \in V\}.$$

Since the ideal $I(q) \subset T(V)$ is generated by elements of even degree, the Clifford algebra $C(V, q) = \mathcal{C}$ inherits a $\mathbb{Z}_2$ grading:

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$$

with $\mathcal{C}_0$ spanned by products of an even number of elements in V and $\mathcal{C}_1$ being spanned by products of an odd number. Further, $\mathcal{C}$ has dimension 2^n and, in some literature, is referred to as the *universal* Clifford algebra.

As we move toward classifying Clifford algebras, we need a few more definitions.

Definition 4.1.7. The **double field** ${}^2\mathbb{K}$ is the $\mathbb{K}$-linear space $\mathbb{K}^2$ assigned the product

$$(a, b)(c, d) = (ac, bd), \quad \text{for all } a, b, c, d, \in \mathbb{K}$$

It is worth noting that a direct sum decomposition $V_0 \oplus V_1$ of a $\mathbb{K}$-linear space V may be regarded as a ${}^2\mathbb{K}$-module structure for V by setting

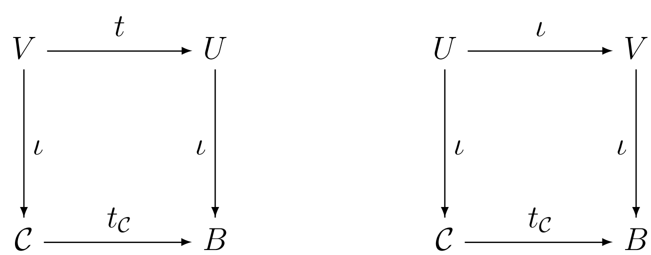
$$(\lambda, \mu)v = \lambda v_0 + \mu v_1, \text{ for all } v \in V \text{ and } (\lambda, \mu) \in {}^2\mathbb{K}.$$

Also, we will be using the notation $\mathbb{K}(n)$ to denote the full matrix algebra of all $n \times n$ matrices with real entries, with matrix multiplication as the product. See that

$^2\mathbb{K}$ may be identified with the subalgebra of $\mathbb{K}(n)$ consisting of the diagonal $n \times n$ matrices.

A 2^n-dimensional real Clifford algebra for an n-dimensional quadratic space V is said to be a *universal* Clifford algebra. Due to our definition of a Clifford algebra, in conjunction with the following theorem and its corollaries, we will focus on these universal Clifford algebras. We will see that for such a space, one can always choose as Clifford algebra the space of endomorphism of some finite-dimensional linear space over $\mathbb{R}, \mathbb{C}, \mathbb{H}, ^2\mathbb{R}$, or $^2\mathbb{H}$, where $\mathbb{H}$ is the set of quaternions.

Theorem 4.1.8 ([24]). *Let $\mathcal{C}$ be a Clifford algebra for an n-dimensional real quadratic space V, with $\dim \mathcal{C} = 2^n$, and let $\mathcal{D}$ be a Clifford algebra for a real quadratic space U, and suppose that $\tau : V \to U$ is an orthogonal map. Then there is a unique algebra map $\tau_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ sending $1_{\mathcal{C}}$ to $1_{\mathcal{D}}$ and a unique algebra-reversing map $(\tau_{\mathcal{C}})^{\sim} : \mathcal{C} \to \mathcal{D}$ sending $1_{\mathcal{C}}$ to $1_{\mathcal{D}}$ such that following diagrams commute.*



Corollary 4.1.9 ([24]). *Let $\mathcal{C}$ and $\mathcal{D}$ be 2^n-dimensional Clifford algebras for an n-dimensional real quadratic space V. Then $\mathcal{C} \cong \mathcal{D}$.*

Corollary 4.1.10 ([24]). *Let $\mathcal{D}$ be a Clifford algebra for an n-dimensional quadratic space V. Then $\mathcal{D}$ is isomorphic to some quotient of any given 2^n-dimensional Clifford algebra $\mathcal{C}$ for V.*

The classification of Clifford algebras for the non-degenerate quadratic spaces $\mathbb{R}^{p,q}$ reduces to the classification of universal Clifford algebras. We recall [24] the construction of the universal algebra $\mathbb{R}_{p+1,q+1}$ for $\mathbb{R}^{p+1,q+1}$ given $\mathbb{R}_{p,q}$ for $\mathbb{R}^{p,q}$.

If V is a linear space over $\mathbb{K}$ or $^2\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and if we take a generating orthonormal subset S of $\operatorname{End}(V)$ of type (p, q), then we can create a subset of

$\mathrm{End}(V^2)$ of type $(p+1, q+1)$ that generates the real algebra $\mathrm{End}(V^2)\otimes_{\mathbb{R}}\mathbb{R}(2)$. Namely, this set is

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \;\middle|\; a \in S \right\} \cup \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Now using the fact that $\mathbb{R}$ is a universal Clifford algebra for $\mathbb{R}^{0,0}$ and applying an induction argument with the above construction, we can state that following proposition.

Proposition 4.1.11 ([24]). *For $n \in \mathbb{Z}_+$, the endomorphism algebra $\mathbb{R}(2^n)$ is a universal Clifford algebra for the space $\mathbb{R}^{n,n}$.*

As $\mathbb{R}_{n,n,} \cong \mathbb{R}(2^n)$, the existence of a universal Clifford algebra for an arbitrary n-dimensional real quadratic space is a direct result of the following two propositions.

Proposition 4.1.12 ([24]). *If V is a non-degenerate n-dimensional real quadratic space, then V is isomorphic to a subspace of $\mathbb{R}^{n,n}$.*

Proposition 4.1.13 ([24]). *If $\mathcal{C}$ is a Clifford algebra for real quadratic space V, and W is a linear subspace of V, then the subalgebra of V generated by W is a Clifford algebra for W.*

So we have outlined the construction of the universal Clifford algebra for any finite-dimensional quadratic space, and low-dimensional Clifford algebras are some well-known spaces. Several of these will be used as the basis for the classification of all finite-dimensional Clifford algebras. For example, $\mathbb{R}$ itself is a Clifford algebra not only $\mathbb{R}^{0,0}$, but also $\mathbb{R}^{1,0}$. Further, if we take $\mathbb{C}$ and $\mathbb{H}$ as real algebras, they are Clifford algebras for $\mathbb{R}^{0,1}$ and $\mathbb{R}^{0,2}$, respectively. If we identify $\mathbb{R}^{0,3}$ with the linear image of $\{i, j, k\}$, the space of pure quaternions, then $\mathbb{H}$ is a Clifford algebra for it.

We refer to [24] for a complete introduction to the classification, and we state several essential results leading up to the table at the end of this section.

Proposition 4.1.14 ([24]). *The universal Clifford algebras $\mathbb{R}_{p+1,q}$ and $\mathbb{R}_{q+1,p}$ are isomorphic.*

Proposition 4.1.15 ([24]). *For $q = 0,1,2,3,4$, the universal Clifford algebra $\mathbb{R}_{0,q}$ is isomorphic to $\mathbb{R}$,$\mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{H}$, $\mathbb{H}(2)$, respectively.*

Proposition 4.1.16 ([24]). *For all p, q,*

$$\mathbb{R}_{p,q+4} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}_{0,4} \cong \mathbb{R}_{p,q} \otimes \mathbb{H}(2)$$

We reference one more result, which will allow us to complete our construction of any $\mathbb{R}^{p,q}$.

Proposition 4.1.17 ([24]). *For all finite p, q,*

$$\mathbb{R}_{p,q+8} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}(16)$$

Now as we collect our previous constructions and results to give the table for low-dimensional Clifford algebras $\mathbb{R}_{p,q}$ below.

Figure 4.1: Classification of Clifford algebras $\mathbb{R}_{p,q}$

$p\downarrow$, $q\rightarrow$	0	1	2	3	4	5	6
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$
1	$^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
2	$\mathbb{R}(2)$	$^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$^2\mathbb{H}(4)$	$\mathbb{H}(8)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$^2\mathbb{H}(8)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$
5	$^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$
6	$\mathbb{H}(4)$	$^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$^2\mathbb{R}(32)$	$\mathbb{R}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$^2\mathbb{R}(64)$

There are a several things here worth noting. The pattern from the top left to the bottom right comes from our construction of $\mathbb{R}_{p+1,q+1}$ from $\mathbb{R}_{p,q}$. Also, the notice the symmetry about the line $q - p = -1$. Clifford algebras have an 8-fold periodicity over the real numbers, which is related to the same periodicities for homotopy groups, and is called Bott periodicity. Clifford algebras also have a 2-fold periodicity over the complex numbers, as every non-degenerate quadratic form on a complex vector space is equivalent to the standard diagonal form.

The table can be summed up in the following theorem.

Theorem 4.1.18 (Classification theorem)**.** . *The Clifford algebra* $\mathbb{R}_{p,q}$ *is isomorphic to the real associative algebras in the following table, where* $n = p + q$*:*

$p-q \bmod(8)$	$\mathbb{R}_{p,q}$
$0,6$	$\mathbb{R}(2^{n/2})$
$1,5$	$\mathbb{C}(2^{(n-1)/2})$
$2,4$	$\mathbb{H}(2^{(n-2)/2})$
3	$\mathbb{H}(2^{(n-3)/2}) \oplus \mathbb{H}(2^{(n-3)/2})$
7	$\mathbb{R}(2^{(n-1)/2}) \oplus \mathbb{R}(2^{(n-1)/2})$

Figure 4.2: Bott Periodicity Table

4.2 Infinitely generated Clifford algebras

One of the early goals of this dissertation was to provide a context - similar to that laid out in Porteous - for which to study Clifford algebras of an infinite-dimensional non-degenerate real quadratic space X. In particular, the central focus was to better understand the corresponding Clifford group and its connection to the group of orthogonal automorphisms on X, or $\mathrm{O}(X)$. Several results were found early in this pursuit, and they are included in this section. Further study is warranted in this area, however current results expressed here lack the complexity of the representation theoretical results previously discussed in this dissertation. Hence, this approach has been waylaid due to interest in other research areas as of late. Regardless, the theory discussed here should connect with existing study of Clifford algebras and infinite-dimensional spin groups [3]. Some basic statements extend from Porteous without alteration - those will not be mentioned here unless they prove vital for a nontrivial result. As mentioned, we look to applications of Clifford algebras to groups of quadratic automorphisms. Here X will be an infinite-dimensional non-degenerate real quadratic space, and A will denote a real Clifford algebra for X. Note that for each $x \in X$, we set $x^{(2)} = x^{-}x = \widehat{x}x = -x^2$.

Proposition 4.2.1. *Let g be an invertible element of A such that, for each $x \in X$,*

(*i*) $gx\widehat{g}^{-1} \in X$, *and*

(*ii*) $g^{-1}x\widehat{g} \in X$

then the map

$$\rho_g : x \mapsto gx\widehat{g}^{-1}$$

is an orthogonal automorphism of X.

Proof. To see that ρ_g is an orthogonal map, see that for each $x \in X$,

$$\begin{aligned}(\rho_g(x))^{(2)} &= (\widehat{gx\widehat{g}^{-1}})gx\widehat{g}^{-1} \\ &= \widehat{g}\widehat{x}g^{-1}gx\widehat{g}^{-1} \\ &= \widehat{x}x \\ &= x^{(2)}\end{aligned}$$

Furthermore, ρ_g is injective as $gx\widehat{g}^{-1} = 0 \implies x = 0$ (this argument does not follow from orthogonality if X is degenerate). Finally, we want to prove ρ_g is surjective. Take $y \in X$, and set $x = g^{-1}y\widehat{g}$. Then $x \in X$ by (ii), and furthermore

$$\rho_g(x) = \rho_g(g^{-1}y\widehat{g}) = g(g^{-1}y\widehat{g})\widehat{g}^{-1} = y$$

Hence ρ_g is also surjective. □

Note that there is no guarantee the condition (i) in the above implies that $g^{-1}x\widehat{g}$ belongs to X, thus each condition is necessary for our purposes. As a result, g will be said to *induce* or *represent* the orthogonal automorphism ρ_g and the set of all such elements g will be denoted by Γ. Thus,

$$\Gamma(X) = \{\ g \mid \rho_g \text{ is an orthogonal automorphism of } X\}$$

Proposition 4.2.2. Γ *is a subgroup in* A

Proof. For closure under multiplication, it is enough to see that $\rho_{gh}(x) = \rho_g(\rho_h(x))$

$$\begin{aligned}\rho_{gh}(x) &= (gh)x(\widehat{gh})^{-1} \\ &= g(hx\widehat{h}^{-1})\widehat{g}^{-1} \\ &= \rho_g(\rho_h(x))\end{aligned}$$

Closure with respect to inversion follows from that for all $g \in \Gamma$, the inverse of ρ_g is $\rho_{g^{-1}}$, and thus is likewise an orthogonal automorphism of X. Finally, it is clear that $1_A \in \Gamma$.

□

Definition 4.2.3. The group $\Gamma(X) = \{\, g \mid \rho_g$ is an orthogonal automorphism of $X\}$ is called the Clifford group for X in the Clifford algebra A.

In order to better represent the Clifford group, we want to expand on its relation to the group of all orthogonal automorphisms on X. To do this will we need several more tools.

Proposition 4.2.4. *Let $W \oplus Y$ be an orthogonal decomposition of X. Then the map $w + y \mapsto w - y$ is an orthogonal map and is a reflection of X in W.*

Definition 4.2.5. A reflection in a linear hyperplane W is said to be a hyperplane reflection. Further, if $\mathbb{R}\{a\}$ is the line in X spanned by a, then $X = \mathbb{R}\{a\} \oplus \mathbb{R}\{a\}^{\perp}$.

One goal is to connect the concept of a hyperplane reflection (or reflection of codimension 1) to the Clifford group

Proposition 4.2.6. *Take an invertible element $a \in X$, Then $a \in \Gamma$, and the map ρ_a is a reflection in $(\mathbb{R}\{a\})^{\perp}$*

Proof. Let a be as stated above, and write $X = \mathbb{R}\{a\} \oplus (\mathbb{R}\{a\})^{\perp}$. Any element of X is then of the form $\lambda a + b$, where $\lambda \in \mathbb{R},\ a \cdot b = 0$.

From Lemma 1.1, we see that $ba = -ab$ and $\widehat{a} = -a$, by $a \in X$. We have:

$$\begin{aligned}\rho_a(\lambda a + b) &= a(\lambda a + b)\widehat{a}^{-1} \\ &= -a(\lambda a + b)a^{-1} \\ &= -a\lambda aa^{-1} + -aba^{-1} \\ &= -a\lambda - (-baa^{-1}) \\ &= -\lambda a + b\end{aligned}$$

which is a reflection in $(\mathbb{R}\{a\})^{\perp}$ □

Proposition 4.2.7. *If $a, b \in X$ are invertible such that $a^{(2)} = b^{(2)}$, then a may be mapped to b either by a single hyperplane reflection or a composite of two reflections.*

Proof. First, we show that either $a - b$ or $a + b$ is invertible, and that they are mutually orthogonal, then we will prove our result.
Clearly, $a^{(2)} = b^{(2)} \implies (a+b)\cdot(a-b) = 0$. Further,

$$\begin{aligned}(a+b)^{(2)} + (a-b)^{(2)} &= 2a^{(2)} + 2b^{(2)} \\ &= 4a^{(2)} \\ &\neq 0\end{aligned}$$

as a being invertible means $a^{(2)} \neq 0$. Hence, at least one of $a + b$ and $a - b$ is invertible, as if both $(a + b)^{(2)}$ and $(a - b)^{(2)}$ are zero, then $a^{(2)}$ must necessarily be zero as well.
Now if $a - b$ is invertible then,

$$\begin{aligned}\rho_{a-b}(a) &= \rho_{a-b}(\frac{1}{2}(a-b) + \frac{1}{2}(a+b)) \\ &= -\frac{1}{2}(a-b) + \frac{1}{2}(a+b) \\ &= b\end{aligned}$$

Whereas if $a + b$ is invertible, then $\rho_b \rho_{a+b}(a) = \rho_b(-b) = b$. □

Now we have the tools to say more about the elements of a Clifford group.

Theorem 4.2.8. *Given an infinite dimensional non-degenerate real quadratic space X, then the orthogonal automorphism ρ_g induced by an element g of $\Gamma(X)$ can be represented as a composite of a finite number of reflections.*

Proof. As before, each element $g \in \Gamma$ induces an orthogonal automorphism of X. Now take $\{\gamma_\lambda : \lambda \in \Omega\}$ as generators for Clifford group $\Gamma(X)$, where Ω is a possibly infinite set. Then for any $g \in \Gamma(X)$,

$$g = \prod_{I \subset \Omega} \gamma_I$$

where this is a *finite* product under the group operation.

Then we examine $\rho_g(x)$ as follows:

$$\begin{aligned}\rho_g(x) &= (\prod_{I\subset J}\gamma_I)x(\widehat{\prod_{I\subset J}\gamma_I})^{-1} \qquad (4.1)\\ &= (\gamma_\alpha\gamma_\beta\ldots\gamma_\omega)x\ (\widehat{\gamma_\alpha\gamma_\beta\ldots\gamma_\omega})^{-1}\\ &= \gamma_\alpha\gamma_\beta\ldots(\gamma_\omega x\widehat{\gamma_\omega}^{-1})\ldots\widehat{\gamma_\beta}^{-1}\widehat{\gamma_\alpha}^{-1}\\ &= \rho_{\gamma_\alpha}(\rho_{\gamma_\beta}(\ldots(\rho_{\gamma_\omega}(x))\ldots))\end{aligned}$$

Where $\gamma_i \in \Gamma \implies \rho_{\gamma_i}$ is an orthogonal automorphism for all $i \in \Omega$.

Now denote $y := \gamma_\omega x\widehat{\gamma_\omega}^{-1}$. Then as $x, y \in X$, and X non-degenerate, we have that x and y are invertible. Further, as

$$x^{(2)} = \rho_\gamma(x))^{(2)} = y^{(2)}$$

By Proposition 4.2.7, we can thus find at most two reflections (codimension 1) taking x to y. We continue this procedure for each ρ_{γ_λ}, where $\gamma_\lambda \in \prod_{I\subset J}\gamma_I = g$, then as each product in (4.1) is finite, we have a finite number of reflections taking x to $\rho_g(x)$. □

As each $g \in \Gamma$ induces an orthogonal automorphism, ρ_g, which is equivalent to the product of finitely many reflections, we say that any element of the Clifford group can be represented by a finite number of reflections.

We will introduce some notation - suppose that $\{e_i : i \in J\}$ is a collection of elements of an associative algebra A. Then, for each naturally ordered subset I of J, e_I will denote the product $\prod_{i\in I} e_i$. Here $e_\emptyset = 1$, where $\emptyset$ denotes the empty set. The next two propositions will play a key role in understanding the structure of both the Clifford algebra and subgroups of the Clifford group. First, we take a moment to mention the fact that a canonical basis can be found for A is due to Gross [11].

Proposition 4.2.9. *Let A be a real associative algebra with unit element* 1 *(identified with $1 \in \mathbb{R}$) and suppose that $\{e_i : i \in J\}$ is a set of elements of A generating*

A such that, for any distinct $i, j \in J$,

$$e_i e_j + e_j e_i = 0$$

Then the set $\{e_I : I \subset J\}$ spans A linearly.

Proposition 4.2.10. *Let A be a Clifford algebra for non-degenerate real quadratic space X. Then the set $\{e_I : I \subset J\}$ is linearly independent.*

Proof. Let $\{e_i : i \in J\}$ be an ordered orthonormal basis for X (thus generating A). Then for each $I \subset J$, e_I is invertible in A and so is non-zero.

To prove that the set $\{e_I : I \subset J\}$ is linearly independent, it is enough to prove that if there are real numbers λ_I, for each $I \subset J$, such that $\sum_{I \subset J} \lambda_I(e_I) = 0$, then for each $K \subset J$, $\lambda_K = 0$. Now since, for any $K \subset J$,

$$\sum_{K \subset J} \lambda_I(e_I) = 0 \iff \sum_{K \subset J} \lambda_I(e_I)(e_J)^{-1} = 0$$

thus making λ_J the coefficient of $e_\emptyset$, it is enough to prove that

$$\sum_{I \subset J} \lambda_I(e_I) = 0 \implies \lambda_\emptyset = 0.$$

Suppose then that $\sum_{I \subset J} \lambda_I(e_I) = 0$. Then for each $i \in J$ and each $I \subset J$, e_i either commutes or anti-commutes with e_I. Then,

$$\sum_{I \subset J} \lambda_I(e_I) = 0 \implies \sum_{I \subset J} \lambda_I e_i(e_I)e_i^{-1} = \sum_{I \subset J} \zeta_{I,i}\lambda_I(e_I) = 0$$

where $\zeta_{I,i} = \pm 1$ depending on whether e_i commutes or anti-commutes with e_I.

It then follows that $\sum_I \lambda_I(e_I) = 0$, with the summation over all I such that e_I commutes with each e_i.

Now there is only one such subset of J, that is $\emptyset$, as $e_\emptyset = 0$. (Note: That in the finite dimensional case e_J would commute with everything if $\#J = \dim X$ were odd). This is because if $I \subset J$ is such that $\#I$ is odd, then e_i commutes with e_I for any $i \in I$ and anti-commutes if $i \notin I$. A similar argument can be made if $\#I$ is even, with any e_i anti-commuting with e_I if $i \in I$ and commuting if $i \notin I$.

So, as $\sum_I \lambda_I(e_I) = 0$, with the summation over all I such that e_I commutes with each e_i, we have that $\lambda_\emptyset = 0$. From this, it follows that the subset $\{e_I : I \subset J\}$ is linearly independent in A.

$\square$

We wish to decompose the Clifford algebra A, and add more machinery to do so. Consider the double field ${}^2\mathbb{K}$ consisting of the $\mathbb{K}$-linear space $\mathbb{K}^2$ assigned the product $(a, b)(c, d) = (ac, bd)$. A direct sum decomposition $X_0 \oplus X_1$ of a $\mathbb{K}$-linear space X may be regarded as a ${}^2\mathbb{K}$-module structure for X by setting

$$(\lambda, \mu)x = \lambda x_0 + \mu x_1, \text{ for all } x \in X \text{ and } (\lambda, \mu) \in {}^2\mathbb{K}.$$

Conversely, any ${}^2\mathbb{K}$-module structure for X determines a direct sum decomposition $X_0 \oplus X_1$ of X as a $\mathbb{K}$-linear space in which $X_0 = (1, 0)X = \{(1, 0)x : x \in X\}$ and $X_1 = (0, 1)X$.

Proposition 4.2.11. *Let $t : X \to X$ be a linear involution of the $\mathbb{K}$-linear space X. Then a ${}^2\mathbb{K}$-module structure, and therefore a direct sum decomposition, is defined for X by setting, for any $x \in X$,*

$$(1, 0)x = \frac{1}{2}(x + t(x)) \text{ and } (0, 1)x = \frac{1}{2}(x - t(x)).$$

Definition 4.2.12. A super field $\mathbb{L}^\alpha$, with fixed field $\mathbb{K}$, consists of a commutative algebra $\mathbb{L}$ with unit 1 over a commutative field $\mathbb{K}$ and an involution α of $\mathbb{L}$, whose set of fixed points is the set of scalar multiples of 1, identified with $\mathbb{K}$.

If you have an associative $\mathbb{L}^\alpha$ algebra A with unit element, then A is an $\mathbb{L}^\alpha$ - Clifford algebra for X if it contains X as a $\mathbb{K}$ - linear subspace in such a way that, for all $x \in X$, $x^{(2)} = -x^2$, provided also that A is generated as a ring by $\mathbb{L}$ and X or, equivalently, as an $\mathbb{L}$-algebra by 1 and X.

If we take a Clifford algebra A for a quadratic space X, then the main involution induces a direct sum decomposition $A^0 \oplus A^1$ of A, where

$$A^0 = \{a \in A : \widehat{a} = a\} \text{ and } A^1 = \{a \in A : \widehat{a} = -a\}$$

Then any element $a \in A$ can be expressed as the sum of its *even* part $a^0 \in A^0$ and its *odd* part $a^1 \in A^1$. For example, take $a = 1 + e_0 + e_1e_2 + e_0e_1e_2$, we have $a^0 = 1 + e_1e_2$ and $a^1 = e_0 + e_0e_1e_2$.

Lemma 4.2.13. *Let $a \in A$ be such that $ax = x\widehat{a}$, for all $x \in X$, A being a Clifford algebra for the non-degenerate real quadratic space X. Then $a \in \mathbb{R}$.*

Proof. Let $a = a^0 + a^1$, where $a^0 \in A^0$ and $a^1 \in A^1$. Then, since $ax = x\widehat{a}$,

$$a^0x = xa^0 \quad \text{and} \quad a^1x = -xa^1 \ ,$$

for all $x \in X$, in particular for each element e_i of some orthonormal basis $\{e_\lambda \ : \ \lambda \in J\}$ for X.

Take

$$a^0 = \sum_{I \in J} \lambda_I(e_I),$$

where it must be that $\#I$ is even (by $a^0 \in A^0$) for each I. Then for a^0 to commute with each e_i, we must have e_I commuting with every e_i. However, as with the proof of Proposition 4.2.10, the only such subset of J for which this can occur is $I = \emptyset$, as $e_\emptyset = 1$. Hence $a^0 = \lambda_\emptyset(e_\emptyset) \in \mathbb{R}$.

We also want to see that if a^1 anti-commutes with each e_i then $a^1 = 0$. By the reasoning similar to the above, there is no such subset of I (as $e_\emptyset = 1$ does not anti-commute with each e_i). Thus $a^1 = 0$. Hence, $a \in \mathbb{R}$. □

Using this lemma, we can strengthen our connection between the Clifford group and the group of orthogonal automorphisms on X.

Theorem 4.2.14. *Given an infinite dimensional non-degenerate real quadratic space X, then the map*

$$\rho_X : \Gamma(X) \to O(X);$$
$$g \ \mapsto \rho_g$$

is a group map with kernel $\mathbb{R}^\times$.

Proof. To prove ρ_X is a group map, let $g, g' \in \Gamma$. Then, for all $x \in X$,

$$\begin{aligned}\rho_{gg'}(x) &= gg'x(\widehat{gg'})^{-1}\\ &= gg'x\widehat{g'}^{-1}\widehat{g}^{-1}\\ &= \rho_g\rho_{g'}(x)\end{aligned}$$

Hence, $\rho_{gg'} = \rho_g\rho_{g'}$, which is what had to be proven.

Now suppose that $\rho_g = \rho_{g'}$, for $g, g' \in \Gamma$. Then, for all $x \in X$, $gx\widehat{g}^{-1} = g'x\widehat{g'}^{-1}$, implying that $(g^{-1}g')x = x(\widehat{g^{-1}g'})$. Therefore, $g^{-1}g' \in \mathbb{R}$ by the previous lemma. Moreover, we have that $g^{-1}g'$ is invertible and is therefore non-zero. □

It is important to note that the map in this theorem is not necessarily surjective. As a result of Theorems 4.2.8 and 4.2.14, any element of Γ is representable as the reflection it induces, and thus by a composition of a finite number of reflections of X. However, the group of orthogonal automorphisms of an infinite real quadratic space X is much larger than the image of ρ_X. If we consider the image of ρ_X, or set of orthogonal automorphism on X induced by $g \in \Gamma$, then we see any element in the image is a composition of finitely many reflections corresponding to a finite product of elements in X, stated as a simple corollary below.

Corollary 4.2.15. *Any element $g \in \Gamma(X)$ is representable as a product (not necessarily unique) of a finite number of elements in X.*

As with the usual study of the orthogonal group, an element $g \in \Gamma$ that can be represented as a product of an even number of elements in X will be called a *rotation* and an element g represents an *inversion* if and only if g is the product of an odd number of elements from X. The set of elements represented as an even product will be denoted as $\Gamma^0(X) = \Gamma^0$. The set of elements represented as an odd product will be denoted as $\Gamma^1(X)$. There are several straight forward consequences of our construction, first noting that $\Gamma^0 = \Gamma \cap A^0$ is a subgroup of Γ

Also, for any $a \in A^0$, $\widehat{a} = a$, then the rotation induced by an element g of Γ^0 is of the form $x \mapsto gxg^{-1}$. Similarly, since for any $a \in A^1$, $\widehat{a} = -a$, the rotation induced by an element g of Γ^1 is of the form $x \mapsto -gxg^{-1}$.

Finally, we want to close this chapter with a look toward future results helping to shape this basic approach. First, we define a useful space,

Definition 4.2.16. Referring to the map from Theorem 4.2.14, we define

$$O_\Gamma(X) = \operatorname{im}(\rho_X) = \{\rho_g \mid g \in \Gamma\}$$

whose elements can be seen from Theorem 4.2.8 to be representable as a finite number of reflections.

The Clifford group $\Gamma(X)$, as constructed, is still quite large. As our goal is now to represent $O_\Gamma(X)$ in a meaningful way, we may wish to find subgroups of Γ that correspond to the known Pin and Spin groups. This is the next step in developing this particular approach, and the next goal would be to develop analogues to these subgroups of the Clifford group, and show they can be identified with another construction of these groups, namely in [3].

In this vein, we may define a quadratic norm N on A as simply

$$N(a) = a^- a$$

for any $a \in A$. Then one expects to find the analogues of the Pin and Spin group in the form, namely

$$\operatorname{Pin}(X) = \{g \in \Gamma \mid |N(g)| = 1\} \quad \text{and}$$
$$\operatorname{Spin}(X) = \{g \in \Gamma^0 \mid |N(g)| = 1\}.$$

However, various issues remain unresolved when concerning N when applied to the Clifford group due to the infinite-dimensional nature of Γ. However, this approach remains viable and accessible due to its intuitive nature. Several open problems and questions make themselves immediately available given the current developments.

Question 4.2.17. Once fully realized, will the defined Pin and Spin groups doubly cover $O_\Gamma(X)$ and $SO_\Gamma(X)$ respectively?

One expects the kernel in each case to be recognizable and possibly equivalent due to the construction of both O_Γ and the corresponding analogue of $SO(X)$ which we have denoted $SO_\Gamma(X)$.

Another offshoot of this approach would be to connect this chapter to the idea of infinite Clifford matrices and further develop such structures and their properties. For example, the following question arises naturally form the developments made here.

Question 4.2.18. If A is an infinitely generated real Clifford algebra for an infinite-dimensional real quadratic space X, and $A(\infty)$ is the real algebra of infinite matrices with entries in A, is there a space for which $A(\infty)$ is a real Clifford algebra?

One may suspect that the potential space in question would take the form of a direct sum between X and a space of the form $\mathbb{R}^{\infty,\infty}$ where there are infinitely many elements of each signature. However, the actual construction of such a space, while showing it generates $A(\infty)$ as a real algebra along with $\{1\}$, remains unrealized.

Appendix A

Young diagrams and $\mathfrak{F}_k^{(m)}$

Given the above, it makes sense for us to re-examine our process and find another way to study $\mathfrak{F}^{(m)}$. By the PBW Theorem, we can write any element in $\mathfrak{F}_k^{(m)}$ in the form

$$\tilde{\psi}_{i_1}^{t_1} \dots \tilde{\psi}_{i_y}^{t_y} \tilde{\psi}_{z_1}^{c_1} \dots \tilde{\psi}_{z_w}^{c_w} \tilde{\psi}_{b_1} \dots \tilde{\psi}_{b_q} \tilde{\psi}_{j_1}^{*} \dots \tilde{\psi}_{j_p}^{*} |m\rangle, \tag{A.1}$$

where $|i_\alpha| = |z_\beta| = 1$, $|b_\gamma| = 0$, and we can require

$$i_{s-1} < i_s < m \quad ; \quad m \leq z_{s+1} < z_s \quad ; \quad m \leq b_{s+1} < b_s \quad ; \quad j_s > j_{s-1}. \tag{A.2}$$

Note that we can also impose that

$$j_s \leq m \quad \text{and } b_\gamma > mB.3) \tag{A.3}$$

to cut out the superfluous contracting of non-present vectors and wedging of present even vectors, both of which would annihilate the monomial. This also removes the case that an even vector is contracted and then immediately wedged back into the monomial.

Given the setup in A.1, the following are true

$$\begin{aligned} &\sum_{\ell=1}^{y} i_\ell t_\ell + \sum_{\ell=1}^{w} z_\ell c_\ell + \sum_{\ell=1}^{q} b_\ell - \sum_{\ell=1}^{p} j_\ell = k, \\ &y + w + q = p \end{aligned} \tag{A.4}$$

The latter condition holds because the monomial must be of charge m - meaning we must contract the same number of vectors that we create. The condition A.4 gives the difference between the sum of all the added vector indices (allowing for multiplicity in the odd cases) and the sum of all the contracted indices, which must be k, the energy of the monomial.

We have arranged the wedging operators in a special way. We place the repeated odd vectors whose indices are less than the charge at the beginning of the monomial in weakly increasing order, followed by odd vectors whose indices are greater than the charge in weakly decreasing order. Next, in strictly decreasing order, are the even vectors whose index is greater than the charge. The remaining positions in our monomial are identical to the tail of $|m\rangle$. In addition, we have arranged the contracting vectors in increasing order. This last choice is simply made for housekeeping purposes. Obviously the order of these operators affects the sign of the monomial, but we are concerned with the linear span of elements of this form, so we are fine.

The motivation for the arrangement in A.1 is that we want to find a way to represent each monomial in $\mathfrak{F}^{(m)}$ that will lend itself to computing $\dim_q \mathfrak{F}^{(m)}$. One way to do this is to compare each monomial in $\mathfrak{F}_k^{(m)}$ to the charge m in a uniform way, similar to the Lie algebra case in Chapter 2 that gave $\dim_q F_k^{(m)}$.

Let $m \geq 0$, then consider the operators

$$\tilde{\psi}_{i_1}^{t_1} \dots \tilde{\psi}_{i_y}^{t_y}$$

where $i_s < i_{s+1} < m$. We construct a weakly decreasing sequence $\lambda^-(\varphi)$ that corresponds to these (possibly repeated) odd vectors in the first positions and is given by

- The first t_1 components of $\lambda^-(\varphi)$ are given by $\lambda_\alpha^- = m - i_1$ for $\alpha = 1, 2, \dots, t_1$.
- The next t_2 components are given by $\lambda_\beta^- = (m - i_2)$ for $\beta = 1, 2, \dots, t_2$.
- This process continues until the last repeated odd vector position with index less than the charge.

Note that as $i_{s-1} < i_s < m$, the result is clearly a strictly decreasing tuple $\lambda^- = (\lambda_1^-, \lambda_2^-, \dots)$ where each component $\lambda_\ell^- \in \frac{1}{2}\mathbb{Z}$. We can double each component of λ^- to find an integer partition and create the corresponding Young diagram. The size of this partition will clearly be twice that of λ^- (which λ^- gives total distance from m of all indices). Also, the length of the partition, or number of parts/rows, is given by

$$\sum_{\ell=1}^{y} t_\ell.$$

Before we continue constructing other Young diagrams for operators of the different types in A.1, we highlight that it is not enough to simply know how many of these odd vectors exist and in what multiplicity. To truly get a correspondence with each element of $\mathfrak{F}_k^{(m)}$ we need to know exactly which of these vectors are present and their multiplicity. This information is now encoded in our Young diagram by relating each vector to m. Once we construct all of the necessary diagrams, we should be able to recover k.

Second, consider the operators

$$\tilde{\psi}_{z_1}^{c_1} \ldots \tilde{\psi}_{z_w}^{c_w}$$

where $z_s > z_{s+1} > m$. We construct the weakly decreasing sequence $\lambda^+(\varphi)$ that corresponds to these (possibly repeated) odd vectors and is given by

- The first c_1 components of $\lambda^+(\varphi)$ are given by $\lambda_\alpha^+ = (z_1 - m)$ for $\alpha = 1, 2, \ldots, c_1$.
- The next c_2 components are given by $\lambda_\beta^+ = (z_2 - m)$ for $\beta = 1, 2, \ldots, c_2$.
- This process continues until the last repeated odd vector position with index greater than or equal to the charge m.

Note that as $z_s > z_{s+1} > m$, the result is clearly a strictly decreasing tuple $\lambda^+ = (\lambda_1^+, \lambda_2^+, \ldots)$ where each component $\lambda_\ell^+ \in \frac{1}{2}\mathbb{Z}$. We can double each component of λ^+ to find an integer partition and create the corresponding Young diagram.

Third, we come to the operators

$$\tilde{\psi}_{b_1} \ldots \tilde{\psi}_{b_q}$$

with $b_s > b_{s+1} > m$. We construct a strictly decreasing sequence using the even vectors, v_i where $i > m$, where our tuple $\mu = (\mu_1, \mu_2, \ldots)$ will be given by

$$\mu_i = b_i - m, \quad \text{for } i = 1, \ldots, q.$$

Then our sequence is decreasing and we again double the components of μ to yield a partition and an associated Young diagram. We have encoded the precise vectors wedged into the monomial, but we now must encode which vectors are contracted.

Hence, look lastly at

$$\tilde{\psi}^*_{j_1} \dots \tilde{\psi}^*_{j_p} |m\rangle$$

with $j_s < j_{s-1} < m$. We again construct a simple decreasing sequence, where our tuple $\zeta = (\zeta_1, \zeta_2, \dots)$ will be given by

$$\zeta_i = m - j_i, \quad \text{for } i = 1, \dots, p.$$

We create our Young diagram as before, and we now have for each monomial a unique collection of four Young diagrams $(\lambda^-, \lambda^+, \mu, \zeta)$. One should note that the number of rows in ζ is equal to the sum of all the rows in λ^-, λ^+, and μ. Further, identifying the exact dimension $\dim \mathfrak{F}_k^{(m)}$ should be possible using the tools we have laid out, and thus so will be $\dim \mathfrak{F}^{(m)}$, however this step remains unresolved.

Appendix B

Computations in $\mathfrak{s}$

Recall from Chapter 3 and [5] that $\widehat{\mathfrak{gl}}_{\infty|\infty}$ is the central extension of $\mathfrak{gl}_{\infty|\infty}$ by the one-dimensional central element c, given by the 2-cocycle.

$$\alpha(A, B) = \mathrm{Str}([J, A]B)$$

where $J = \sum_{r \leq 0} E_{rr}$ and the supertrace of a matrix $C = (c_{rs})$ is defined by

$$\mathrm{Str}\ C = \sum_{r \in \frac{1}{2}\mathbb{Z}} (-1)^{|r|} c_{rr}.$$

Then the bracket inherited by $\mathfrak{s}$ is given by

$$[X, Y] = XY - (-1)^{|X||Y|} YX + \alpha(X, Y)$$

for $X, Y \in \mathfrak{s}$. We refer to [5] and note that for $A, B \in \widehat{\mathfrak{gl}}_{\infty|\infty}$, we have

$$\mathrm{Str}(AB) = (-1)^{|A||B|} \mathrm{Str}(AB).$$

so for our 2-cocycle, we see

$$\alpha(X, Y) = \mathrm{Str}([J, A]B) = \ \mathrm{Str}(JAB - AJB). \tag{B.1}$$

We want to compute some of the commutation relations of the subsuperalgebra $\mathfrak{s}$ (stated in). First recall the basis elements,

$$\lambda_\Psi(n) = \sum_{k \in \Psi} E_{k,k+n}, \qquad \mu_\Psi(n) = \sum_{k \in \Psi} E_{k-\frac{1}{2},k+n-\frac{1}{2}},$$
$$e_\Psi(n) = \sum_{k \in \Psi} E_{k-\frac{1}{2},k+n}, \qquad f_\Psi(n) = \sum_{k \in \Psi} E_{k,k+n-\frac{1}{2}},$$

along with the central element c.

There are a few notes to make before we begin. First, μ and λ are clearly even elements (and parity preserving) and e and f are odd elements (parity reversing). Recall that for the below computations, $\gamma(n)$ is the number of elements in $(\Psi_1 \cap \Psi_2) \cap \{1, 2, \ldots, n\}$. The need for this identification only arises when $m = -n$, under which we always assume without loss of generality that $n > 0$. In these times we are computing the supertrace, and $\gamma(n)$ simply accounts for if there are zero entries resulting from Ψ_1 and Ψ_2 being **finite** subsets of $\mathbb{Z}$. Note that each of the above elements are described in sums over $\mathbb{Z}$ as well. We be showing the following computations hold:

(1) $[\lambda_{\Psi_1}(n), e_{\Psi_2}(m)] = -e_{\Psi_1\cap\Psi_2}(m+n)$,

(2) $[\lambda_{\Psi_1}(n), f_{\Psi_2}(m)] = f_{\Psi_1\cap\Psi_2}(m+n)$,

(3) $[\mu_{\Psi_1}(n), e_{\Psi_2}(m)] = e_{\Psi_1\cap\Psi_2}(m+n)$,

(4) $[\mu_{\Psi_1}(n), f_{\Psi_2}(m)] = -f_{\Psi_1\cap\Psi_2}(m+n)$,

(5) $[\lambda_{\Psi_1}(n), \lambda_{\Psi_2}(m)] = \gamma(n)\delta_{m,-n}c$,

(6) $[\mu_{\Psi_1}(n), \mu_{\Psi_2}(m)] = -\gamma(n)\delta_{m,-n}c$,

(7) $[\lambda_{\Psi_1}(n), \mu\Psi_2(m)] = 0$,

(8) $[e_{\Psi_1}(n), f_{\Psi_2}(m)] = \lambda_{\Psi_1\cap\Psi_2}(m+n) + \mu_{\Psi_1\cap\Psi_2}(m+n) - \gamma(n)\delta_{m,-n}c$.

We see that whenever we are looking at

$$\mathrm{Str}([J,A]B) = JAB - AJB$$

for λ, μ, e, and f, the only non-zero cases occur when the rows of A and the columns of B are of the same parity, as that is the only time non-zero diagonal entries are possible. We will refer to this condition as (†) to limit unnecessary computations.

Proof. (1) We want to show that $[\lambda_{\Psi_1}(n), e_{\Psi_2}(m)] = -e_{\Psi_1\cap\Psi_2}(m+n)$.

Consider first our 2-cocycle,

$$\begin{aligned}
&\mathrm{Str}([J, \lambda_{\Psi_1}(n)]e_{\Psi_2}(m)) \\
&\quad = \mathrm{Str}\left(\sum_{\substack{k\in\Psi_1\\ k\leq 0}} E_{k,k+n} \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} - \sum_{\substack{k\in\Psi_1\\ k\leq -n}} E_{k,k+n} \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m}\right) \\
&\quad = 0 - 0
\end{aligned}$$

Thus we have remaining

$\lambda_{\Psi_1}(n)e_{\Psi_2}(m) - (-1)^{|e||\lambda|}e_{\Psi_2}(m)\lambda_{\Psi_1}(n)$

$$= \sum_{k\in\Psi_1} E_{k,k+n} \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} - \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} \sum_{k\in\Psi_1} E_{k,k+n}$$

$$= -\sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} \sum_{k\in\Psi_1} E_{k,k+n}$$
$$= -\sum_{i\in\Psi_1\cap\Psi_2} E_{i-\frac{1}{2},i+m+n}$$
$$= -e_{\Psi_1\cap\Psi_2}(m+n)$$

(2) We want to show that$[\lambda_{\Psi_1}(n), f_{\Psi_2}(m)] = f_{\Psi_1\cap\Psi_2}(m+n)$,

Note that by (†) our 2-cocyle is 0, Then we have the remaining difference

$\lambda_{\Psi_1}(n)f_{\Psi_2}(m) - (-1)^{|f||\lambda|}f_{\Psi_2}(m)\lambda_{\Psi_1}(n)$
$$= \sum_{k\in\Psi_1} E_{k,k+n} \sum_{j\in\Psi_2} E_{j,j+m-\frac{1}{2}}$$
$$- \sum_{j\in\Psi_2} E_{j,j+m-\frac{1}{2}} \sum_{k\in\Psi_1} E_{k,k+n}$$

$$= \sum_{k\in\Psi_1} E_{k,k+n} \sum_{j\in\Psi_2} E_{j,j+m-\frac{1}{2}}$$
$$= -\sum_{i\in\Psi_1\cap\Psi_2} E_i, i+m+n-\frac{1}{2}$$
$$= f_{\Psi_1\cap\Psi_2}(m+n)$$

(3) We want to show that $[\mu_{\Psi_1}(n), e_{\Psi_2}(m)] = -e_{\Psi_1\cap\Psi_2}(m+n)$.

Note we can again apply (†) when computing the 2-cocycle. So we are left simply with the term

$$\mu_{\Psi_1}(n)e_{\Psi_2}(m) - (-1)^{|e||\mu|}e_{\Psi_2}(m)\mu_{\Psi_1}(n),$$

which is rewritten as

$$\begin{aligned}\sum_{k\in\Psi_1} E_{k-\frac{1}{2},k+n-\frac{1}{2}} \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} - \sum_{j\in\Psi_2} E_{j-\frac{1}{2},j+m} \sum_{k\in\Psi_1} E_{k-\frac{1}{2},k+n-\frac{1}{2}} \\ = \sum_{i\in\Psi_1\cap\Psi_2} E_{i-\frac{1}{2},i+m+n} \\ = e_{\Psi_1\cap\Psi_2}(m+n)\end{aligned}$$

(4) The computations are similar to (1).

(5) We want to show that $[\lambda_{\Psi_1}(n), \lambda_{\Psi_2}(m)] = \gamma(n)\delta_{m,-n}c$

Finally, our 2-cocycle will yield a nonzero term.

$$\begin{aligned}&\mathrm{Str}([J, \lambda_{\Psi_1}(n)]\lambda_{\Psi_2}(m)) \\ &\quad = \mathrm{Str}\left(\sum_{\substack{k\in\Psi_1\\k\leq 0}} E_{k,k+n} \sum_{j\in\Psi_2} E_{j,j+m} - \sum_{\substack{k\in\Psi_1\\k\leq -n}} E_{k,k+n} \sum_{j\in\Psi_2} E_{j,j+m}\right) \\ &\quad = \mathrm{Str}\left(\sum_{\substack{i\leq 0\\i\in\Psi_1\cap\Psi_2}} E_{i,i+n+m} - \sum_{\substack{i\in\Psi_1\cap\Psi_2\\i\leq n}} E_{i,i+n+m}\right) \\ &\quad = \mathrm{Str}\left(\sum_{\substack{i=1\\i\in\Psi_1\cap\Psi_2}}^{n} E_{i,i+n+m}\right) \\ &\quad = \delta_{m,-n} \sum_{\substack{i=1\\i\in\Psi_1\cap\Psi_2}}^{n} (-1)^{|}i|E_{ii} \qquad \text{where we take } n=-m>0 \\ &\quad = \gamma(n)\delta_{m,-n}\end{aligned}$$

Thus we have $\gamma(n)\delta_{m,-n}c$ resulting from our 2-cocycle. We also see that,

$$\begin{aligned}
&\lambda_{\Psi_1}(n)\lambda_{\Psi_2}(m) - \lambda_{\Psi_2}(m)\lambda_{\Psi_1}(n) \\
&\quad = \sum_{k\in\Psi_1} E_{k,k+n} \sum_{j\in\Psi_2} E_{j,j+m} - \sum_{j\in\Psi_2} E_{j,j+m} \sum_{k\in\Psi_1} E_{k,k+n} \\
&\quad = \sum_{i\in\Psi_1\cap\Psi_2} E_{i,i+m+n} - \sum_{i\in\Psi_1\cap\Psi_2} E_{i,i+m+n} \\
&\quad = 0
\end{aligned}$$

(6) The computations are similar to (5)

(7) The fact that $[\lambda_{\Psi_1}(n), \mu_{\Psi_2}(m)] = 0$ is clear from considering their possible non-zero entries and the definition of $[\cdot,\cdot]$.

(8) Our last relation to show is

$$[e_{\Psi_1}(n), f_{\Psi_2}(m)] = \lambda_{\Psi_1\cap\Psi_2}(m+n) + \mu_{\Psi_1\cap\Psi_2}(m+n) - \gamma(n)\delta_{m,-n}c.$$

For our 2-cocycle, we compute the supertrace,

$$\begin{aligned}
&\mathrm{Str}([J, e_{\Psi_1}(n)] f_{\Psi_2}(m)) \\
&\quad = \mathrm{Str}\left(\sum_{\substack{k\in\Psi_1 \\ k\leq 0}} E_{k-\frac{1}{2},k+n} \sum_{j\in\Psi_2} E_{j,j+m-\frac{1}{2}} - \sum_{\substack{k\in\Psi_1 \\ k\leq -n}} E_{k-\frac{1}{2},k+n} \sum_{j\in\Psi_2} E_{j,j+m-\frac{1}{2}} \right) \\
&\quad = \mathrm{Str}\left(\sum_{\substack{i=1 \\ i\in\Psi_1\cap\Psi_2}}^{n} E_{i-\frac{1}{2},i-\frac{1}{2}} \right) \qquad \text{where we take } n = -m > 0. \\
&\quad = \sum_{\substack{i=1 \\ i\in\Psi_1\cap\Psi_2}}^{n} (-1)^i E_{i-\frac{1}{2},i-\frac{1}{2}}
\end{aligned}$$

So we have the term $-\gamma(n)\delta_{m,-n}c$ combined with

$$\begin{aligned}
&e_{\Psi_1}(n)f_{\Psi_2}(m) - (-1)^{|e||f|}f_{\Psi_2}(m)e_{\Psi_1}(n) \\
&\quad = \sum_{i\in\Psi_1\cap\Psi_2} E_{i-\frac{1}{2},i+m+n-\frac{1}{2}} + \sum_{i\in\Psi_1\cap\Psi_2} E_{i,i+m+n} \\
&\quad = \mu(m+n) + \lambda(m+n)
\end{aligned}$$

Thus we have our result. □

Now we turn our attention to the computations involving our representation $\hat{r}$ when restricted to the space $\mathfrak{s}$. These are the remaining proofs from ( ). We look at the following relations:

(1) $\hat{r}_m(f_\Psi(n))|m\rangle = 0, \quad \text{for } n \geq 1,$

(2) $\hat{r}_m(\lambda_\Psi(n))|m\rangle = 0 \quad \text{for } n > 0,$

(3) $\hat{r}_m(c) = I,$

(4) $\hat{r}_m(\lambda_\Psi(0))|m\rangle = 0,$

(5) $r_m(\mu_\Psi(0))|m\rangle = (m(k - \frac{1}{2}) - 1)|m\rangle,$

Proof. (1) This is proven in Chapter 3.

(2) Compute

$$\begin{aligned}
\hat{r}_m(\lambda_\Psi(n))|m\rangle &= \hat{r}_m(\sum_{k\in\Psi}(E_{k,k+n}))|m\rangle \\
&= v_k \wedge v_m \wedge \cdots \wedge \widehat{v_{k+n}} \wedge \ldots
\end{aligned} \tag{B.2}$$

as $|k| = |k+n| = \bar{0}$. Consider that the sum in (B.) will be 0 when $k+n > m$ due to v_{k+n} not being in $|m\rangle$. Also, the sum in (3.11) is 0 when $k < m$ as a result of v_k being even and present in $|m\rangle$. Combining these two inequalities we see the sum is zero when $n > 0$, for $n \in \mathbb{Z}$.

(3) This is immediate from the definition of $\hat{r}$.

(4) For this case, recall that $\hat{r}_m(E_{ii}) = (r_m - \mathrm{I})(E_{ii})$. Then

$$\begin{aligned}\hat{r}_m(\lambda_\Psi(0))|m\rangle &= (r_m(\sum_{k\in\Psi} E_{k,k}) - \mathrm{I})|m\rangle \\ &= \sum_{k\in\Psi}(-1)^k v_k \wedge v_m \wedge \cdots \wedge \widehat{v_k} \wedge \cdots - |m\rangle \\ &= 0\end{aligned}$$

where the last step is made by simply rearranging the even vector v_k in the left term back into the position in which it began.

(5) We again note that $\hat{r}_m(E_{ii}) = r_m(E_{ii}) - \mathrm{I}$. Then

$$\begin{aligned}\hat{r}_m(\mu_\Psi(0))|m\rangle &= (r_m(\sum_{k\in\Psi} E_{k-\frac{1}{2},k-\frac{1}{2}}) - \mathrm{I})|m\rangle \\ &= \sum_{k\in\Psi}\sigma(k-\frac{1}{2})v_{k-\frac{1}{2}} \wedge v_m \wedge \cdots \wedge \widehat{v_{k-\frac{1}{2}}} \wedge \cdots - |m\rangle \qquad \text{(B.3)}\end{aligned}$$

where $\sigma(k-\frac{1}{2}) = m(k-\frac{1}{2})(-1)^q(-1)^{|k-\frac{1}{2}|\sum_{\ell=0}^{q-1}|i_\ell}$ from Proposition 3.3.2. Consider that the sum in (3.11) will be 0 when $k-\frac{1}{2} > m$ due to $v_{k+n-\frac{1}{2}}$ not being in $|m\rangle$. Also, when $k-\frac{1}{2} \leq m$, we can move $v_{k-\frac{1}{2}}$ in the left term back where it was in $|m\rangle$ and have

$$\text{(B.3)} = m(k-\frac{1}{2})|m\rangle - |m\rangle = (m(k-\frac{1}{2}) - 1)|m\rangle.$$

□

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