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Relative Divergence, Subgroup Distortion, and Geodesic Divergence

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RELATIVE DIVERGENCE, SUBGROUP DISTORTION,
AND GEODESIC DIVERGENCE

by

Hung C. Tran

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ABSTRACT

RELATIVE DIVERGENCE, SUBGROUP DISTORTION, AND GEODESIC DIVERGENCE

by

Hung C. Tran

The University of Wisconsin-Milwaukee, 2015
Under the Supervision of Professor Christopher Hruska

In the first part of this dissertation, we generalize the concept of divergence of finitely generated groups by introducing the upper and lower relative divergence of a finitely generated group with respect to a subgroup. Upper relative divergence generalizes Gersten's notion of divergence, and lower relative divergence generalizes a definition of Cooper-Mihalik. While the lower divergence of Cooper-Mihalik can only be linear or exponential, relative lower divergence can be any polynomial or exponential function. In this dissertation, we examine the relative divergence (both upper and lower) of a group with respect to a normal subgroup or a cyclic subgroup. We also explore relative divergence of $CAT(0)$ groups and relatively hyperbolic groups with respect to various subgroups to better understand geometric properties of these groups.

In the second part of this dissertation, we answer the question of Behrstock and Druţu about the existence of Morse geodesics in $CAT(0)$ spaces with divergence function strictly greater than r^n and strictly less than r^{n+1} , where n is an integer greater than 1. We show that for each real number $s > 2$, there is a $CAT(0)$ space X with a proper and cocompact action of some finitely generated group such that X contains a Morse bi-infinite geodesic with the divergence equivalent to r^s .

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Introduction

Geometric group theory is a branch of mathematics that studies finitely generated groups by using topological and geometric methods. In the early 1980s, M. Gromov gave prominence to the idea of considering a finitely generated group as a geometric object. He also proposed a program to classify all metric spaces up to *large scale geometry* by so-called *quasi-isometry* and he suggested using this program to classify all finitely generated groups.

Motivated by Gromov, a lot of group properties that are invariant under quasi-isometry have been studied such as the growth rate of a finitely generated group, the number of ends of a group, the isoperimetric function of a finitely presented group, the divergence of a finitely generated group, and others. Among these quasi-isometric invariants, I am especially interested in divergence of finitely generated groups. Moreover, I realized that the concept of divergence is also useful to study the embedding of a group as a subgroup into a finitely generated group. Therefore, I introduced two types of *relative divergence* of a finitely generated group with respect to a subgroup: upper relative divergence and lower relative divergence (see Chapter 1). In this chapter, I also introduced the concept of lower subgroup distortion which parallels the standard notion of subgroup distortion to study the connection between the geometry of a finitely generated group with the geometry of some finitely generated subgroup.

In addition to relative divergence, I also studied the notion of bi-infinite geodesics divergence. I proved the the existence of Morse geodesics in $CAT(0)$ spaces with divergence function r^s , where s is an arbitrary real number greater than 2 (see Chapter 2).

The material in Chapter 1 was first published, in a slightly different form, in *Algebraic & Geometric Topology* in Volume 15, issue 3 (2015), published by Mathematical Sciences Publishers. The material in Chapter 2 has been accepted for publication, in a slightly different form, in *Geometriae Dedicata*, published by Springer.

Chapter 1

Relative divergence and subgroup distortion in finitely generated groups

1.1 Introduction

Two different notions of divergence of a finitely generated group are introduced by Cooper-Mihalik [ABC⁺91] and Gersten [Ger94]. We refer to Cooper-Mihalik's notion as lower divergence and Gersten's notion as upper divergence. The lower divergence of a one-ended group G is exponential if G is hyperbolic and linear otherwise (see Cooper-Mihalik [ABC⁺91] and Sisto [Sis]). Therefore, lower divergence only detects hyperbolicity. Upper divergence is more diverse since the upper divergence of a finitely generated group can be any polynomial or exponential function (see Macura [Mac13] and Sisto [Sis]). Upper divergence has been studied by Macura [Mac13], Behrstock-Charney [BC12], Duchin-Rafi [DR09], Druţu-Mozes-Sapir [DMS10], Sisto [Sis] and others. Moreover, upper divergence is a quasi-isometry invariant, and it is therefore a useful tool to classify finitely generated groups up to quasi-isometry. Motivated by Gersten and Cooper-Mihalik's notions, we introduce two types of relative divergence of a finitely generated group with respect to a subgroup: upper relative divergence and lower relative divergence.

We now introduce some notations and we will work on them for the concept of rel-

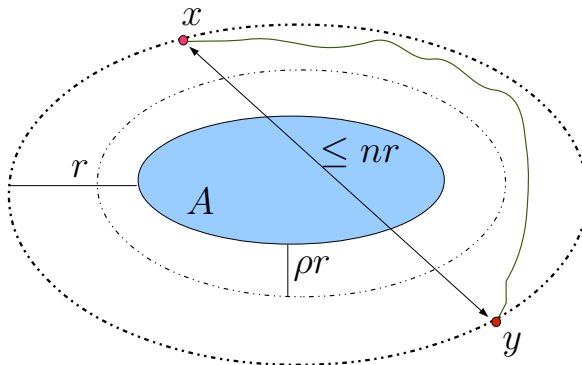


Figure 1.1: The picture illustrates the idea of upper relative divergence of a geodesic space X with respect to a subspace A . The picture for lower relative divergence is almost identical except the distance between x and y is greater than or equal nr .

ative divergence. Let (X, d) be a geodesic space and A a subspace. For each positive r , let $d_{r,A}$ be the induced length metric on the complement of the r -neighborhood of A in X . We now define the relative divergence of the space X with respect to the subspace A (both upper relative divergence and lower relative divergence). For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define two functions δ_ρ^n and σ_ρ^n from $[0, \infty)$ to $[0, \infty]$ as follows:

For each $r \in [0, \infty)$, let $\delta_\rho^n(r) = \sup d_{\rho r, A}(x, y)$ where the supremum is taken over all x, y which lie in $\partial N_r(A)$ such that $d_{r,A}(x, y) < \infty$ and $d(x, y) \leq nr$.

Similarly, let $\sigma_\rho^n = \inf d_{\rho r, A}(x, y)$ where the infimum is taken over all x, y which lie in $\partial N_r(A)$ such that $d_{r,A}(x, y) < \infty$ and $d(x, y) \geq nr$.

The family of functions $\{\delta_\rho^n\}$ is the upper relative divergence of the pair (X, A) , denoted $Div(X, A)$, and the family of functions $\{\sigma_\rho^n\}$ is the lower relative divergence of the pair (X, A) , denoted by $div(X, A)$.

In Section 1.4, we show that both upper relative divergence and lower relative divergence depend only on the quasi-isometry type of (X, A) . Therefore, we can define both the upper and the lower relative divergence of a pair (G, H) , denoted by $Div(G, H)$ and $div(G, H)$, where G is a finitely generated group and H is a

subgroup. While upper relative divergence generalizes upper divergence introduced by Gersten [Ger94], lower relative divergence generalizes lower divergence defined by Cooper-Mihalik [ABC⁺91]. The relative divergence of a pair (G, H) measures the distance distortion of the complement of the r -neighborhood of H in the Cayley graph of G when r increases.

1.1.1 Upper relative divergence

The following theorem describes the upper relative divergence of a finitely generated group with respect to a finitely generated normal subgroup.

Theorem 1.1.1. *Let G be a finitely generated group and H a finitely generated normal subgroup of G . Then*

$$Div(G/H, e) \preceq Div(G, H) \preceq Dist_G^H \circ \overline{Div(G/H, e)}$$

where $Dist_G^H$ is the upper distortion of H in G and $\overline{Div(G/H, e)}$ is a slight modification of $Div(G/H, e)$. Moreover, if G/H is one-ended and H is undistorted in G , then $Div(G, H) \sim Div(G/H, e)$.

In the above theorem, we use the well-known concept of distortion of subgroups. In some sense, this measures the “upper bound” of the distance distortion of a subgroup in comparison with the distance of a whole group. However, we also need the concept of “lower bound” of the distance distortion of subgroups to better understand how a subgroup is embedded into a whole group. Therefore, we introduce the concept of lower distortion and we refer to the traditional concept of distortion as upper distortion (see Section 1.3). The above theorem also helps us find a pair of groups (G, H) , where G is a CAT(0) group and H is a normal subgroup of G , such that $Div(G, H)$ can be any polynomial or exponential function (see Remark 1.5.3).

The upper divergence of a one-ended relative hyperbolic group is at least exponential by Sisto [Sis]. The following theorem strengthens the result of Sisto.

Theorem 1.1.2. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H a subgroup of G such that $0 < \tilde{e}(G, H) < \infty$, where $\tilde{e}(G, H)$ is the number of filtered ends of H in G . We assume that H is not conjugate to an infinite index subgroup of any peripheral subgroup. Then $\text{Div}(G, H)$ is at least exponential.*

We refer the readers to Section 1.2.3 for the definition of the number of filtered ends.

1.1.2 Lower relative divergence

As mentioned earlier, the lower divergence of a finitely generated group is either linear or exponential. The lower relative divergence of a pair of groups, on the other hand, is more diverse.

Theorem 1.1.3. *Let f be any polynomial function or exponential function. There is a pair of groups (G, H) , where G is a CAT(0) group (i.e. the group that acts properly and cocompactly on some CAT(0) space) and H is an infinite cyclic subgroup of G , such that $\text{div}(G, H)$ is f .*

We compute the lower relative divergence of a pair of groups (G, H) when H is an infinite normal subgroup. The following theorem helps us find the upper bound of the relative lower divergence of a pair of groups (G, H) when H is an infinite normal subgroup of G .

Theorem 1.1.4. *Let G be a finitely generated group and H an infinite normal subgroup of G . Let K be any finitely generated infinite subgroup of H . Then, the relative lower divergence $\text{div}(G, H)$ is dominated by the lower distortion of K in G . In particular, if H is finitely generated, then the relative lower divergence $\text{div}(G, H)$ is dominated by the lower distortion of H in G .*

In order to measure the lower relative divergence of a finitely generated group with respect to a normal subgroup, we use the concept of lower distortion of a subgroup (which was mentioned earlier). Although the idea of lower distortion is implicit in works of Gromov [Gro93], Ol'shanskii [Ol'99] and many others, the exact

concept does not seem to be recorded in the literature. Applying the above theorem in the case of CAT(0) groups, we can show that the relative lower divergence of a CAT(0) group G with respect to a normal subgroup H containing at least one infinite order element is linear (see Theorem 1.7.8).

We also examine the lower relative divergence of a relatively hyperbolic group with respect to a subgroup. While the upper relative divergence of a finitely generated relatively hyperbolic group with respect to almost all subgroups is at least exponential (See Theorem 1.1.2), its lower relative divergence can be linear (see Theorem 1.8.25 and Theorem 1.8.35). Moreover, we also examine the lower relative divergence of a finitely generated relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup in the following theorem.

Theorem 1.1.5. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite fully relatively quasiconvex subgroup of G such that $0 < \tilde{e}(G, H) < \infty$, where $\tilde{e}(G, H)$ is the number of filtered ends of H in G . Then $\text{div}(G, H)$ is at least exponential.*

In the above theorem, if we drop the condition “fully relative quasiconvexity” of the subgroup H , the conclusion of the theorem is no longer true (see Theorem 1.8.35).

1.1.3 Lower distortion

In this dissertation, we also introduce *lower distortion* as a new invariant for pairs (G, H) of finitely generated groups. As we mentioned earlier, upper distortion only measures the “upper bound” of the distance distortion of a subgroup in comparison with the distance of a whole group and we need “lower bound” of the distance distortion of subgroups. We refer the readers to Definition 1.3.3 for the concept of lower distortion. In general, the lower and upper distortion of a pair of groups are not the same (see Example 1.3.17). We also show some properties of lower distortion as well as its relation with upper distortion (see Theorem 1.3.6, Proposition 1.3.9 and Proposition 1.3.5). Moreover, we use one result of Ol’shanskii [Ol’99] to construct examples of pairs of groups with a large class of lower and upper distortion functions.

Theorem 1.1.6. *Let $f: [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing function such that $f(0) = 0$ and f^{-1} is subadditive. Suppose that there is a positive integer C such that $f(n) \leq C^n$ for every positive n . Let H be a finitely generated group such that its growth is bounded by some polynomial function. Then there is a finitely generated group G containing H such that the upper and lower divergence of the pair (G, H) are both equivalent to f .*

1.1.4 Overview

In Section 1.2, we prepare some preliminary knowledge for the main part of the first part of this dissertation. This knowledge will be used to define the concept of relative divergence and compute relative divergence of certain pairs of groups.

In Section 1.3, we recall the concept of distortion of a subgroup, which we call upper distortion and introduce the related concept of lower distortion. Together with upper distortion, lower distortion helps us understand the connection between the geometry of a group and the geometry of its subgroups. We also carefully investigate this new concept although it is not the main part of this dissertation. Finally, we give the proof of Theorem 1.1.6 and discuss an example of Gromov to show the difference between upper and lower distortion.

In Section 1.4, we give precise definitions of upper and lower divergence of a pair (X, A) , where X is a geodesic space and A is a subspace. We use these concepts to define the upper and lower divergence of a pair (G, H) , where G is a finitely generated group and H is a subgroup. We also investigate some key properties of relative divergence.

In Sections 1.5 and 1.6, we investigate the divergence of a finitely generated group with respect to a normal subgroup or a cyclic subgroup. In Section 1.5, the proof of Theorems 1.1.1 and 1.1.4 are also shown.

In Section 1.7, we examine relative divergence of some $CAT(0)$ groups. We also investigate a family of groups studied by Macura [Mac13] to show that relative lower divergence can be a polynomial function with arbitrary degree. In this section, readers can find the proof of Theorem 1.1.3 for the case the lower divergence is

polynomial.

In Section 1.8, we examine the relative divergence of a relatively hyperbolic group. We also investigate the lower relative divergence of a relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup and use this fact to show that the lower divergence of a pair of groups can be at least exponential. In this section, we prove Theorem 1.1.2 and Theorem 1.1.5. Moreover, readers can see the proof of Theorem 1.1.3 for the case the lower divergence is exponential in this section.

1.2 Preliminaries

In this section, we discuss some preliminary background before discussing the main part of the dissertation. We first construct the notions of domination and equivalence. We review some concepts in geometric group theory: geodesic spaces, quasi-geodesics, quasi-isometry and quasi-isometric embedding, and the number of filtered ends of pairs of groups. We also introduce the concept of quasi-isometry between two pairs of metric spaces.

1.2.1 The notions of domination and equivalence

In this section, we build the notions of domination and equivalence on the set of some certain families of functions. These notions are the tool to measure the relative divergence of a finitely generated group with respect to a subgroup.

Definition 1.2.1. Let \mathcal{M} be the collection of all functions from $[0, \infty)$ to $[0, \infty]$. Let f and g be arbitrary elements of \mathcal{M} . *The function f is dominated by the function g* , denoted $f \preceq g$, if there are positive constants A, B, C and D such that $f(x) \leq Ag(Bx) + Cx$ for all $x > D$. Two function f and g are *equivalent*, denoted $f \sim g$, if $f \preceq g$ and $g \preceq f$. *The function f is strictly dominated by the function g* , denoted $f \prec g$, if f is dominated by g and they are not equivalent.

Remark 1.2.2. The relations \preceq and \prec are transitive. The relation \sim is an equivalence relation on the set \mathcal{M} .

Let f and g be two polynomial functions in the family \mathcal{M} . We observe that f is dominated by g iff the degree of f is less than or equal to the degree of g and they are equivalent iff they have the same degree. All exponential functions of the form a^{bx+c} , where $a > 1, b > 0$ are equivalent. Therefore, a function f in \mathcal{M} is *linear, quadratic or exponential...* if f is respectively equivalent to any polynomial with degree one, two or any function of the form a^{bx+c} , where $a > 1, b > 0$.

Definition 1.2.3. Let $\{\delta_\rho^n\}$ and $\{\delta_\rho^m\}$ be two families of functions of \mathcal{M} , indexed over $\rho \in (0, 1]$ and positive integers $n \geq 2$. The family $\{\delta_\rho^n\}$ is dominated by the family $\{\delta_\rho^m\}$, denoted $\{\delta_\rho^n\} \preceq \{\delta_\rho^m\}$, if there exists constant $L \in (0, 1]$ and a positive integer M such that $\delta_{L\rho}^n \preceq \delta_\rho^{Mn}$. The notions of strict domination and equivalence can be defined as above.

Remark 1.2.4. The relations \preceq and \prec are transitive. The relation \sim is an equivalence relation.

If f is an element in \mathcal{M} , we could represent f as a family $\{\delta_\rho^n\}$ for which $\delta_\rho^n = f$ for all ρ and n . Therefore, the family $\{\delta_\rho^n\}$ is dominated by (or dominates) a function f in \mathcal{M} if $\{\delta_\rho^n\}$ is dominated by (or dominates) the family $\{\delta_\rho^m\}$ where $\delta_\rho^m = f$ for all ρ and n . The equivalence between a family $\{\delta_\rho^n\}$ and a function f in \mathcal{M} can be defined similarly. Thus, a family $\{\delta_\rho^n\}$ is linear, quadratic, exponential, etc if $\{\delta_\rho^n\}$ is equivalent to the function f where f is linear, quadratic, exponential, etc.

1.2.2 Geodesic spaces, quasigeodesics, quasi-isometry

In this section, we review the concepts of geodesic spaces, quasigeodesics, quasi-isometry and quasi-isometric embedding, and we introduce the concept of quasi-isometry between two pair of metric spaces. These concepts play an important role in defining the concept of upper relative divergence and lower relative divergence of a finitely generated group with respect to a subgroup. Most of information in this section is cited from [GdlH90].

Remark 1.2.5. For each path with finite length α in a geodesic space X , we denote the endpoints of α by α_+, α_- and the length of α by $\ell(\alpha)$. For each ray α in a space

X , we denote the initial point of α by α_+ .

Definition 1.2.6. Let (X, d) be a metric space.

1. A path p in X is an (L, C) -quasigeodesic for some $L \geq 1$ and $C \geq 0$, if for every subpath q of p the inequality $\ell(q) \leq L d(q_+, q_-) + C$ holds.
2. A path p in X is a quasigeodesic if it is (L, C) -quasigeodesic for some $L \geq 1$ and $C \geq 0$.
3. A path p in X is an L -quasigeodesic if it is (L, L) -quasigeodesic for some $L \geq 1$.
4. A path p in X is a geodesic if it is $(1, 0)$ -quasigeodesic.
5. Two quasigeodesics are *equivalent* if the Hausdorff distance between their images is finite.
6. The metric space X is a *geodesic space* if any pair of points in X can be joined by a geodesic segment.

Definition 1.2.7. Let (X, d_X) and (Y, d_Y) be two metric spaces. The map Φ from X to Y is a *quasi-isometry* if there is a constant $K \geq 1$ and a function Ψ from Y to X such that the following holds:

$$d_Y(\Phi(x_1), \Phi(x_2)) \leq K d_X(x_1, x_2) + K \text{ for all } x_1, x_2 \text{ in } X \quad (1.1)$$

$$d_X(\Psi(y_1), \Psi(y_2)) \leq K d_Y(y_1, y_2) + K \text{ for all } y_1, y_2 \text{ in } Y \quad (1.2)$$

$$d_Y(\Phi \circ \Psi(y), y) \leq K \text{ for all } y \text{ in } Y \quad (1.3)$$

$$d_X(\Psi \circ \Phi(x), x) \leq K \text{ for all } x \text{ in } X \quad (1.4)$$

The proof of the following lemma is obvious, and we leave it to the reader.

Lemma 1.2.8. *Let (X, d_X) and (Y, d_Y) be two geodesic spaces and the map Φ from X to Y a quasi-isometry. Then there is a constant $C \geq 1$ such that the following hold:*

1. $(1/C) d_X(x_1, x_2) - 1 \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq C d_X(x_1, x_2) + C$, for all x_1, x_2 in X
2. $N_C(\Phi(X)) = Y$
3. If α is a path connecting two points x_1 and x_2 in X , then there is a path β connecting $\Phi(x_1)$ and $\Phi(x_2)$ in Y such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most C . Moreover, $|\beta| \leq C|\alpha| + C$.
4. If β is a path connecting two points $\Phi(x_1)$ and $\Phi(x_2)$ for some $x_1, x_2 \in X$, then there is a path α connecting x_1 and x_2 in X such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most C . Moreover, $|\alpha| \leq C|\beta| + C$.

Definition 1.2.9. Let (X, d_X) and (Y, d_Y) be two geodesic spaces and the map Φ from X to Y a quasi-isometric embedding if

$$(1/C) d_X(x_1, x_2) - 1 \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq C d_X(x_1, x_2) + C$$

for all x_1, x_2 in X .

Remark 1.2.10. Throughout this dissertation, we denote (X, A) to be a pair of metric spaces, where X is a geodesic space and A is a subspace of X .

Definition 1.2.11. Two pairs of spaces (X, A) and (Y, B) are *quasi-isometric* if there is a quasi-isometry Φ from X to Y such that the Hausdorff distance between $\Phi(A)$ and B is finite.

It is not hard to prove the following proposition and we leave it to the reader.

Proposition 1.2.12. *Quasi-isometry of pairs of metric spaces is an equivalence relation.*

1.2.3 Filtered ends of pairs of groups

In this section, we review the concepts of the number of ends of groups and the number of filtered ends of pairs of groups. We refer the readers to Chapter 14

in [Geo08] for the proof of all the statements on these concepts. We also prove the lemma on the existence of subgroup perpendicular ray which is defined below.

We now define the concept of the number of filtered ends of a pair of groups and we will see that this concept generalizes the concept of the number of ends of a group.

Definition 1.2.13. Let G be a group with a finite generating set S and H a subgroup of G . For each positive r we denote $C_r(H)$ to be the complement of the r -neighborhood of H in the Cayley graph $\Gamma(G, S)$. A connected component U of $C_r(H)$ is *deep* if U does not lie in the s -neighborhood of H for any positive s . Let $\tilde{e}_r(G, H)$ be the number of deep components of $C_r(H)$. We note that $\tilde{e}_r(G, H) \geq \tilde{e}_s(G, H)$ if $r > s$. The number of *filtered ends of the pair* (G, H) , denoted $\tilde{e}(G, H)$, is the supremum of the set $\{\tilde{e}_r(G, H) \mid r > 0\}$.

Remark 1.2.14. Let G be a finitely generated group and H a subgroup.

1. The number $\tilde{e}(G, H)$ does not depend on the choice of finite generating set S of G and $\tilde{e}(G, H) = 0$ iff H is a finite index subgroup of G .
2. If $\tilde{e}(G, H) = m < \infty$, then there is a positive number r_0 such that $C_r(H)$ has exactly m deep components for each $r > r_0$.
3. When H is the trivial subgroup, $\tilde{e}(G, H)$ is *the number of ends of G* , denoted $\tilde{e}(G)$. A finitely generated group is *one-ended* if $\tilde{e}(G) = 1$.

Theorem 1.2.15. [Geo08, Proposition 14.5.9] *If H is a finitely generated normal subgroup of G then $\tilde{e}(G, H)$ equals the number of ends of G/H .*

Definition 1.2.16. Let G be a group with a finite generating set S and H an infinite index subgroup of G . A geodesic ray γ in the Cayley graph $\Gamma(G, S)$ is *H -perpendicular* if the initial point h of γ lies in H and $d_S(\gamma(r), H) = r$ for all positive r .

The following lemma shows the existence of many H -perpendicular geodesic rays.

Lemma 1.2.17. *Let G be a group with a finite generating set S and H an infinite index subgroup of G . Then for each element h in H , there is an H -perpendicular geodesic ray with the initial point h .*

Proof. For each positive integer n , there is a vertex g_n in $C_n(H)$. Let k_n be an element in H and α_n a geodesic segment connecting g_n and k_n such that the length of α_n is equal to the distance between g_n and H . We define $\gamma_n = (hk_n^{-1})\alpha_n$, then γ_n is a geodesic segment with the initial point h and $d_S(\gamma_n(r), H) = r$ for all positive r less than the length of γ_n . By the Arzela-Ascoli theorem, there is a geodesic ray γ with the initial point h such that $d_S(\gamma(r), H) = r$ for all positive r . \square

1.3 Distortion of subgroups

In this section, we will review the concept of distortion of a subgroup, which we call upper distortion. This concept of distortion will later help us compute relative divergence of a large class of pairs of groups. We also introduce the concept of lower distortion of a subgroup. This new concept is also a tool to compute relative divergence. We investigate some key properties of lower distortion and the relation between lower distortion and upper distortion.

First of all, we will review the concept of upper distortion.

Definition 1.3.1. Let G be a group with a finite generating set S and H a subgroup of G with a finite generating set T . The *upper subgroup distortion* of H in G is the function $Dist_G^H: (0, \infty) \rightarrow (0, \infty)$ defined as follows:

$$Dist_G^H(r) = \max\{ |h|_T \mid h \in H, |h|_S \leq r \}.$$

Remark 1.3.2. It is well-known that the concept of upper distortion does not depend on the choice of finite generating sets S and T . More precisely, the functions $Dist_G^H$ are equivalent for all pairs of finite sets (S, T) generating (G, H) respectively.

The function $Dist_G^H$ is non-decreasing, and dominates a linear function.

A finitely generated subgroup H of G is *undistorted* if $Dist_G^H$ is linear.

We now introduce the concept of lower distortion.

Definition 1.3.3. Let G be a group with a finite generating set S and H a subgroup of G with a finite generating set T . The *lower distortion* of H in G is the function $dist_G^H: (0, \infty) \rightarrow (0, \infty)$ defined as follows:

$$dist_G^H(r) = \min\{ |h|_T \mid h \in H, |h|_S \geq r \}.$$

We use the convention that the minimum of the empty set is 0.

Remark 1.3.4. Similar to the concept of upper distortion, the concept of lower distortion also does not depend on the choice of generating sets. When H is an infinite subgroup, the function $dist_G^H$ is non-decreasing and dominates a linear function.

The following proposition shows a relation between upper distortion and lower distortion.

Proposition 1.3.5. *Let G be a finitely generated group and H a finitely generated subgroup of G . Then $dist_G^H \preceq Dist_G^H$.*

Proof. Let S be a finite generating set of G and we assume that S contains the finite generating set T of the subgroup H . Thus, we could consider $\Gamma(H, T)$ as a subgraph of $\Gamma(G, S)$. If H is a finite subgroup then $dist_G^H$ is a bounded function and the proof follows easily. Thus, we assume H is an infinite subgroup.

For each $r > 1$, we could chose an element k in H such that $|k|_S \geq 2r$. We connect the identity element e and k by a geodesic α in $\Gamma(H, T)$. Thus, we can choose h to be an element in α such that $r \leq |h|_S \leq 2r$. Since h is also an element of H , then $dist_G^H(r) \leq |h|_T \leq Dist_G^H(2r)$. Thus, $dist_G^H \preceq Dist_G^H$. \square

We now investigate some key properties of lower distortion:

Theorem 1.3.6. *Suppose that G, H, K are all infinite finitely generated groups and $K \leq H \leq G$. Then:*

1. $dist_H^K \circ dist_G^H \preceq dist_G^K$
2. $dist_H^K \preceq dist_G^K$

3. $dist_G^H \preceq dist_G^K$
4. If $|G : H| < \infty$, then $dist_G^K \sim dist_H^K$
5. If $|H : K| < \infty$, then $dist_G^K \sim dist_G^H$
6. If H_1 and H_2 are two commensurable finitely generated subgroups, then $dist_G^{H_1} \sim dist_G^{H_2}$

Proof. We call S_1 , S_2 and S_3 finite generating sets of G , H and K respectively. We can assume that $S_3 \subset S_2 \subset S_1$. We now prove that

$$dist_H^K \circ dist_G^H(n) \leq dist_G^K(n) \text{ for all } n.$$

For any positive number n , we choose $k_0 \in K$ such that $|k_0|_{S_1} \geq n$ and $|k_0|_{S_3} = dist_G^K(n)$. Since $k_0 \in H$ and $|k_0|_{S_1} \geq n$, then $|k_0|_{S_2} \geq dist_G^H(n)$. Therefore, $|k_0|_{S_3} \geq dist_H^K(dist_G^H(n))$. Thus,

$$dist_H^K \circ dist_G^H(n) \leq dist_G^K(n) \text{ for all } n.$$

Statements (2) and (3) are immediate results of (1) since the lower distortion functions $dist_G^H$ and $dist_H^K$ are non-decreasing and at least linear.

We now prove Statement (4). Since $dist_H^K \preceq dist_G^K$, then we only need to prove $dist_G^K \preceq dist_H^K$. Since $|G : H| < \infty$, then there is a positive integer C such that

$$d_{S_2}(h_1, h_2) \leq C d_{S_1}(h_1, h_2) + C \text{ for all } h_1 \text{ and } h_2 \text{ in } H.$$

We now prove that

$$dist_G^K(n) \leq dist_H^K(2Cn) \text{ for all } n.$$

For any positive number $n > 1$, we choose $k_0 \in K$ such that $|k_0|_{S_2} \geq 2Cn$ and $|k_0|_{S_3} = dist_H^K(2Cn)$. Thus,

$$|k_0|_{S_1} \geq \frac{|k_0|_{S_2} - C}{C} \geq 2n - 1 \geq n.$$

Therefore, $dist_G^K(n) \leq dist_H^K(2Cn)$. In particular, $dist_G^K \preceq dist_H^K$.

We now prove Statement (5). Since $dist_G^H \preceq dist_G^K$, then we only need to prove $dist_G^K \preceq dist_G^H$. Since $|H : K| < \infty$, then there is a positive integer C such that

$$d_{S_3}(k_1, k_2) \leq C d_{S_2}(k_1, k_2) + C \text{ for all } k_1 \text{ and } k_2 \text{ in } K,$$

and $H \subset N_C(K)$ with respect to metric d_{S_2} . We now show that

$$dist_G^K(n) \leq C dist_G^H(2n) + C^2 + C \text{ for all } n \geq C.$$

For any positive number n greater than C , we choose $h_0 \in H$ such that $|h_0|_{S_1} \geq 2n$ and $|h_0|_{S_2} = dist_G^H(2n)$. Since $H \subset N_C(K)$ with respect to metric d_{S_2} , then there is $k_0 \in K$ such that $d_{S_2}(k_0, h_0) \leq C$. In particular, $d_{S_1}(k_0, h_0) \leq C$. Thus,

$$|k_0|_{S_1} \geq |h_0|_{S_1} - C \geq 2n - C \geq n.$$

Thus, $|k_0|_{S_3} \geq dist_G^K(n)$

Also

$$|k_0|_{S_3} \leq C|k_0|_{S_2} + C \leq C(|h_0|_{S_2} + C) + C$$

and

$$|h_0|_{S_2} = dist_G^H(2n)$$

Therefore, $dist_G^K(n) \leq C dist_G^H(2n) + C^2 + C$. In particular, $dist_G^K \preceq dist_G^H$.

We easily obtain (6) from (5) by observing that $|H_1 : (H_1 \cap H_2)| < \infty$ and $|H_2 : (H_1 \cap H_2)| < \infty$. \square

We now explain the relationship between the lower distortion and the growth of a finitely generated group. We will see that the growth function will be an upper bound of the lower distortion. Before showing this fact, we need to review the concept of growth of groups.

Definition 1.3.7. Let G be a group with a finite set of generators S . The growth of G , denoted by $Growth_G$, is a function $f : [0, \infty) \rightarrow [0, \infty)$ to itself defined by letting $f(r)$ to be the number of elements of G that lie in the ball $B(e, r)$ for each $r \geq 0$.

Remark 1.3.8. It is well-known that the growth of a finitely generated group does not depend on the choice of finite generating set (the proof is almost identical to the case of upper distortion). More precisely, the functions $Growth_G$ are equivalent for all finite sets S of generators of G . Moreover, the function $Growth_G$ is dominated by the exponential function.

Proposition 1.3.9. *Let G be a finitely generated group and H a finitely generated subgroup of G . Then the lower distortion $dist_G^H$ is dominated by the growth function $Growth_G$ of G . In particular, the lower distortion $dist_G^H$ is dominated by the exponential function.*

Proof. Let S be a finite generating set of G . We will assume that S contains the finite generating set T of the subgroup H . Thus, we could consider $\Gamma(H, T)$ as a subgraph of $\Gamma(G, S)$. If H is finite, then $dist_G^H$ is bounded and the proof follows easily. Thus, we assume H is an infinite subgroup.

For each $r > 1$, we could choose an element h in H such that $|h|_S \geq r$. We connect the identity element e and h by a geodesic α in $\Gamma(H, T)$. Let h' be a vertex in α such that $|h'|_S \geq r$ and the subpath α' of α connecting e and h' must lie in the closed ball with center e and radius $2r$ of $\Gamma(G, S)$. Thus, the length of α' is bounded by the number of vertices in this ball. Therefore, $|h'|_T$ is bounded by the number of vertices of the closed ball with center e and radius $2r$ in $\Gamma(G, S)$. Thus, $dist_G^H(r) \leq Growth_G(2r)$. Therefore, $dist_G^H \preceq Growth_G$. \square

We now find some examples of finitely generated groups and its finitely generated subgroups to see their lower distortion. The following theorem can be deduced from the work of Milnor (see the proof of in [Mil68, Lemma 4]). We just use the new concept of lower distortion to interpret a part of Milnor's work.

Theorem 1.3.10. *Let $G = \langle a, b, c | bab^{-1}a^{-1} = c, ac = ca, bc = cb \rangle$ be the Heisenberg group and H the cyclic group generated by c . Then $dist_G^H$ and $Dist_G^H$ are both quadratic.*

Remark 1.3.11. In [Tit81], Tits investigates the growth of a finitely generated virtually nilpotent group. We can use a part of his work to find a pair (G, H) , where

G is a finitely generated nilpotent group and H is a finitely generated subgroup, such that $dist_G^H$ and $Dist_G^H$ can be equivalent to the same polynomial with arbitrary degree.

In [Osi01], Osin also gives a formula to compute upper distortion of arbitrary subgroups of nilpotent groups.

Before studying more examples about lower distortion, we need to review the concept of length functions and a key theorem.

Definition 1.3.12. Let G be a group with a finite generating set S and H a subgroup of G . The *length function* ℓ of H inside G is the function from the group H to the set of natural numbers as follows:

$$\ell(h) = |h|_S \text{ for } h \in H .$$

Remark 1.3.13. In some sense, the concept of length function can give us more information than the concepts of upper and lower distortion when we investigate an embedding of a subgroup.

Theorem 1.3.14. (*[Ol'99]*) Let ℓ be the length function of group H inside some finitely generated group G . Then the following conditions hold:

1. $\ell(h) = \ell(h^{-1})$ for every $h \in H$; $\ell(h) = 0$ iff $h = e$.
2. $\ell(h_1 h_2) \leq \ell(h_1) + \ell(h_2)$ for every $h_1, h_2 \in H$.
3. There is a positive integer C such that the cardinality of the set $\{ h \in H \mid \ell(h) \leq n \}$ does not exceed C^n for every natural number n

Conversely for every group H and every function ℓ from H to the set of natural numbers satisfying (1)–(3), there exists an embedding of H into a 2-generated group G with a finite generating set $S = \{g_1, g_2\}$ such that the length function ℓ_1 of H inside G is equivalent to ℓ (i.e. there exists a positive integer B such that $(1/B)\ell(h) \leq \ell_1(h) \leq B\ell(h)$).

Definition 1.3.15. A function $f : [0, \infty) \rightarrow [0, \infty)$ is subadditive if $f(i + j) \leq f(i) + f(j)$ for every positive numbers i and j .

We now apply Theorem 1.3.14 to show that any finitely generated group H can be a subgroup of a finitely generated group G such that the lower distortion and the upper distortion of H in G can be both equivalent to any element of some large class of functions.

Theorem 1.3.16. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing function such that $f(0) = 0$ and f^{-1} is subadditive. Suppose that there is a positive integer C such that $f(n) \leq C^n$ for every positive n . Let H be a finitely generated group such that its growth is bounded by some polynomial function. Then there is a finitely generated group G containing H such that $\text{dist}_G^H \sim \text{Dist}_G^H \sim f$.*

Proof. We fix some finite generating set T for H . Let A and m be a positive integers such that the number of group elements in a ball with radius n is bounded by An^m for every positive integer n . For each nonnegative number x , we define $\lceil x \rceil$ to be the smallest integer that is greater than or equal to x . We now define the length function $\ell : H \rightarrow \mathbb{N}$ as follows:

$$\ell(h) = \lceil f^{-1}(|h|_T) \rceil \text{ for every } h \in H .$$

We will check ℓ satisfies Conditions (1)–(3) in Theorem 1.3.14. Obviously, $\ell(h) = \ell(h^{-1})$ for every $h \in H$ and $\ell(h) = 0$ iff $h = e$. We now check ℓ satisfies Condition (2). Indeed, for every $h_1, h_2 \in H$

$$\begin{aligned} \ell(h_1 h_2) &= \lceil f^{-1}(|h_1 h_2|_T) \rceil \\ &\leq \lceil f^{-1}(|h_1|_T + |h_2|_T) \rceil \\ &\leq \lceil f^{-1}(|h_1|_T) + f^{-1}(|h_2|_T) \rceil \\ &\leq \lceil f^{-1}(|h_1|_T) \rceil + \lceil f^{-1}(|h_2|_T) \rceil \\ &\leq \ell(h_1) + \ell(h_2). \end{aligned}$$

Finally, we need to check ℓ satisfies Condition (3). Since for each nonnegative integer n

$$\begin{aligned} \{h \in H \mid \ell(h) \leq n\} &= \{h \in H \mid \lceil f^{-1}(|h|_T) \rceil \leq n\} \\ &= \{h \in H \mid f^{-1}(|h|_T) \leq n\} \\ &= \{h \in H \mid |h|_T \leq f(n)\} \\ &\subset \{h \in H \mid |h|_T \leq C^n\} \end{aligned}$$

and the cardinality of the set $\{h \in H \mid |h|_T \leq C^n\}$ is bounded by $A(C^m)^n$, then the cardinality of the set $\{h \in H \mid \ell(h) \leq n\}$ is bounded by $A(C^m)^n$.

By Theorem 1.3.14, the group H is a subgroup of some finitely generated group G with a finite generating set S such that the function ℓ is equivalent to ℓ_1 , where $\ell_1(h) = |h|_S$ for every $h \in H$. Therefore, there is a positive integer B such that $(1/B)\ell(h) \leq \ell_1(h) \leq B\ell(h)$ for every $h \in H$.

We now show that the upper distortion $Dist_G^H$ is dominated by f . For each positive number n and any $h \in H$ such that $|h|_S \leq n$, we see that

$$f^{-1}(|h|_T) \leq \ell(h) \leq B\ell_1(h) \leq Bn.$$

Thus, $|h|_T \leq f(Bn)$. Therefore, $Dist_G^H(n) \leq f(Bn)$. In particular, the upper distortion $Dist_G^H$ is dominated by f .

We finish the proof of the theorem by showing that the lower distortion $dist_G^H$ dominates f . For each positive number n and any $h \in H$ such that $|h|_S \geq Bn + B$, we see that

$$f^{-1}(|h|_T) \geq \ell(h) - 1 \geq \frac{1}{B}\ell_1(h) - 1 \geq n.$$

Thus, $|h|_T \geq f(n)$. Therefore, $dist_G^H(Bn + B) \geq f(n)$. In particular, the lower distortion $dist_G^H$ dominates f . \square

We now show one pair of groups (G, H) such that $dist_G^H$ and $Dist_G^H$ are not equivalent. The following example is defined by Gromov [Gro93]

Example 1.3.17. Let $G = \langle a, b, c \mid bab^{-1} = a^2, cbc^{-1} = b^2 \rangle$ and let H the cyclic subgroup generated a . Observe that

$$a^{2^{2^n}} = b^{2^n} a b^{-2^n} = c^n b c^{-n} a c^n b^{-1} c^{-n}$$

Thus, $Dist_G^H(4n + 2) \geq 2^{2^n}$ for each positive number n . Therefore, the upper distortion $Dist_G^H$ is super-exponential. However, the lower distortion $dist_G^H$ is at most exponential by Proposition 1.3.9. Therefore, two functions $dist_G^H$ and $Dist_G^H$ are not equivalent.

1.4 Relative divergence of geodesic spaces and finitely generated groups

1.4.1 Relative upper divergence

In this section, we introduce the concept of relative upper divergence of geodesic spaces as well as finitely generated groups. We also prove that upper relative divergence is a quasi-isometry invariant.

Definition 1.4.1. Let X be a geodesic space and A a subspace of X . Let r be any positive number.

1. $N_r(A) = \{ x \in X \mid d_X(x, A) < r \}$
2. $\partial N_r(A) = \{ x \in X \mid d_X(x, A) = r \}$
3. $C_r(A) = X - N_r(A)$.
4. Let $d_{r,A}$ be the induced length metric on the complement of the r -neighborhood of A in X . If the subspace A is clear from context, we can use the notation d_r instead of using $d_{r,A}$.

Definition 1.4.2. Let (X, A) be a pair of metric spaces. For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define a function $\delta_\rho^n: [0, \infty) \rightarrow [0, \infty]$ as follows:

For each r , let $\delta_\rho^n(r) = \sup d_{\rho r}(x_1, x_2)$ where the supremum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \leq nr$.

The family of functions $\{\delta_\rho^n\}$ is the *relative upper divergence* of X with respect to A , denoted $Div(X, A)$.

Before defining the upper relative divergence of a finitely generated group with respect to a subgroup, we need the following proposition.

Proposition 1.4.3. *If two pairs of spaces (X, A) and (Y, B) are quasi-isometric, then $Div(X, A) \sim Div(Y, B)$.*

Before proving the above proposition, we need the following lemmas.

Lemma 1.4.4. *Let X, Y be geodesic spaces and A a subspace of X . Let Φ be a quasi-isometry from X to Y . Then $Div(X, A) \preceq Div(Y, \Phi(A))$.*

Proof. Let $B = \Phi(A)$. Let $Div(X, A) = \{\delta_\rho^n\}$ and $Div(Y, B) = \{\delta'_\rho^n\}$. Let K be the number provided by Lemma 1.2.8. Let $L = 1/8K^2$ and $M = [2K(2K+1)+1]+1$. We will prove that $\delta_{L\rho}^n \preceq \delta'_\rho^{Mn}$. More precisely, we define $r_0 = 3K(1+K) + 8K^2/\rho$ and we are going to show that for each $r > r_0$

$$\delta_{L\rho}^n(r) \leq K \delta'_\rho^{Mn} \left(\frac{r}{2K} \right) + (2K^2 + 1)r.$$

Indeed, let x_1 and x_2 be arbitrary points in $\partial N_r(A)$ such that $d_X(x_1, x_2) \leq nr$ and $d_{r,A}(x_1, x_2) < \infty$. Thus, there is a path α in $C_r(A)$ connecting x_1 and x_2 . By Lemma 1.2.8, there is a path β connecting $\Phi(x_1), \Phi(x_2)$ such that the Hausdorff distance between $\Phi(\alpha)$ and β is at most K . Thus,

$$\begin{aligned} d_Y(\beta, B) &\geq d_Y(\Phi(\alpha), B) - K \\ &\geq \frac{1}{K} d_X(\alpha, A) - 1 - K \\ &\geq \frac{r}{K} - 1 - K \geq \frac{r}{2K} \end{aligned}$$

Thus, we could choose y_1 in $\partial N_{r/2K}(B)$ and a geodesic β_1 in $C_{r/2K}(B)$ connecting $\Phi(x_1)$ and y_1 such that the length of β_1 is bounded above by the distance between $\Phi(x_1)$ and B . Also, $d_Y(\Phi(x_1), B) \leq K d_X(x_1, A) + K \leq Kr + K$. Therefore, the length of β_1 is at most $Kr + K$. Similarly, we could choose y_2 in $\partial N_{r/2K}(B)$ and

a geodesic β_2 in $C_{r/2K}(B)$ connecting $\Phi(x_2)$ and y_2 such that the length of β_2 is bounded above by $Kr + K$.

We define $\beta_3 = \beta_1 \cup \beta \cup \beta_2$, then β_3 is a path in $C_{r/2K}(B)$ connecting y_1 and y_2 . Thus, $d_{r/2K,B}(y_1, y_2) < \infty$

Also

$$\begin{aligned} d_Y(y_1, y_2) &\leq d_Y(y_1, \Phi(x_1)) + d_Y(\Phi(x_1), \Phi(x_2)) + d_Y(\Phi(x_2), y_2) \\ &\leq (Kr + K) + (K d_X(x_1, x_2) + K) + (Kr + K) \\ &\leq 2Kr + 3K + Knr \leq (2K + 1)nr \leq Mn\left(\frac{r}{2K}\right) \end{aligned}$$

We are now going to show that

$$d_{L\rho r,A}(x_1, x_2) \leq K d_{\rho(r/2K),B}(y_1, y_2) + (2K^2 + 1)r.$$

Indeed, let β' be an arbitrary path in $C_{\rho(r/2K)}(B)$ connecting y_1 and y_2 . We define $\gamma = \beta_1 \cup \beta' \cup \beta_2$, then γ is a path in $C_{\rho(r/2K)}(B)$ connecting $\Phi(x_1)$, $\Phi(x_2)$ and the length of γ is bounded above by $2Kr + 2K + |\beta'|$.

By Lemma 1.2.8, there is a path α' connecting x_1 and x_2 in X such that the Hausdorff distance between $\Phi(\alpha')$ and γ is at most K . Moreover, $|\alpha'| \leq K|\gamma| + K$. Since

$$\begin{aligned} d_Y(\Phi(\alpha'), B) &\geq d_Y(\gamma, B) - K \\ &\geq \frac{\rho r}{2K} - K \geq \frac{\rho r}{4K} \end{aligned}$$

then

$$\begin{aligned} d_X(\alpha', A) &\geq \frac{1}{K} d_Y(\Phi(\alpha'), B) - 1 \\ &\geq \frac{\rho r}{4K^2} - 1 \geq \frac{\rho r}{8K^2} \geq L\rho r \end{aligned}$$

Thus, α' is a path in $C_{L\rho r}(A)$ connecting x_1 and x_2 . Therefore, the distance in $C_{L\rho r}(A)$ between x_1 and x_2 is bounded above by the length of α' .

Also

$$\begin{aligned} |\alpha'| &\leq K|\gamma| + K \\ &\leq K(2Kr + 2K + |\beta'|) + K \\ &\leq K|\beta'| + (2K^2 + 1)r, \end{aligned}$$

and β' is an arbitrary path in $C_{\rho(r/2K)}(B)$ connecting y_1 and y_2 .

Thus,

$$d_{L\rho r,A}(x_1, x_2) \leq K d_{\rho(r/2K),B}(y_1, y_2) + (2K^2 + 1)r.$$

Therefore,

$$\delta_{L\rho}^n(r) \leq K \delta_{\rho}^{Mn} \left(\frac{r}{2K} \right) + (2K^2 + 1)r.$$

Thus, $\delta_{L\rho}^n \preceq \delta_{\rho}^{Mn}$. □

Lemma 1.4.5. *Let X be a geodesic space. Let A and B be two subspaces such that the Hausdorff distance between them is finite. Then $Div(X, A) \sim Div(X, B)$.*

Proof. We only need to prove $Div(X, A) \preceq Div(X, B)$ since the argument for the other direction is almost identical. There is a positive number r_0 such that A lies in the r_0 -neighborhood of B and B also lies in the r_0 -neighborhood of A . Thus, $\overline{N_r(A)} \subset N_{r+r_0}(B)$ and $\overline{N_r(B)} \subset N_{r+r_0}(A)$ for each positive r . Let $Div(X, A) = \{\delta_{\rho}^n\}$ and $Div(X, B) = \{\delta_{\rho}^{6n}\}$. We will to show $\delta_{\rho/4}^n \preceq \delta_{\rho}^{6n}$. More precisely, we are going to prove that for each $r > 4r_0/\rho$

$$\delta_{\rho/4}^n(r) \leq \delta_{\rho}^{6n} \left(\frac{r}{2} \right) + 4r.$$

Let x_1, x_2 be arbitrary points in $\partial N_r(A)$ such that $d_X(x_1, x_2) \leq nr$ and $d_{r,A}(x_1, x_2) < \infty$. Thus, there is a path α in $C_r(A)$ connecting x_1 and x_2 . Therefore, α lies in $C_{r-r_0}(B)$. Thus, α also lies in $C_{r/2}(B)$ because $r/2 > r_0$. Moreover, x_1 and x_2 lies in $N_{r+r_0}(B)$. Therefore, we could choose y_1, y_2 in $\partial N_{r/2}(B)$ and two geodesics β_1, β_2 in $C_{r/2}(B)$ connecting x_1, y_1 and x_2, y_2 respectively such that the length of β_1 and β_2 are at most $r + r_0$. Since the distance between x_1 and x_2 is bounded above by nr , then the distance between y_1 and y_2 is at most $nr + 2r + 2r_0$. Thus, $d_X(y_1, y_2) \leq (n + 4)r \leq 3nr \leq 6n(r/2)$. We define $\alpha' = \beta_1 \cup \alpha \cup \beta_2$, then α' is a path in $C_{r/2}(B)$ connecting y_1 and y_2 . Thus, $d_{r/2,B}(y_1, y_2) < \infty$.

We are now going to show that

$$d_{\rho r/4,A}(x_1, x_2) \leq d_{\rho(r/2),B}(y_1, y_2) + 4r.$$

Indeed, let γ be an arbitrary path in $C_{\rho(r/2)}(B)$ connecting y_1 and y_2 . Then γ also lies in $C_{\rho(r/2)-r_0}(A)$. Therefore, γ lies in $C_{\rho r/4}(A)$. Since β_1 and β_2 lies in $C_{r/2}(B)$, then they also lies in $C_{r/2-r_0}(A)$. Thus, β_1 and β_2 lies in $C_{\rho r/4}(A)$. We define $\gamma' = \beta_1 \cup \gamma \cup \beta_2$, then γ' is a path in $C_{\rho r/4}(A)$ connecting x_1 and x_2 . Thus, $d_{\rho r/4, A}(x_1, x_2) \leq |\gamma'|$

Also

$$\begin{aligned} |\gamma'| &\leq |\beta_1| + |\gamma| + |\beta_2| \\ &\leq (r + r_0) + |\gamma| + (r + r_0) \\ &\leq |\gamma| + 4r \end{aligned}$$

and γ is an arbitrary path in $C_{\rho(r/2)}(B)$ connecting y_1, y_2 .

Thus,

$$d_{\rho r/4, A}(x_1, x_2) \leq d_{\rho(r/2), B}(y_1, y_2) + 4r.$$

Therefore,

$$\delta_{\rho/4}^n(r) \leq \delta_{\rho}^{6n} \left(\frac{r}{2} \right) + 4r.$$

Thus, $\delta_{\rho/4}^n \preceq \delta_{\rho}^{6n}$. □

We now finish the proof of Proposition 1.4.3.

Proof. Let Φ be a map from X to Y such that the Hausdorff distance between $\Phi(A)$ and B is finite. Then $Div(X, A) \preceq Div(Y, \Phi(A))$ by Lemma 1.4.4 and $Div(Y, \Phi(A)) \sim Div(Y, B)$ by Lemma 1.4.5. Thus, $Div(X, A) \preceq Div(Y, B)$. Similarly, $Div(Y, B) \preceq Div(X, A)$. Therefore, $Div(X, A) \sim Div(Y, B)$. □

We are now ready to define the concept of relative upper divergence of a finitely generated group with respect to a subgroup.

Definition 1.4.6. Let G be a finitely generated group and H its subgroup. We define *the relative upper divergence* of G with respect to H , denoted $Div(G, H)$ to be the relative upper divergence of the Cayley graph $\Gamma(G, S)$ with respect to H for some finite generating set S .

Remark 1.4.7. If H is the trivial subgroup, then $\delta_\rho^n = \delta_\rho^2$ for all $n \geq 2$. Thus, we can ignore the parameter n in the family $\{\delta_\rho^n\}$ and consider that $Div(G, e)$ is characterized by the one-parametrized family of functions $\{\delta_\rho\}$. By this way, the upper relative divergence $Div(G, e)$ is the same as the upper divergence $Div(G)$ of the group G in terms of Gersten [Ger94]

1.4.2 Relative lower divergence

In this section, we introduce the concept of relative lower divergence of geodesic spaces as well as finitely generated groups. Similar to upper divergence, this concept is also a quasi-isometry invariant.

Definition 1.4.8. Let (X, A) be a pair of spaces. For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define a function $\sigma_\rho^n: [0, \infty) \rightarrow [0, \infty]$ as follows:

For each positive r , if there is no pair of $x_1, x_2 \in \partial N_r(A)$ such that $d_X(x_1, x_2) \geq nr$ and $d_r(x_1, x_2) < \infty$, we define $\sigma_\rho^n(r) = \infty$.

Otherwise, we define $\sigma_\rho^n(r) = \inf d_{\rho r}(x_1, x_2)$ where the infimum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$.

The family of functions $\{\sigma_\rho^n\}$ is *the relative lower divergence* of X with respect to A , denoted $div(X, A)$.

By using the same argument from the previous section, we have the following proposition.

Proposition 1.4.9. *If two pairs of spaces (X, A) and (Y, B) are quasi-isometric, then $div(X, A) \sim div(Y, B)$.*

We now define the concept of relative lower divergence of a finitely generated group with respect to a subgroup.

Definition 1.4.10. Let G be a finitely generated group and H its subgroup. We define *the relative lower divergence* of G with respect to H , denoted $div(G, H)$, to be the relative lower divergence of the Cayley graph $\Gamma(G, S)$ with respect to H for some finite generating set S .

Before moving on other section, we would like to discuss the concept of lower divergence of a geodesic ray in Charney-Sultan [CS15]; and the connection between this concept and the concept of lower relative divergence. We first recall the concept of lower divergence of a geodesic ray in Charney-Sultan [CS15].

Definition 1.4.11. Let γ be a geodesic ray in a geodesic space X . For any $t > r > 0$, let $\rho_\gamma(r, t)$ denote the infimum of the lengths of all paths from $\gamma(t - r)$ to $\gamma(t + r)$ which lie outside the open ball of radius r about $\gamma(t)$. Define *the lower divergence of γ* to be the growth rate of the following function:

$$ldiv_\gamma(r) = \inf_{t > r} \rho_\gamma(r, t).$$

The following theorem shows the concept of lower relative divergence generalizes the concept of lower divergence of a geodesic ray.

Theorem 1.4.12. *Let γ be a geodesic ray in a geodesic space X . Then*

$$div(X, \gamma) \sim ldiv_\gamma.$$

The proof of the above theorem is similar to the proof we are going to give for Proposition 1.6.6 and we leave it to the reader.

1.4.3 Some Properties of Relative Divergence of finitely generated groups

In this section, we examine some key properties of relative divergence and we compare upper and lower relative divergence.

Theorem 1.4.13. *Let G be a finitely generated group and H a subgroup of G . Suppose that $Div(G, H) = \{\delta_\rho^n\}$ and $div(G, H) = \{\sigma_\rho^n\}$.*

1. *If H is an infinite index subgroup of G , then $\delta_\rho^n(r) < \infty$ for every $r > 0$.*
2. *If H is infinite and $0 < \tilde{e}(G, H) < \infty$, then $\sigma_\rho^n(r) < \infty$ for every $r > 0$.*

Proof. Fix a finite set S of generators of G .

First, we will prove that $\delta_\rho^n(r) < \infty$ for every $r > 0$. We define

$$A = S(e, r) \cap \partial N_r(H).$$

Obviously, A is a non-empty finite set. We define

$$B = \{ (x, y) \mid x \in A, y \in \partial N_r(H), d_r(x, y) < \infty \text{ and } d_S(x, y) \leq nr \}.$$

Therefore, B is also a non-empty finite set. Define $M = \{ d_{\rho r}(x, y) \mid (x, y) \in B \}$ and we will show $\delta_\rho^n(r) \leq M$.

Indeed, let x, y be arbitrary points in $\partial N_r(H)$ such that $d_r(x, y) < \infty$ and $d_S(x, y) \leq nr$. Let h be an element in H such that $d_S(x, H) = d_S(x, h) = r$. Thus, $(h^{-1}x, h^{-1}y) \in B$ and $d_{\rho r}(x, y) = d_{\rho r}(h^{-1}x, h^{-1}y)$. Thus, $d_{\rho r}(x, y) \leq M$. It follows that $\delta_\rho^n(r) \leq M$.

We now assume that $0 < \tilde{e}(G, H) < \infty$ and we will prove $\sigma_\rho^n(r) < \infty$ for all $r > 0$. Let $m = \tilde{e}(G, H)$. For each $i \in \{0, 1, 2, \dots, m\}$ we could choose h_i in H such that the distance between h_i and h_j is at least $(n+2)r$ whenever $i \neq j$. By Lemma 1.2.17, for each $i \in \{0, 1, 2, \dots, m\}$ we could choose an H -perpendicular ray γ_i with the initial point h_i . Thus, there are at least two different rays γ_i and γ_j such that $\gamma_i \cap C_r(H)$ and $\gamma_j \cap C_r(H)$ lie in the same component of $C_r(H)$. We define $u = \gamma_i(r)$ and $v = \gamma_j(r)$. Then u, v lie in $\partial N_r(H)$, the distance $d_r(u, v) < \infty$ and $d_S(u, v) \geq nr$. Thus,

$$\sigma_\rho^n(r) \leq d_{\rho r}(x, y) < \infty.$$

□

Theorem 1.4.14. *Let G be an infinite finitely generated group and H an infinite finitely generated subgroup of G . If $0 < \tilde{e}(G, H) < \infty$, then $\text{div}(G, H) \preceq \text{Div}(G, H)$*

Proof. Fix a finite generating set S of G such that $T = S \cap H$ generates H . We could consider $\Gamma(H, T)$ as a subgraph of $\Gamma(G, S)$. We denote $\text{Div}(G, H) = \{\delta_\rho^n\}$ and $\text{div}(G, H) = \{\sigma_\rho^n\}$. Let $m = \tilde{e}(G, H)$ and $M = 4(2m+1)$. We will show $\sigma_\rho^n \preceq \delta_\rho^{Mn}$. More precisely, we are going to prove that for each $r > 2$

$$\sigma_\rho^n(r) \leq \delta_\rho^{Mn}(r).$$

For each $i \in \{0, 1, 2, \dots, m\}$ we choose h_i in H such that $4nir \leq |h_i|_S < 4nir + 1$ and γ_i to be an H -perpendicular geodesic ray with the initial point h_i . Since $m = \tilde{e}(G, H)$, then there are two different geodesics γ_i and γ_j ($i < j$) such that $\gamma_i \cap C_r(H)$ and $\gamma_j \cap C_r(H)$ lie in the same component of $C_r(H)$. We define $x = \gamma_i(r)$ and $y = \gamma_j(r)$, then x and y lie in $\partial N_r(H)$ and $d_r(x, y) < \infty$. Also,

$$\begin{aligned} d_S(x, y) &\leq d_S(x, h_i) + d_S(h_i, h_j) + d_S(h_j, y) \\ &\leq r + 4n(i + j)r + 2 + r \leq 8mnr + 4r \leq (Mn)r \end{aligned}$$

and

$$\begin{aligned} d_S(x, y) &\geq d_S(h_i, h_j) - d_S(h_i, x) - d_S(h_j, y) \\ &\geq 4njr - 4nir - 1 - r - r \geq 4nr - 3r \geq nr \end{aligned}$$

Thus,

$$\sigma_\rho^n(r) \leq d_{\rho r}(x, y) \leq \sigma_\rho^{Mn}(r).$$

Therefore, $\sigma_\rho^n \preceq \delta_\rho^{Mn}$. □

Theorem 1.4.15 (Commensurability). *Let G be a finitely generated group.*

1. *If $K \leq H \leq G$ and $[H : K] < \infty$, then $Div(G, H) \sim Div(G, K)$ and $div(G, H) \sim div(G, K)$.*
2. *If H_1 and H_2 are two commensurable subgroups of G . Then, $Div(G, H_1) \sim Div(G, H_2)$ and $div(G, H_1) \sim div(G, H_2)$.*
3. *If $K \leq H \leq G$ and $[G : H] < \infty$, then $Div(G, K) \sim Div(H, K)$ and $div(G, K) \sim div(H, K)$.*
4. *For any conjugate gHg^{-1} of H , $Div(G, gHg^{-1}) \sim Div(G, H)$ and $div(G, gHg^{-1}) \sim div(G, H)$.*

Proof. The theorem follows immediately from Proposition 1.4.3 and Proposition 1.4.9. □

1.5 Relative divergence of finitely generated groups with respect to their normal subgroups

In this section, we investigate the upper and lower divergence of a finitely generated group relative to a normal subgroup.

Lemma 1.5.1. *Let G be a group with a finite generating set S and H a normal subgroup of G . Suppose g_1H, g_2H are arbitrary left cosets of H and the distance between them is n . Then for any element g_1h in g_1H the distance between g_1h and g_2H is also n .*

Proof. Obviously, the distance between g_1h and g_2H is at least n . Thus, we only need to show this distance is bounded above by n . Choose g_1h_1 in g_1H and g_2h_2 in g_2H such that the distance between them is n . Define $g = g_1hh_1^{-1}g_1^{-1}$. Since H is a normal subgroup, then g lies in H and $g' = g(g_2h_2)$ is an element in g_2H . Also, $d_S(g_1h, g') = d_S(gg_1h_1, gg_2h_2) = d_S(g_1h_1, g_2h_2) = n$. Therefore, the distance between g_1h and g_2H is at most n . \square

Theorem 1.5.2. *Let G be a finitely generated group and H a finitely generated normal subgroup of G . Suppose that $Div(G, H) = \{\delta_\rho^n\}$ and $Div(G/H, e) = \{\delta_\rho\}$. Let $\overline{\delta}_\rho^n(r) = \delta_\rho(r) + nr$ for each positive r and $\overline{Div}(G/H, e) = \{\overline{\delta}_\rho^n\}$. Then $Div(G/H, e) \preceq Div(G, H) \preceq Dist_G^H \circ \overline{Div}(G/H, e)$. Moreover, if G/H is one-ended and H is undistorted in G , then $Div(G, H) \sim Div(G/H, e)$.*

Proof. Let S be a finite generating set of G and assume that $T = G \cap S$ generates H . Moreover, the image \overline{S} of S under the quotient map is a finite generating set of the quotient group G/H . We see that the Cayley graph $\Gamma(G/H, \overline{S})$ is the quotient graph of the Cayley graph $\Gamma(G, S)$ under the action of H .

We will first show that $\delta_\rho^n \preceq Dist_G^H \circ \overline{\delta}_\rho^n$. More precisely, we will show that $\delta_\rho^n(r) \leq 2Dist_G^H \circ \overline{\delta}_\rho^n(r)$ for all positive r .

Indeed, let x, y be arbitrary points in $\partial N_r(H)$ such that $d_{r,H}(x, y) < \infty$ and $d_S(x, y) \leq nr$. We assume that r is an integer and x, y are vertices. Thus, there is a path in $C_r(H)$ connecting x and y . Let \overline{x} and \overline{y} be the associated points of x and

y respectively in $\Gamma(G/H, \bar{S})$. Thus, \bar{x} and \bar{y} lie in the sphere $S_r(\bar{e})$ and there is a path outside the ball $B_r(\bar{e})$ connecting them.

Since $d_{\rho r, \bar{e}}(\bar{x}, \bar{y}) \leq \delta_\rho(r)$, then there is a path α in $C_{\rho r}(\bar{e})$ connecting \bar{x}, \bar{y} such that the length of α is bounded above by $\delta_\rho(r)$. Thus, there is a path β in $C_{\rho r}(H)$ connecting x and some point y' in $\partial N_r(H)$. Moreover, $y' = hy$ for some h , and α, β have the same length. Thus, the length of β is also bounded above by $\delta_\rho(r)$. Thus, the distance between x and y' is also bounded above by $\delta_\rho(r)$ with respect to the metric d_S . Therefore, the distance between y and y' is bounded above by $\delta_\rho(r) + nr$ with respect to the metric d_S . Since y and y' lie in the same left coset gH , then there is a path γ with vertices in gH connecting y and y' . Thus, the path γ must lie in $C_r(H)$ by Lemma 1.5.1. Moreover, the path γ can be chosen with the length bounded above by $Dist_G^H(\delta_\rho(r) + nr)$. We define $\beta' = \beta \cup \gamma$ then β' is a path in $C_{\rho r}(H)$ connecting x, y and the length of β' is bounded above by $Dist_G^H(\delta_\rho(r) + nr) + \delta_\rho(r)$. Thus

$$d_{\rho r, H}(x, y) \leq Dist_G^H(\delta_\rho(r) + nr) + \delta_\rho(r) \leq 2Dist_G^H \circ \bar{\delta}_\rho^n(r).$$

Therefore,

$$\delta_\rho^n(r) \leq 2Dist_G^H \circ \bar{\delta}_\rho^n(r).$$

Thus,

$$\delta_\rho^n \preceq Dist_G^H \circ \bar{\delta}_\rho^n.$$

We now show $\delta_\rho \preceq \delta_\rho^n$. More precisely, we are going to show that $\delta_\rho(r) \leq \delta_\rho^n(r)$ for all positive r .

Indeed, let u and v be arbitrary points in $S_r(\bar{e})$ of $\Gamma(G/H, \bar{S})$ and $d_{r, \bar{e}}(u, v) < \infty$. We assume that r is an integer and u, v are vertices. Choose x_1 and y_1 be lifting points of u and v respectively such that $d_S(x_1, y_1) = d_{\bar{S}}(u, v) \leq 2r \leq nr$. Obviously, x_1 and y_1 lie in $\partial N_r(H)$. We will show $d_{r, H}(x_1, y_1) < \infty$.

Indeed, since there is a path in $C_r(\bar{e})$ connecting u and v , then there is a path α_1 in $C_r(H)$ connecting two points x_1 and some point y'_1 , where $y'_1 = h'y_1$ for some h' in H . Since y_1 and y'_1 lie in the same left coset $g'H$, then there is a path α_2 with vertices in $g'H$ connecting y_1 and y'_1 . By Lemma 1.5.1, the path α_2 also lies in

$C_r(H)$. By concatenating α_1 and α_2 , we have a path in $C_r(H)$ connecting x_1 and y_1 . Thus, $d_{r,H}(x_1, y_1) < \infty$.

We now prove that $d_{\rho r, \bar{e}}(u, v) \leq d_{\rho r, H}(x_1, y_1)$. Indeed, for any path γ' in $C_{\rho r}(H)$ connecting x_1 and y_1 , there is a path $\bar{\gamma}'$ connecting u, v such that the length of $\bar{\gamma}'$ is less than or equal to the length of γ' . Thus, $d_{\rho r, \bar{e}}(u, v) \leq d_{\rho r, H}(x_1, y_1)$. Therefore, $\delta_\rho(r) \leq \delta_{\rho r, \bar{e}}^n(r)$. Thus, $\delta_\rho \preceq \delta_{\rho r, \bar{e}}^n$.

If a quotient group G/H is one-ended, then $\delta_\rho(r) \geq 2r$ for each $r > 0$. Thus,

$$\bar{\delta}_\rho^n(r) = \delta_\rho(r) + nr \leq (n+1)\delta_\rho(r).$$

Therefore,

$$\delta_\rho^n(r) \leq 2Dist_G^H \circ \bar{\delta}_\rho^n(r) \leq 2Dist_G^H((n+1)\delta_\rho(r)).$$

So $\delta_\rho^n \preceq \delta_\rho$ if $Dist_G^H$ is dominated by a linear function.

Thus, $Div(G, H) \sim Div(G/H, e)$ if G/H is one-ended and H is undistorted in G .

□

Remark 1.5.3. If $G = H \times K$ and K is a one-ended group, then $Div(G, H) \sim Div(K, e)$. Thus, we could have any desired relative upper divergence $Div(G, H)$ by controlling the divergence $Div(K, e)$. In particular, any finitely generated group H could be embedded as a subgroup of a larger finitely generated group G such that $Div(G, H)$ is any polynomial functions or exponential function. Indeed, we only need to choose K to be a one-ended hyperbolic group to have the upper relative divergence $Div(G, H)$ as the exponential function. Similarly, we can choose a one-ended group K such that $Div(K, e)$ is equivalent to a desired polynomial (for example, see [Mac13]) and $Div(G, H)$ is also equivalent to this desired polynomial.

Theorem 1.5.4. *Let G be a finitely generated group and H an infinite normal subgroup of G . Let K be any finitely generated infinite subgroup of H . Then, $div(G, H) \preceq dist_G^K$. In particular, if H is finitely generated, then $div(G, H) \preceq dist_G^H$.*

Proof. Let S be a finite generating set of G and assume that $T = K \cap S$ generates K . Thus, $\Gamma(K, T)$ is a subgraph of $\Gamma(G, S)$. Denote $div(G, H) = \{\sigma_\rho^n\}$. We will prove that $\sigma_\rho^n \preceq dist_G^K$. More precisely, $\sigma_\rho^n(r) \leq dist_G^K(nr)$.

For each $r > 0$, we assume that r is an integer. Since $dist_G^K(nr) = \min\{|k|_T \mid |k|_S \geq nr\}$, then there is an element k_0 in K such that $|k_0|_S \geq nr$ and $|k_0|_T \leq dist_G^K(nr)$. Let α be a geodesic in $\Gamma(K, T)$ connecting the identity element e and k_0 . Thus, all vertices of α lie in H , and the length of α is bounded above by $dist_G^K(nr)$. Choose any element g in G such that $d_S(g, H) = r$ and define $x = g$ and $y = gk_0$. By Lemma 1.5.1, the points x and y lie in $\partial N_r(H)$ and $g\alpha$ is a path in $C_r(H)$ connecting x and y . Moreover, $d_S(x, y) = |k_0|_S \geq nr$. Thus,

$$\sigma_\rho^n(r) \leq d_{\rho r}(x, y) \leq \ell(g\alpha) \leq \ell(\alpha) \leq dist_G^K(nr).$$

Therefore, $\sigma_\rho^n \preceq dist_G^K$. □

Corollary 1.5.5. *Let G be a finitely generated group and H an infinite normal subgroup of G . If H contains some infinite finitely generated subgroup, then $div(G, H)$ is dominated by the growth of G . In particular, $div(G, H)$ is at most exponential.*

Remark 1.5.6. In Corollary 1.5.5, it is unknown whether or not $div(G, H)$ is dominated by the exponential function when every finitely generated subgroup of H is finite.

In Theorem 1.5.4, the relative lower divergence $div(G, H)$ can be strictly dominated by $dist_G^H$. Similarly, $Div(G, H)$ could be strictly dominated by $Dist_G^H \circ \overline{Div(G/H, e)}$ in Theorem 1.5.2. We now compute the relative divergence of the Heisenberg group with respect to some cyclic subgroup to show these facts.

Before computing the relative divergence of the Heisenberg group with respect to some cyclic subgroup, we need some results about this group.

Lemma 1.5.7. *Let $G = \langle a, b, c \mid bab^{-1}a^{-1} = c, ac = ca, bc = cb \rangle$ be the Heisenberg group and H the cyclic subgroup generated by c . Then*

1. *Each element of G could be written uniquely in the form $a^k b^\ell c^p$, where k, ℓ, p are integers.*

2.

$$(a^k b^\ell c^p)a = a^{k+1} b^\ell c^{p+1}$$

$$(a^k b^\ell c^p)b = a^k b^{\ell+1} c^p$$

$$(a^k b^\ell c^p)c = a^k b^\ell c^{p+1}$$

3. H is a normal subgroup of G , and $G/H = \mathbb{Z}^2$ is one-ended.

4. If $|a^k b^\ell c^p| \leq N$, then $|k| \leq N$, $|\ell| \leq N$, $|p| \leq N^2$

5. $d_S(a^k b^\ell c^p, H) = |k| + |\ell|$

Proof. For the facts (1), (2), (3) and (4), we refer the reader to Examples 1.5 and 1.18 in [GdlH90]. We now prove the fact (5).

First we observe that c commutes with every element of group G . Since $d_S(a^k b^\ell c^p, c^p) = d_S(c^p a^k b^\ell, c^p) = |a^k b^\ell|_S \leq |k| + |\ell|$ and $c^p \in H$, then $d_S(a^k b^\ell c^p, H) \leq |k| + |\ell|$. Let $c^{p'}$ be an element in H such that $d_S(a^k b^\ell c^p, H) = d_S(a^k b^\ell c^p, c^{p'})$. Thus, $d_S(a^k b^\ell c^p, H) = |c^{-p'} a^k b^\ell c^p|_S = |a^k b^\ell c^{p-p'}|_S$. Let w be the shortest word such that $a^k b^\ell c^{p-p'} \equiv_G w$. We could write w in the form $w = a^{k_1} b^{\ell_1} c^{p_1} a^{k_2} b^{\ell_2} c^{p_2} \dots a^{k_n} b^{\ell_n} c^{p_n}$ and $|w|_S = \sum_{i=1}^n (|k_i| + |\ell_i| + |p_i|)$. We note that the values of k_i, ℓ_i, p_i can be zero. Thus,

$$d_S(a^k b^\ell c^p, H) = \sum_{i=1}^n (|k_i| + |\ell_i| + |p_i|).$$

Also, there is p'' such that $w \equiv_G a^{k_1+k_2+\dots+k_n} b^{\ell_1+\ell_2+\dots+\ell_n} c^{p''}$

Thus, $a^k b^\ell c^{p-p'} \equiv_G a^{k_1+k_2+\dots+k_n} b^{\ell_1+\ell_2+\dots+\ell_n} c^{p''}$

By (1), it implies that $k = k_1 + k_2 + \dots + k_n$ and $\ell = \ell_1 + \ell_2 + \dots + \ell_n$

Then,

$$d_S(a^k b^\ell c^p, H) = \sum_{i=1}^n (|k_i| + |\ell_i| + |p_i|) \geq |k| + |\ell|.$$

Therefore, $d_S(a^k b^\ell c^p, H) = |k| + |\ell|$. \square

Theorem 1.5.8. *Let $G = \langle a, b, c \mid bab^{-1}a^{-1} = c, ac = ca, bc = cb \rangle$ be the Heisenberg group and H the cyclic group generated by c . Then*

1. dist_G^H and Dist_G^H are both quadratic.
2. $\text{div}(G, H)$ and $\text{Div}(G, H)$ are both linear.

Proof. The fact that dist_G^H and Dist_G^H are both quadratic could be seen in Theorem 1.3.10. We see that $\tilde{e}(G, H) = e(G/H) = 1$ by Theorem 1.2.15. Thus, $\text{div}(G, H) \preceq \text{Div}(G, H)$ by Theorem 1.4.14. Therefore, it is sufficient to show $\text{Div}(G, H)$ is linear.

Denote $\text{Div}(G, H) = \{\delta_\rho^n\}$. We will show that $\delta_\rho^n \preceq r$. More precisely, we are going to show that $\delta_\rho^n(r) \leq 50nr$ for all positive r .

Indeed, let x and y be arbitrary points in $\partial N_r(H)$ such that $d_r(x, y) < \infty$ and $d_S(x, y) \leq nr$. Assume that r is an integer and x, y are vertices. Write $x = a^k b^\ell c^p$ and $y = a^{k'} b^{\ell'} c^{p'}$. Thus, $|k| + |\ell| = r$ and $|k'| + |\ell'| = r$ by Lemma 1.5.7(5).

By Lemma 1.5.7(2) and the fact that c commutes with any element of group G , we compute

$$x^{-1}y = a^{k'-k} b^{\ell'-\ell} c^{(p'-p)-\ell(k'-k)}.$$

Also,

$$|x^{-1}y|_S = d_S(x, y) \leq nr.$$

Thus, $|k' - k| \leq nr$, $|\ell' - \ell| \leq nr$ and $|(p' - p) - \ell(k' - k)| \leq n^2 r^2$.

Therefore,

$$|p' - p| \leq |(p' - p) - \ell(k' - k)| + |\ell(k' - k)| \leq n^2 r^2 + nr^2 \leq 2n^2 r^2.$$

Let ℓ_1 be a number such that $\ell \ell_1 \geq 0$ and $|\ell_1| = r$. Let $x_1 = x b^{\ell_1 - \ell}$; $x_2 = x_1 a^{r-k}$ and $x_3 = x_2 b^{13nr - \ell_1}$. By Lemma 1.5.7(2), we see that $x_3 = a^r b^{13nr} c^{p + \ell_1(r-k)}$.

Since $x_1 = x b^{\ell_1 - \ell}$ and $|\ell_1 - \ell| \leq r$; then there is a path α_1 with edges labeled by b connecting x and x_1 such that the length of α_1 is less than or equal to r . Similarly, there is a path α_2 with edges labeled by a connecting x_1, x_2 such that the length of α_2 is less than $2r$ and a path α_3 with edges labeled by b connecting x_2, x_3 such that the length of α_3 is less than $14nr$. Let $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$. We see that each vertex of α is of the form $x = a^{k_1} b^{\ell_1} c^{p_1}$ where $|k_1| + |\ell_1| \geq r$. Therefore, α is a path

in $C_r(H)$ by Lemma 1.5.7(5) and α connects x and x_3 , where $x_3 = a^r b^{13nr} c^{p+\ell_1(r-k)}$ and $|\ell_1| = r$. Moreover, the length of α is bounded above by $17nr$.

By a similar argument, there is a path β in $C_r(H)$ connecting y and y_3 , where $y_3 = a^r b^{13nr} c^{p'+\ell'_1(r-k')}$ and $|\ell'_1| = r$. Moreover, the length of β is bounded above by $17nr$.

We now try to connect x_3 and y_3 by a path γ in $C_r(H)$ with length bounded above by $14nr$. Indeed, let $p_1 = p + \ell_1(r - k)$ and $p'_1 = p' + \ell'_1(r - k')$. If $p_1 = p'_1$ (i.e. $x_3 = y_3$), then we can consider γ is a trivial path connecting x_3 and y_3 with length 0. If $p_1 \neq p'_1$, then we assume that $p_1 < p'_1$. Thus,

$$|p'_1 - p_1| \leq |p' - p| + |\ell_1(r - k)| + |\ell'_1(r - k')| \leq 2n^2r^2 + 2r^2 + 2r^2 \leq 4n^2r^2.$$

Thus, $0 < p'_1 - p_1 \leq 4n^2r^2$.

Let t be a positive number such that $t^2 \leq (p'_1 - p_1) < (t + 1)^2$ and let $t_1 = (p'_1 - p_1) - t^2$. Then $t \leq 2nr$ and $t_1 \leq (t + 1)^2 - t^2 \leq 2t + 1 \leq 5nr$. Also, $c^{p'_1-p_1} = c^{t^2} c^{t_1} = b^t a^t b^{-t} a^{-t} c^{t_1}$ and $y_3 = x_3 c^{p'_1-p_1}$. Thus, we could connect x_3, y_3 by a path γ such that the length of γ is bounded above by $4t + t_1$. Therefore, this length is bounded above by $13nr$. Also, the distance between x_3 and H is $(13n+1)r$. Thus, γ must lie in $C_r(H)$. Let $\bar{\gamma} = \alpha \cup \gamma \cup \beta$ then $\bar{\gamma}$ is a path in $C_r(H)$ connecting x, y and the length of $\bar{\gamma}$ is bounded above by $50nr$. Thus, $d_{pr}(x, y) < 50nr$. Therefore, $\delta_\rho^n(r) \leq 50nr$. Thus, $\delta_\rho^n \leq r$. \square

1.6 Relative divergence of finitely generated groups with respect to their cyclic subgroups

In this section, we investigate the upper and lower divergence of a finitely generated group relative to an infinite cyclic subgroup.

Definition 1.6.1. Let G be a group with finite generating set S and H an infinite cyclic subgroup of G generated by some element h in S . Let e_h be the edge with the identity vertex as the initial point and labeled by h in $\Gamma(G, S)$. A bi-infinite arc $\alpha = \cup_{n \in \mathbb{Z}} h^n e_h$ is *the axis* of H .

Suppose G is a finitely generated one-ended group and H is an infinite cyclic subgroup of G in this section. Let h be a generator of H and assume that the finite generating set S of G contains h . Let α be the axis of H . Thus, α is a bi-infinite arc with all vertices in H .

We now define the concept of divergence of a bi-infinite arc in a one-ended geodesic space. This concept will play an important role for investigating the lower divergence of a one-ended group G with respect to an infinite cyclic subgroup.

Definition 1.6.2. Let X be a one-ended geodesic space and β a proper bi-infinite arc. Let c be one point on β . *The divergence of (β, c)* , denoted $div(\beta, c)$, is the function $f: (0, \infty) \rightarrow (0, \infty)$ as follows:

For each positive r , we define

$$f(r) = \inf \{ |\gamma| \mid \gamma \text{ is a path in } X - B(c, r) \text{ with endpoints on } \beta \text{ and on different sides of } c \}.$$

Remark 1.6.3. Observe that $div(\beta, c)$ is a non-decreasing function.

Let α be the axis of the infinite cyclic subgroup H , which is defined in Definition 1.6.1. Then $div(\alpha, h^i) = div(\alpha, e)$ in the Cayley graph $\Gamma(G, S)$ for any element h^i in H and let $div_\alpha = div(\alpha, e)$.

For each x in $\Gamma(G, S) - \alpha$ and u a point in α such that $d_S(x, \alpha) = d_S(x, u)$, the point u must be a vertex of $\Gamma(G, S)$. Thus, $N_r(\alpha) = N_r(H)$ for each $r > 1$. Therefore, $\partial N_r(\alpha) = \partial N_r(H)$ and $C_r(\alpha) = C_r(H)$ for each $r > 1$.

Definition 1.6.4. Let c be an arc in $\Gamma(G, S)$. If c_0 is any subset of c , the *Hull* of c_0 in c , denoted $Hull_c(c_0)$, is the smallest connected subspace of c containing c_0 .

Lemma 1.6.5. *Choose $r > 1$ and let n be a positive integer. Choose $s \geq 3Dist_G^H((n+2)r)$. Let a, b, c be three different points in α such that c lies between a, b . Assume that a, b lie outside the ball $B(c, s)$. Let γ be an arc outside $B(c, s)$ connecting a and b . Then there are two points x, y in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x, y) \geq nr$ and the segment of γ connecting x and y lies in $C_r(\alpha)$.*

Proof. First, we will show that γ does not lie in the r -neighborhood of α . Assume by way of contradiction that γ lies in the r -neighborhood of α . For each G -vertex

v of γ , let

$$c_v = \text{Hull}_\alpha(\alpha \cap \overline{B(v, r)}).$$

For each edge e of γ with G -endpoints v and w , let

$$c_e = \text{Hull}_\alpha(c_v \cup c_w).$$

We see that the subsegment $[a, b]$ of α is covered by the sets c_e for all edges e of γ . In particular, c lies in some c_e , where e is an edge of γ . Therefore, c lies between two vertices u_1 and v_1 of α whose distance from vertices of e is at most r . Thus, the distance between u_1 and v_1 is less than $2r + 1$. Therefore, the length of the subsegment $[u_1, v_1]$ of α is less than $\text{Dist}_G^H(2r + 1)$. Thus,

$$d_S(c, \gamma) \leq \text{Dist}_G^H(2r + 1) + r < 2\text{Dist}_G^H((n + 2)r) < s,$$

which is a contradiction. Thus, γ does not lie in the r -neighborhood of α .

Let $M = \{x_i \mid i \in \{0, 1, 2, \dots, n\}\}$ be the set of points of γ that satisfies the following conditions:

1. We have $x_0 = a$ and $x_n = b$.
2. For each $i \in \{1, 2, \dots, n - 1\}$, the distance between x_i and α is r .
3. For each $i \in \{0, 1, 2, \dots, n - 1\}$, the open segment (x_i, x_{i+1}) does not contain any point in $\partial N_r(\alpha)$

For each $i \in \{1, 2, \dots, n - 1\}$, let x'_i be a vertex of α such that $d_S(x_i, x'_i) = r$. We again assign $x'_0 = a$ and $x'_n = b$. For each $i \in \{0, 1, 2, \dots, n - 1\}$, we define d_i to be the subsegment of α that connect x'_i and x'_{i+1} . Thus, c must lie in some d_{i_0} . Since $(x_{i_0}, x_{i_0+1}) \cap \partial N_r(\alpha) = \emptyset$, then either $(x_{i_0}, x_{i_0+1}) \subset N_r(\alpha)$ or $(x_{i_0}, x_{i_0+1}) \cap N_r(\alpha) = \emptyset$

If $(x_{i_0}, x_{i_0+1}) \subset N_r(\alpha)$, we can use the same argument as above to show that $d_S(c, \gamma) < s$, which is a contradiction. Thus, $(x_{i_0}, x_{i_0+1}) \cap N_r(\alpha) = \emptyset$ or $(x_{i_0}, x_{i_0+1}) \subset C_r(\alpha)$.

Since the distance between x_{i_0} , c is at least s and the distance between x'_{i_0} , x_{i_0} is r , then the distance between x'_{i_0} and c is at least $s - r$. Thus, the length of the

segment of α connecting x'_{i_0} and c is at least $s - r$. Similarly, the length of the segment of α connecting x'_{i_0+1} and c is also at least $s - r$. Thus, the length of the segment of α connecting x'_{i_0} and x'_{i_0+1} is also at least $2s - 2r$. Therefore, this length is strictly bounded below by $Dist_G^H((n+2)r)$. Thus, the distance in H between x'_{i_0} and x'_{i_0+1} is strictly greater than $Dist_G^H((n+2)r)$. Therefore, the distance in G between x'_{i_0} and x'_{i_0+1} is at least $(n+2)r$. Also, the distances $d_S(x'_{i_0}, x_{i_0})$ and $d_S(x'_{i_0+1}, x_{i_0+1})$ are both r . Thus, the distance between x_{i_0} and x_{i_0+1} is at least nr . We let $x = x_{i_0}$ and $y = x_{i_0+1}$. \square

Proposition 1.6.6. *Let G be a one-ended group with a finite generating set S . Let H be an infinite cyclic subgroup generated by some element in S and α the axis of H . Then,*

$$div_\alpha \preceq div(G, H) \preceq div_\alpha \circ (3Dist_G^H).$$

Proof. Denote $div(G, H) = \{\sigma_\rho^n\}$.

We will first show that $\sigma_\rho^n \preceq div_\alpha \circ (3Dist_G^H)$. More precisely, we are going to show that $\sigma_\rho^n(r) \leq div_\alpha \circ (3Dist_G^H)((n+2)r)$ for all numbers $r > 1$.

Indeed, let $s = 3Dist_G^H((n+2)r)$. Let γ be any arc outside the ball $B(e, s)$ connecting two points u and v on α such that e lies between u and v . By Lemma 1.6.5, there are two points x and y in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x, y) \geq nr$ and the segment of γ connecting x and y lies in $C_r(\alpha)$. By Remark 1.6.3, two points x and y also lie in $\partial N_r(H)$. Then $d_{\rho r}(x, y)$ is bounded above by the length of γ . Therefore, $\sigma_\rho^n(r)$ is bounded above by the length of γ . Thus,

$$\sigma_\rho^n(r) \leq div_\alpha(s).$$

Therefore,

$$\sigma_\rho^n(r) \leq div_\alpha \circ (3Dist_G^H)((n+2)r).$$

We now will show that $div_\alpha \preceq \sigma_\rho^n$ for each $n \geq 20$. More precisely, we are going to show that for each $r > 3$

$$div_\alpha(\rho r) \leq \sigma_\rho^n(r) + 2r.$$

Indeed, let x_1 and y_1 be arbitrary points in $\partial N_r(H)$ such that $d_X(x_1, y_1) \geq nr$ and $d_r(x_1, y_1) < \infty$. Let β be any arc in $C_{\rho r}(H)$ connecting x_1 and y_1 . Let x_2 and y_2 be vertices in α such that $d_S(x_1, \alpha) = d_S(x_1, x_2) = r$ and $d_S(y_1, \alpha) = d_S(y_1, y_2) = r$. Let β_1 be a geodesic connecting x_1 and x_2 and β_2 a geodesic connecting y_1 and y_2 . Since the distance between x_1 and y_1 is bounded below by nr , then the distance between x_2 and y_2 is bounded below by $(n-2)r$. Let h^i be a vertex of α such that h^i lies between x_2, y_2 such that x_2, y_2 do not lie in the ball of center h^i with radius $5r$. Let $\bar{\beta} = \beta_1 \cup \beta \cup \beta_2$. Thus, $\bar{\beta}$ is a path outside the ball $B(h^i, \rho r)$ connecting the two points x_2, y_2 in α and h^i lies between x_2, y_2 . Therefore, we could have an arc β' from $\bar{\beta}$ connecting two points x_2 and y_2 . Thus, $div_\alpha(\rho r)$ is bounded above by the length of $\bar{\beta}$. Therefore, $div_\alpha(\rho r)$ is bounded above by $|\beta| + 2r$. Therefore, $div_\alpha(\rho r)$ is bounded above by $d_{\rho r}(x_1, y_1) + 2r$. Thus,

$$div_\alpha(\rho r) \leq \sigma_\rho^n(r) + 2r.$$

Therefore,

$$div_\alpha \preceq \sigma_\rho^n.$$

□

Theorem 1.6.7. *Let G be a one-ended finitely generated group and H an infinite cyclic subgroup of G . Suppose that $div(G, H) = \{\sigma_\rho^n\}$ and $Div(G, e) = \{\delta_\rho\}$. Then $\sigma_\rho^n \preceq \delta_\rho \circ ((3/\rho)Dist_G^H)$. In particular, $div(G, H) \preceq Div(G, e)$ if H is an undistorted subgroup.*

Proof. We will show that $\sigma_\rho^n(r) \leq \delta_\rho \circ ((3/\rho)Dist_G^H)((n+2)r)$ for all number $r > 1$.

Indeed, let $s = (3/\rho)Dist_G^H((n+2)r)$. Choose x and y in $\alpha \cap S(e, s)$ such that e lies between x and y . Let γ be an arbitrary arc outside $B_{\rho s}(e)$ connecting x and y . Since $\rho s = 3Dist_G^H((n+2)r)$, then there are two points x_1 and y_1 in $\gamma \cap \partial N_r(\alpha)$ such that $d_S(x_1, y_1) \geq nr$ and the segment of γ connecting x_1 and y_1 lies in $C_r(\alpha)$ by Lemma 1.6.5. Thus, the two points x_1 and y_1 also lie in $\partial N_r(H)$ and the segment of γ connecting x_1 and y_1 also lies in $C_r(H)$ by Remark 1.6.3. Thus, the distance $d_{\rho r}(x_1, y_1)$ is bounded above by the length of γ . Therefore, $\sigma_\rho^n(r)$ is also bounded

above by the length of γ . Thus,

$$\sigma_\rho^n(r) \leq \delta_\rho(s).$$

Therefore,

$$\sigma_\rho^n(r) \leq \delta_\rho \circ \left(\frac{3}{\rho} \text{Dist}_G^H\right)((n+2)r).$$

Thus, $\sigma_\rho^n \preceq \delta_\rho \circ ((3/\rho)\text{Dist}_G^H)$. □

Remark 1.6.8. In Theorem 1.6.7, we could not replace $\text{div}(G, H)$ by $\text{Div}(G, H)$. For example, let $H = \mathbb{Z}$ and K be any one-ended finitely generated group such that $\text{Div}(K, e)$ is super-linear. We define $G = H \times K$. Thus, G is a one-ended finitely generated group and H is an infinite cyclic subgroup of G . Then, Dist_G^H is linear and $\text{Div}(G, H) = \text{Div}(K, e)$ is super-linear. Also the divergence $\text{Div}(G, e)$ is linear (see Theorem 4.1 in [Ger94]). Thus, Theorem 1.6.7 is no longer true if we replace $\text{div}(G, H)$ by $\text{Div}(G, H)$.

Moreover, the two functions σ_ρ^n and $\delta_\rho \circ ((3/\rho)\text{Dist}_G^H)$ in Theorem 1.6.7 can be equivalent in some cases (for example: $G = \mathbb{Z}^2$ and H any cyclic subgroup of G), and σ_ρ^n can be strictly dominated by $\delta_\rho \circ ((3/\rho)\text{Dist}_G^H)$ in some other cases (see Theorem 1.5.8).

1.7 Relative divergence of CAT(0) groups

In this section, we investigate the relative divergence of (G, H) where G is a CAT(0) group. We use Theorem 1.5.2 to build CAT(0) groups with arbitrary polynomial upper relative divergences with respect to some subgroup (see Theorem 1.7.7). We also examine the class of groups defined by Macura [Mac13] to obtain arbitrary polynomial lower relative divergence (see Corollary 1.7.12).

We now review some concepts and some basic properties of a CAT(0) group. We refer the reader to [BH99] for studying more on CAT(0) groups.

Definition 1.7.1. Let X be a geodesic space. A *geodesic triangle* Δ in X consists of three points p, q, r in X and three geodesic segments $[p, q], [q, r], [r, p]$. A *comparison*

triangle for Δ in \mathbb{E}^2 is a geodesic triangle $\bar{\Delta}$ in \mathbb{E}^2 with vertices $\bar{p}, \bar{q}, \bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$ and $d(r, p) = d(\bar{r}, \bar{p})$. A point \bar{x} in $[\bar{q}, \bar{r}]$ is called a *comparison point* for x in $[q, r]$ if $d(q, x) = d(\bar{q}, \bar{x})$. Comparison points on $[p, q]$ and $[p, r]$ are defined in the same way.

Definition 1.7.2. A geodesic triangle Δ in a geodesic space X satisfies *the* CAT(0) *inequality* if $d(x, y) \leq d(\bar{x}, \bar{y})$ for all points x and y on Δ and corresponding points \bar{x}, \bar{y} on the comparison triangle $\bar{\Delta}$ in Euclidean space \mathbb{E}^2 .

Definition 1.7.3. A geodesic space X is CAT(0) *space* if every triangle in X satisfies the CAT(0) inequality.

A group is CAT(0) if it acts properly and cocompactly on some proper CAT(0) space.

The proof of the following proposition can be found in [BH99].

Proposition 1.7.4. *Let (X_1, d_1) and (X_2, d_2) be CAT(0) spaces. Then the Cartesian product $X_1 \times X_2$ endowed with the metric d defined by $d^2 = d_1^2 + d_2^2$ is a CAT(0) space.*

The following corollary is an immediate result of the above proposition.

Corollary 1.7.5. *The direct product of two CAT(0) groups is a CAT(0) group.*

The following theorem is a direct result from Corollary III.F.4.8 and Theorem III.F.4.10 in [BH99].

Theorem 1.7.6. *Every finitely generated abelian subgroup of a CAT(0) group is undistorted.*

Theorem 1.7.7. *Let f be any polynomial function or exponential function. There is a pair of groups (G, H) where G is a CAT(0) group and H is a normal infinite cyclic subgroup of G such that $\text{Div}(G, H) \sim f$.*

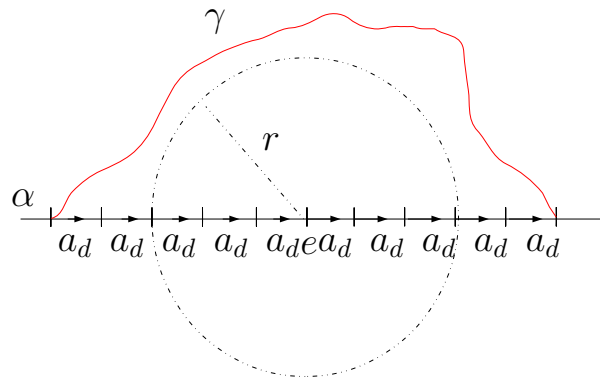


Figure 1.2: The path γ lies outside $B(e, r)$ with endpoints on α and on different sides of e

Proof. We will build the group G of the form $G = K \times \mathbb{Z}$ and we choose a suitable one-ended CAT(0) groups K . We choose H to be the \mathbb{Z} factor of G . Thus, we observe that $Div(G, H) = Div(G/H, e) = Div(K, e)$ by Theorem 1.5.2.

If f is a polynomial of degree d , then we choose a subgroup K such that $Div(K, e)$ is equivalent to f (see [Mac13] for example). If f is the exponential function, we choose K to be a surface group of genus $g \geq 2$. Since a surface group of genus $g \geq 2$ is a CAT(0) group, then the group G is also a CAT(0) group by Corollary 1.7.5. Moreover, K is a one-ended hyperbolic group, then the upper divergence of K is exponential. Thus, the relative upper divergence $Div(G, H)$ is also exponential. \square

Theorem 1.7.8. *Let G be a CAT(0) group and H a normal subgroup of G that contains at least one infinite order element. Then $div(G, H)$ is linear.*

Proof. By Theorem 1.7.6, there is an undistorted cyclic subgroup K in H . By Theorem 1.5.4, we observe that $div(G, H)$ is linear. \square

We now investigate relative lower divergence of a class of CAT(0) groups introduced by Macura in [Mac13]. First, we will review this class of groups.

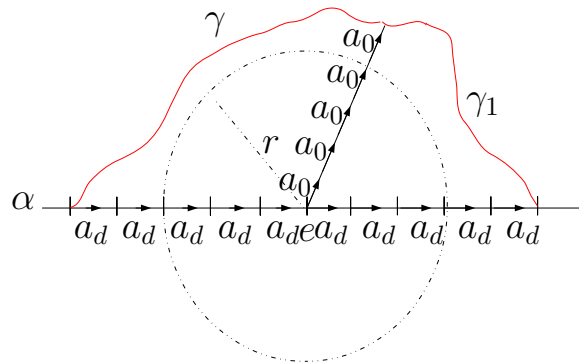


Figure 1.3: The subsegment γ_1 of γ connecting two points of γ_0 and γ_d , where γ_0 and γ_d are two geodesic rays issuing from e such that they are infinite concatenations of edges a_0 and a_d respectively

For each integer $d \geq 2$, we define

$$G_d = \langle a_0, a_1, \dots, a_d \mid a_0 a_1 = a_1 a_0, a_i^{-1} a_0 a_i = a_{i-1}, \text{ for } 2 \leq i \leq d \rangle$$

and H_d to be the cyclic subgroup generated by a_d .

Let X_d be the presentation complex of G_d and \tilde{X}_d is the universal cover of X_d . The space \tilde{X}_d is a CAT(0) square complex (see Macura [Mac13]). Moreover, G_d is one-ended and we could consider the 1-skeleton $\tilde{X}_d^{(1)}$ of \tilde{X}_d as the Cayley graph of G_d . Let α be the axis of the infinite cyclic subgroup of H_d as in Definition 1.6.1. By Proposition 1.6.6 and Theorem 1.7.6, we can investigate the divergence div_α of α in \tilde{X}_d to understand the lower divergence $div(G_d, H_d)$. Before computing div_α , we need to review some results from [Mac13].

Proposition 1.7.9. [Mac13, Proposition 4.4] *There is a polynomial q_d , of degree d , such that for any point O in \tilde{X}_d and any two points P, Q on the sphere $S(O, r) \subset \tilde{X}_d$, there is a path γ in $\tilde{X}_d - B(O, r)$ connecting P and Q such that the length of γ is at most $q_d(r)$*

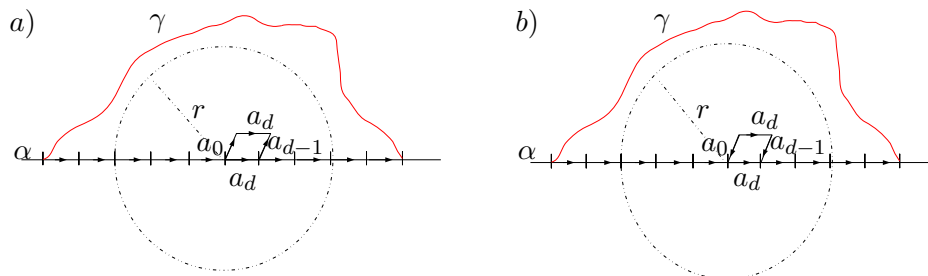


Figure 1.4: The position of 2-cell c_1 in the diagram D

Proposition 1.7.10. [Mac13, Theorem 5.3] *There is a polynomial p_d , of degree d , such that the following holds. Let T be any vertex on \tilde{X}_d . Let γ_0 be a geodesic ray which is the infinite concatenation of edges a_0 , and γ_d a geodesic ray which is the infinite concatenation of edges a_d . We assume that γ_0 and γ_d have the same initial point T . For each path β outside the ball $B(T, r)$ connecting $P \in \gamma_d$ and $Q \in \gamma_0$, the length of β is bounded below by $p_d(r)$.*

Proposition 1.7.11. *The divergence div_α is polynomial of degree d .*

Proof. By Proposition 1.7.9, there is a polynomial q_d , of degree d such that the following holds: Let r be any positive number and u, v two points in $S(e, r) \cap \alpha$ such that e lies between u, v . There is a path outside $B(e, r)$ of length at most $q_d(r)$ connecting u and v . Therefore, div_α is bounded above by q_d .

We now prove that div_α has some polynomial of degree d as a lower bound. Let p_d be the polynomial of degree d in Proposition 1.7.10. We will show div_α is bounded below by this polynomial. Indeed, for each positive r , let γ be any path outside $B(e, r)$ with endpoints on α and on different sides of e (see Figure 1.2).

We are going to show that there exists a subsegment γ_1 of γ connecting two points of γ_0 and γ_d , where γ_0 and γ_d are two geodesic rays issuing from e such that they are infinite concatenations of edges a_0 and a_d respectively (see Figure 1.3).

We will use the same technique as in [Ger94] for this argument. We observe that the path γ and the subsegment of α between two endpoints of γ form a loop in \tilde{X}_d

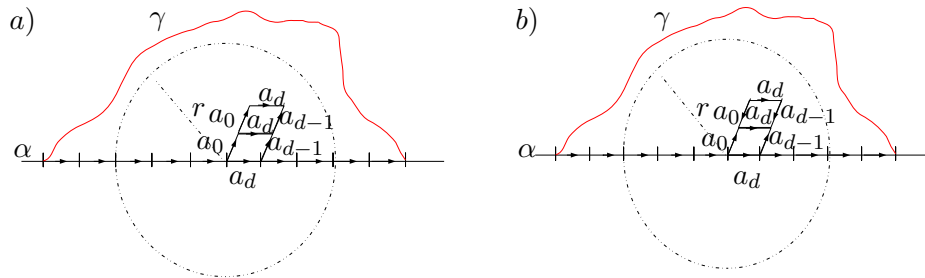


Figure 1.5: The position of 2-cell c_2 in the diagram D

which may fill in with a reduced Van Kampen diagram D (see [LS77]). Since the path γ lies outside the ball $B(e, r)$, the edge $a_d^{(1)}$ of α with the initial point e must lie in some 2-cell of D . By the presentation of G_d , the edge $a_d^{(1)}$ must lie in a 2-cell c_1 labeled by $a_d^{-1} a_0 a_d a_{d-1}^{-1}$. There are two cases for c_1 depending on its orientation in D (see Figure 1.4).

We now only argue on the first case (see Figure 1.4.a) and the argument of the second case (see Figure 1.4.b) is almost identical. If the edge $a_d^{(2)}$ that is opposite to $a_d^{(1)}$ in c_1 lies in the path γ , it is obvious that there exist a subsegment γ_1 of γ connecting two points of γ_0 and γ_d . Otherwise, $a_d^{(2)}$ must lie in some 2-cell c_2 labeled by $a_d^{-1} a_0 a_d a_{d-1}^{-1}$ of D . Again, there are two possibilities for c_2 depending on the orientation of c_2 in D (see Figure 1.5).

In the second case (see Figure 1.5.b), we see that the two 2-cells c_1 and c_2 form a cancellable pair in D . This is impossible since the diagram D is reduced. Thus, the second possibility is ruled out. By arguing inductively, we obtain a corridor that is a concatenation of 2-cells labeled by $a_d^{-1} a_0 a_d a_{d-1}^{-1}$ such that one edge $a_d^{(n)}$ labeled by a_d of the last 2-cell in the corridor must lie in the boundary of D . If $a_d^{(n)}$ is an edge of α , the diagram D would not be planar topologically. Thus, $a_d^{(n)}$ must be an edge of γ (see Figure 1.6).

Therefore, there exists a subsegment γ_1 of γ connecting two points of γ_0 and γ_d . Since the length of γ_1 is bounded below by $p_d(r)$ by Proposition 1.7.10, then the length of γ is also bounded below by $p_d(r)$. Therefore, the divergence div_α must be

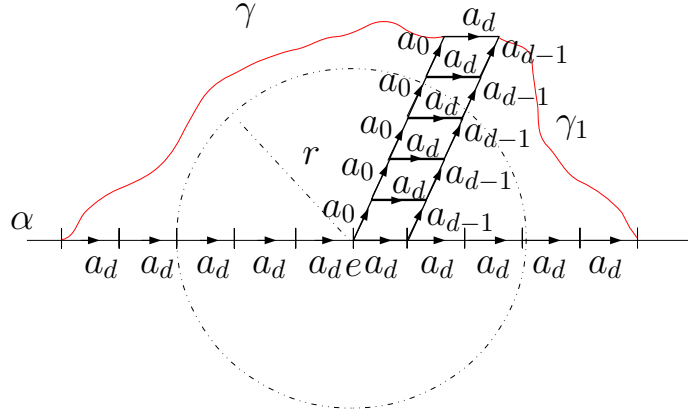


Figure 1.6: The corridor that is a concatenation of 2-cells labeled by $a_d^{-1}a_0a_da_{d-1}^{-1}$ in the diagram D

dominated the polynomial $p_d(r)$.

□

Corollary 1.7.12. *Let H_d be a cyclic subgroup of G_d generated by a_d . Then the relative lower divergence $\text{div}(G_d, H_d)$ is polynomial function of degree d .*

Proof. This is an immediate consequence of Proposition 1.6.6 and Proposition 1.7.11.

□

1.8 Relative divergence of relatively hyperbolic groups

We now investigate the relative divergence of a relatively hyperbolic group with respect to a subgroup.

Definition 1.8.1. A geodesic metric space (X, d) is δ -hyperbolic if every geodesic triangle with vertices in X is δ -thin in the sense that each side lies in the δ -neighborhood of the union of other sides.

A finitely generated group G is *hyperbolic* if the Cayley graph $\Gamma(G, S)$ is a hyperbolic space for some finite set of generators S .

Definition 1.8.2. A subspace Y of a geodesic metric space X is *quasiconvex* when there exists some $k > 0$ such that every geodesic in X that connects a pair of points in Y lies inside the k -neighborhood of Y .

Suppose G is a hyperbolic group with a finite generating set S . A subgroup H of a group G is quasiconvex if it is quasi-convex in the Cayley graph $\Gamma(G, S)$.

Remark 1.8.3. The concepts of hyperbolic groups and quasiconvex subgroups do not depend on the choice of finite set of generators (see [GdlH90] and [ABC⁺91]).

We now discuss a generalization of the concepts of hyperbolic groups and quasiconvex subgroups. They are relatively hyperbolic groups and relatively quasiconvex subgroups.

Definition 1.8.4. Given a finitely generated group G with Cayley graph $\Gamma(G, S)$ equipped with the path metric and a finite collection \mathbb{P} of subgroups of G , one can construct the *coned off Cayley graph* $\hat{\Gamma}(G, S, \mathbb{P})$ as follows: For each left coset gP where $P \in \mathbb{P}$, add a vertex v_{gP} , called a *peripheral vertex*, to the Cayley graph $\Gamma(G, S)$ and for each element x of gP , add an edge $e(x, gP)$ of length $1/2$ from x to the vertex v_{gP} . This results in a metric space that may not be proper (i.e. closed balls need not be compact).

Remark 1.8.5. Throughout this section, we denote the metric in $\Gamma(G, S)$ by d_S and the metric in $\hat{\Gamma}(G, S, \mathbb{P})$ by d .

Definition 1.8.6 (Relatively hyperbolic group). A finitely generated group G is *hyperbolic relative to a finite collection \mathbb{P} of subgroups of G* if the coned off Cayley graph is δ -hyperbolic and *fine* (i.e. for each positive number n , each edge of the coned off Cayley graph is contained in only finitely many circuits of length n).

Each group $P \in \mathbb{P}$ is a *peripheral subgroup* and its left cosets are *peripheral left cosets* and we denote the collection of all peripheral left cosets by Π .

An element g of G is *hyperbolic* if g is not conjugate to any element of any peripheral subgroups.

Lemma 1.8.7. [Hru10, Proposition 9.4] *Let G be a group with a finite generating set S . Suppose xH and yK are arbitrary left cosets of subgroups of G . For each constant L there is a constant $L' = L'(G, S, xH, yK)$ so that in the metric space $(\Gamma(G, S), d_S)$ we have*

$$N_L(xH) \cap N_L(yK) \subset N_{L'}(xHx^{-1} \cap yKy^{-1}).$$

Definition 1.8.8. Let (G, \mathbb{P}) be a relatively hyperbolic group. A subgroup H of G is *relatively quasiconvex* if the following holds. Let S be some (any) finite generating set for G . Then there is a constant $\kappa = \kappa(S)$ such that for each geodesic \bar{c} in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting two points of H , every G -vertex of \bar{c} lies within a d_S -distance κ of H .

Remark 1.8.9. We note that the concepts of relative hyperbolicity and relative quasiconvexity do not depend on the choice of finite set of generators (see [Osi06]).

Definition 1.8.10. Let (G, \mathbb{P}) be a relatively hyperbolic group.

1. A relatively quasiconvex subgroup H of G is *strongly relatively quasiconvex* if for each conjugate $g^{-1}Pg$ of any peripheral subgroup P and $H \cap g^{-1}Pg$ is a finite subgroup of $g^{-1}Pg$.
2. A relatively quasiconvex subgroup H of G is *fully relatively quasiconvex* if for each conjugate $g^{-1}Pg$ of any peripheral subgroup P , $H \cap g^{-1}Pg$ is a finite subgroup or finite index subgroup of $g^{-1}Pg$.

Lemma 1.8.11. [Osi06, Theorem 4.13] *Let (G, \mathbb{P}) be a relatively hyperbolic group. Let H be a subgroup of G . Then the following conditions are equivalent:*

1. H is *strongly relatively quasiconvex*.
2. H is generated by a finite set T such that the natural map $(H, d_T) \rightarrow \hat{\Gamma}(G, S, \mathbb{P})$ is a *quasi-isometric embedding*.

Lemma 1.8.12. [Osi06, Theorem 1.14] *Let (G, \mathbb{P}) be a relatively hyperbolic group with a finite generating set S . Then for any hyperbolic element $h \in G$ of infinite order, there exist $\lambda > 0$ and $c \geq 0$ such that $d(e, h^n) > \lambda|n| - c$. In particular, the cyclic subgroup H generated by h is undistorted with respect to (G, d_S) and strongly relatively quasiconvex.*

The following lemma is an immediate result of Proposition 2.36 in [Osi06].

Lemma 1.8.13. *Let (G, \mathbb{P}) be a relatively hyperbolic group. Then the following conditions hold:*

1. $g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1}$ is finite, where P_1 and P_2 are two different peripheral subgroups.
2. $g P g^{-1} \cap P$ is finite, where P is a peripheral subgroup and $g \notin P$.

Theorem 1.8.14. [Gro87, Section 8.2] *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite subgroup of G . If H is not conjugate to a subgroup of any peripheral subgroup, H contains a hyperbolic element.*

Lemma 1.8.15. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite index, infinite normal subgroup of G . Then H contains at least one infinite order hyperbolic element.*

Proof. If H is not conjugate to a subgroup of any peripheral subgroup, H contains a hyperbolic element by Theorem 1.8.14. Suppose that H is a subgroup of some conjugate $g P g^{-1}$ of some peripheral subgroup P , then $H = g^{-1} H g$ is a subgroup of P . Let g_1 be an element in $G - P$, then $H = g_1^{-1} H g_1$ is also a subgroup of $g_1^{-1} P g_1$. Then, $|P \cap g_1^{-1} P g_1| = \infty$, which contradicts Lemma 1.8.13. Therefore, H is not a subgroup of any conjugate of any peripheral subgroup. \square

Lemma 1.8.16. [Osi06, Theorem 3.26] *There is a positive constant σ such that the following holds. Let $\Delta = pqr$ be a triangle whose sides p, q, r are geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$. Then for each G -vertex v on p , there is a G -vertex u in the union $q \cup r$ such that $d_S(u, v) \leq \sigma$.*

The following lemma is an immediate result of Lemma 1.8.16.

Lemma 1.8.17. *There is a positive constant σ such that the following holds. Let pqr be a quadrilateral whose sides p, q, r, s are geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$. Then for each G -vertex v on p , there is a G -vertex u in the union $q \cup r \cup s$ such that $d_S(u, v) \leq 2\sigma$.*

Lemma 1.8.18. *[DS05, Lemma A.3] Let (G, \mathbb{P}) be a relatively hyperbolic group with a finite generating set S . Then there is a constant $K > 1$ such that the following holds. Let p and q be paths in $\hat{\Gamma}(G, S, \mathbb{P})$ such that $p^- = q^-$, $p^+ = q^+$, and q is geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$. Then for any vertex $v \in q$, there exists a vertex $w \in p$ such that $d_S(w, v) \leq K \log_2 |p|$.*

Lemma 1.8.19. *[DS05, Lemma 4.15] Let (G, \mathbb{P}) be a relatively hyperbolic group with a finite generating set S . For each A_0 there is a constant $A_1 = A_1(A_0)$ such that the following holds in $\text{Cayley}(G, S)$. Let c be a geodesic segment whose endpoints lie in the A_0 -neighborhood of a peripheral left coset gP . Then c lies in the A_1 -neighborhood of gP .*

Lemma 1.8.20. *[DS05, Theorem 4.1] Suppose (G, \mathbb{P}) is relatively hyperbolic with a finite generating set S . For each $M, M' < \infty$ there is a constant $\iota = \iota(M, M') < \infty$ so that for any two peripheral cosets $gP \neq g'P'$ we have*

$$\text{diam}(\mathcal{N}_M(gP) \cap \mathcal{N}_{M'}(g'P')) < \iota.$$

with respect to the metric d_S .

The following concepts are introduced by Hruska (see [Hru10, Definition 8.9]) and he used it to describe the connection between geodesics in $\Gamma(G, S)$ and geodesics in $\hat{\Gamma}(G, S, \mathbb{P})$.

Definition 1.8.21. Let c be a geodesic of $\Gamma(G, S)$, and let ϵ, R be positive constants. A point $x \in c$ is (ϵ, R) -deep in a peripheral left coset gP (with respect to c) if x is not within a distance R of an endpoint of c and $B(x, R) \cap c$ lies in $\mathcal{N}_\epsilon(gP)$. A point $x \in c$ is (ϵ, R) -deep if x is (ϵ, R) -deep in some peripheral left coset gP . If x is not (ϵ, R) -deep in any peripheral left coset gP then x is an (ϵ, R) -transition point of c

Lemma 1.8.22. [Hru10, Lemma 8.10] *Let (G, \mathbb{P}) be relatively hyperbolic with a finite generating set S . For each ϵ there is a constant $R = R(\epsilon)$ such that the following holds. Let c be any geodesic of $\Gamma(G, S)$, and let \bar{c} be a connected component of the set of all (ϵ, R) -deep points of c . Then there is a peripheral left coset gP such that each $x \in \bar{c}$ is (ϵ, R) -deep in gP and is not (ϵ, R) -deep in any other peripheral left coset.*

Lemma 1.8.23. [Hru10, Proposition 8.13] *Let (G, \mathbb{P}) be relatively hyperbolic with a finite generating set S . There exist constants ϵ , R and L such that the following holds. Let c be any geodesic of $\Gamma(G, S)$ with endpoints in G , and let \hat{c} be a geodesic of $\hat{\Gamma}(G, S, \mathbb{P})$ with the same endpoints as c . Then in the metric d_S , the set of G -vertices of \hat{c} is at a Hausdorff distance at most L from the set of (ϵ, R) -transition points of c . Furthermore, the constants ϵ and R satisfy the conclusion of Lemma 1.8.22.*

Lemma 1.8.24. [DS05, Lemma 4.12] *Let (G, \mathbb{P}) be relatively hyperbolic with a finite generating set S . Then for each $\theta \in [0, 1/2)$ there exist a number $M = M(\theta) > 0$ such that for every geodesic q of length ℓ and every peripheral left coset gP with $q(0), q(\ell) \in N_{\theta\ell}(gP)$ we have $q \cap N_M(gP) \neq \emptyset$.*

Theorem 1.8.25. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite index, infinite normal subgroup of G . Then $\text{div}(G, H)$ is linear.*

Proof. The proof of this theorem follows from Theorem 1.5.4, Lemma 1.8.12 and Lemma 1.8.15. □

Proposition 1.8.26. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H a subgroup of G for which H contains at least one infinite order hyperbolic element. If $0 < \bar{e}(G, H) < \infty$, then $\text{Div}(G, H)$ is at least exponential.*

Proof. Suppose that H contains an infinite order hyperbolic element h and assume that h is an element of the finite generating set S of G . By Lemma 1.8.12, there is a positive integer L such that $d(1, h^n) \geq (n/L) - L$. Moreover, the subgroup H_1 generated by h is strongly relatively quasiconvex. Thus, there is a constant

$A > 1$ such that the set of G -vertices of any geodesic β in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting two element of H_1 must lie in the A -neighborhood of H_1 with respect to the metric d_S .

We define $m = \tilde{e}(G, H)$ and $M = L(12m + 2L + 2)$. Let $K > 1$ be the constant in Lemma 1.8.18 and let σ the constant in Lemma 1.8.17. Denote $Div(G, H) = \{\delta_\rho^n\}$. We will prove that $e^r \preceq \delta_\rho^{Mn}$. More precisely, we define $r_0 = 2\sigma + (2/\rho)(A + 2\sigma) + L + 1$ and we will prove $2^{pr/2K} \leq \delta_\rho^{Mn}(r)$ for each $r > r_0$. We assume r is an integer.

Indeed, for each $i \in \{0, 1, 2, \dots, m\}$ we define γ_i to be an H -perpendicular geodesic ray with the initial point $k_i = h^{L(6inr + L)}$. Since $m = \tilde{e}(G, H)$, then there are two different geodesics γ_i and γ_j ($i < j$) such that $\gamma_i \cap C_r(H)$ and $\gamma_j \cap C_r(H)$ lie in the same component of $C_r(H)$. We define $x = \gamma_i(r)$ and $y = \gamma_j(r)$, then x, y lie in $\partial N_r(H)$ and $d_r(x, y) < \infty$. Also,

$$\begin{aligned} d_S(x, y) &\leq d_S(x, k_i) + d_S(k_i, e) + d_S(e, k_j) + d_S(h_j, y) \\ &\leq r + L(6inr + L) + L(6jnr + L) + r \\ &\leq L(12mnr + 2L) + 2r \\ &\leq L(12m + 2L + 2)nr \leq (Mn)r \end{aligned}$$

and

$$\begin{aligned} d(k_i, k_j) &= d(e, h^{6L(j-i)nr}) \\ &\geq 6(j-i)nr - L \geq 12r - L \geq 6r \end{aligned}$$

Let α_1 be a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting k_i, k_j and let α_2 a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting x, y . Let β_1 be a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting x, k_i and β_2 a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ connecting y and k_j . Let u be a point in α_1 such that $d(u, k_i) > 2r$ and $d(u, k_j) > 2r$. Thus, there is a G -vertex v in $\beta_1 \cup \alpha_2 \cup \beta_2$ such that $d_S(u, v) \leq 2\sigma$.

If v lies in β_1 , then the distance in $\hat{\Gamma}(G, S, \mathbb{P})$ between u and k_i is bounded above by $r + 2\sigma$. Thus, this distance is at most $2r$ which contradicts the choice of u . Thus, v does not lie in β_1 . Similarly, v does not lie in β_2 . Thus, v must lie in α_2 . Also, u lies in the A -neighborhood of the subgroup H_1 with respect to the metric d_S . Thus, v lies in the $(A + 2\sigma)$ -neighborhood of H_1 with respect to the metric d_S . Therefore, the distance in $\Gamma(G, S)$ between v and H is bounded above by $(A + 2\sigma)$.

We now prove that $d_{\rho r}(x, y) \geq 2^{\rho r/2K}$. Indeed, let γ be any path in $C_{\rho r}(H)$ connecting x and y . By Lemma 1.8.18, there exists a vertex $w \in \gamma$ such that $d_S(w, v) \leq K \log_2 |\gamma|$. Since

$$d_S(w, v) \geq d_S(w, H) - d_S(v, H) \geq \rho r - A - 2\sigma \geq \frac{\rho r}{2},$$

then

$$K \log_2 |\gamma| \geq \frac{\rho r}{2}.$$

Thus, $|\gamma| \geq 2^{\rho r/2K}$. Therefore, $d_{\rho r}(x, y) \geq 2^{\rho r/2K}$. Therefore, $2^{\rho r/2K} \leq \delta_\rho^{Mn}(r)$. Thus, $e^r \leq \delta_\rho^{Mn}$. \square

The following is a key lemma we are going to use to investigate the lower divergence of a relatively hyperbolic group with respect to a fully relatively quasiconvex subgroup.

Lemma 1.8.27. *Let (G, \mathbb{P}) be relatively hyperbolic with a finite generating set S . There exist constants ϵ, R, σ, K and A such that the following hold:*

1. *A subgroup H is relatively quasiconvex if and only if there is a constant κ such that for each geodesic c in $\Gamma(G, S)$ joining points in H , the set of (ϵ, R) -transition points of c lies in the κ -neighborhood of H .*
2. *Let $\Delta = pqr$ be a triangle whose sides p, q, r are geodesic in $\Gamma(G, S)$. Then for each (ϵ, R) -transition point v on p , there is an (ϵ, R) -transition point u in the union $q \cup r$ such that $d_S(u, v) \leq \sigma$.*
3. *Let p and q be paths in $\Gamma(G, S)$ such that $p^- = q^-$, $p^+ = q^+$ and q is geodesic in $\Gamma(G, S)$. For any (ϵ, R) -transition point $v \in q$, there exists a vertex $w \in p$ such that $d_S(w, v) \leq K \log_2 |p| + K$.*
4. *For each peripheral left coset gP and any geodesic c with endpoints outside $N_A(gP)$. If $\ell(c) > 9 \max\{d_S(c^+, gP); d_S(c^-, gP)\}$, then the path c contains an (ϵ, R) -transition point w which lies in the A -neighborhood of gP .*

Furthermore, the constants ϵ and R satisfy the conclusion of Lemma 1.8.22.

We now give the proof for the above lemma. The reader can also find the proof of Statement (1) in [Hru10].

Proof. Let ϵ and R be constants in Lemma 1.8.23. Statements (1), (2), and (3) are immediate results of Definition 1.8.8, Lemma 1.8.16, Lemma 1.8.18 and Lemma 1.8.23. We now focus on proving Statement (4).

Let

$$\begin{aligned} A_0 &= A_0\left(\frac{1}{3}\right) \text{ be the constant in Lemma 1.8.24} \\ A_1 &= A_1(A_0) \text{ be the constant in Lemma 1.8.19} \\ A_2 &= A_2(A_1, \epsilon) \text{ be the constant in Lemma 1.8.20} \\ A &= A_0 + A_1 + A_2 + \epsilon + 1 \end{aligned}$$

Let gP be any peripheral left coset. Let c be any geodesic with endpoints outside $N_A(gP)$ such that $\ell(c) > 9 \max\{d_S(c^+, gP), d_S(c^-, gP)\}$. Let $r = \max\{d_S(c^+, gP), d_S(c^-, gP)\}$. Thus, the length of c is greater than $9r$ and $r > A$. Since $\ell(c) > 9 \max\{d_S(c^+, gP), d_S(c^-, gP)\}$, then $c \cap N_{A_0}(u_1P) \neq \emptyset$ by Lemma 1.8.24. Let a_1 and a_2 be the first vertex and the last vertex in $c \cap N_{A_0}(gP)$. Thus, the subsegment $[a_1, a_2]$ of c connecting a_1 and a_2 must lie in the A_1 -neighborhood of gP . Let a'_1 and a'_2 the vertices in $c - [a_1, a_2]$ such that $d_S(a_1, a'_1) \leq 1$ and $d_S(a_2, a'_2) \leq 1$. We assume that a'_1 lies between c^+ , a_1 and that a'_2 lies between c^- , a_2 . Obviously, a'_1 and a'_2 must lie in the $(A_0 + 1)$ -neighborhood of gP . In particular, they lie in the r -neighborhood of gP . If the distance between c^+ and a_1 is greater than $4r$, then the distance in between c^+ and a'_1 is greater than $3r$. Thus, the subsegment of c connecting c^+ and a'_1 must intersect the A_0 -neighborhood of gP which contradicts to the choice of a_1 . Thus, $d_S(c^+, a_1) \leq 4r$. Similarly, $d_S(c^-, a_2) \leq 4r$. Also, the length of c is at least $9r$. Thus, the length of $[a_1, a_2]$ is at least r . In particular, this length is bounded below by A_2 .

We now show that c contains an (ϵ, R) -transition point w in the A -neighborhood of gP . Indeed, if $[a_1, a_2]$ contains an (ϵ, R) -transition point w , then w must lie in

the A_1 -neighborhood of gP . In particular, w lies in the A -neighborhood of gP and we are done.

We now consider the case that $[a_1, a_2]$ contains only (ϵ, R) -deep points. Therefore, $[a_1, a_2]$ lies in some ϵ -neighborhood of some peripheral left coset $g'P'$. Thus,

$$[a_1, a_2] \subset N_{A_1}(gP) \cap N_\epsilon(g'P').$$

Also, the length of $[a_1, a_2]$ is at least r . Thus, the length of $[a_1, a_2]$ is bounded below A_2 . Therefore, $\text{diam}(N_{A_1}(gP) \cap N_\epsilon(g'P'))$ is strictly greater than A_2 . Thus, $gP = g'P'$. It follows that $[a_1, a_2]$ lies in the ϵ -neighborhood of gP . Also, the endpoints of c both lie outside the ϵ -neighborhood of gP . Thus, we could find an (ϵ, R) -transition point w in c such that $d_S(w, gP) \leq \epsilon + 1$. In particular, w lies in the A -neighborhood of gP . \square

Theorem 1.8.28. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite fully relatively quasiconvex subgroup of G . If $0 < \tilde{e}(G, H) < \infty$, then $\text{div}(G, H)$ is at least exponential.*

Remark 1.8.29. Before giving the proof of the theorem, we would like to discuss a large class of groups and their subgroups to which the theorem applies. More precisely, we are going to discuss different pairs of groups (G, H) , where G is a relatively hyperbolic group and H is an infinite fully relatively quasiconvex subgroup of G with $0 < \tilde{e}(G, H) < \infty$.

Let G be the fundamental group of some hyperbolic surface and H an infinite cyclic subgroup of G . Thus, G is a hyperbolic group and H is an infinite malnormal quasiconvex subgroup of G . In particular, G is a relatively hyperbolic group and H is an infinite fully relatively quasiconvex subgroup. Obviously, the number of filtered ends $\tilde{e}(G, H) = 2$.

We now come up with other example. Let G be the fundamental group of some hyperbolic finite volume three manifold with cusps. Therefore, G is relatively hyperbolic with respect to the collection of its cusp subgroups. Let H be any cusp subgroup of G . We can see that H is an infinite fully relatively quasiconvex subgroup of G and $\tilde{e}(G, H) = 1$.

We now discuss the case H is a strongly relatively quasiconvex subgroup with finite number of filtered ends $\tilde{e}(G, H)$. We can choose G be the fundamental group of some hyperbolic finite volume three manifold with cusps as above. Again, G is relatively hyperbolic with respect to the collection of its cusp subgroups. Let H be a cyclic subgroup generated by a hyperbolic element. It is obvious that H is a strongly relatively quasiconvex subgroup and the number of filtered ends $\tilde{e}(G, H) = 1$

Now, we come up with a pair of groups (G, H) that satisfy all conditions in Theorem 1.8.28 and H is neither strongly relative quasiconvex nor a subgroup of some peripheral subgroup. Let G be the fundamental group of some hyperbolic finite volume three manifold with more than one cusp. We can pick up any cusp subgroup P and any cyclic subgroup K of G generated by some hyperbolic element. By Theorem 2 in [MPS12], it is obvious that we can choose some finite index subgroup P_1 of P and some finite index subgroup K_1 of K such that the subgroup H generated by P_1 and K_1 is isomorphic to their free product and H is also a fully relatively quasiconvex subgroup. It is not hard to see that the number of filtered ends $\tilde{e}(G, H) = 1$.

Proof. Let ϵ, R, σ, K and A be the constants in Lemma 1.8.27

Let κ be the constant such that for each geodesic c in $\Gamma(G, S)$ joining points in H , the set of (ϵ, R) -transition points of c lies in the κ -neighborhood of H .

By Lemma 1.8.7, we observe that the diameter of the set $(N_\kappa(H) \cap N_\epsilon(tP))$ is finite whenever $|tPt^{-1} \cap H| < \infty$. Also, the number of peripheral left cosets tP , where $|t|_S \leq \kappa + \epsilon$ and $P \in \mathbb{P}$, is finite. Thus, the number $B = \max\{\text{diam}(N_\kappa(H) \cap N_\epsilon(tP) \mid |t|_S \leq \kappa + \epsilon, P \in \mathbb{P} \text{ and } |tPt^{-1} \cap H| < \infty\}$ is finite. Similarly, we could choose a finite number C such that the C -neighborhood of H contains all peripheral left cosets tP where $|t|_S \leq \kappa + \epsilon$ and $|tPt^{-1} : (tPt^{-1} \cap H)| < \infty$.

Denote $\text{div}(G, H) = \{\sigma_\rho^n\}$. We will prove that $e^r \preceq \sigma_\rho^{27n}$. More precisely, we define

$$r_0 = \frac{4C}{\rho}(\kappa + K + A + B + C + 2\sigma)$$

and we will prove $2^{\rho r/4K} \leq \sigma_\rho^{27n}(r)$ for each $r > r_0$. We assume r is an integer.

Let x and y be arbitrary points in $\partial N_r(H)$ such that $d_S(x, y) \geq (27n)r$ and $d_r(x, y) < \infty$. (The existence of x and y is guaranteed by the condition $0 < \tilde{e}(G, H) < \infty$.) Let x_1 and y_1 be points in H such that $d_S(x, x_1) = d_S(x, H) = r$ and $d_S(y, y_1) = d_S(y, H) = r$

Let γ be any path in $C_{\rho r}(H)$ connecting x and y . Let c be a geodesic in $\Gamma(G, S)$ connecting x and y and c_1 a geodesic in $\Gamma(G, S)$ connecting x_1 and y_1 . Let β_1 be a geodesic in $\Gamma(G, S)$ connecting x and x_1 and β_2 a geodesic in $\Gamma(G, S)$ connecting y and y_1 .

By Lemma 1.8.27, for each (ϵ, R) -transition point u in c_1 there is an (ϵ, R) -transition point v_u in $\beta_1 \cup c \cup \beta_2$ such that $d_S(u, v_u) \leq 2\sigma$. We have two main cases:

Case 1: Suppose that v_u lies in c for some u in c_1 .

Since u lies in the κ -neighborhood of H , then v_u lies in the $(\kappa + 2\sigma)$ -neighborhood of H . By Lemma 1.8.27, there exists a vertex $w \in \gamma$ such that $d_S(w, v_u) \leq K \log_2 |\gamma| + K$. Since w lies outside $N_{\rho r}(H)$, then the distance $d_S(w, v_u)$ is bounded below by $\rho r - \kappa - 2\sigma$. Thus, $K \log_2 |\gamma| \geq \rho r - \kappa - 2\sigma - K \geq \rho r/4$ by the choice of r . Thus, the length of γ is bounded below by $2^{\rho r/4K}$.

Case 2: Suppose that v_u lies in $\beta_1 \cup \beta_2$ for all (ϵ, R) -transition point u in c_1 .

We could choose u_1 and u_2 in c_1 such that $v_{u_1} \in \beta_1$, $v_{u_2} \in \beta_2$ and all points in the geodesic c_1 lies between u_1 and u_2 are (ϵ, R) -deep points with respect to some peripheral left coset gP . In particular, the two points u_1, u_2 lie in the ϵ -neighborhood gP . Since v_{u_1} lies in β_1 and the length of β_1 is r , then the distance between u_1 and x_1 is bounded above by $r + 2\sigma$. Thus, the distance between u_1 and x_1 is bounded above by $2r$ by the choice of r . Similarly, the distance between u_2 and y_1 is bounded above by $2r$ with respect to the metric d_S . By the same argument, the distances $d_S(u_1, x)$ and $d_S(u_2, y)$ are also bounded above by $2r$. Also, the distance between x and y is at least $(27n)r$. Thus, the distance between u_1 and u_2 is bounded below by $(27n - 4)r$. Therefore, this distance is bounded below by $(23)r$ by the choice of n .

Since the distance $d_S(H, gP) \leq d_S(H, u_1) + d_S(u_1, gP) \leq \kappa + \epsilon$, then there are

some h_1 in H and t in G such that $|t|_S \leq \kappa + \epsilon$ and $gP = h_1tP$. Thus,

$$\begin{aligned} \text{diam}(N_\epsilon(tP) \cap N_\kappa(H)) &= \text{diam}(N_\epsilon(h_1tP) \cap N_\kappa(h_1H)) \\ &= \text{diam}(N_\epsilon(gP) \cap N_\kappa(H)). \end{aligned}$$

Since u_1 and u_2 lie in $N_\epsilon(gP) \cap N_\kappa(H)$, then

$$\text{diam}(N_\epsilon(gP) \cap N_\kappa(H)) \geq d_S(u_1, u_2) \geq (23)r > 23r > r_0 > B$$

Thus,

$$\text{diam}(N_\epsilon(tP) \cap N_\kappa(H)) > B.$$

Therefore, $|tPt^{-1} \cap H| = \infty$ by the choice of B . It follows that

$$|tPt^{-1} : (tPt^{-1} \cap H)| < \infty$$

since H is a fully relatively quasiconvex subgroup. Therefore, $tP \subset N_C(H)$. Thus,

$$gP = h_1tP \subset h_1N_C(H) = N_C(H).$$

Therefore, γ lies outside the $(\rho r - C)$ -neighborhood of gP . Thus, γ lies outside the $(\rho r/2)$ -neighborhood of gP by the choice of r .

We now show that there is an (ϵ, R) -transition point w in c such that $d_S(w, gP) \leq A$. Since gP lies in the C -neighborhood of H and the distance between x and H is r , then x lies outside the $(r - C)$ -neighborhood of gP . In particular, x lies outside the A -neighborhood of gP . Similarly, y also lies outside the A -neighborhood of gP . Since the distance between x and u_1 is bounded above by $2r$ and u_1 lies in the ϵ -neighborhood of gP , then x lies in the $(2r + \epsilon)$ -neighborhood of gP . In particular, x lies in the $3r$ -neighborhood of gP . Similarly, y also lies in the $3r$ -neighborhood of gP . Since x and y lies in the $3r$ -neighborhood of gP and the distance between x and y is greater than $27r$, then $\ell(c) > 9 \max\{d_S(c^+, gP), d_S(c^-, gP)\}$, then c contains an (ϵ, R) -transition point w in the A -neighborhood of gP by Lemma 1.8.27.

We now prove that the length of γ is bounded below by $2^{\rho r/4K}$. Indeed, by Lemma 1.8.27, there exists a vertex $v \in \gamma$ such that $d_S(v, w) \leq K \log_2 |\gamma| + K$. Also

$$d_S(v, w) \geq d_S(v, gP) - d_S(gP, w) \geq \frac{\rho r}{2} - A.$$

Thus,

$$K \log_2 |\gamma| \geq \frac{\rho r}{2} - A - K \geq \frac{\rho r}{4}.$$

Therefore, the length of γ is bounded below by $2^{pr/4K}$. Thus, $d_{pr}(x, y) \geq 2^{pr/4K}$. Thus, $2^{pr/4K} \leq \sigma_\rho^{27n}$. Therefore, $e^r \preceq \sigma_\rho^{27n}$. \square

Question 1.8.30. For the pair (G, H) as in Theorem 1.8.28, is the relative lower divergence $div(G, H)$ exactly exponential? What conditions do we need to put on the pair (G, H) to force the lower relative divergence $div(G, H)$ to be exactly exponential?

Corollary 1.8.31. *Let G be a hyperbolic group and H an infinite quasiconvex subgroup of G . If $0 < \tilde{e}(G, H) < \infty$, then $div(G, H)$ is at least exponential.*

Corollary 1.8.32. *Let (G, \mathbb{P}) be a relatively hyperbolic group and P an infinite peripheral subgroup. If $0 < \tilde{e}(G, P) < \infty$, then $div(G, P)$ is at least exponential.*

Corollary 1.8.33. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H an infinite strongly relatively quasiconvex subgroup. If $0 < \tilde{e}(G, H) < \infty$, then $div(G, H)$ is at least exponential.*

Remark 1.8.34. From the results of Corollary 1.8.31 and Theorem 1.6.7, we could extend the result of Corollary 1.7.12. More precisely, there is a pair of groups (G, H) , where G is a one-ended CAT(0) group and H is an infinite cyclic subgroup of G such that $div(G, H)$ is exponential. For example, let G be the fundamental group of a hyperbolic surface M and H the fundamental group of a closed essential curve C of M . Then G is a one-ended CAT(0) group and it is also hyperbolic. Since the infinite cyclic subgroup H is also quasiconvex, then $div(G, H)$ is at least exponential. Also, $div(G, H)$ is dominated by the upper divergence of G (see Theorem 1.6.7) and the upper divergence of one-ended finitely presented group is at most exponential (see Lemma 6.15 in [Sis]). Thus, $div(G, H)$ is at most exponential. Therefore, $div(G, H)$ is exactly exponential.

In Theorem 1.8.28, we could not replace the condition “fully relative quasiconvexity” of the subgroup H by the condition “relative quasiconvexity”. Readers could look at the following theorem as a counter example.

Theorem 1.8.35. *Let $G = \langle a_1, a_2, a_3, b, c \mid [a_1, a_2][a_3, b] = e, [b, c] = e \rangle$ and H be the cyclic subgroup of G generated by c . Let P be the subgroup generated by b and c . Then, G is a relatively hyperbolic group with respect to the subgroup P , $0 < \tilde{e}(G, H) < \infty$, H is a relatively quasiconvex subgroup and $\text{div}(G, H)$ is linear.*

Before giving the proof of Theorem 1.8.35, we need to review a result in Hruska [Hru04]

Definition 1.8.36. [Hru04, Definition 5.1] A CAT(0) 2-complex X has the *Isolated Flats Property* if there is a function $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every pair of distinct flat planes $F_1 \neq F_2$ in X and for every $k \geq 0$, the intersection $N_k(F_1) \cap N_k(F_2)$ of k -neighborhoods of F_1 and F_2 has diameter at most $\Phi(k)$.

Theorem 1.8.37. [Hru04, Theorem 1.6] *Suppose a group G acts properly and cocompactly by isometry on a CAT(0) 2-complex with the Isolated Flats Property. Then G is hyperbolic relative to the collection of maximal virtually abelian subgroups of rank two.*

We now give the proof for Theorem 1.8.35.

Proof. We are going to show that G acts properly and cocompactly by isometry on a CAT(0) 2-complex with the Isolated Flats Property. It is obvious that $G = G_1 *_{\langle b_1 \rangle = \langle b_2 \rangle} P$ where $G_1 = \langle a_1, a_2, a_3, b_1 \mid [a_1, a_2][a_3, b_1] = e \rangle$ and $P = \langle b_2, c \mid [b_2, c] = e \rangle$. Let X_1 be the presentation 2-complex of G_1 and X_2 the presentation 2-complex of P . We build the 2-complex presentation for G by identifying the 1-cell b_1 of X_1 and the 1-cell b_2 of X_2 into one 1-cell called b . Let \tilde{X}_1 and \tilde{X}_2 be the universal covers of X_1 and X_2 respectively. It is well-known that we can put a metric on \tilde{X}_1 such that \tilde{X}_1 becomes the 2-dimensional hyperbolic plane and G_1 acts properly and cocompactly on \tilde{X}_1 by isometry. Similarly, we can put a metric on \tilde{X}_2 such that \tilde{X}_2 becomes the 2-dimensional flat and P acts properly and cocompactly on \tilde{X}_2 by isometry. It is obvious that the universal cover \tilde{X} of X is the union of copies of \tilde{X}_1 and \tilde{X}_2 such that a copy of \tilde{X}_1 intersects a copy of \tilde{X}_2 in a bi-infinite arc labeled by b . Thus, \tilde{X} is a CAT(0) 2-complex with the Isolated Flats Property. Moreover,

the group G acts properly and cocompactly by isometry on \tilde{X} . Therefore, G is a relatively hyperbolic group with respect to the subgroup P by Theorem 1.8.37.

By examining the construction of \tilde{X} , we can see that $\tilde{e}(G, H) = 1$. Moreover, H is a relatively quasiconvex subgroup since it is a subgroup of peripheral subgroup P . We now show that the relative lower divergence $div(G, H)$ is linear.

First we show that $|b^n|_S = |n|$. Let $m = |b^n|_S$. Obviously, $m \leq |n|$. There is a homomorphism Φ from G to \mathbb{Z} that maps every element in S to the generator of \mathbb{Z} . Since $m = |b^n|_S$, then there is a word w in $S \cup S^{-1}$ with the length m such that $b^n \equiv_G w$. Therefore,

$$b^n \equiv_G s_1 s_2 \cdots s_m \text{ where } s_i \in S \cup S^{-1}.$$

Thus,

$$\Phi(b^n) = \Phi(s_1) + \Phi(s_2) + \cdots + \Phi(s_m).$$

Since $\Phi(b^n) = n$ and $\Phi(s_i) = -1$ or 1 , then $|n| \leq m$. Thus, $|b^n|_S = m = |n|$. Similarly, $|c^n|_S = |n|$

We now show that $d_S(b^m c^n, H) = |m|$. Denote $d_S(b^m c^n, H) = \ell$. Obviously, $\ell \leq |m|$. There is a group homomorphism Ψ from G to \mathbb{Z} that maps b to the generators of \mathbb{Z} and the remaining elements in S to 0 . Suppose that $d_S(b^m c^n, H) = d_S(b^m c^n, c^{n'})$ for some $c^{n'}$ in H . Thus, there is a word w' with the length ℓ such that $b^m c^n \equiv_G c^{n'} w'$. Therefore,

$$b^m c^n \equiv_G c^{n'} s'_1 s'_2 \cdots s'_\ell \text{ where } s'_i \in S \cup S^{-1}.$$

Thus,

$$\Psi(b^m) + \Psi(c^n) = \Psi(c^{n'}) + \Psi(s'_1) + \Psi(s'_2) + \cdots + \Psi(s'_\ell).$$

Since $\Psi(b^m) = m$, $\Psi(c^n) = \Psi(c^{n'}) = 0$ and $\Psi(s_i) = -1, 0$ or 1 , then $|m| \leq \ell$. Thus, $d_S(b^m c^n, H) = |m|$.

Denote $div(G, H) = \{\sigma_\rho^n\}$. We will prove that σ_ρ^n is bounded above by a linear function. More precisely, we will show $\sigma_\rho^n \leq nr$ for each $r > 0$.

We assume r is an integer. Let $x = b^r$ and $y = b^r c^{nr}$. Then x and y lie in $\partial N_r(H)$ and $d_S(x, y) \geq nr$. Let γ be the path with vertices $\{b^r, b^r c, b^r c^2, \dots, b^r c^{nr}\}$.

Then, γ is a path in $C_r(H)$ connecting x and y . Thus, $d_r(x, y) < \infty$. Moreover, $d_{\rho r}(x, y) \leq nr$ since the length of γ is nr . Thus, $\sigma_\rho^n \leq nr$. Therefore, σ_ρ^n is bounded above by a linear function.

□

Theorem 1.8.38. *Let (G, \mathbb{P}) be a relatively hyperbolic group and H a subgroup of G such that $0 < \tilde{e}(G, H) < \infty$. We assume that H is not conjugate to any infinite index subgroup of any peripheral subgroup. Then $Div(G, H)$ is at least exponential.*

Proof. If H is a finite subgroup, then the relative upper divergence $Div(G, H)$ is equivalent to the upper divergence of G by Theorem 1.4.15 and Remark 1.4.7. Also, the upper divergence of G is at least exponential by Sisto [Sis]. Thus, $Div(G, H)$ is at least exponential.

In the case that H is conjugate to a finite index subgroup of some peripheral subgroup. We assume that H is a finite index subgroup of some peripheral subgroup by Theorem 1.4.15. Thus, $div(G, H)$ is at least exponential by Theorem 1.8.28. Also, $div(G, H)$ is dominated by $Div(G, H)$ by Theorem 1.4.14. Therefore, the upper relative divergence $Div(G, H)$ is also at least exponential.

If H is an infinite subgroup that is not conjugate to any subgroup of any peripheral subgroup, H contains a hyperbolic element by Theorem 1.8.14. Thus, $Div(G, H)$ is at least exponential by Proposition 1.8.26.

□

Remark 1.8.39. In Theorem 1.8.38, if the group G is finitely presented, then the upper divergence of G is exactly exponential. Therefore, the upper relative divergence $Div(G, H)$ is also exponential when the subgroup H is finite. However, it is still unknown whether the upper relative divergence $Div(G, H)$ is exactly exponential in general; or what conditions we need to put on the pair (G, H) to make the relative upper divergence $Div(G, H)$ to be exactly exponential.

Chapter 2

Divergence of Morse geodesics

2.1 Introduction

The *divergence* of two geodesic rays α and β with the same initial point x_0 in a one-ended geodesic space X , denoted $\text{Div}_{\alpha,\beta}$, is a function $g : (0, \infty) \rightarrow (0, \infty)$ which for each positive number r the value $g(r)$ is the infimum on the lengths of all paths outside the open ball with radius r about x_0 connecting $\alpha(r)$ and $\beta(r)$. Consequently, the *divergence of a bi-infinite geodesic* γ , denoted Div_γ , is the divergence of the two geodesic rays obtained from γ with the initial point $\gamma(0)$. A (quasi-)geodesic γ is *Morse* if for any constants $K > 1$ and $L > 0$, there is a constant $M = M(K, L)$ such that every (K, L) -quasi-geodesic σ with endpoints on γ lies in the M -neighborhood of γ . In [BD], Behrstock and Druţu asked a question:

Question 2.1.1. (see Question 1.5, [BD]) Can the divergence function of a Morse geodesic in a CAT(0) space be greater than r^{m-1} and less than r^m for each $m \geq 3$?

The following theorem gives a positive answer to the above question as follows:

Theorem 2.1.2. *For each integer $m \geq 3$, there is a CAT(0) space X_m with a proper, cocompact action of some finitely generated group such that for each s in $(m-1, m]$ there is a Morse geodesic in X with the divergence function equivalent to r^s .*

In the above theorem, we refer the reader to Convention 2.3.1 for the concept of equivalence.

The existence of CAT(0) spaces containing Morse geodesics with polynomials divergence function of degree greater than one can be deduced from the work of Macura [Mac13], Dani-Thomas [DT15], Charney-Sultan [CS15], and Chapter 1 of this dissertation. However, the existence of Morse geodesic with divergence greater than r^{m-1} and less than r^m for $m \geq 3$ was still mysterious.

In [Ger94], Gersten introduced the divergence of a space. Consequently, he defined the divergence of a finitely generated group to be the divergence of its Cayley graph. This concept has been studied by Macura [Mac13], Behrstock-Charney [BC12], Duchin-Rafi [DR09], Druţu-Mozes-Sapir [DMS10], Sisto [Sis] and others. Moreover, the divergence of a space is a quasi-isometry invariant, and it is therefore a useful tool to classify finitely generated groups up to quasi-isometry. In the concept of the divergence of a space, Gersten used the concept of the divergence of two geodesic rays as the main idea to define it. Therefore, if we understand the divergence of bi-infinite geodesic, we may also understand the divergence of the whole space as well as a group that acts properly, cocompactly on the space. In particular, we hope that Theorem 2.1.2 can shed a light for the positive answer to the following question:

Question 2.1.3. (see Question 1.3, [BD]) Are there examples of CAT(0) groups whose divergence in the sense of Gersten is strictly between r^{m-1} and r^m for some m ?

The Morse property of quasi-geodesics is a quasi-isometry invariant. Therefore, it is a useful tool to classify geodesic spaces as well as finitely generated groups up to quasi-isometry. Theorem 2.1.2 reveals a geometric aspect of Morse geodesics. Therefore, Theorem 2.1.2 reveals a geometric aspect of Morse quasi-geodesics since each Morse quasi-geodesic has a finite Hausdorff distance from some Morse geodesic. Moreover, Theorem 2.1.2 helps us come up with a new quasi-isometry invariant of spaces, called spectrum divergence. The *spectrum divergence* of a geodesic space is a family S of functions from positive reals to positive reals such that a function f belongs to S if there is a bi-infinite Morse geodesic in the space with divergence function equivalent to f . Theorem 2.1.2 reveals that the spectrum divergence of a

CAT(0) space can contain uncountably many of power functions. Moreover, it is still unknown that what kind of function besides power functions can belongs to the spectrum of a CAT(0) space.

2.2 Right-angled Coxeter groups

Definition 2.2.1. Given a finite, simplicial graph Γ , the associated right-angled Coxeter group G_Γ has generating set S the vertices of Γ , and relations $s^2 = 1$ for all s in S and $st = ts$ whenever s and t are adjacent vertices.

Definition 2.2.2. Given a nontrivial, connected, finite, simplicial, triangle-free graph Γ with the set S of vertices, we may define the *Davis complex* $\Sigma = \Sigma_\Gamma$ to be the Cayley 2-complex for the presentation of the Coxeter group G_Γ , in which all disks bounded by a loop with label s^2 for s in S have been shrunk to an unoriented edge with label s . Then the vertex set of Σ is G_Γ and the 1-skeleton of Σ is the Cayley graph C_Γ of G_Γ with respect to the generating set S . Since all relators in this presentation other than $s^2 = 1$ are of the form $stst = 1$, then Σ is a square complex. The Davis complex Σ_Γ is a CAT(0) space and the group G_Γ acts properly and cocompactly on the Davis complex Σ_Γ (see [Dav08]).

Definition 2.2.3. Let Γ be a nontrivial, connected, finite, simplicial, triangle-free graph and $\Sigma = \Sigma_\Gamma$ the associated Davis complex. We observe that each edge of Σ is on the boundary of a square. We define a *midline* of a square in Σ to be a geodesic segment in the square connecting two midpoints of its opposite edges. We define a *hyperplane* to be a connected subspace that intersects each square in Σ in empty set or a midline. Each hyperplane divides the square complex Σ into two components. We define *the support* of a hyperplane H to be the union of squares which contain edges of H .

Since each square in Σ has the label of the form $stst$, then each midline in each square of Σ connects two midpoints of edges with the same label. Thus, each hyperplane is a graph and vertices are the midpoints of edges with the same label. Therefore, we define *the type* of a hyperplane H to be the label of edges containing

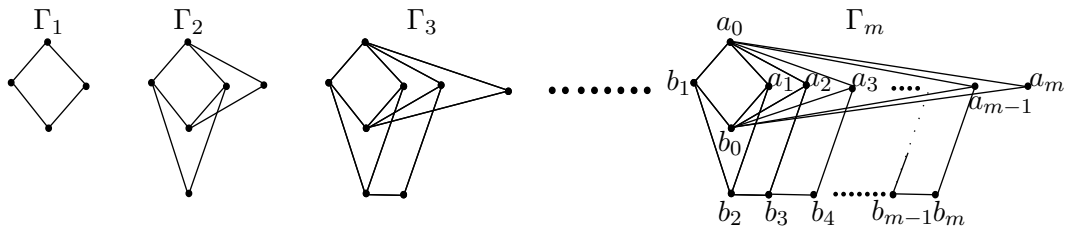


Figure 2.1: A collection of nontrivial, connected, finite, simplicial, triangle-free graphs.

vertices of H . Obviously, if two hyperplanes with the types a and b intersect, then a and b commute.

Remark 2.2.4. The length of a path α in C_Γ is equal to its number of hyperplane-crossings. A path is a geodesic if and only if it does not cross any hyperplane twice (see Lemma 3.2.14 [Dav08]).

If Γ' is a full subgraph of Γ , then the Davis complex $\Sigma_{\Gamma'}$ with respect to the graph Γ' embeds isometrically in the Davis complex Σ_Γ with respect to the graph Γ .

2.3 Proof of Theorem 2.1.2

Convention 2.3.1. Let f and g be two functions from positive reals to positive reals. We say that f is *dominated* by g , denoted $f \preceq g$, if there are positive constants A, B, C such that $f(x) \leq g(Ax) + Bx$ for all $x > C$. We say that f is *equivalent* to g , denoted $f \sim g$, if $f \preceq g$ and $g \preceq f$.

For each integer m , let Γ_m be the graph shown in Figure 2.1 and X_m be the Davis complex with respect to Γ_m . Let S_m denote the set of all vertices of Γ_m . For each integer $m \geq 3$, integer $i \geq 1$ and real number $t > 1$, let $w_{m,i,t}$ be the word $(a_m b_m)(a_m b_2)^{\lfloor i^{t-1} \rfloor}$. If m and t are clear from context, we can use the notation w_i instead of $w_{m,i,t}$. Let $\gamma_{t,m}$ be the bi-infinite path in X_m which passes through e and

labeled by $\cdots (a_m b_2)(a_m b_2)(a_m b_2)w_1 w_2 w_3 \cdots$, such that $\gamma_{t,m}(0) = e$, $\gamma_{t,m}(4) = w_1$, and $\gamma_{t,m}(-2i) = (a_m b_2)^{-i}$ for each positive integer i .

We observe that the labels of two consecutive edges of $\gamma_{t,m}$ do not commute. Thus, $\gamma_{t,m}$ is a bi-infinite geodesic (see Theorem 3.4.2, [Dav08]). We define a function f_t on the set of positive integers as follow:

$$f_t(n) = \lfloor 1^{t-1} \rfloor + \lfloor 2^{t-1} \rfloor + \lfloor 3^{t-1} \rfloor + \lfloor 4^{t-1} \rfloor + \lfloor 5^{t-1} \rfloor + \cdots + \lfloor n^{t-1} \rfloor.$$

There are constants $0 < h_t \leq 1/2$ and $n_t > 0$ such that for each $n > n_t$ the following holds:

$$h_t n^t \leq f_t(n) \leq n^t.$$

We are going to use the constants h_t and n_t many times in the rest of the dissertation.

Remark 2.3.2. For each nontrivial, connected, finite, simplicial, triangle-free graph Γ , the associated Davis complex Σ_Γ is a CAT(0) space. The first skeleton C_Γ of Σ_Γ is a Cayley graph of the group G_Γ with the word metric. It is not hard to see that the natural embedding of C_Γ in to Σ_Γ is a quasi-isometry. Moreover, we can see easily that the divergence function of a pair geodesic rays with the same initial point in C_Γ with respect to the word metric is equivalent to the divergence function of this pair of rays with respect to the CAT(0) metric.

We remark that Theorem 2.1.2 is stated under the CAT(0) metric and most results about divergence of pairs of geodesic rays we are going to use from [DT15] are stated under the word metric. However, we can apply these results to the case of the CAT(0) metric by the above observation.

Remark 2.3.3. Let α and β be two geodesic rays in a CAT(0) space with the same initial point x_0 . Assume that $\text{Div}_{\alpha,\beta}(r) \geq f(r)$. Using the fact that the projection onto a closed ball does not increase distances, we can show that if η is a path outside $B(x_0, r)$ connecting two points on α and β , then $\ell(\eta) \geq f(r)$. These observations will be used sometimes in the rest of the dissertation.

We now recall Gersten's definition of divergence from [Ger94] and get more results on the divergence of pairs of geodesic rays from [DT15].

Let X be a geodesic space and x_0 one point in X . Let d_{r,x_0} be the induced length metric on the complement of the open ball with radius r about x_0 . If the point x_0 is clear from context, we can use the notation d_r instead of using d_{r,x_0} .

Definition 2.3.4. Let X be a geodesic space with geodesic extension property and x_0 one point in X . We define the *divergence* of X , denoted Div_X , as a function $\delta : [0, \infty) \rightarrow [0, \infty)$ as follows:

For each r , let $\delta(r) = \sup d_r(x_1, x_2)$ where the supremum is taken over all $x_1, x_2 \in S_r(x_0)$ such that $d_r(x_1, x_2) < \infty$.

Remark 2.3.5. We remark that the above definition is only applied to geodesic spaces with geodesic extension property (i.e. any finite geodesic segment can be extended to an infinite geodesic ray) and it is a simplified version of the concept of the divergence of geodesic spaces in general (see [Ger94]). Since the spaces X that we will consider (Cayley graphs of right-angled Coxeter groups or Davis complexes of right-angled Coxeter groups) have the geodesic extension property, then the above definition works well with the purpose of this dissertation. Moreover, it is not hard to see the divergence of each pair of rays with the same initial point in X must be dominated by the divergence of X .

Lemma 2.3.6. *Let m be an arbitrary positive integer. For $1 \leq n \leq m$, let α_n and β_n be any geodesic rays in X_{m+2} satisfying the following conditions:*

1. α_n emanates from e and travels along $H_{b_{n+1}}$, and
2. β_n emanates from e and travels along one of H_{a_n} , H_{b_n} , or $H_{b_{n+2}}$.

Then the divergence of the pair of rays (α_n, β_n) dominates r^n in X_{m+2} .

The proof of the above lemma was shown in the proof of Proposition 5.3 in [DT15].

Lemma 2.3.7. *Let $m \geq 3$ be an integer. In the CAT(0) space X_m , let α be an arbitrary geodesic ray emanating from e that travels along the support of the hyperplane labeled by a_m and let β be a path emanating from e that travels along the support of the hyperplane labeled by b_m . Then the divergence of α and β is equivalent to r^{m-1} in X_m .*

Proof. We know that the space X_m embeds isometrically in X_{m+2} . We observe that the ray α is labeled by $a_0b_0a_0b_0 \cdots$ or $b_0a_0b_0a_0 \cdots$ in X_m . Similarly, the ray β is labeled by $a_{m-1}b_{m-1}a_{m-1}b_{m-1} \cdots$ or $b_{m-1}a_{m-1}b_{m-1}a_{m-1} \cdots$ in X_m . Therefore, the ray α also travels along $H_{a_{m-1}}$ in X_{m+2} . Moreover, the ray β travels along H_{b_m} in X_{m+2} . Thus, the divergence of α and β dominates r^{m-1} in X_{m+2} by Lemma 2.3.6. Since any path that avoids a ball centered at some point x_0 with some radius r in X_m still avoids the ball at x_0 with the same radius r in X_{m+2} , then the divergence of α and β also dominates r^{m-1} in X_m .

Since α and β are also geodesics rays in X_{m-1} and the divergence of X_{m-1} is r^{m-1} (see [DT15]), then the divergence of α and β dominates r^{m-1} in X_{m-1} . Again, the space X_{m-1} embeds isometrically in X_m . Therefore, any path that avoids a ball centered at some point x_0 with some radius r in X_{m-1} still avoids the ball at x_0 with the same radius r in X_m . This implies that the divergence of α and β also dominates r^{m-1} in X_m . Therefore, the divergence of α and β is equivalent to r^{m-1} in X_m . \square

Lemma 2.3.8. *Let $m \geq 3$ be an integer. In the CAT(0) space X_m , let α be an arbitrary geodesic ray emanating from e that travels along the support of the hyperplane labeled by a_m and let α' be a path emanating from e consisting of a geodesic segment labeled $a_m b_m$ followed by an arbitrary geodesic ray emanating from $a_m b_m$ that travels along the support of the hyperplane labeled by a_m . Then α' is a geodesic ray and the divergence of the pair (α, α') is equivalent to r^{m-1} .*

Proof. It is obvious that each pair of two consecutive generators of α' do not commute. Therefore, α' is a geodesic ray. Let α_1 be the geodesic ray emanating from a_m with the same label with α . Therefore, two rays α and α_1 both lie in the support of the hyperplane labeled by a_m and the Hausdorff distance between them is

exactly 1. Let α_2 be a sub-ray of α' with the initial point $a_m b_m$. Let β_1 and β_2 be two arbitrary geodesic rays in the support of the hyperplane labeled by b_m such that the initial point of β_1 is a_m and the initial point of β_2 is $a_m b_m$. Therefore, the Hausdorff distance between the two rays β_1 and β_2 is exactly 1. By Lemma 2.3.7, the divergence functions of the two pairs (α_1, β_1) and (α_2, β_2) are both equivalent to r^{m-1} . Therefore, the proof of the lemma follows easily. We note that we need Remark 2.3.3 and the fact that each path connecting α and α_2 must intersect the hyperplane labeled by the edge b_m of α' . \square

Lemma 2.3.9. *Let $m \geq 3$ be an integer. In the CAT(0) space X_m , let α be an arbitrary geodesic ray emanating from e that travels along the support of the hyperplane labeled by a_m and let α' be a path emanating from e consisting of a geodesic segment labeled $a_m b_2$ followed by an arbitrary geodesic ray emanating from $a_m b_2$ that travels along the support of the hyperplane labeled by a_m . Then α' is a geodesic ray and the divergence of the pair (α, α') is linear. Moreover, the union of the two rays α and α' is a quasi-geodesic.*

Proof. We observe that each pair of two consecutive generators of α' do not commute. Thus, α' is a geodesic ray. Let α_1 be an arbitrary geodesic ray emanating from a_m with the same label with α . Therefore, two rays α and α_1 both lie in the support of the hyperplane labeled by a_m and the Hausdorff distance between them is exactly 1. Let α_2 be a sub-ray of α' with the initial point $a_m b_2$. Let β_1 and β_2 be two geodesic rays labeled by $b_1 a_1 b_1 a_1 \cdots$ such that the initial point of β_1 is a_m and the initial point of β_2 is $a_m b_2$. Thus, two rays β_1 and β_2 both lie in the support of the hyperplane labeled by b_2 and the Hausdorff distance between them is exactly 1. Since the subgroup generated by $\{a_0, b_0, a_1, b_1\}$ is one-ended virtually abelian, then the associated Davis complex Σ' has linear divergence. Also the pair of rays (α_1, β_1) lies in some translation of Σ' , then the divergence of this pair is at most linear. Similarly, the divergence of the pair (α_2, β_2) is also at most linear. Therefore, the divergence of the pair (α, α') is dominated by a linear function.

Let α_3 be a union of the ray α_1 , the edge b_2 , and the ray α_2 . We observe that each pair of two consecutive generators of α_3 do not commute. Therefore, α_3 is a

geodesic ray. Therefore, it is not hard to see the union of the two rays α and α' is a quasi-geodesic. \square

Proposition 2.3.10. *For each integer $m \geq 3$ and real number $t > 1$, the divergence $\text{Div}_{\gamma_{t,m}} \leq r^{m-1+\frac{1}{t}}$ in X_m .*

Proof. For each number r large enough, we can choose an integer $n > n_t$ such that

$$r \leq h_t n^t \leq f_t(n) \leq n^t \leq \left(\frac{2}{h_t}\right)r.$$

Let $x = w_1 w_2 w_3 w_4 \cdots w_n$. Then

$$|x| = 2((\lfloor 1^{t-1} \rfloor + 1) + (\lfloor 2^{t-1} \rfloor + 1) + (\lfloor 3^{t-1} \rfloor + 1) + (\lfloor 4^{t-1} \rfloor + 1) + (\lfloor 5^{t-1} \rfloor + 1) + \cdots \\ \cdots + (\lfloor n^{t-1} \rfloor + 1)) = 2n + 2f_t(n).$$

Thus, $2f_t(n) \leq |x| \leq 4f_t(n)$. Therefore, we can connect x and $\gamma_{t,m}(r)$ by a path β_1 outside $B(e, r)$ such that

$$\ell(\beta_1) \leq 4f_t(n) - r \leq \left(\frac{8}{h_t} - 1\right)r.$$

We now try to connect $\gamma_{t,m}(-r)$ and x by a path β_2 outside $B(e, r)$ such that $\ell(\beta_2) \leq Mr^{m-1+\frac{1}{t}}$ for some constant M not depending on r and which completes the proof of the proposition.

Let $k = t - 1$, $\ell = |x|/2$, and $s_i = \gamma_{t,m}(2i)$ for each $0 \leq i \leq \ell$. Let H_{a_m} be the support of the hyperplane that crosses the edge a_m with one endpoint e . For $0 \leq i \leq \ell$, let u_i be a geodesic ray which run along the support $s_i H_{a_m}$ with the initial point s_i . We can choose u_i such that they have the same label for all i . Obviously, s_i and s_{i+1} are endpoints of the subsegment v_i of $\gamma_{t,m}$ labeled by $a_m b_m$ or $a_m b_2$ for $0 \leq i \leq \ell - 1$. Let T_1 be the set of index i such that v_i is labeled by $a_m b_m$ and T_2 be the set of index i such that v_i is labeled by $a_m b_2$. Since each w_i only contains one subword labeled by $a_m b_m$, then T_1 contains n elements and T_2 contains $(\ell - n)$ elements. For $0 \leq i \leq \ell - 1$, let m_i be a geodesic with the initial point s_i which runs along v_i followed by u_{i+1} . (The fact that m_i is a geodesic is guaranteed by Lemmas 2.3.8 and 2.3.9.)

For each i in T_1 , we can connect $u_i(8f_t(n))$ and $m_i(8f_t(n))$ by a path η_i outside $B(s_i, 8f_t(n))$ with length bounded above by $M_1(f_t(n))^{m-1} + N_1$ for some constants M_1 and N_1 not depending on r and n by Lemma 2.3.8. For each i in T_2 , we can connect $u_i(8f_t(n))$ and $m_i(8f_t(n))$ by a path η_i outside $B(s_i, 8f_t(n))$ with length bounded above by $M_2f_t(n) + N_2$ for some constants M_2 and N_2 not depending on r and n by Lemma 2.3.9. Since the distance between e and s_i is bounded above by $4f_t(n)$ and $f_t(n) \geq r$, then each η_i also lies outside $B(e, r)$ for $0 \leq i \leq \ell - 1$. Let η_ℓ be a subsegment of u_ℓ connecting x and $u_\ell(8f_t(n))$. Moreover, the ray σ with the initial point e which runs along a geodesic segment between e and s_ℓ followed by u_ℓ is a geodesic since each pair of consecutive edges of σ_i are labeled by two group generators which do not commute. Therefore, η_ℓ lies outside $B(e, r)$.

For $0 \leq i \leq \ell - 1$, we have $m_i(8f_t(n)) = u_{i+1}(8f_t(n) - 2)$. Thus, we can connect $m_i(8f_t(n))$ and $u_{i+1}(8f_t(n))$ by a path η'_i with length 2. Obviously, η'_i lies outside $B(e, r)$.

Let $\eta = (\eta_0\eta'_0)(\eta_1\eta'_1)(\eta_2\eta'_2) \cdots (\eta_{\ell-1}\eta'_{\ell-1})\eta_\ell$. Thus, η is a path outside $B(e, r)$ connecting $u_0(8f_t(n))$ and x . Moreover,

$$\ell(\eta) \leq n \left(M_1(f_t(n))^{m-1} + N_1 + 2 \right) + (\ell - n) \left(M_2f_t(n) + N_2 + 2 \right) + 8f_t(n).$$

It follows that there is some constant M_3 not depending on r and n , such that the length of η is bounded above by $M_3n^{(m-1)t+1}$. Therefore, there is some constant M_4 not depending on r and n , such that the length of η is bounded above by $M_4r^{(m-1)+\frac{1}{t}}$.

By Lemma 2.3.9 and the similar argument as above, we can connect $\gamma_{t,m}(-r)$ and $u_0(8f_t(n))$ by a path α outside $B(e, r)$ with length bounded above by $M_5r^2 + N_5$ for some constants M_5 and N_5 not depending on r . Let $\beta_2 = \alpha\eta$. Then β_2 lies outside $B(e, r)$ and connects $\gamma_{t,m}(-r)$ and x . Moreover, the length of β_2 is bounded above by $Mr^{m-1+\frac{1}{t}}$ for some constant M not depending on r . \square

Proposition 2.3.11. *For each integer $m \geq 3$ and real number $t > 1$, we have $r^{m-1+\frac{1}{t}} \preceq \text{Div}_{\gamma_{t,m}}$ in X_m .*

Proof. For each number r large enough, we can choose an integer $n > n_t$ such that

$$r \leq 10h_t n^t \leq 10f_t(n) \leq 10n^t \leq \left(\frac{2}{h_t}\right)r.$$

Let η be any path outside $B(e, r)$ connecting $\gamma_{t,m}(-r)$ and $\gamma_{t,m}(r)$. Since $\gamma_{t,m}$ restricted to $[-r, r]$ is a geodesic and η is a path with the same endpoints, η must cross each hyperplane crossed by $\gamma_{t,m}([-r, r])$ at least once. Let $s_0 = e$ and $s_i = w_1 w_2 w_3 w_4 \cdots w_i$ for $1 \leq i \leq \lfloor h_t n \rfloor$. Thus,

$$\begin{aligned} |s_i| &= 2((\lfloor 1^{t-1} \rfloor + 1) + (\lfloor 2^{t-1} \rfloor + 1) + (\lfloor 3^{t-1} \rfloor + 1) + (\lfloor 4^{t-1} \rfloor + 1) + \cdots \\ &\quad \cdots + (\lfloor i^{t-1} \rfloor + 1)) = 2i + 2f_t(i) \leq 4i^t. \end{aligned}$$

Let H_{a_m} be the support of the hyperplane that crosses the edge a_m with one endpoint e . For $0 \leq i \leq \lfloor h_t n \rfloor$, it is not hard to see $|s_i| \leq r$. Thus, the path η must cross $s_i H_{a_m}$ for $0 \leq i \leq \lfloor h_t n \rfloor$. Let g_i be the point at which η first crosses $s_i H_{a_m}$, where g_i lies in the component of the complement of the hyperplane in $s_i H_{a_m}$ containing e . Let u_i denote the geodesic connecting s_i and g_i which runs along $s_i H_{a_m}$. Similarly, for $0 \leq i \leq \lfloor h_t n \rfloor$, let h_i be the point at which η first crosses $s_i a_m b_m H_{a_m}$, where h_i lies in the component of the complement of the hyperplane in $s_i a_m b_m H_{a_m}$ containing e . Let v_i denote the geodesic connecting $s_i a_m b_m$ and h_i which runs along $s_i a_m b_m H_c$. For $0 \leq i \leq \lfloor h_t n \rfloor$, let η_i be a subsegment of η connecting g_i and h_i . Let m_i be a geodesic with the initial point s_i which runs along $a_m b_m$ followed by v_i . (The fact that m_i is a geodesic is guaranteed by Lemma 2.3.9.) Since

$$d(g_i, s_i) \geq d(g_i, e) - d(s_i, e) \geq r - 4i^t \geq 5h_t n^t - 4(h_t n)^t \geq h_t n^t$$

and

$$d(h_i, s_i) \geq d(h_i, e) - d(s_i, e) \geq r - 4i^t \geq 5h_t n^t - 4(h_t n)^t \geq h_t n^t$$

then

$$\ell(\eta_i) \geq M_1 (h_t n^t)^{m-1} - N_1$$

for some constants M_1 and N_1 not depending on r by Lemma 2.3.8 and Remark 2.3.3. Thus,

$$\ell(\eta) \geq (M_1 (h_t n^t)^{m-1} - N_1)(h_t n - 1) \geq M r^{m-1+\frac{1}{t}}$$

for some constant M not depending on r , which proves the proposition. \square

Before showing that each bi-infinite geodesic $\gamma_{t,m}$ is Morse, we would like to mention the concept of lower divergence as follows. The *lower divergence* of a bi-infinite ray γ in a one-ended geodesic space, denoted ldiv_γ , is a function $h : (0, \infty) \rightarrow (0, \infty)$ defined by $h(r) = \inf_t \rho_\gamma(r, t)$, where $\rho_\gamma(r, t)$ is the infimum of the lengths of all paths from $\gamma(t - r)$ to $\gamma(t + r)$ which lie outside the open ball of radius r about $\gamma(t)$.

The following proposition characterize Morse geodesics by using the concept of lower divergence.

Proposition 2.3.12 (Theorem 2.14, [CS15]). *Let X be a CAT(0) space and $\gamma \subset X$ a bi-infinite geodesic. Then the bi-infinite geodesic in X is Morse iff it has superlinear divergence.*

Remark 2.3.13. Let X be a one-ended geodesic space and γ a bi-infinite periodic geodesic (i.e. there is some isometry of X acts on γ by translation). It is not hard to see that the divergence of γ and the lower divergence of γ are equivalent.

Proposition 2.3.14. *For each integer $m \geq 3$ and real number $t > 1$, the lower divergence of the geodesic $\gamma_{t,m}$ is at least quadratic and the geodesic $\gamma_{t,m}$ is Morse.*

Proof. For each u and each r large enough, let η be any path outside $B(\gamma_{t,m}(u), r)$ connecting $\gamma_{t,m}(u - r)$ and $\gamma_{t,m}(u + r)$. Since $\gamma_{t,m}[u - r, u + r]$ is a geodesic and η is a path with the same endpoints, η must cross each hyperplane crossed by $\gamma_{t,m}[u - r, u + r]$ at least once. Without the loss of generality, we can assume $\gamma_{t,m}(u)$ is a group element and $\gamma_{t,m}(u + 2) = \gamma_{t,m}(u)a_m b_*$, where b_* is either b_m or b_2 .

Let $s_i = \gamma_{t,m}(u + 2i)$ for each integer i between 0 and $r/2$. Let H_{a_m} be the support of the hyperplane that crosses the edge a_m with one endpoint e . Let g_i be the point at which η first crosses $s_i H_{a_m}$, where g_i lies in the component of the complement of the hyperplane in $s_i H_{a_m}$ containing $\gamma_{t,m}(u)$. Let u_i denote the geodesic connecting s_i and g_i which runs along $s_i H_{a_m}$. Let m_i be a geodesic with the initial point s_i which runs along subsegment of $\gamma_{t,m}$ connecting s_i, s_{i+1} followed by u_{i+1} for each

integer i between 0 and $r/2 - 1$. Let η_i be a subsegment of η connecting g_i and g_{i+1} for each integer i between 0 and $r/2 - 1$.

Since

$$d(g_i, s_i) \geq d(g_i, \gamma_{t,m}(u)) - d(s_i, \gamma_{t,m}(u)) \geq r - 2i$$

and

$$d(g_{i+1}, s_i) \geq d(g_{i+1}, \gamma_{t,m}(u)) - d(s_i, \gamma_{t,m}(u)) \geq r - 2i$$

then

$$\ell(\eta_i) \geq M_1(r - 2i) - N_1$$

for some constants M_1 and N_1 not depending on r by Lemma 2.3.8, Lemma 2.3.9 and Remark 2.3.3.

Thus,

$$\ell(\eta) \geq \sum_i \ell(\eta_i) \geq \sum_i (M_1(r - 2i) - N_1) \geq Mr^2 - N$$

for some constant M not depending on r and u . Therefore, $\rho_{\gamma_{t,m}}(r, u)$ is bounded below by $Mr^2 - N$ for all u . Thus, $\gamma_{t,m}$ has at least quadratic lower divergence. Thus, the geodesic $\gamma_{t,m}$ is Morse for each integer $m \geq 3$ and $t > 1$ by Proposition 2.3.12. \square

Thus, for each integer $m \geq 3$ and each s in $(m - 1, m)$ the geodesic $\gamma_{t,m}$ is Morse and has the divergence function equivalent to r^s , where $t = 1/(s - m + 1)$. Finally, we finish the proof of Theorem 2.1.2 by the following proposition.

Proposition 2.3.15. *For each $m \geq 2$, let α_m be a bi-infinite geodesic containing e and labeled by $\cdots a_m b_m a_m b_m \cdots$. Then α_m is a Morse bi-infinite geodesic and the divergence of α_m is equivalent to r^m in X_m .*

Proof. We first show that the divergence of α_m is equivalent to r^m in X_m . Since the divergence of X_m is equivalent to r^m (see Section 5 in [DT15]), then the divergence of α_m is dominated by r^m . We only need to show that $r^m \preceq \text{Div}_{\alpha_m}$.

For each $r > 0$, let γ be an arbitrary path from $\alpha_m(-r)$ to $\alpha_m(r)$ which lies outside the open ball with radius r about e .

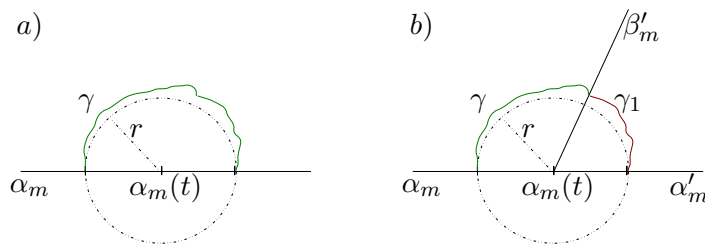


Figure 2.2: The path γ lies outside $B(\alpha_m(t), r)$ with endpoints $\alpha_m(t-r)$, $\alpha_m(t+r)$ and the subpath γ_1 of γ connecting two points of α'_m and β'_m .

We are going to show that there exists a subpath γ_1 of γ connecting two points of α'_m and β'_m , where α'_m and β'_m are two geodesic rays issuing from e , α'_m is labeled by $b_m a_m b_m a_m \cdots$ or $a_m b_m a_m b_m \cdots$ and β'_m is labeled by $b_{m-1} a_{m-1} b_{m-1} a_{m-1} \cdots$ or $a_{m-1} b_{m-1} a_{m-1} b_{m-1} \cdots$ (see Figure 2.2.b).

We will use the same technique as in [Ger94] for this argument. We observe that the path γ and the subsegment of α_m between $\alpha_m(-r)$, $\alpha_m(r)$ form a loop in X_m which may fill in with a reduced Van Kampen diagram D (see [LS77]). We want to obtain a corridor in D that is a concatenation of 2-cells labeled by $b_m a_{m-1} b_m a_{m-1}$ or $b_m b_{m-1} b_m b_{m-1}$ alternately such that the first edge labeled by b_m of the corridor must lie in α_m with one endpoints e and the last edge labeled by b_m of the corridor must lie in the boundary of D (see Figure 2.3).

Since the path γ lies outside the ball $B(e, r)$, the edge $b_m^{(1)}$ of α_m with one endpoint e and labeled by b_m must lie in some 2-cell of D . By the presentation of G_{Γ_m} , the edge $b_m^{(1)}$ must lie in a 2-cell c_1 labeled by $b_m a_{m-1} b_m a_{m-1}$ or $b_m b_{m-1} b_m b_{m-1}$. If the edge $b_m^{(2)}$ that is opposite to $b_m^{(1)}$ in c_1 lies in the the boundary of D , it is obvious that we can find the corridor we need. Otherwise, $b_m^{(2)}$ must lie in some 2-cell c_2 labeled by $b_m a_{m-1} b_m a_{m-1}$ or $b_m b_{m-1} b_m b_{m-1}$ of D . Since D is reduced, then the labels of c_1 and c_2 must be different. By arguing inductively, we obtain a corridor in D that is a concatenation of 2-cells labeled by $b_m a_{m-1} b_m a_{m-1}$ or $b_m b_{m-1} b_m b_{m-1}$ alternately such that the first edge labeled by b_m of the corridor lie in α_m with one endpoints e and the last edge $b_m^{(n)}$ labeled by b_m of the corridor lie in the boundary of D . If $b_m^{(n)}$ is an edge of α , the diagram D would not be planar topologically. Thus, $b_m^{(n)}$ must

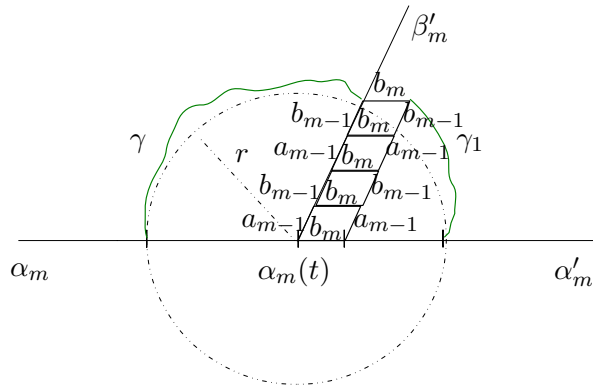


Figure 2.3:

be an edge of γ .

Therefore, there exists a subsegment γ_1 of γ connecting two points of α'_m and β'_m , where α'_m and β'_m are two geodesic rays issuing from e , α'_m is labeled by $b_m a_m b_m a_m \cdots$ or $a_m b_m a_m b_m \cdots$ and β'_m is labeled by $b_{m-1} a_{m-1} b_{m-1} a_{m-1} \cdots$ or $a_{m-1} b_{m-1} a_{m-1} b_{m-1} \cdots$. Therefore, the divergence of the pair (α'_m, β'_m) is dominated by the divergence α_m . Also, the divergence of the pair (α'_m, β'_m) is equivalent to r^m (see the proof of Proposition 5.3 in [DT15]). Therefore, the divergence of α_m is equivalent to r^m . We observe that α_m is a periodic geodesic. Therefore, the lower divergence of α_m and the divergence of α_m are equivalent. Thus, the lower divergence of α_m is super linear. This implies that α_m is a Morse geodesic. \square

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