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Extensions of Enveloping Algebras Via Anti-cocommutative Elements

Daniel Owen Yee
University of Wisconsin-Milwaukee

EXTENSIONS OF ENVELOPOING ALGEBRAS VIA ANTI-COCOMMUTATIVE ELEMENTS

by

Daniel Yee

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ABSTRACT

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Daniel Yee

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We know that given a connected Hopf algebra H, the universal enveloping algebra $U(P(H))$ embeds in H as a Hopf subalgebra. Depending on $P(H)$, we show that there may be another enveloping algebra (not as a Hopf subalgebra) within H by using anti-cocommutative elements. Thus, this is an extension of enveloping algebras with regards to the Hopf structure. We also use these discoveries to apply to global dimension, and finish with antipode behavior and future research projects.

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Chapter 1

Introduction

1.1 Preview

Connected Hopf algebras are a generalization of universal enveloping algebras. There has been a significant amount of research in universal enveloping aglebras in both past and present. We ask the same questions of connected Hopf algebras, and attempt to answer some of these questions under specific conditions.

Chapter 1 is introductory. We cover filtered algebras, Lie algebras along with their enveloping algebras, and the Gelfand-Kirillov Dimension. We provide definitions and state results that will be used throughout Chapters 2 and 3. More importantly, we introduce examples that will gain additional algebraic structure within Chapter 2 and will be used extensively in Chapter 3.

Chapter 2 introduces coalgebras and Hopf algebras. Due to the amount of research into these algebraic structures, we will focus on results that define connectedness in Hopf algebras, as well as certain properties. One important result is the connected version of the Taft-Wilson Theorem, which states that a cocommutative connected Hopf algebra is an universal enveloping algebra. Lastly, we present newer elements, namely the anti-cocommutative elements which were covered in the Wang, Zhang, Zhuang 2015 paper [29]. These elements are pivotal for the next chapter, and provide an extension of universal enveloping algebras with respect to the Hopf structure.

Chapter 3 and 4 covers the new results. We use the tools and definitions mentioned in Chapters 1 and 2 to prove some results concerning connected Hopf algebras. In particular, we will be focusing on a particular class of connected Hopf algebras, precisely those which are generated by anti-cocommutative elements. A motivation for this research is finding the elusive Noetherian condition within said algebras. One way to find a Noetherian subalgebra is to search for a subalgebra that is algebra-isomorphic to a universal enveloping algebra of some Lie algebra. A more general technique is to measure the growth of the algebra via GK-dimension. Simultaneously, we take a look at the properties of these connected Hopf algebras that are analogous to properties of universal enveloping algebras. And finally, we ask questions for future research.

In recent discovery there has been a few overlapping results between this thesis and the paper [3] written by Brown, Gilmartin, Zhang.

1.2 Description of Results

As the reader will see, the class of algebras we focus on, connected Hopf algebras, are generalizations of universal enveloping algebras via with respect to their Hopf structure. Since there has been many results concerning universal enveloping algebras, we try to search for universal enveloping algebras embedded in connected Hopf algebras.

Given a connected Hopf algebra H, we know $\mathfrak{g} = P(H)$ is a Lie subalgebra of H, we show that there could be (or not) a Lie algebra extension containing $P(H)$. When this happens, we say $P(H)$ satisfying the ALE property.

Proposition 1.2.1. *If $\mathfrak{g}$ is a finite dimensional completely solvable Lie algebra, then $\mathfrak{g}$ satisfies the ALE property.*

Corollary 1.2.2. *If $\mathfrak{g}$ is a finite dimensional simple Lie algebra, then $\mathfrak{g}$ does not satisfy the*

ALE property.

Furthermore, given $\mathfrak{g}$ satisfying the ALE property, what is the structure of the Lie algebra extension?

Proposition 1.2.3. *If $\mathfrak{g}$ is a finite dimensional nilportent Lie algebra, then $\mathfrak{g}$ satisfies the ALE property and any ALE of $\mathfrak{g}$ is a completely solvable Lie algebera.*

ALE are extensions of $\mathfrak{g}$ obtained by adding an anti-cocommutative element to $\mathfrak{g}$. However, if one wants more anti-cocommutative elements but wants to have similar properties to enveloping algebras, one can check normality.

Theorem 1.2.4. *Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra and $A \in \mathcal{A}(\mathfrak{g})$. If $U(\mathfrak{g})$ is a normal Hopf subalgebra of A, then $GK.dim(A) = \dim_F P_2(A)$.*

The notation $\mathcal{A}(\mathfrak{g})$ will be givin in section 4.1.

The universal enveloping algebras satisfies the property that global dimension is exactly the dimension of the Lie algebra generating it. We ask if the global dimension matches, then do we have a universal enveloping algebra?

Theorem 1.2.5. *If H is any connected Hopf algebra such that*

$$r.gl.dim(H) = \dim_F P(H) < \infty,$$

and $P(H)$ is completely solvable, then $H = U(P(H))$.

Theorem 1.2.6. *Suppose H is a connected Hopf algebra with*

$$r.gl.dim(H) = \dim_F P(H) < \infty,$$

and $U(P(H))$ is a normal Hopf subalgebra of H, then $H = U(P(H))$.

Lastly we know that the antipode of any enveloping algebra is involutive, that is S^2 is the identity map. However, that is not the case for all connected Hopf algebras.

Proposition 1.2.7. *Let $\mathfrak{g}$ be any Lie algebra, and consider $A \in \mathcal{A}(\mathfrak{g})$. If S is the antipode of A, then either $S^2 = id_A$, or $S^k \neq id_A$ for any $k \in \mathbb{Z} - 0$. In other words, either A is involutive or S has infinite order.*

1.3 Notation & Setup

Throughout this paper we will consider all vector spaces, linear maps, tensor prodcuts, algebras, and algebra homomorphisms over an algebraically closed field F of characteristic zero, e.g. $F = \mathbb{C}$. Furthermore, we denote $F\{x_1, ..., x_n\}$ as a vector space over F spanned by $x_1, ..., x_n$, and $\dim_F$ as the vector space dimension. We also denote $\implies$ as implies.

In an algebra A, we assume that the bracket $[a, b]$ denotes $ab - ba$ in A. Furthermore, if $\mathfrak{g}$ is a Lie algebra within an algebra A, then the bracket on $\mathfrak{g}$ is assumed to be the natural bracket in A.

Additionally, if C is a coalgebra, we denote the maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to F$ to be the comultiplication and counit, respectively. Furthermore, if B is a bialgebra denoted $(B, \mu, \imath, \Delta, \varepsilon)$, where (B, Δ, ε) is the coalgebra, and $\mu : B \otimes B \to B$ and $\imath : F \to B$ denote multiplication and unit, respectively, thus the triple $(B, \mu, \imath)$ is an algebra. Lastly, H is a Hopf algebra can be denoted by the sextuple $(H, \mu, \imath, \Delta, \varepsilon, S)$, where $(H, \mu, \imath)$ is the algebra, (H, Δ, ε) is the coalgebra, and $S : H \to H$ is the antipode of H. If necessary, we denote $S_H = S$ to emphasize the antipode of a Hopf algebra H.

Chapter 2

Background

2.1 Algebra Filtrations

First we recall vector space and algebra filtrations.

Definition 2.1.1. A **vector space filtration** of a vector space V, is a collection of vector subspaces $\{V_k \subseteq V : k \in \mathbb{Z}\}$ such that

$$V_k \subseteq V_{k+1} \text{ for all } k \in \mathbb{Z}, \text{ and } V = \bigcup_{k=1}^{\infty} V_k.$$

An **algebra filtration** of an algebra A is a vector space filtration $\{A_k \subseteq A : k \in \mathbb{Z}\}$ of A such that

$$1 \in A_0, \text{ and } A_i A_j \subseteq A_{i+j} \text{ for all } i, j \in \mathbb{Z},$$

where $A_i A_j$ is multiplication in A. If an algebra filtration exists on A, then we say that A is a $\mathbb{Z}$**-filtered algebra**.

Additionally, we say that vector space or algebra filtration $\{A_k : k \in \mathbb{Z}\}$ is **discrete** if $A_k = 0$ for all $k < 0$, and a filtration is **locally finite** if it is discrete and $\dim_F A_k < \infty$ for all $k \geq 0$.

Example 2.1.2. The following algebras are filtered algebras with filtration.

1. The base field F with $F_k = F$ for all $k \geq 0$.

2. The commutative polynomial ring $A = F[x, y]$ and its discrete filtration $A_k = \bigoplus_{i=0}^{k} A_i$, where $A_0 = F$, $A_1 = F\{x, y\}$ and $A_i A_j = A_{i+j}$ for any $i, j \in \mathbb{N}_0$. Hence $A_2 = F\{x^2, y^2, xy\} = A_1 A_1$.

3. The Laurent extension $L = F[x^{\pm 1}, y^{\pm 1}]$ and its $\mathbb{Z}$-filtration $A_k = \bigoplus_{i=-k}^{k} V_i$, where $V_0 = F$, $V_1 = F\{x\}$, $V_{-1} = F\{x^{-1}\}$, and $V_i V_j = V_{i+j}$ for any $i, j \in \mathbb{Z}$.

Extending 2 and 3, every graded algebra can be a filtered algebra.

Because we are working with discrete filtered algebras throughout the paper, we will be assuming that filtered algebras are discrete from here on out. Every filtered algebra induces another algebra called the associated graded algebra.

Definition 2.1.3. Suppose $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is an algebra filtration on an algebra A. The **associated graded algebra** of A (with respect to the filtration $\mathcal{A}$) is the vector space

$$\operatorname{gr} A := \bigoplus_{n=0}^{\infty} A_n / A_{n-1},$$

with $A_{-1} = 0$ and multiplication defined by

$$(x + A_{i-1})(y + A_{j-1}) = xy + A_{i+j-1}.$$

Hence it is an (discretely graded) algebra.

We will soon see in the next section an important example of a filtered algebra and its associated graded algebra.

Further studies have been made on filtered algebras and their associated graded algebras, see [19]. One important result is that the associated graded algebra carries their ring theoretic properties to the corresponding filtered algebra.

Proposition 2.1.4. *Suppose A is a filtered algebra.*

1. *If $gr\,A$ is a domain then so is A.*

2. *If $gr\,A$ is right Noetherian, then so is A.*

2.2 Lie Algebras

Lie algebras have been extensively researched for the past century, see [12] for more details. We will define basic necessities here.

Definition 2.2.1. A vector space $\mathfrak{g}$ with a bilinear form $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a **Lie algebra** if the following properties are satisfied:

1. $[x, x] = 0$,

2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (called the **Jacobi Identity**),

for all $x, y, z \in \mathfrak{g}$. A vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a **Lie subalgebra** if it is also a Lie algebra with the same $[,]$.

Furthermore, we say that a subspace $\mathfrak{j}$ of a Lie algebra $\mathfrak{g}$ is an **ideal** if for any $a \in \mathfrak{j}$ implies $[a, b] \in \mathfrak{j}$ for all $b \in \mathfrak{g}$. It is clear that every ideal is also a Lie subalgebra.

One important example of a Lie algebra derives from algebras: if A is an algebra, then the vector space

$$\{ab - ba : a, b \in A\}$$

is a Lie algebra with $[a, b] = ab - ba$.

Definition 2.2.2. Let $\mathfrak{g}$ be a Lie algebra.

1. If $[x, y] = 0$ for all $x, y \in \mathfrak{g}$, then we say that $\mathfrak{g}$ is **Abelian**.

2. The vector space

$$Z(\mathfrak{g}) := F\{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$$

is called the **center** of $\mathfrak{g}$. Notice that $Z(\mathfrak{g})$ is an Abelian ideal of any Lie algebra $\mathfrak{g}$.

3. If $\mathfrak{g}$ has no proper nonzero ideals, i.e. 0 and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$, and $\dim_F \mathfrak{g} \geq 3$, then we say that $\mathfrak{g}$ is **simple**.

4. To further 3, we say that a Lie algebra is **semismple** if it is a direct sum of simple Lie subalgebras.

5. If $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$ for all $i \in \mathbb{N}_0$ with $\mathfrak{g}_0 = \mathfrak{g}$, and $\mathfrak{g}_k = 0$ for some $k \in \mathbb{N}$, then we say that $\mathfrak{g}$ is **nilpotent**.

6. If $\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i]$ for all $i \in \mathbb{N}_0$ with $\mathfrak{g}_0 = \mathfrak{g}$, and $\mathfrak{g}_k = 0$ for some $k \in \mathbb{N}$, then we say that $\mathfrak{g}$ is **solvable**.

7. If there exist ideals

$$\mathfrak{g} = \mathfrak{j}_0 \supsetneq \mathfrak{j}_1 \supsetneq \cdots \supsetneq \mathfrak{j}_n = 0,$$

such that $\dim_F(\mathfrak{j}_i/\mathfrak{j}_{i+1}) = 1$ for all $i \leq n-1$, then we say that $\mathfrak{g}$ is completely solvable.

Since ideals are subalgebras themselves, we may use these adjectives to describe an ideal, such as Abelian ideal.

Example 2.2.3. Let $\mathfrak{g}$ be any Lie algebra.

1. Every Abelian Lie algebra is nilpotent.

2. Every nilpotent Lie algebra is completely solvable.

3. Every completely solvable Lie algebra is solvable.

4. Every solvable Lie algebra over an algebraically closed field of characteristic zero is completely solvable.

5. If $\mathfrak{g} = F\{x, y\}$ with $[x, y] = x$, then $\mathfrak{g}$ is completely solvable.

6. If $\mathfrak{g} = F\{x, y, z\}$ with $[x, y] = z$ and $z \in Z(\mathfrak{g})$, then $\mathfrak{g}$ is a nilpotent Lie algebra. In particular, this Lie algebra is called the **Heisenberg algebra**.

7. If $\mathfrak{g} = F\{e, f, h\}$ with $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$, then $\mathfrak{g}$ is a simple Lie algebra, namely $\mathfrak{g} = \mathfrak{sl}_2(F)$.

The examples mentioned, though low dimensional, will be the primary examples throughout this paper.

There is one particular result that we need to consider: Levi's Decomposition. This result explains that semisimple and solvable parts are disjoint.

Theorem 2.2.4. *Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra. Then*

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$, i.e. $\mathfrak{r}$ is an ideal that is also a solvable Lie subalgebra.

2.3 Universal Enveloping Algebras

Recall that if V is a vector space, the tensor algebra generated by V is

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}, \text{ where } V^{\otimes n} = \overbrace{V \otimes \cdots \otimes V}^{n} \text{ and } V^{\otimes 0} = F.$$

In the tensor algebra, multiplication is defined as

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m,$$

for any $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$and any $w_1 \otimes \cdots \otimes w_m \in V^{\otimes m}$.

Definition 2.3.1. Let $\mathfrak{g}$ be any Lie algebra. The **universal enveloping algebra** of $\mathfrak{g}$ is the algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where I is the ideal generated by $\{x \otimes y - y \otimes x - [x, y] : x, y \in \mathfrak{g}\}$.

Intuitively the adjective, "universal," would imply that this algebra would satisfy a universal property.

Lemma 2.3.2. *For any Lie algebra $\mathfrak{g}$ with $U(\mathfrak{g})$ as its universal enveloping algebra, and any algebra A with a Lie algebra homomorphism $\theta : \mathfrak{g} \to A$, there exists a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to A$ such that $\theta = \phi \circ \imath$, where $\imath : \mathfrak{g} \to U(\mathfrak{g})$ is the natural Lie algebra homomorphism.*

The previous lemma also implies that given a Lie subalgebra, there is a corresponding universal enveloping algebra contained within a universal enveloping algebra.

Lemma 2.3.3. *If $\mathfrak{h}$ is a Lie subalgebra of a Lie algebra $\mathfrak{g}$, then $U(\mathfrak{h})$ is a subalgebra of $U(\mathfrak{g})$.*

We will take a look at a few examples, given that the reader is familiar with Ore extensions.

Example 2.3.4. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra.

1. If $\mathfrak{g} = F\{x_1, ..., x_m\}$ is Abelian, then $U(\mathfrak{g}) = F[x_1, ..., x_m]$ the commutative polynomial algebra in m variables.

2. If $\mathfrak{g} = F\{x, y\}$ with $[x, y] = x$, then $U(\mathfrak{g}) = F[x][y; \alpha]$ the Ore extension with $\alpha(x) = x + 1$.

3. If $\mathfrak{g} = \mathfrak{sl}_2(F)$ then $U(\mathfrak{g}) = F[e][h;\delta_1][f;\delta_2]$ the iterated Ore extension, where $\delta_1(e) = 2e$, $\delta_2(h) = 2f$, and $\delta_2(e) = h$.

4. If $\mathfrak{g}$ is a free Lie algebra on two variables X, Y, then $U(\mathfrak{g}) = F\langle X, Y\rangle$, the free algebra in two variables ([12, Theorem 5.4.7]).

We will need to state the obligatory basis theorem for universal enveloping algebras.

Theorem 2.3.5. *[13, Theorem 6.8][Poincaré-Birkhoff-Witt Theorem] Let $\mathfrak{g}$ be any Lie algebra and B be an ordered basis for $\mathfrak{g}$. Define the following vector subspaces of $U(\mathfrak{g})$*

$$U_d = F\left\{\prod_{i=1}^{t} x_i^{e_i} : e_i \in \mathbb{N}_0,\ \sum_{i=1}^{t} e_i = d,\ \textit{and } x_i \in B \textit{ with } x_1 < x_2 < \cdots < x_t\right\}$$

with $U_0 = F$, for all $d \in \mathbb{N}$. Then $\bigcup_{i=0}^{\infty} U_i$ is a basis for $U(\mathfrak{g})$.

Moreover, $gr\,U(\mathfrak{g}) \cong S(\mathfrak{g})$ as algebras, where $S(\mathfrak{g})$ is the symmetric algebra (commutative polynomial) on $\mathfrak{g}$.

Corollary 2.3.6. *For any finite dimensional Lie algebra, its universal enveloping algebra is a Noetherian domain.*

A consequence of the PBW-Theorem is the fact that the natural Lie algebra homomorphism is a monomorphism.

Corollary 2.3.7. *If $\mathfrak{g}$ is any Lie algebra then the natural Lie algebra homomorphism $\imath : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

[12] has mentioned many properties about the universal enveloping algebra.

Theorem 2.3.8. *[12, Theorem 5.1.1] Let $\mathfrak{g}$ be any Lie algebra and $U := U(\mathfrak{g})$ be its universal enveloping algebra. Then*

1. *There is a unique algebra homomorphism $\Delta : U \to U \otimes U$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$.*

2. There is a unique algebra anti-automorphism $S : U \to U$, i.e. $S(ab) = S(b)S(a)$, such that $S(x) = -x$ for all $x \in \mathfrak{g}$.

Additionally, Δ is a monomorphism ([12, Corollary 5.2.5]).

Lastly we would like to restate a result about the global dimension, denoted gl.dim of universal enveloping algebras.

Theorem 2.3.9. *[8] If $\mathfrak{g}$ is a finite dimensional Lie algebra, then*

$$gl.dim(U(\mathfrak{g})) = \dim_F \mathfrak{g}.$$

2.4 Gelfand-Kirillov Dimension

The Gelfand-Kirillov Dimension measures the growth of an algebra. In this section we will be briefly mentioning such concepts. We can find many definitions, examples, and results from [13] and [16].

Definition 2.4.1. Let V be a vector subspace of an algebra A. We say that V is a **generating space** if $A = \bigcup_{n=1}^{\infty} \sum_{k=0}^{n} V^k$, where $V^0 = F$ and $V^k = \prod_{i=1}^{k} V$ is multiplication of vector spaces in A.

Now let A be an affine algebra and V be a finite dimensional generating space. The **Gelfand-Kirillov dimension** of A, denoted GK.dim(A), is

$$\text{GK.dim}(A) = \limsup_{n\to\infty} \log_n(\dim_F \sum_{i=1}^{n} V^i).$$

In general, if A is any algebra, then we define its GK dimension by

$$\text{GK.dim}(A) = \sup\{\text{GK.dim}(B) : B \text{ is an affine subalgebra of } A\}.$$

We must check that the definition for GK dimension is well-defined, in other words, regardless of choice of generating space, GK dimension is the same.

Lemma 2.4.2. *[13, Lemma 1.1] Suppose A is an algebra and V and W are generating spaces of A. Then*

$$\limsup_{n\to\infty} \log_n(\dim_F \sum_{i=1}^{n} V^i) = \limsup_{n\to\infty} \log_n(\dim_F \sum_{j=1}^{n} W^j).$$

The following properties of GK dimension can be left as a straightforward exercise:

Lemma 2.4.3. *[13, Lemma 3.1] Suppose A is an algebra.*

1. *If B is a subalgebra, then GK.dim$(B) \leq$ GK.dim(A).*

2. *If B is an algebra and $f : A \to B$ is a surjective algebra homomorphism, then GK.dim$(B) \leq$ GK.dim(A).*

Due to the definition, there is a possibility that the GK-dimension of some algebra is a non-integer real number that is greater than 2.

Theorem 2.4.4. *[13, Theorem 2.5][Bergman's Gap Theorem] If A is any algebra with $1 \leq$ GK.dim$(A) \leq 2$, then GK.dim$(A) = 1$ or GK.dim$(A) = 2$.*

Proposition 2.4.5. *[16, Proposition 8.1.18] For any $r \in \mathbb{R}$ with $r \geq 2$, there exists an algebra A such that GK.dim$(A) = r$.*

Fortunately, under the right circumstances the GK-dimension will be an integer.

Theorem 2.4.6. *[13, Theorem 4.5] If A is a commutative algebra, then GK.dim(A) is an integer or GK.dim$(A) = \infty$.*

Let's consider several examples:

Example 2.4.7. Assume that A is an algebra.

1. If A is finite dimensional, then GK.dim$(A) = 0$.

2. If $A = F[x_1, ..., x_m]$ a commutative polynomial algebra, then $\text{GK.dim}(A) = m$.

3. More generally, if $A = U(\mathfrak{g})$ and $\mathfrak{g}$ is any Lie algebra, then $\text{GK.dim}(A) = \dim_F \mathfrak{g}$.

4. Also if A is any algebra, then $\text{GK.dim}(A[x_1, ..., x_m]) = \text{GK.dim}(A) + m$.

5. If $A = F[x_1^{\pm 1}, ..., x_m^{\pm 1}]$ a commutative Laurent extension, then $\text{GK.dim}(A) = m$.

6. Combining 1 and 5, we see that if G is a finitely generated Abelian group and $A = FG$, then $\text{GK.dim}(A) = m$, where m is the number of copies of the group $\mathbb{Z}$ in G.

7. Let $A = F[x][y; \frac{d}{dx}]$ the Weyl algebra. Then $\text{GK.dim}(A) = 2$

8. Expanding on 7, if $A_n(F)$ is the n-th Weyl algebra with $2n$ variables, then we have $\text{GK.dim}(A_n(F)) = 2n$.

9. If $A = F\langle X, Y\rangle$ a free algebra, then $\text{GK.dim}(A) = \infty$.

In the examples above, we see that many are iterative Ore or Laurent extensions and that for each extension we increase the GK dimension by one. However, this is generally not the case, especially with derivations.

Proposition 2.4.8. *[13, Proposition 3.9] Let $n \in \mathbb{N}$. Then there exists algebras A and B with $GK.dim(A) = GK.dim(B) = 0$ and F-derivations δ_A and δ_B such that*

1. *$GK.dim(A[t; \delta_A]) = n$.*

2. *$GK.dim(B[t; \delta_B]) = \infty$.*

Since almost all of the algebras we will be working with will be filtered, there is a result regarding the GK dimension associated graded algebra of a filtered algebra.

Proposition 2.4.9. *[13, Lemma 6.5] If A is a filtered algebra, then*

$$GK.dim(gr\, A) \leq GK.dim(A).$$

For algebras that are finitely generated as a module over a subalgebra, the GK dimension does not change.

Proposition 2.4.10. *[13, Proposition 5.5] Suppose $B \subseteq A$ are algebras and A is a finitely generated right (or left) B-module. Then*

$$GK.dim(A) = GK.dim(B).$$

Additionally, there is an effect on domains when asssuming finite GK-dimension.

Corollary 2.4.11. *[16, Corollary 8.1.21] If R is a domain that is also an algebra, and $GK.dim(R) < \infty$, then R is an Ore domain.*

Chapter 3

Coalgebras & Hopf Structures

3.1 Coalgebras

In an algebra (A, μ), μ satisfies the associative property

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \\ \Big\downarrow{\scriptstyle \mu \otimes \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

If we reverse the arrows, we achieve a new algebraic structure.

Definition 3.1.1. A vector space C is called a **coalgebra** if there are linear maps $\varepsilon : C \to F$ a **counit**, and $\Delta : C \to C \otimes C$ a **comultiplication** such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \mathrm{id}_C \otimes \varepsilon} \\ C \otimes C & \xrightarrow[\varepsilon \otimes \mathrm{id}_C]{} & F \otimes C \cong C \otimes F \cong C \end{array}$$

Coassociativity:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \mathrm{id}_H \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes \mathrm{id}_H]{} & C \otimes C \otimes C \end{array}$$

Naturally, we say that a subspace V of a coalgebra (C, Δ, ε) is a **subcoalgebra** if the restrictions $\Delta|_V$ and $\varepsilon|_V$ are comultiplication and counit on V. Additionally we say that a coalgebra is **simple** if it has no proper subcoalgebra except the trivial coalgebra F.

Example 3.1.2.

1. The base field F is a coalgebra with $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. In fact, F is a simple coalgebra.

2. Given any group G, the group algebra FG is a coalgebra with $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$.

3. Additionally, for any $g \in G$, the vector space $F\{g\}$ is a simple coalgebra.

4. Given any Lie algebra $\mathfrak{g}$, the vector space $F \oplus \mathfrak{g}$ is a coalgebra with $\Delta(1) = 1 \otimes 1$, $\varepsilon(1) = 1$, and $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, for all $x \in \mathfrak{g}$.

5. Let $T_2 = F\{1, g, x, gx\}$ and define the following

$$\begin{aligned}
\Delta(1) &= 1 \otimes 1 & \varepsilon(1) &= 1, \\
\Delta(g) &= g \otimes g & \varepsilon(g) &= 1, \\
\Delta(x) &= x \otimes 1 + g \otimes x & \varepsilon(x) &= 0, \\
\Delta(gx) &= gx \otimes g + 1 \otimes gx & \varepsilon(gx) &= 0.
\end{aligned}$$

   Then T_2 is a coalgebra called the **Taft algebra**. (It's also a Hopf algebra with the relations $g^2 = 1$, $x^2 = 0$, $xg = -gx$.)

We will start with the finiteness theorem for coalgebras.

Theorem 3.1.3. *[18, 5.1.1] Let C be any coalgebra. Given any $c \in C$ there exists a finite dimensional subcoalgebra D of C such that $c \in D$.*

Corollary 3.1.4. *[18, 5.1.2] Every simple coalgebra is finite dimensional.*

Definition 3.1.5. Suppose C is a coalgebra.

1. We say that $g \in C$ is **group-like** if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Denote the set of all group-like elements of C by $G(C)$.

2. Assuming $1 \in G(C)$, we say that $x \in C$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Denote the set of all primitive elements of C by $P(C)$.

3. Assuming $g, h \in G(C)$, we say that $v \in C$ is **skew-primitive** if $\Delta(v) = v \otimes g + h \otimes x$, where $g, h \in C$ are group-like. Denote the set of all g, h-skew primitive elements of C by $P_{g,h}(C)$.

4. We say that $c \in C$ is **cocommutative** whenever $\Delta(c) = \tau \circ \Delta(c)$, where $\tau : a \otimes b \mapsto b \otimes a$. Furthermore we say that the coalgebra is **cocommutative** if every element is cocommutative.

Example 3.1.6. Recall the previous collection of examples.

1. Every $g \in G$ is a group-like element in the group algebra FG.

2. Every $x \in \mathfrak{g}$ is a primitive element in the universal enveloping algebra $U(\mathfrak{g})$.

3. Moreover, all elements in both FG and $U(\mathfrak{g})$ are cocommutative, since every group-like and primitive element is cocommutative.

4. The elements $x, gx \in T_2$ are skew primitive elements.

3.2 Coradical Filtration

With a coalgebra, there exists a vector space filtration which is unique. But first, we must define the zero-th filter.

Definition 3.2.1. Let C be a coalgebra. We call the sum of simple subcoalgebras of C the **coradical**, denoted C_0. Additionally if every simple subcoalgebra of C is one dimensional, then we say that C is **pointed**. If C_0 is one dimensional, i.e. $C_0 = F$, then we say that C is **connected**.

Example 3.2.2. 1. The base field F is a connected coalgebra. In fact F is a simple connected coalgebra.

2. For any group G, FG is a pointed coalgebra, i.e. $(FG)_0 = FG$.

3. Moreover, for any $g \in G$, the vector space $F\{g\}$ is a simple connected coalgebra.

4. The polynomial ring $R = F[x]$ is a connected coalgebra with $R_0 = F$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$.

It follows by definition, that the coradical of any nonzero coalgebra is a subcoalgebra.

Definition 3.2.3. Given a coalgebra C with coradical C_0, we define the following:

$$C_{n+1} = \Delta^{-1}(C_n \otimes C + C \otimes C_0) \text{ for all } n \in \mathbb{N}_0.$$

We call the sequence of vector spaces C_n the **coradical filtration** of C.

As the name states, a coradical filtration of any coalgebra is a filtration of vector spaces similar to an algebra filtration. Since coalgebras are not algebras, their coradical filtrations are typically not algebra filtrations. However, this filtration satisfies, "coalgebra," properties dual to algebra properties.

Theorem 3.2.4. *[18, Theorem 5.2.2] Given the coalgebra C with coradical filtration C_n, the following conditions holds:*

1. $C_i \subseteq C_{i+1}$ *for all* $i \in \mathbb{N}_0$.

2. $C = \bigcup_{i \in \mathbb{N}_0} C_i$.

3. $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}$.

Moreover each C_i is a subcoalgebra of C.

Definition 3.2.5. A sequence of vector subspaces of a coalgebra that satisfy the previous proposition is called a **coalgebra filtration**.

We will compare the zero-th filter of any coradical filtration on a coalgebra and its coradical with the next lemma.

Lemma 3.2.6. *[18, Lemma 5.3.4] If C is any coalgebra and $\{B_n\}$ is a coalgebra filtration on C, then $B_0 \supseteq C_0$, where C_0 is the coradical of C.*

In addition, we would say that the coradical filtration of any coalgebra is a unique filtration. Next, we want to compare the coradical filtration of subcoalgebra of any coalgebra.

Corollary 3.2.7. *[18, Lemma 5.2.12] If D is a subcoalgebra of a coalgebra C, and D_n and C_n are the coradical filtrations of D and C respectively, then $D_n = D \cap C_n$, for all $n \in \mathbb{N}_0$.*

For coalgebra homomorphisms, we gain a stronger morphism when assuming connectedness. But to prove this, we need the result known as the Taft-Wilson Theorem.

Theorem 3.2.8. *[18, Theorem 5.4.1] Let C be a pointed coalgebra. Then*

1. $C_1 = FG(C) \oplus (\bigoplus_{g,h \in G(C)} P'_{g,h}(C))$, *and*

2. *for any $n \geq 1$ and $c \in C_n$,*

$$c = \sum_{g,h \in G} c_{g,h} \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w,$$

   for some $w \in C_{n-1} \otimes C_{n-1}$,

where $P'_{g,h}(C)$ is the vector space $P_{g,h}(C)/F(g-h)$.

The Taft-Wilson Theorem tells us how the elements of any pointed coalgebra can be written.

Corollary 3.2.9. *[18, Lemma 5.3.2] Suppose C is a connected coalgebra with $G(C) = \{1\}$. Then*

1. $C_1 = F1 \oplus P(C)$, *and*

2. for any $n \in \mathbb{N}$ *and* $c \in C_n$

$$\Delta(c) = c \otimes 1 + 1 \otimes c + w$$

where $w \in C_{n-1} \otimes C_{n-1}$*.*

Corollary 3.2.10. *If* C *is a connected coalgebra with* $G(C) = \{1\}$*, and* D *is a subcoalgebra, then* $P(D) = D \cap P(C)$*.*

Now we can state the result that we would be applying in the next section.

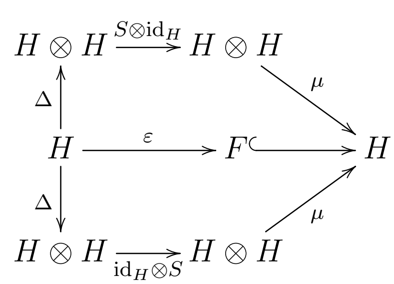
Theorem 3.2.11. *[1, Theorem 2.4.11] Let* C, D *be coalgebras and* $f : C \to D$ *be a coalgebra homomorphism. Then* f *is a monomorphism if and only if* $f|_{P(C)}$ *is injective; namely* $\ker f \cap P(C) = 0$*.*

3.3 Bialgebras & Hopf Algebras

Since algebras and coalgebras are mostly mutually exclusive, we focus on the algebras (or coalgebras) that have a compatible coalgebra structure (respectively algebra structure).

Definition 3.3.1. Let A be an algebra with multiplication μ and a coalgebra structure (A, Δ, ε). We say that A is a **bialgebra** if both Δ and ε are algebra homomorphisms, or equivalently μ is a coalgebra homomorphism.

Now let H be a bialgebra. An **antipode** on H is a linear map $S : H \to H$ such that the following diagram commutes:



A bialgebra with an antipode is called a **Hopf algebra**.

Example 3.3.2. The following are Hopf algebras:

1. The field F since $\Delta(1) = 1 \otimes 1$.
2. The group algebra FG for any group G.
3. Any commutative polynomial ring $F[X]$ where X is either finite or infinite.
4. A universal enveloping algebra $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.

As one would expect, not every algebra can be a bialgebra, and therefore a Hopf algebra. However, if the algebra is embedded into a bialgebra or Hopf algebra, we can test whether that algebra can be a bialgebra.

Lemma 3.3.3. *Let H be a Hopf algebra and A be a subalgebra of H. Set $K = A \cap \ker \varepsilon$. Then A is a Hopf subaglebra if and only if $\Delta(K) \subseteq K \otimes A + A \otimes K$ and $S(K) \subseteq K$.*

Proof. One direction is obvious. Suppose that $\Delta(K) \subseteq K \otimes A + A \otimes K$ and $S(K) \subseteq K$. Note that $\Delta(K) \subseteq A \otimes A$. For any $a \in A$ we have $\varepsilon(a - \varepsilon(a)) = 0$ whence $a - \varepsilon(a) \in K$. Since $\Delta(a - \varepsilon(a)) \in A \otimes A$, and $\Delta(a - \varepsilon(a)) = \Delta(a) - \varepsilon(a)(1 \otimes 1)$, then $\Delta(a - \varepsilon(a)) + \varepsilon(a)(1 \otimes 1) = \Delta(a) \in A \otimes A$. Additonally, as $S(a - \varepsilon(a)) = S(a) - \varepsilon(a)$, we have $S(a - \varepsilon(a)) + \varepsilon(a) = S(a) \in A$. Therefore A is a Hopf subalgebra of H. □

With a bialgebra B the collection of primitive elements in B has an additional structure.

Lemma 3.3.4. *[1, Theorem 2.1.3] If B is a bialgebra, then $P(B)$ is a Lie algebra with $\varepsilon(P(B)) = 0$.*

If we recall, when N is a normal subgroup of a group G, then G/N is a group. In the language of Hopf algebras, we have that $FG/(FN \cap \ker \varepsilon)$ is a Hopf algebra. However, not every Hopf subalgebra can be modded out, which is analogous to not every subgroup of a group having the ability to be modded out.

Definition 3.3.5. Let H be any Hopf algebra and K be any Hopf subalgebra.

1. We say that K is **left normal** if $\mathrm{ad}_l[H](K) \subseteq K$, where

$$\mathrm{ad}_l[h](k) = \sum_h h_1 k S(h_2),$$

for all $k \in K$ and all $h \in H$.

2. We say that K is **right normal** if $\mathrm{ad}_r[H](K) \subseteq K$, where

$$\mathrm{ad}_r[h](k) = \sum_h S(h_1) k h_2,$$

for all $k \in K$ and all $h \in H$.

3. We say that K is a **normal Hopf subalgebra** if K is both left normal and right normal.

A simple example of a normal Hopf subalgebra is a Hopf subalgebra in the center of the Hopf algebra. A trivial example is that the base field is a normal Hopf subalgebra of any Hopf algebra.

We ask does a normal Hopf subalgebra of an universal enveloping algebra look like? The question was answered in [17] but will be restated here.

Lemma 3.3.6. *Let $\mathfrak{g}$ be any Lie algebra. If B is a normal Hopf subalgebra of $U(\mathfrak{g})$, then $P(B)$ is an ideal of $\mathfrak{g}$.*

Proof. Let B be a normal Hopf subalgebra of $U(\mathfrak{g})$, hence $P(B) \subseteq \mathfrak{g}$. Let $b \in P(B)$, then for any $g \in \mathfrak{g}$ we have $\mathrm{ad}_r[g](b) = -gb + bg = [b, g]$. Since $\mathrm{ad}_r[g](b) \in B$ and $[b, g] \in \mathfrak{g}$, then $[b, g] \in B \cap P(U(\mathfrak{g})) = P(B)$, whence $P(B)$ is an ideal of $\mathfrak{g}$. □

Proposition 3.3.7. *If $\mathfrak{g}$ is any Lie algebra and $\mathfrak{j}$ is an ideal of $\mathfrak{g}$, then $U(\mathfrak{j})$ is a normal Hopf subalgebra of $U(\mathfrak{g})$.*

Proof. Set $T = U(\mathfrak{j})$. Since ad_r satisfies $\mathrm{ad}_r[a+b] = \mathrm{ad}_r[a]+\mathrm{ad}_r[b]$ and $\mathrm{ad}_r[ab] = \mathrm{ad}_r[b]\circ\mathrm{ad}_r[a]$ for all $a, b \in U(\mathfrak{g})$, then without loss of generality, we only need to show that $\mathrm{ad}_r[\mathfrak{g}](T) \subseteq T$. Since $\mathrm{ad}_r[g]$ acts as a derivation on T for any $g \in \mathfrak{g}$, thus for any $t_1, ..., t_k \in \mathfrak{j}$ and any $k \in \mathbb{N}$,

$$\mathrm{ad}_r[g](t_1 \cdots t_k) = \mathrm{ad}_r[g](t_1)(t_2 \cdots t_k) + \cdots + (t_1 t_2 \cdots t_{k-1})\mathrm{ad}_r[g](t_k).$$

Since $\mathfrak{j}$ is an ideal of $\mathfrak{g}$, hence $\mathrm{ad}_r[g](t_j) \in \mathfrak{j}$, then $\mathrm{ad}_r[g](t_1 \cdots t_k) \in T$. Since $\mathrm{ad}_l[g] = -\mathrm{ad}_r[g]$ for all $g \in \mathfrak{g}$, then T is a normal Hopf subalgebra. □

3.4 Connected Hopf Algebras

From here on out, we focus on a certain family of Hopf algebras: connected. The adjective stems from the coalgebra structure and not any vector space filtration.

Definition 3.4.1. We say that a Hopf algebra is **connected** if the underlying coalgebra is connected.

As previously stated, not every bialgebra is a Hopf algebra. However, if we define a connected bialgebra as a bialgebra with a connected coalgebra, the bialgebra will gain an antipode.

Lemma 3.4.2. *[18, Lemma 5.2.10] Every connected bialgebra is a connected Hopf algebra.*

Example 3.4.3. 1. Clearly F is a connected Hopf algebra.

2. For any Lie algebra $\mathfrak{g}$, the enveloping algebra $U(\mathfrak{g})$ is a connected Hopf algebra.

3. A group algebra FG is not connected unless G is the trivial group.

In fact, the only Artinian connected Hopf algebra over a field of characteristic zero is the trivial Hopf algebra.

Theorem 3.4.4. *[14] If H is an Artinian connected Hopf algebra, then $H = F$.*

Since Hopf algebras are coalgebras, every Hopf algebra will have a filtration of vector spaces, namely the coradical filtration. However, not every coradical filtration is an algebra filtration. The following proposition tells us when such a condition is satisfied.

Proposition 3.4.5. *[18] Let H_n be the coradical filtration of a Hopf algebra H. Then H_0 is a Hopf subalgebra of H if and only if H_n is an algebra filtration, i.e. $H_m H_n \subseteq H_{m+n}$.*

It easily follows that the coradical filtration of a connected Hopf algebra is an algebra filtration, since F is the trivial Hopf algebra.

Since $P(H)$ is a Lie algebra, then there exists a corresponding universal enveloping algebra $U(P(H))$. We can place the enveloping algebra within the Hopf algebra H.

Lemma 3.4.6. *For every pointed or connected bialgebra H, there exists a Hopf monomorphism $U(P(H)) \to H$.*

Thus, we will state that $U(P(H))$ is a Hopf subalgebra of H instead of referring to the natural Hopf monomorphsim.

Since we are working with characteristic zero and $U(P(H))$ is a cocommutative Hopf algebra, we can classify all cocommutative connected Hopf algebras.

Theorem 3.4.7. *[1, Theorem 2.5.3] If H is a cocommutative connected Hopf algebra then $H = U(P(H))$.*

Corollary 3.4.8. *If H is a connected Hopf algebra, then*

1. *$U(P(H))$ is the largest cocommutative Hopf subalgebra of H,*

2. *$U(P(H))$ is the smallest Hopf subalgebra of H containing $P(H)$ as a Lie algebra.*

Proof. 1. Suppose that A is a cocommutative Hopf subalgebra of H. Since the characteristic of F is zero, then $A = U(P(A))$. Since $P(A)$ is a Lie subalgebra of $P(H)$, then A is a Hopf subalgebra of U.

2. Suppose that B is a Hopf subalgebra of H such that $P(H) \subseteq B$, whence $P(H) = P(B)$. Let $i_B : P(H) \to B$ and $i_U : P(H) \to U$ be the inclusion maps. Then there exists a Hopf algebra homomorphism $\beta : U \to B$ such that $\beta \circ i_U = i_B$. Since both i_U and i_B are injective then so is β. □

Now we apply Lemma 3.3.6, Proposition 3.3.7, and Corollary 3.4.8 to the following statement.

Corollary 3.4.9. *Let $\mathfrak{g}$ be any Lie algebra. Then a Hopf subalgebra B of $U(\mathfrak{g})$ is normal if and only if $P(B)$ is an ideal of $\mathfrak{g}$. In this case, $B = U(P(B))$.*

Proof. One direction is immediate from Lemma 3.3.6. Assume $P(B)$ is an ideal of $\mathfrak{g}$. By Corollary 3.4.8, $U(P(B)) = B$. Applying Proposition 3.3.7 gives us the desired result. □

Now we see that every connected Hopf algebra H is an algebra extension of the enveloping algebra $U(P(H))$. Additionally, many properties of the universal enveloping algebra carry over to the Hopf algebra.

There have been many papers describing the antipode of Hopf algebras. Thus, we would like to state how the antipode is effected by connectedness (or pointedness), and mimics the anti-automorphism property given by the universal enveloping algebra.

Corollary 3.4.10. *[18, Corollary 5.2.11] Let H be a Hopf algebra with a cocommutative coradical. Then the antipode of H is bijective.*

Since the universal enveloping algebra is a domain and has a commutative associated graded algebra, or more precisely a polynomial algebra, then we would like to know if these properties hold for connected Hopf algebras.

Proposition 3.4.11. *[30, Proposition 6.4] If H is a connected Hopf algebra then $gr\,H$ is commutative.*

Proposition 3.4.12. *[30, Propostion 6.5] If K is an affine, coradically graded Hopf algebra, i.e. the associated graded algebra of a connected Hopf algebra, then K is algebra-isomorphic to the commutative polynomial ring in $l > 0$ variables.*

Theorem 3.4.13. *[30, Proposition 6.6] If H is a connected Hopf algebra then H is a domain.*

We will continually use these facts in the next chapter without reference.

3.5 Anti-Cocommutative Elements

Definition 3.5.1. Let C be a connected coalgebra and $\tau : C \otimes C \to C \otimes C$ be the twist map, i.e. $\tau : a \otimes b \mapsto b \otimes a$. We say that $c \in C$ is **anti-cocommutative** or **anti-symmetric**, if $\tau \circ \delta(c) = -\delta(c)$, where $\delta(c) = \Delta(c) - (c \otimes 1 + 1 \otimes c)$.

We denote the space of all anti-cocommutative elements of C as $P_2(C)$, i.e.

$$P_2(C) = \{c \in C : \tau \circ \delta(c) = -\delta(c)\}.$$

The notion of anti-cocommutative elements was presented in [30] and [29]. Therefore, the following properties about anti-cocommutative elements were given in the referenced papers.

Lemma 3.5.2. *[29, Lemma 2.5] Suppose C is a connected coalgebra.*

1. *Then $P(C)$ is a subcoalgebra of $P_2(C)$.*

2. *$P_2(C) = \{x \in C : \tau \circ \delta(x) = -\delta(x)$ and $\delta(x) \in P(C) \otimes P(C)\}$.*

3. *Then $P_2(C)$ is the largest subcoalgebra of C consisting of anti-cocommutative elements of C.*

Example 3.5.3. Let $C = F\{1, x, y, t\}$ be a coalgebra with $\Delta(1) = 1 \otimes 1$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\Delta(y) = y \otimes 1 + 1 \otimes y$ and $\Delta(t) = t \otimes 1 + 1 \otimes t + x \otimes y - y \otimes x$. We see that C is a connected coalgebra with $C_0 = FG(C) = F\{1\}$, $P(C) = F\{x, y\}$, and $P_2(C)$ contains $P(C)$ and t, since $\delta(t) = x \otimes y - y \otimes x$.

We need to switch from a general coaglebra to Hopf algebras.

Lemma 3.5.4. *[29, Lemma 2.5] Suppose H is any Hopf algebra.*

1. *Then $P_2(H)$ is a Lie subalgebra of H if and only if $[\delta(x), \delta(y)] = 0$ in $H \otimes H$, for all $x, y \in P_2(H)$.*

2. *Then $P_2(H)$ is a $P(H)$-module.*

3. *If $P(H)$ is Abelian, then $P_2(H)$ is a Lie subalgebra of H and in H, we have $[P_2(H), P_2(H)] \subseteq P(H)$.*

4. *If $\dim_F(P_2(H)/P(H)) = 1$, then $P_2(H)$ is a Lie subalgebra of H.*

5. *Then $\dim_F(P_2(H)/P(H)) \leq \binom{\dim_F P(H)}{2}$.*

Note that by [29, Lemma 2.5], $P_2(C)$ is the largest subcoalgebra containing anti-cocommutative elements which is similar to $U(P(H))$ as the largest subcoalgebra containing cocommutative elements.

On the other hand, there are other properties that are parallel to properties of the enveloping algebra.

Lemma 3.5.5. *[29, Lemma 2.6] Suppose H is a connected Hopf algebra.*

1. *If $H = U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$, then $P_2(H) = \mathfrak{g}$.*

2. *If $P(H) \neq P_2(H)$ then $U(P(H)) \neq H$ and $\dim_F P(H) < GK.dim(H)$.*

3. *$P_2(H) \cong P_2(gr\, H)$ as coalgebras, and $P_2(gr\, H) \oplus P(gr\, H)^2 = P(gr\, H) \oplus H_2/H_1$, where H_n is the coradical filtration of H.*

Finally, if the GK-dimension of a connected Hopf algebra is finite and is close to the dimension of the space of primitive elements, then said Hopf algebra is an enveloping algebra.

Theorem 3.5.6. *[29, Theorem 2.7] Suppose H is a connected Hopf algebra. If $GK.dim(H) \leq \dim_F P(H) + 1 < \infty$, then $H \cong U(L)$ as algebras for some finite dimensional Lie algebra L.*

We will see more examples pertaining to connected Hopf algebras with anti-cocommutative elements in the next chapter.

3.6 Additional Properties

We now tie in some of the concepts together that were shown in various papers.

[18, Question 3.5.4] asks whether every Hopf algebra is left and right faithfully flat over any Hopf subalgebra. This question was partially answered by Masuoka.

Theorem 3.6.1. *[15] Suppose the coradical H_0 of a Hopf algebra H is cocommutative. If K is a right coideal coalgebra (e.g. Hopf subalgebra) such that $S(K_0) = K_0$, then H is a left and right faithfully flat K-module.*

We will be using this result in the next chapter. Moreover, there have been recent studies in Hopf algebras with certain GK-dimension.

Theorem 3.6.2. *[30, Propostion 3.6] Let H be a pointed, or connected Hopf algebra. Then*

$$GK.dim(H) = \sup\{GK.dim(K) : K \text{ is an affine Hopf subalgebra of } H\}.$$

Theorem 3.6.3. *[30, Theorem 6.9] Given a connected Hopf algebra H, the following statements are equivalent:*

1. $GK.dim(H) < \infty$,
2. $GK.dim(gr\,H) < \infty$,
3. $gr\,H$ *is an affine algebra,*
4. $gr\,H$ *is algebra-isomorphic to the polynomial ring of* $l > 0$ *variables.*

In this case, $GK.dim(H) = GK.dim(gr\,H)$ *which is a positive integer.*

The next lemma gives us a comparison between connected Hopf algebras via GK-dimension, and the following corollary tells us how Hopf subalgebras cannot be close to each other. The lemma also motivates one of the sections in the next chapter.

Lemma 3.6.4. *[30, Lemma 7.2] If K is a Hopf subalgebra of a connected Hopf algebra H, and if $GK.dim(K) = GK.dim(H) < \infty$, then $K = H$.*

Corollary 3.6.5. *If K is a Hopf subalgebra of a connected Hopf algebra H with finite GK-dimension, and H is a left (or right) finitely generated K-module, then $H = K$.*

Proof. Using GK-dimension [13, Proposition 5.5], $\text{GK.dim}(H) = \text{GK.dim}(K)$. Applying [30, Lemma 7.2] forces $H = K$, as claimed. □

Proposition 3.6.6. *[30, Proposition 7.4] Let H be a connected Hopf algebra and $d = GK.dim(H)$.*

1. *If $d = 0$ then $H = F$, the trivial Hopf algebra.*
2. *If $d = 1$ then $H = F[x]$ with x being a primitive element.*
3. *If $d = 2$ then $H \cong U(\mathfrak{g})$ as Hopf algebras, where $\mathfrak{g}$ is either the 2-dimensional Abelian Lie algebra or the 2-dimensional solvable Lie algebra.*

Finally, there exists a classification of connected Hopf algebras with low GK-dimension.

Theorem 3.6.7. *[28, Theorem 1.2] Let H be a connected Hopf algebra with GK-dimension 4, and let $p = \dim_F P(H)$. Then one of the following occurs:*

1. *If $p = 4$ then $H = U(P(H))$.*
2. *If $p = 3$ then $H \cong U(L)$ where L is an anti-cocommutative Lie algebra of dimension 4.*
3. *If $p = 2$ then H is not isomorphic to some universal enveloping algebra.*

Chapter 4

Main Results

4.1 Anti-Cocommutative Lie Extensions

In this section we construct not a single algebra but a class of connected Hopf algebras with a fixed Lie algebra $\mathfrak{g}$. We also investigate specific subalgebras within these connected Hopf algebras.

To start, pick any Lie algebra $\mathfrak{g}$. We let $\mathcal{A}(\mathfrak{g})$ denote the class of locally finite connected Hopf algebras A, i.e. its coradical filtration is a locally finite filtration, such that $P(A) = \mathfrak{g}$, A is generated by $P_2(A)$ as an algebra, and $U(\mathfrak{g}) \neq A$.

Because $P_2(A)/\mathfrak{g}$ is isomorphic to some subspace of $\mathfrak{g} \wedge \mathfrak{g}$, we will use the wedge notation $[z, x] \wedge y$ which is equivalent to $[z, x] \otimes y - y \otimes [z, x]$ in $A \otimes A$ in example 4.1.12 and example 4.1.10.

Now for each $A \in \mathcal{A}(\mathfrak{g})$ one would assume that A is unique up to $\dim_F P_2(A)$. However that is not the case as example 4.1.1 will show.

Example 4.1.1. Let $\mathfrak{g} = F\{x, y\}$ be a Lie algebra, $A \in \mathcal{A}(\mathfrak{g})$ and $s = s_{xy} \in P_2(A)$ with $\Delta(s) = s \otimes 1 + 1 \otimes s + x \otimes y - y \otimes x$. In particular $P_2(A) = \mathfrak{g} \oplus F\{s\}$.

1. [29, Lemma 3.2] If $[x, y] = 0$ then it follows that

$$\delta([x, s]) = x \otimes [x, y] - [x, y] \otimes x = 0,$$

$$\delta([y, s]) = [y, x] \otimes y - y \otimes [y, x] = 0.$$

   This implies that $[x, s]$ and $[y, s]$ are primitive elements of A, thus set

$$[x, s] = \alpha_{11}x + \alpha_{12}y,$$

$$[y, s] = \alpha_{21}x + \alpha_{22}y,$$

   where $\alpha_{ij} \in F$. Now we consider the matrix $\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$. Now for example we could set every $\alpha_{ij} = 0$ which would imply that A is a commutative connected Hopf algebra. Or we could set $\alpha_{12} = 1$, $\alpha_{11} = \alpha_{22} = 0$ which would imply that $[x, s] = y$ and $[y, s] = 0$. With $[x, y] = 0$ we get that $\mathfrak{g} \oplus F\{s\}$ is isomorphic to the 3-dimensional Heisenberg algebra as Lie algebras.

2. If $[x, y] = x$ then it follows that

$$\delta([x, s]) = x \otimes [x, y] - [x, y] \otimes x = 0,$$

$$\delta([y, s]) = [y, x] \otimes y - y \otimes [y, x] = -(x \otimes y - y \otimes x) = \delta(-s).$$

   Since $[x, s]$ and $[y, s] + s$ are primitive elements of A, then we have

$$[x, s] = \beta_{11}x + \beta_{12}y,$$

$$[y, s] = -s + \beta_{21}x + \beta_{22}y,$$

   where $\beta_{ij} \in F$. First note that $P_2(A)$ is a Lie algebra containing $\mathfrak{g}$ as a subalgebra.

Then by the Jacobi identity

$$\begin{aligned}
0 =& [x,[y,s]] + [y,[s,x]] + [s,[x,y]] \\
&= [x, -s + \beta_{21}x + \beta_{22}y] - [y, \beta_{11}x + \beta_{12}y] + [s,x] \\
&= 2[s,x] + \beta_{22}x + \beta_{11}x \\
&= -2\beta_{11}x - 2\beta_{12}y + (\beta_{22} + \beta_{11})x \\
&= (\beta_{22} - \beta_{11})x - 2\beta_{12}y = 0,
\end{aligned}$$

which forces $\beta_{12} = 0$ and $\beta_{22} = \beta_{11}$. Further calculation shows that $\beta_{11} = 0$ whence $\beta_{22} = 0$ (see [29, Lemma 3.2]). Therefore $[x,s] = 0$ and $[y,s] = -s + \beta_{21}x$. Moreover, we are free to choose $\beta_{21} \in F$, so regardless of whether β_{21} is zero, s cannnot commute with y, hence $P_2(A)$ is not an Abelian extension of $\mathfrak{g}$.

For examples with $\dim_F \mathfrak{g} \geq 3$, we have $\dim_F P_2(A)/\mathfrak{g} \geq 3$. In this case, given linearly independent $s,t \subseteq P_2(A)$, there might be a relation between s and t.

Additionally, we will be looking at $A \otimes A$, so it's handy to keep in mind the following small shortcuts.

Lemma 4.1.2. *For any Lie algebra $\mathfrak{g}$ with $A \in \mathcal{A}(\mathfrak{g})$, the following conditions are equivalent for any $h \in P_2(A)$ with $\delta(h) = x \otimes y - y \otimes x$, and $x,y,z \in \mathfrak{g}$:*

1. *$ad[z](h) \in \mathfrak{g}$,*
2. *$[z,x] \otimes y + x \otimes [z,y] = y \otimes [z,x] + [z,y] \otimes x$,*
3. *$\delta(h)\Delta(z) = \Delta(z)\delta(h)$ in $A \otimes A$.*

Proof. 2 $\iff$ 3. This derives from the following two calculations in $A \otimes A$:

$$[\Delta(z), \delta(h)] = [z,x] \otimes y - y \otimes [z,x] + x \otimes [z,y] - [z,y] \otimes x,$$

$$[\Delta(z), h \otimes 1 + 1 \otimes h] = [z,h] \otimes 1 + 1 \otimes [z,h].$$

Thus, $\Delta(z)\delta(h) = \delta(h)\Delta(z)$ if and only if $[z, x] \otimes y + x \otimes [z, y] = y \otimes [z, x] + [z, y] \otimes x$.

$1 \iff 3$. By definition $\text{ad}[z](h) = zh - hz = [z, h]$. Applying the calculations above yields,

$$\Delta(\text{ad}[z](h)) = [\Delta(z), \Delta(h)] = [\Delta(z), h \otimes 1 + 1 \otimes h] + [\Delta(z), \delta(h)].$$

Therefore $\text{ad}[z](h) \in \mathfrak{g}$ if and only if $\Delta(z)\delta(h) = \delta(h)\Delta(z)$. □

Sometimes there is a Lie algebra in $P_2(A)$ properly containing the Lie algebra $\mathfrak{g} = P(A)$ as a Lie subalgebra. So we introduce a definition which describes this property.

Definition 4.1.3. Let $\mathfrak{g}$ be a Lie F-algebra and H be a connected Hopf algebra with $P(H) = \mathfrak{g}$. An **anti-cocommutative Lie extension** (or **ALE** for short) of $\mathfrak{g}$ is a vector space $L \subseteq P_2(H)$ such that L is an anti-cocommutative coassociative Lie algebra (defined in [29]), and $\dim_F L = \dim_F \mathfrak{g} + 1$. When an ALE of $\mathfrak{g}$ exists, we say that $\mathfrak{g}$ satisfies the *ALE property.*

We provide a few small examples of ALE of a given Lie algebra.

Example 4.1.4. Suppose we have a Lie algebra $\mathfrak{g}$ with $A \in \mathcal{A}(\mathfrak{g})$.

1. If $\mathfrak{g}$ is any Abelian Lie algebra, then $P_2(A)$ is an ALE of $\mathfrak{g}$ since $(x \otimes y)(a \otimes b) = (a \otimes b)(x \otimes y)$. Hence $s_{[a,x]y} \in \mathfrak{g}$ for all $a, b, x, y \in \mathfrak{g}$.

2. If $\mathfrak{g}$ is a 2-dimensional Lie algebra then $\dim_F P_2'(A) = 1$. We see that $P_2(A)$ is an ALE of $\mathfrak{g}$ (see Example 4.1.1).

3. From [29, Theorem 2.7]: if H is a connected Hopf F-algebra with

$$\text{GK.dim}(H) = \dim_F P(H) + 1 < \infty,$$

then H is an enveloping algebra of some ALE of the Lie algebra $P(H)$.

4. Suppose $F = \mathbb{R}$ and $\mathfrak{g} = F\{x, y, z\}$ with $[x, y] = z$, $[z, x] = y$, and $[z, y] = 0$. Clearly $\mathfrak{g}$ is a solvable Lie algebra but not completely solvable since $F\{y, z\}$ is a proper ideal of $\mathfrak{g}$. We see that in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$

$$[\Delta(x), y \otimes z - z \otimes y] = [x, y] \otimes z - z \otimes [x, y] + y \otimes [x, z] - [x, z] \otimes y = 0,$$
$$[\Delta(y), y \otimes z - z \otimes y] = y \otimes [y, z] - [y, z] \otimes y = 0,$$
$$[\Delta(z), y \otimes z - z \otimes y] = [z, y] \otimes z - z \otimes [z, y] = 0.$$

Now let $A \in \mathcal{A}(\mathfrak{g})$ with $s_{yz} \in P_2(A)$ and $\delta(s_{yz}) = y \otimes z - z \otimes y$. The calculation has shown that $[\Delta(\mathfrak{g}), \delta(s_{yz})] = 0$, which implies that

$$\Delta([g, s_{yz}]) = [g, s_{yz}] \otimes 1 + 1 \otimes [g, s_{yz}],$$

for any $g \in \mathfrak{g}$. Thus we have that $[\mathfrak{g}, s_{yz}] \subseteq \mathfrak{g}$ say $[g, s_{yz}] = a_g$. So setting $\mathfrak{h} = \mathfrak{g} \oplus F\{s_{yz}\}$ and define $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ by $[g, s_{yz}] = a_g$ for all $g \in \mathfrak{g}$, and $(\mathfrak{g}, [,])$ is the Lie algebra $\mathfrak{g}$. Hence $\mathfrak{h}$ is a ALE of $\mathfrak{g}$.

In the last example, note that $\mathfrak{h}$ is a solvable Lie algebra.

As we can see from the examples that Lie algebras which are at least solvable seem to have the ALE property. In fact that is what the next proposition will demonstrate.

Proposition 4.1.5. *If $\mathfrak{g}$ is a finite dimensional completely solvable Lie algebra, then $\mathfrak{g}$ satisfies the ALE property.*

Proof. Let $A \in \mathcal{A}(\mathfrak{g})$. Applying of [6, Corollary 2.4.3], we see that there exists $v \in P_2(A)/\mathfrak{g}$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$, where $\lambda : \mathfrak{g} \to F$ is an F-linear map. Since $x(v) = [x, v]$ in A then $\mathfrak{g} \oplus F\{v\}$ is an ALE of $\mathfrak{g}$. □

Proposition 4.1.5 only states the existence of an ALE but does not address which anti-cocommutative element contributes towards an ALE.

Recall that a submodule N of a module M is essential if every nonzero submodule intersects N nontrivially.

Proposition 4.1.6. *Let H be a connected Hopf algebra such that $H \neq U(P(H))$. If V is an essential $U(P(H))$-submodule of $P(H) \wedge P(H)$, then $V \cap P_2(H) \supsetneq P(H)$.*

Proof. Set $\mathfrak{g} = P(H)$. Naturally there is a coalgebra map $\phi : F \oplus P_2(H) \to F \oplus \mathfrak{g} \oplus (\wedge^2\mathfrak{g})$ with $\phi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$. By [18, Lemma 5.3.3], ϕ is a coalgebra monomorphism. Since V is essential in $\wedge^2\mathfrak{g}$ and $\mathfrak{g} \subsetneq P_2(H)$, then $\phi(F \oplus P_2(H)) \cap V \neq 0$. Again using the fact that ϕ is injective and $P_2(H) \neq \mathfrak{g}$, we have,

$$V \cap P_2(H) = \phi^{-1}(V) \cap P_2(H) \supsetneq \mathfrak{g},$$

as desired. □

For an ALE to exist, the Lie algebra must have a 2-dimensional ideal.

Proposition 4.1.7. *Fix a Lie algebra $\mathfrak{g}$ and set $A = A(\mathfrak{g})$. Let $t \in P_2(A)$ non-primitve with $\delta(t) = x \otimes y - y \otimes x \in \mathfrak{g} \otimes \mathfrak{g}$. Then $\mathfrak{g} \oplus F\{t\}$ is an ALE if and only if $F\{x, y\}$ is a two-dimensional ideal of $\mathfrak{g}$.*

Proof. Set $\mathfrak{n} = F\{x, y\}$ and assume that $\mathfrak{n}$ is a 2-dimensional ideal of $\mathfrak{g}$. Then for any $g \in \mathfrak{g}$, we have $[g, x], [g, y] \in \mathfrak{n}$ so set $[g, x] = \alpha_1 x + \beta_1 y$ and $[g, y] = \alpha_2 x + \beta_2 y$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$. It follows that

$$\begin{aligned}
\delta([g,t]) &= [g,x] \otimes y - y \otimes [g,x] + x \otimes [g,y] - [g,y] \otimes x \\
&= (\alpha_1 x + \beta_2 y) \otimes y - y \otimes (\alpha_1 x + \beta_2 y) + x \otimes (\alpha_2 x + \beta_2 y) - (\alpha_1 x + \beta_2 y) \otimes x \\
&= \alpha_1 (x \otimes yy - y \otimes x) + \beta_2 (x \otimes y - y \otimes x) \\
&= (\alpha_1 + \beta_2)(x \otimes y - y \otimes x) \\
&= (\alpha_1 + \beta_2)\delta(t).
\end{aligned}$$

This shows that $[g,t] = t + g_0$ for some $g_0 \in \mathfrak{g}$, whence $[g,t] \in \mathfrak{g} \oplus F\{t\}$, whence $\mathfrak{g} \oplus F\{t\}$ is an ALE.

Now let $\mathfrak{n}$ be the ideal in $\mathfrak{g}$ generated by $\{x,y\}$, but assume that $\mathfrak{g} \oplus F\{t\}$ is an ALE. Suppose that $\dim_F \mathfrak{n} > 2$. Then there exists $g \in \mathfrak{g}$ and $z, w \in \mathfrak{n}$ such that the dimension of the vector space $F\{x,y,z,w\}$ is at least 3, and

$$[g,x] = \alpha_1 x + \beta_1 y + \gamma_1 z,$$
$$[g,y] = \alpha_2 x + \beta_2 y + \gamma_2 w,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in F$ and either $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$. (Otherwise $[g,x], [g,y] \in \mathfrak{n}$ for all $g \in \mathfrak{g}$ would imply that $\dim_F \mathfrak{n} = 2$.) Let $s_{yz}, s_{xw} \in P_2(A)$ with $\delta(s_{yz}) = y \otimes z - z \otimes y$ and $\delta(s_{xw}) = x \otimes w - w \otimes x$. It follows that

$$\begin{aligned}
\delta([g,t]) &= [g,x] \otimes y - y \otimes [g,x] + x \otimes [g,y] - [g,y] \otimes x \\
&= (\alpha_1 x + \beta_1 y + \gamma_1 z) \otimes y - y \otimes (\alpha_1 x + \beta_1 y + \gamma_1 z) \\
&\qquad + x \otimes (\alpha_2 x + \beta_2 y + \gamma_2 w) - (\alpha_2 x + \beta_2 y + \gamma_2 w) \otimes x \\
&= (\alpha_1 + \beta_2)(x \otimes y - y \otimes x) - \gamma_1 (y \otimes z - z \otimes y) + \gamma_2 (x \otimes w - w \otimes x) \\
&= (\alpha_1 + \beta_2)\delta(t) - \gamma_1 \delta(s_{yz}) + \gamma_2 \delta(s_{xw}).
\end{aligned}$$

This shows that $[g,t] = (\alpha_1 + \beta_2)t - \gamma_1 s_{yz} + \gamma_2 s_{xw} + g_0$, for some $g_0 \in \mathfrak{g}$. Since either $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$, then $[g,t] \notin \mathfrak{g} \oplus F\{t\}$, a contradiction. Therefore, we must have $\dim_F \mathfrak{n} = 2$. $\square$

Remark 4.1.8. Note that in the proof of Lemma 4.1.7, if $\mathrm{ad}_{\mathfrak{n}}(g)$ is represented by a 2×2 matrix, for any $g \in \mathfrak{g}$, then $\alpha_1 + \beta_2$ is the trace of of $\mathrm{ad}_{\mathfrak{n}}(g)$.

Of course Proposition 4.1.7 would force simple Lie algebras to have no ALE.

Corollary 4.1.9. *If $\mathfrak{g}$ is a simple Lie algebra, then $\mathfrak{g}$ does not satisfy the ALE property.*

Proof. Ideals of $\mathfrak{g}$ are either 0 or $\mathfrak{g}$ itself. Since $\dim_F \mathfrak{g} \geq 3$, then by Proposition 4.1.7, there are no ALE for $\mathfrak{g}$. $\square$

To emphasize Corollary 4.1.9, we take a look at the smallest simple Lie algebra, $\mathfrak{sl}_2$.

Example 4.1.10. Let $\mathfrak{g} = \mathfrak{sl}_2(F) = F\{e, f, h\}$ with $[e, f] = h$, $U = U(\mathfrak{g})$, and $A \in \mathcal{A}(\mathfrak{g})$. Applying the idea that the vector space of anti-cocomutative elements in $P_2(A)$, namely $F\{s_{ef}, s_{eh}, s_{fh}\}$ is isomorphic to $\mathfrak{g} \wedge \mathfrak{g}$ as $\mathfrak{g}$-modules, and thus reverting to the $\wedge$ notation, we have that

$$
\begin{aligned}
e(e \wedge f) = e \wedge h &\implies [e, s_{ef}] = s_{eh} + g_0 \\
f(e \wedge f) = f \wedge h &\implies [f, s_{ef}] = s_{fh} + g_1 \\
h(e \wedge f) = -4e \wedge f &\implies [h, s_{ef}] = -4s_{ef} + g_2 \\
e(e \wedge h) = 0 &\implies [e, s_{eh}] = g_3 \\
f(e \wedge h) = 2e \wedge f &\implies [f, s_{eh}] = 2s_{ef} + g_4 \\
h(e \wedge h) = 2e \wedge h &\implies [h, s_{eh}] = 2s_{eh} + g_5 \\
e(f \wedge h) = 2e \wedge f &\implies [e, s_{fh}] = 2s_{eh} + g_6 \\
f(f \wedge h) = 0 &\implies [f, s_{fh}] = g_7 \\
h(f \wedge h) = -2f \wedge h &\implies [h, s_{fh}] = -2s_{fh} + g_8,
\end{aligned}
$$

where $g_0, ..., g_8 \in \mathfrak{g}$.

Now if H is a Hopf subalgebra of A properly containing U as a Hopf subalgebra, then it follows that $H = A$. To see this, we have $P_2(H) \neq \mathfrak{g}$ thus $P_2(H)$has a nontrivial anti-cocommutative element, say $g = q_1 s_{ef} + q_2 s_{eh} + q_3 s_{fh}$, where $q_1, q_2, q_3 \in F$. Without loss of generality asssume that q_1, q_2, q_3 are nonzero. Then in A, $[e, g] = 2q_2 s_{eh} + 2q_3 s_{ef}$ and so $[e, [e, g]] = 2q_3 s_{eh}$, which forces $s_{eh} \in P_2(H)$. Furthermore $\frac{1}{2}[f, s_{eh}] = s_{ef} + \frac{1}{2}g_4$ and since $\frac{1}{2}g_4 \in \mathfrak{g} \subseteq P_2(H)$, then $s_{ef} \in P_2(H)$. Finally seeing $[f, s_{ef}] = s_{ef} + g_2$ we get $s_{ef} \in P_2(H)$. Therefore $P_2(H) = P_2(A)$, and since A is generated by the coalgebra $F \oplus P_2(A)$, this forces $H = A$.

Remark 4.1.11. In example 4.1.10 we could have used the fact that $P_2(A)/\mathfrak{g} \cong \mathfrak{g} \wedge \mathfrak{g}$ is a finite dimensional simple $\mathfrak{g}$-module, whence $\mathrm{Soc}(P_2(A)/\mathfrak{g}) = P_2(A)/\mathfrak{g}$, and so any nontrivial anti-cocommutative element in $P_2(A)$ can generate $P_2(A)/\mathfrak{g}$ as a $\mathfrak{g}$-module which produces the basis $\{s_{ef}, s_{eh}, s_{fh}\}$, whence $s_{ef}, s_{eh}, s_{fh} \in P_2(H)$.

On rare occasions we do not need to have an algebraically closed field for a finite dimensional solvable Lie algebra to satisfy the ALE property. But for the next example, that is not the case.

Example 4.1.12. Suppose that $F = \mathbb{R}$, $a \in F - 0$, and $\mathfrak{g} = F\{x_1, x_2, x_3, x_4\}$ where

$$[x_4, x_1] = x_1 + ax_3,\ [x_4, x_2] = x_2,$$

$$[x_4, x_3] = x_1,\ [x_3, x_1] = x_2,$$

$$[x_3, x_2] = [x_2, x_1] = 0.$$

We see that $F\{x_1, x_2, x_3\}$ is a proper ideal of $\mathfrak{g}$, whence $\mathfrak{g}$ is not completely solvable over F. Consider $A \in \mathcal{A}(\mathfrak{g})$. To shorten the calculation, we use the fact that $\mathfrak{g}$ acting on anti-cocommutative elements in $P_2(A)$ is the same as $\mathfrak{g}$ acting on $\wedge^2\mathfrak{g}$. So set $t_{ij} = x_i \wedge x_j \in \mathfrak{g} \wedge \mathfrak{g}$, which corresponds to $s_{x_i x_j} \in P_2(A)$, for all $i < j \leq 4$, then it follows that

$$x_1(t_{12}) = x_2(t_{12}) = x_3(t_{12}) = 0,$$

$$x_4(t_{12}) = [x_4, x_1] \wedge x_2 + x_1 \wedge [x_4, x_2] = 2x_1 \wedge x_2 + ax_3 \wedge x_2 = 2t_{12} - at_{23},$$

$$x_1(t_{13}) = x_1 \wedge [x_1, x_3] = -t_{12},\ x_2(t_{13}) = 0,$$

$$x_3(t_{13}) = [x_3, x_1] \wedge x_3 = t_{23},\ x_4(t_{13}) = [x_4, x_1] \wedge x_3 + x_1 \wedge [x_4, x_3] = t_{13},$$

$$x_1(t_{23}) = x_2(t_{23}) = x_3(t_{23}) = 0,$$

$$x_4(t_{23}) = [x_4, x_2] \wedge x_3 + x_2 \wedge [x_4, x_3] = t_{23} - t_{12},$$

while

$$x_1(t_{14}) = x_1 \wedge [x_1, x_4] = -at_{13},\ x_2(t_{14}) = x_1 \wedge [x_2, x_4] = t_{12},$$
$$x_3(t_{14}) = [x_3, x_1] \wedge x_4 + x_1 \wedge [x_3, x_4] = t_{24},$$
$$x_4(t_{14}) = [x_4, x_1] \wedge x_4 = t_{14} + at_{34},$$

$$x_1(t_{24}) = x_2 \wedge [x_1, x_4] = t_{12} - at_{23},\ x_2(t_{24}) = x_2 \wedge [x_2, x_4] = 0,$$
$$x_3(t_{24}) = x_2 \wedge [x_3, x_4] = t_{12},\ x_4(t_{24}) = [x_4, x_2] \wedge x_4 = t_{24},$$

$$x_1(t_{34}) = [x_1, x_3] \wedge x_4 + x_1 \wedge [x_3, x_4] = t_{24},\ x_2(t_{34}) = x_3 \wedge [x_2, x_4] = -t_{23},$$
$$x_3(t_{34}) = x_3 \wedge [x_3, x_4] = t_{13},\ x_4(t_{34}) = [x_4, x_3] \wedge x_4 = -t_{14}.$$

We see that the submodule $F\{t_{12}, t_{23}\}$ is an essential module in $\mathfrak{g} \wedge \mathfrak{g}$. Define $s_{ij} \in P_2(A)$ with $\delta(s_{ij}) = x_i \otimes x_j - x_j \otimes x_i$. Now there are two cases to consider: when $a = 2$ and when $a \neq 2$.

Case $a = 2$. $[x_4, (s_{12} + 2s_{23})] = 0$. This shows that $F\{s_{12} + 2s_{23}\}$ is a proper (simple) submodule of $F\{s_{12}, s_{23}\}$, and hence $F\{s = s_{12} + 2s_{23}\}$ is a simple submodule of C_2 such that $x_i(s) \in \mathfrak{g}$ for all $i \leq 4$. Moreover, the ideal in $\mathfrak{g}$ generated by $F\{x_2, 2x_3 - x_1\}$ is 2-dimensional. Therefore $L = \mathfrak{g} \oplus F\{s\}$ is an ALE of $\mathfrak{g}$ as well as a solvable Lie algebra.

Case $a \neq 2$. It follows that $W = F\{s_{12}, s_{23}\}$ is a 2-dimensional simple submodule of $P_2(A)$, since $F\{s_{12}, s_{23}\}$ is simple. Notice that

$$\delta(s_{12})\delta(s_{23}) - \delta(s_{23})\delta(s_{12}) = (x_2 \otimes x_2)\Delta(x_2)$$

in A which implies that $[s_{12}, s_{23}] \notin \mathfrak{g} \oplus W$, whence $\mathfrak{g}$ does not satisfy the ALE property (and so $\mathfrak{g}$ does not have any 2-dimensional proper ideal).

Remark 4.1.13. In the last example, 4.5.1, it is unusual that the 4-dimensional Lie algebra does not have the ALE property when $F = \mathbb{R}$, but when $F = \mathbb{C}$ then it does satisfy the ALE property regardless of whether $a = 2$ or not for any $a \in F$ by Proposition 4.1.5.

Additionally there is more structure to the example 4.1.12 beyond ALE. See section 4.2 Extending Further with Anti-Cocommutative Elements.

Back to Proposition 4.1.5, if we drop the condition that F is algebraically closed then we need the Lie algebra to have a richer structure. But in return ALE's of these Lie algebras will receive a nice structure as well.

Proposition 4.1.14. *If $\mathfrak{g}$ is a finite dimensional nilpotent Lie algebra, then $\mathfrak{g}$ satisfies the ALE property, and any ALE of $\mathfrak{g}$ is a completely solvable Lie algebra.*

Proof. Because $\mathfrak{g}$ is nilpotent, by Engel's Theorem there exists $s \in P_2(A)$ such that $x(s) = 0$ for all $x \in \mathfrak{g}$. By Lemma 4.1.2, $x(s) = [x, s] \in \mathfrak{g}$, therefore $\mathfrak{g} \oplus F\{s\}$ is an ALE of $\mathfrak{g}$. □

In the Abelian case, there is always a nontrivial tower of Lie algebras in $P_2(A)$ assuming that there are enough elements.

Corollary 4.1.15. *If $\mathfrak{g}$ is a finite dimensional Abelian Lie algebra with $A \in \mathcal{A}(\mathfrak{g})$, then*

1. *every subspace C of $P_2(A)$ satisfying $\dim_F C = \dim_F \mathfrak{g} + 1$ is an ALE.*
2. *any ALE L_1 nilpotent,*
3. *any subspace $L_2 \supsetneq \mathfrak{g}$ of $P_2(A)$ with $\dim_F L_2 = \dim_F \mathfrak{g} + 2$ is completely solvable.*

Proof. For any $s \in P_2(A)$, we have $L_1 = \mathfrak{g} \oplus F\{s\}$ is a nilpotent Lie algebra. Moreover, if $L_2 = \mathfrak{g} \oplus F\{s, t\}$ for any 2-dimensional subspace $\{s, t\} \subseteq FS_2$, and since L_2 is an ALE of $\mathfrak{g} \oplus F\{s\}$, then by Proposition 4.1.14, L_2 is a completely solvable Lie algebra. □

Since finite dimensional nilpotent Lie F-algebras induce ALEs that are completely solvable, not all ALE satisfy the ALE property. Take for example the 3-dimensional Heisenberg algebra.

Example 4.1.16. Consider $\mathfrak{h} = F\{x, y, z\}$, where $[x, y] = z$ and $[z, x] = [z, y] = 0$. In $U(\mathfrak{h}) \otimes U(\mathfrak{h})$, it follows that

$$[\Delta(x), (x \otimes y - y \otimes x)] = x \otimes z - z \otimes x,$$

$$[\Delta(y), (x \otimes y - y \otimes x)] = y \otimes z - z \otimes y,$$

$$[\Delta(z), (x \otimes y - y \otimes x)] = 0,$$

$$[\Delta(g), (x \otimes z - z \otimes x)] = 0,$$

$$[\Delta(g), (y \otimes z - z \otimes y)] = 0,$$

for all $g \in \mathfrak{g}$. So if $A \in \mathcal{A}(\mathfrak{g})$ with $s_{yz} \in P_2(A)$ and $\delta(s_{yz}) = y \otimes z - z \otimes y$, the calculation shows that $L = \mathfrak{h} \oplus F\{s_{yz}\}$ is a ALE which is completely solvable. Without removing the coalgebra structure on L, we see that

$$[\delta(s_{xz}), \delta(s_{yz})] = (z \otimes z)\Delta(z),$$

$$[\delta(s_{xy}), \delta(s_{yz})] = (z \otimes z)\Delta(y),$$

which are both nonzero. This shows that both vector spaces $L \oplus F\{s_{xz}, s_{yz}\}$ and $L \oplus F\{s_{xy}, s_{yz}\}$ cannnot be Lie algebras.

4.2 Further Extensions with Anti-Cocommutative Elements

In many cases you'll have more than one anti-cocommutative element to consider. In this section we consider this case and ask when the algebra is "nice", i.e. having finite Gelfand-Kirillov dimension, Noetherian, etc.

We start with an example that would pave the way for more general techniques. It would also show that there exists, under certain conditions, extensions beyond an ALE, and that these extensions are not Lie algebras themselves. In example 4.1.16, adjoining $U(\mathfrak{h})$ with the set $\{s_{xz}, s_{yz}\}$ does not make $\mathfrak{h} \oplus F\{s_{xz}, s_{yz}\}$ a Lie algebra, but both $\mathfrak{h} \oplus F\{s_{xz}\}$ and $\mathfrak{h} \oplus F\{s_{yz}\}$ are Lie algebras (ALE).

Example 4.2.1. Suppose $\mathfrak{h} = F\{x, y, z\}$ is the 3-dimensional Heisenberg algebra over F with $[x, y] = z$. Consider $A \in \mathcal{A}(\mathfrak{h})$ with $\dim_F P_2(A) = \binom{\dim_F P(H)}{2}$. If $s_{xz} \in P_2(A)$ is anti-cocomutative with $\delta(s_{xz}) = x \otimes z - z \otimes x$, then the subalgebra X generated by the vector space $\mathfrak{h} \oplus F\{s_{xz}, s_{yz}\}$ is a Hopf subalgebra of GK-dimension 5.

Denote s, t by $s_{xz}, s_{yz} \in A := A(\mathfrak{h})$, respectively. Then in $A \otimes A$, we have

$$\begin{aligned}
[\delta(s), \delta(t)] &= (x \otimes z - z \otimes x)(y \otimes z - z \otimes y) - (y \otimes z - z \otimes y)(x \otimes z - z \otimes x) \\
&= xy \otimes z^2 - xz \otimes zy - zy \otimes xz + z^2 \otimes xy \\
&\qquad - yx \otimes z^2 + yz \otimes zx + zx \otimes yz - z^2 \otimes yx \\
&= (xy - yx) \otimes z^2 + z^2 \otimes (xy - yx) \\
&= (z \otimes z)\Delta(z) \\
&= \tfrac{1}{3}\delta(z^3).
\end{aligned}$$

Additionally we have

$$\begin{aligned}
[s \otimes 1 + 1 \otimes s, \delta(t)] &= [s, y] \otimes z - z \otimes [s, y] + [z, s] \otimes y - y \otimes [z, s] \\
[\delta(s), t \otimes 1 + 1 \otimes t] &= [x, t] \otimes z - z \otimes [x, t] + [t, z] \otimes x - x \otimes [t, z].
\end{aligned}$$

Before computing $\Delta([s, t])$ we must first compute $[\mathfrak{h}, s]$ and $[\mathfrak{h}, t]$. So let $\alpha_{ij} \in F$ for all

$i, j \leq 3$, and set

$$[x, s] = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z,$$

$$[y, s] = \alpha_{21}x + \alpha_{22}y + \alpha_{23}z,$$

$$[z, s] = \alpha_{31}x + \alpha_{32}y + \alpha_{33}z.$$

Because $F\{x, z, s\}$ is a Lie subalgebra of the Lie algebra $\mathfrak{h} \oplus F\{s\}$, then $\alpha_{12} = \alpha_{32} = 0$. Additionally, by the Jacobi identity,

$$\begin{aligned} 0 &= [x, [y, s]] + [s, [x, y]] + [y, [s, x]] \\ &= [x, \alpha_{21}x + \alpha_{22}y + \alpha_{23}z] + [s, z] - [y, \alpha_{11}x + \alpha_{13}z] \\ &= \alpha_{22}z - (\alpha_{31}x + \alpha_{33}z) \\ &= (\alpha_{22} - \alpha_{33})z - \alpha_{31}x, \end{aligned}$$

which implies that $\alpha_{31} = 0$, and $\alpha_{22} = \alpha_{33}$. Similarly, if $[x, t] = \lambda_{11}x + \lambda_{12}y + \lambda_{13}z$, then it follows that $\lambda_{21} = \lambda_{31} = 0$ since $F\{y, z, t\}$ is a Lie subalgebra of $\mathfrak{h} \oplus F\{t\}$, and $\lambda_{32} = 0$ and $\lambda_{11} = \lambda_{33}$ by the Jacobi identity.

Back to computing $\Delta([s,t])$, we see that

$$\begin{aligned}[s,y]\otimes z - z\otimes[s,y] + [z,s]\otimes y - y\otimes[z,s]\\
&= (\alpha_{21}x + \alpha_{22}y + \alpha_{23}z)\otimes z - z\otimes(\alpha_{21}x + \alpha_{22}y + \alpha_{23}z)\\
&\quad + \alpha_{33}(x\otimes z - z\otimes x)\\
&= (\alpha_{21}+\alpha_{33})(x\otimes z - z\otimes x) + \alpha_{22}(y\otimes z - z\otimes y)\\
&= (\alpha_{21}+\alpha_{33})\delta(s) + \alpha_{22}\delta(t),\\
[x,t]\otimes z - z\otimes[x,t] + [t,z]\otimes x - x\otimes[t,z]\\
&= (\lambda_{11}x + \lambda_{12}y + \lambda_{13}z)\otimes z - z\otimes(\lambda_{11}x + \lambda_{12}y + \lambda_{13}z)\\
&\quad + \lambda_{33}(z\otimes x - x\otimes z)\\
&= (\lambda_{11}-\lambda_{33})(x\otimes z - z\otimes x) + \lambda_{12}(y\otimes z - z\otimes y)\\
&= -\lambda_{22}\delta(s) + \lambda_{12}\delta(t).\end{aligned}$$

Now we see that

$$\begin{aligned}\delta([s,t]) &= [s\otimes 1 + 1\otimes s, \delta(t)] + [\delta(s), t\otimes 1 + 1\otimes t] + [\delta(s),\delta(t)]\\
&= (\alpha_{21}+\alpha_{33}-\lambda_{22})\delta(s) + (\alpha_{22}+\lambda_{12})\delta(t) + \tfrac{1}{3}\delta(z^3).\end{aligned}$$

Finally, if $\eta = \alpha_{21}+\alpha_{33}-\lambda_{22}$ and $\gamma = \alpha_{22}+\lambda_{12}$, then

$$\begin{aligned}\Delta([s,t]-\eta s-\gamma t-\tfrac{1}{3}z^3) &= [s,t]\otimes 1 + 1\otimes[s,t] + \delta([s,t])\\
&\quad - \eta(s\otimes 1 + 1\otimes s - \delta(s)) - \gamma(t\otimes 1 + 1\otimes t + \delta(t))\\
&\quad - \tfrac{1}{3}(z^3\otimes 1 + 1\otimes z^3 + \delta(z^3))\\
&= ([s,t]-\eta s-\gamma t-\tfrac{1}{3}z^3)\otimes 1 + 1\otimes([s,t]-\eta s-\gamma t-\tfrac{1}{3}z^3),\end{aligned}$$

whence $[s,t]-\eta s-\gamma t-\frac{1}{3}z^3 \in P(A) = \mathfrak{h}$. This implies that $[s,t] = \frac{1}{3}z^3+\eta s+\gamma t+a_1x+a_2y+a_3z$

for some $a_1, a_2, a_3 \in F$. Furthermore, in A,

$$\begin{aligned}
[z,[s,t]] &= [z, \tfrac{1}{3}z^3 + \eta s + \gamma t + a_1 x + a_2 y + a_3 z] \\
&= \eta(\alpha_{33} z) + \gamma(\lambda_{33} z) \\
[s,[t,z]] &= -[s, \lambda_{33} z] = -\alpha_{33}\lambda_{33} z \\
[t,[z,s]] &= [s, \alpha_{33} z] = \lambda_{33}\alpha_{33} z
\end{aligned}$$

And so the Jacobi identity yields $\eta\alpha_{33} + \gamma\lambda_{33} = 0$.

Since X is a connected Hopf subalgebra, then its associated graded algebra $\operatorname{gr} X = F[\overline{x}, \overline{y}, \overline{z}, \overline{s}, \overline{t}]$, since both $\overline{s}, \overline{t}$ are of degree 2, whence $\overline{s}\overline{t} = \overline{t}\overline{s}$. This shows that the Hopf subalgebra X is Noetherian of GK-dimension 5, as claimed.

To generalize example 4.2.1, we first decompose the linear map $\delta : H \to H \otimes H$.

Definition 4.2.2. For any connected coalgebra C, define the linear maps $\delta_{ac}, \delta_{cc} : C \to C \otimes C$ by

$$\delta_{ac} = \tfrac{1}{2}(\delta - \tau \circ \delta), \qquad \delta_{cc} = \tfrac{1}{2}(\delta + \tau \circ \delta).$$

Notice that $\delta = \delta_{ac} + \delta_{cc}$.

Lemma 4.2.3. *Suppose H is any connected Hopf algebra $P = P_2(H)$, and $U = U(P(H))$. Then*

1. $\delta_{cc}([s,t]) = [\delta(s), \delta(t)]$ *in* $H \otimes H$, *where* $s, t \in P$.

2. $\delta_{ac}|_U = 0$ *while* $\delta_{cc}|_U = \delta|_U$.

3. $\delta_{ac}|_P = \delta|_P$ *while* $\delta_{cc}|_P = 0$.

Proof. 1. In $H \otimes H$ notice that

$$\delta([s,t]) = [(s \otimes 1 + 1 \otimes s), \delta(t)] + [\delta(s), (t \otimes 1 + 1 \otimes t)] + [\delta(s), \delta(t)].$$

Applying the twist map τ yields

$$\tau \circ \delta([s,t]) = -[(s \otimes 1 + 1 \otimes s), \delta(t)] - [\delta(s), (t \otimes 1 + 1 \otimes t)] + [\delta(s), \delta(t)].$$

Therefore $(\delta + \tau \circ \delta)[s,t] = 2[\delta(s), \delta(t)]$, and $(\delta - \tau \circ \delta)[s,t] = 2[(s \otimes 1 + 1 \otimes s), \delta(t)] + 2[\delta(s), (t \otimes 1 + 1 \otimes t)]$, whence $\delta_{cc}([s,t]) = [\delta(s), \delta(t)]$.

The rest is straightforward. □

In short, δ_{cc} preserves the cocommutative part of δ, while δ_{ac} preserves the anti-cocommutative part.

Intuitively one would suspect that the anti-cocommutative part would belong to the largest anti-cocommutative subcoalgebra $P_2(A)$, and the cocommutative part would belong to the largest cocommutative subcoalgebra, the universal enveloping algebra.

Proposition 4.2.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $A \in \mathcal{A}(\mathfrak{g})$. Suppose $s, t \in P_2(A)$ are non-primitive, and U_n is the coradical filration of the Hopf subalgebra $U(\mathfrak{g})$. Then $\delta_{cc}([s,t]) \in \delta(U_3)$ if and only if $\delta_{ac}([s,t]) \in \delta(P_2(A))$.*

Proof. Assume that $\delta_{cc}([s,t]) \subseteq \delta(U_3)$, i.e. $\delta_{cc}([s,t]) = \delta(w)$ for some $w \in U_3$. By Lemma 4.2.3 we have

$$\Delta([s,t] - w) = ([s,t] - w) \otimes 1 + 1 \otimes ([s,t] - w) + \delta_{ac}([s,t]),$$

hence $\delta([s,t] - w) = \delta_{ac}([s,t])$. Since $\tau \circ \delta_{ac} = -\delta_{ac}$ then $[s,t] - w$ is anti-cocommutative. Thus by definition $[s,t] - w \in P_2(A)$. Therefore $\delta([s,t] - w) = \delta_{ac}([s,t]) \in \delta(P_2(A))$.

Now let $\delta_{ac}([s,t]) = \delta(v)$ for some $v \in P_2(A)$. By Lemma 4.2.3 we have

$$\Delta([s,t] - v) = ([s,t] - v) \otimes 1 + 1 \otimes ([s,t] - v) + \delta_{cc}([s,t]),$$

which implies that $[s,t] - v$ is cocommutative. Since U is the largest cocommutative subcoalgebra in A by Corollary 3.4.8, then $[s,t] - v \in U(\mathfrak{g})$. If A_n is the coradical filtration on

A, then $st \in A_4$, hence

$$[s,t] - p \in A_4 \cap U(\mathfrak{g}) = U_4,$$

Since $\delta_{cc}(v) = 0$, then we have $\delta_{cc}([s,t]) = \delta([s,t] - v) \in \delta(U_4)$. To show that $\delta_{cc}([s,t]) \in \delta(U_3)$, consider $\delta(s) = x \otimes y - y \otimes x$ and $\delta(t) = a \otimes b - b \otimes a$. Then

$$\begin{aligned}
\delta_{cc}([s,t]) &= [\delta(s), \delta(t)] \\
&= xa \otimes yb - xb \otimes ya - ya \otimes xb + yb \otimes xa - ax \otimes by \\
&\qquad + ay \otimes bx + bx \otimes ay - by \otimes ax \\
&= ax \otimes [y,b] + [y,b] \otimes ax + by \otimes [x,a] + [x,a] \otimes by + [x,a] \otimes [y,b] \\
&\quad + [y,b] \otimes [x,a] - (xb \otimes [y,a] + [y,a] \otimes xb + ya \otimes [x,b] + [x,b] \otimes ya) \\
&\qquad\qquad + [x,b] \otimes [y,a] + [y,a] \otimes [x,b] \\
&\in (U_2/U_1) \otimes U_1 + U_1 \otimes (U_2/U_1).
\end{aligned}$$

Since $[s,t] - p \in U_4$, if $[s,t] - v \notin U_3$, then $\delta_{cc}([s,t]) = v_1 + v_2 + u$ with $v_1 \in (U_3/U_2) \otimes U_1$, $p_2 \in U_1 \otimes (U_3/U_2)$ are both nonzero, and $u \in U_2 \otimes U_2$. But this is absurd, therefore $[s,t] - v \in U_3$, whence $\delta_{cc}([s,t]) \in \delta(U_3)$. □

In other words, Proposition 4.2.4 states that instead of looking at $\delta([,])$ as a whole, we may observe either $\delta_{cc}([,])$ or $\delta_{ac}([,])$. If the computation allows us to pullback to some anti-cocommutative or cocommutative element, then there is a possibility of a "nice" Hopf subalgebra that is not the enveloping algebra.

Theorem 4.2.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $A \in \mathcal{A}(\mathfrak{g})$. Suppose $U = U(\mathfrak{g})$ and $t_1, ..., t_n \in P_2(A)$ are non-primitive elements satisfying the following conditions:*

1. *$V = F\{t_1, ..., t_n\}$ is an n-dimensional vector space,*

2. *for every $i, j \leq n$, $\delta_{ac}([t_i, t_j]) \in \delta(P_2(A))$, and*

3. $\mathfrak{g} \oplus V$ *is a* $\mathfrak{g}$*-module.*

Then GK.dim(B) $= n + \dim_F \mathfrak{g}$*, where B is the Hopf subalgebra of A generated by* $\mathfrak{g} \oplus V$*.*

Proof. By Proposition 4.2.4, $\delta_{cc}([t_i, t_j]) \in \delta(U_3)$ for all $i, j \leq n$. Applying Lemma 4.2.3 shows that

$$\Delta([t_i, t_j] - w_{ij} - u_{ij}) = ([t_i, t_j] - w_{ij} - u_{ij}) \otimes 1 + 1 \otimes ([t_i, t_j] - w_{ij} - u_{ij}),$$

i.e. $[t_i, t_j] - w_{ij} - u_{ij} \in \mathfrak{g}$ where $w \in P_2(A)$ with $\delta_{ac}(w_{ij}) = \delta_{ac}([t_i, t_j])$ and $u_{ij} \in U_3$ with $\delta_{cc}(u_{ij}) = \delta_{cc}([t_i, t_j])$. Without loss of generality, assume that $[t_i, t_j] = w_{ij} + u_{ij}$. By the hypothesis $[\mathfrak{g} \oplus V, \mathfrak{g}] \in \mathfrak{g} \oplus V$ in A, therefore in $\operatorname{gr} A$, we have that $[\overline{t_i}, \overline{x}] = 0$ for any $x \in \mathfrak{g}$ and any $i \leq n$, and $[\overline{t_i}, \overline{t_j}] = 0$ for all $i, j \leq n$, since $\overline{w_{ij} + u_{ij}} \in A_3/A_2$ and $\overline{t_i t_j} \in A_4/A_3$ (as A_n represents the coradical filtration on A). This shows that if B is the Hopf subalgebra of A generated by $\mathfrak{g} \oplus V$, we have that $\operatorname{gr} B$ is exactly the commutative polynomial algebra $F[\mathfrak{g} \oplus V]$. Hence $\text{GK.dim}(\operatorname{gr} B) = n + \dim_F \mathfrak{g}$ and so by [30, Theorem 6.9], $\text{GK.dim}(B) = n + \dim_F \mathfrak{g}$. □

Corollary 4.2.6. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra, and* $A \in \mathcal{A}(\mathfrak{g})$*. Suppose* $U = U(\mathfrak{g})$ *and* $t_1, ..., t_n \in P_2(A)$ *are non-primitive elements satisfying the following conditions:*

1. $V = F\{t_1, ..., t_n\}$ *is an n-dimensional vector space,*

2. for every $i, j \leq n$*,* $\delta_{ac}([t_i, t_j]) \in \delta(P_2(A))$*, and*

3. for every $i \leq n$*, the vector space* $\mathfrak{g} \oplus F\{t_i\}$ *is an ALE.*

Then GK.dim(B) $= n + \dim_F \mathfrak{g}$*, where B is the Hopf subalgebra generated of A by* $\mathfrak{g} \oplus V$*.*

Proof. Having $\mathfrak{g} \oplus F\{t_i\}$ informs us that $[\mathfrak{g}, t_i] \subseteq \mathfrak{g} \oplus F\{t_i\}$ for all $i \leq n$, whence $U \oplus V$ is a (left) U-module. Now apply Theorem 4.2.5. □

Example 4.2.1 satisfies Corollary 4.2.6, and hence the desired result.

Now we add normality in these algebras. In particular, if the largest cocommutative Hopf algebra is also a normal Hopf subalgebra in a connected Hopf algebra we would see a concept that was mentioned many times over.

Proposition 4.2.7. *Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra and $A \in \mathcal{A}(\mathfrak{g})$. Suppose $t \in P_2(A)$ is non-primitive, and let B be the Hopf subalgebra of A generated by the $\mathfrak{g} \oplus F\{t\}$. Then $U(\mathfrak{g}) \subseteq B$ is a normal Hopf subalgebra of B if and only if $[\mathfrak{g}, t] \subseteq \mathfrak{g}$ in B.*

Proof. Set $\delta(t) = x \otimes y - y \otimes x$. Denote $\mathrm{ad}_r[t]$ with $\mathrm{ad}[t]$. It follows that $S(t) = -t + [x, y]$, and in A we have that

$$\begin{aligned} \mathrm{ad}[t](g) &= S(t)g + gt - xgy + ygx \\ &= -tg + xyg - yxg + gt + ygx - xgy \\ &= [t, g] + y[g, x] + x[y, g]. \end{aligned}$$

So if $U(\mathfrak{g})$ is a normal Hopf subalgebra of B, then $[t, g] \in U(\mathfrak{g})$, and since $[t, g] \in P_2(A)$ we have that $[t, g] \in U(\mathfrak{g}) \cap P_2(A) = P_2(U(\mathfrak{g})) = \mathfrak{g}$. And conversely the assumption $[t, \mathfrak{g}] \subseteq \mathfrak{g}$ forces $\mathrm{ad}[t](\mathfrak{g}) \subseteq U(\mathfrak{g})$. Since $\mathrm{ad}[ba] = \mathrm{ad}[a] \circ \mathrm{ad}[b]$ for any $a, b \in A$, we have that $\mathrm{ad}[A](U(\mathfrak{g})) \subseteq U(\mathfrak{g})$. This argument holds for the left adjoint,

$$\mathrm{ad}_l[t](g) = [t, g] + [g, x]y + [y, g]x,$$

therefore $U(\mathfrak{g})$ is a normal Hopf sublagebra of B. □

Recall that since $P_2(H)/\mathfrak{g}$ can be embedded in $\mathfrak{g} \wedge \mathfrak{g}$, there exists a connected Hopf algebra $A \in \mathcal{A}(\mathfrak{g})$ such that $P_2(A)/\mathfrak{g}$ is isomorphic to $\mathfrak{g} \wedge \mathfrak{g}$.

Corollary 4.2.8. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with $\dim_F Z(\mathfrak{g}) \geq 3$. There exists $A \in \mathcal{A}(\mathfrak{g})$ such that $U(\mathfrak{g})$ is a normal Hopf subalgebra of A and*

$$\dim_F(P_2(A)/\mathfrak{g}) \geq 3.$$

Proof. Set $F\{x_1, ..., x_n\} = Z(\mathfrak{g})$. We know that there exists $H \in \mathcal{A}(\mathfrak{g})$ with

$$\dim_F(P_2(H)/\mathfrak{g}) = \binom{\dim_F \mathfrak{g}}{2}.$$

So consider $s_{ij} \in P_2(H)$ so that $\delta(s_{ij}) = x_i \otimes x_j - x_j \otimes x_i$ with $i < j \leq n$. It follows that $[\delta(s_{ij}), \Delta(g)] = 0$ for all $i < j \leq n$, hence $[s_{ij}, g] \subseteq \mathfrak{g}$ for all $g \in \mathfrak{g}$. Therefore if A is Hopf algebra generated by $\mathfrak{g} \oplus F\{s_{ij} : i < j \leq n\}$, then $A \in \mathcal{A}(\mathfrak{g})$ and $P_2(A) = F\{s_{ij} : i < j \leq n\}$. Moreover, by Proposition 4.2.7, $U(\mathfrak{g})$ is a normal Hopf subalgebra of B. □

To apply Proposition 4.2.7, when working with certain Lie algebras its enveloping algebra cannot achieve normality in any connected Hopf algebra.

Corollary 4.2.9. *Suppose $\mathfrak{g}$ is a finite dimensional simple Lie algebra, then $U(\mathfrak{g})$ cannot be a normal Hopf subalgebra of A, for any $A \in \mathcal{A}(\mathfrak{g})$.*

Proof. If $U(\mathfrak{g})$ is a normal Hopf subalgebra, then $\mathfrak{g}$ satisfies the ALE property which contradicts Corollary 4.1.9. □

Additionally if $U(\mathfrak{g})$ is a normal Hopf subalgebra, then $\mathfrak{g} \oplus F\{t\}$ is an ALE. Thus under normality we can achieve one of the main results.

Theorem 4.2.10. *Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra and $A \in \mathcal{A}(\mathfrak{g})$. If $U(\mathfrak{g})$ is a normal Hopf subalgebra of A, then GK.dim$(A) = \dim_F P_2(A)$.*

Proof. With $U := U(\mathfrak{g})$ normal in A, we have that $\mathfrak{g} \oplus F\{t\}$ is an ALE and $[t, \mathfrak{g}] \subseteq \mathfrak{g}$ for all non-primitive $t \in P_2(A)$ by Proposition 4.2.7, respectively. So consider $t_{12}, t_{34} \in P_2(A)$ where $\delta(t_{12}) = x_1 \otimes x_2 - x_2 \otimes x_1$ and $\delta(t_{34}) = x_3 \otimes x_4 - x_4 \otimes x_3$. Then by Lemma 4.2.3, in

$A \otimes A$ we have

$$\begin{aligned}\delta_{ac}([t_{12}, t_{34}]) &= \delta([t_{12}, t_{34}]) - [\delta(t_{12}), \delta(t_{34})] \\ &= [t_{12} \otimes 1 + 1 \otimes t_{12}, \delta(t_{34})] + [\delta(t_{12}), t_{34} \otimes 1 + 1 \otimes t_{34}] \\ &= [t_{12}, x_3] \otimes x_4 - x_4 \otimes [t_{12}, x_3] + x_3 \otimes [t_{12}, x_4] - [t_{12}, x_4] \otimes x_3] \\ &\qquad\qquad + [x_1, t_{34}] \otimes x_2 - x_2 \otimes [x_1, t_{34}] + x_1 \otimes [x_2, t_{34}] - [x_2, t_{34}] \otimes x_1 \\ &= b_3 \otimes x_4 - x_4 \otimes b_3 + x_3 \otimes b_4 - b_4 \otimes x_3 \\ &\qquad + a_1 \otimes x_2 - x_2 \otimes a_1 + x_1 \otimes a_2 - a_2 \otimes x_1,\end{aligned}$$

where $b_i = [t_{12}, x_i] \in \mathfrak{g}$ with $i = 3, 4$ and $a_j = [x_j, t_{34}] \in \mathfrak{g}$ with $j = 1, 2$. This shows that $\delta_{ac}([t_{12}, t_{34}]) \in \delta(P_2(A))$, or in particular

$$\delta_{ac}([t_{12}, t_{34}]) = \delta(s_{b_3x_4} + s_{x_3b_4} + s_{a_1x_2} + s_{x_1a_2}),$$

where $\delta(s_{b_3x_4}) = b_3 \otimes x_4 - x_4 \otimes b_3$. Therefore Corollary 4.2.6 implies that $\text{GK.dim}(A) = \dim_F(P_2(A)/\mathfrak{g}) + \dim_F \mathfrak{g} = \dim_F P_2(A)$, as desired. □

Thus with normality there exists a "nice" Hopf algebra.

Corollary 4.2.11. *If H is a connected Hopf algebra and $U(P(H))$ is a normal Hopf subalgebra of H, then GK.dim$(A) = \dim_F P_2(H)$, where A is the Hopf subalgebra of H generated by $P_2(H)$. Moreover A is a Noetherian (Auslander-regular) algebra.*

Proof. Immediately follows from Theorem 4.2.10 and the Noetherian condition follows from [30, Corollary 6.10]. □

4.3 Application: Global Dimension

In this section, we focus on the global dimension of connected Hopf algebras, and apply the ideas of anti-cocommutative Lie extensions.

By comparison, the papers [30] and [29], the authors use the Gelfand-Kirillov dimension (GK-dim) to characterize and classify certain connected Hopf algebras. While [29, Corollary 6.10] does mention global dimension when having finite GK-dimension, our main focus will be on global dimension in this section.

Lemma 4.3.1. *Suppose H is a connected Hopf algebra and A is any Hopf subalgebra of H. Then it follows that*

1. *r.gl.dim*$(A) \leq$ *r.gl.dim*(H)*, when A is right Noetherian with finite right global dimension.*

2. *gl.dim*$(U(P(H))) \leq$ *r.gl.dim*(H) *when* $\dim_F P(H) < \infty$.

Proof. 1. Since A is a Hopf subalgebra of H, then by [15, Theorem 1.3], H is a faithfully flat left and right A-module. Applying [16, Theorem 7.2.6] yields r.gl.dim$(A) \leq$ r.gl.dim(H).

2. Given $\dim_F P(H) < \infty$ then U is Noetherian with gl.dim$(U) = \dim_F P(H)$. Apply part 1. □

Theorem 4.3.2. *If H is any connected Hopf algebra such that*

$$r.gl.dim(H) = \dim_F P(H) < \infty,$$

and $P(H)$ is completely solvable, then $H = U(P(H))$.

Proof. Assume that $H \neq U(P(H))$, then by [29, Lemma 2.4], $P_2(H) \neq P(H)$. Let A be the subalgebra of H generated by the coalgebra $P_2(H)$. Clearly $A \in \mathcal{A}(P(H))$ with $P_2(A) = P_2(H)$. By Proposition 4.1.5, there exists $t \in P_2(A)$ such that $P(H) \oplus F\{t\}$ is a finite dimensional ALE of $P(H)$. Thus if A' is the sublagebra of A generated by the coalgebra $P(H) \oplus F\{t\}$, then $A' \cong U(\mathfrak{g})$ as algebras, for some finite dimensional Lie algebra $\mathfrak{g}$ with $\dim_F \mathfrak{g} > \dim_F P(H)$. Since A' is a Noetherian Hopf subalgebra of H, then by Lemma 4.3.1,

$$\text{r.gl.dim}(H) \geq \text{gl.dim}(A') > \dim_F P(H) = \text{gl.dim}(U(P(H))),$$

which is absurd. Therefore we have $H = U(P(H))$. $\square$

We may want to replace $P(H)$ being completely solvable with $U(P(H))$ being a normal Hopf subalgebra to achieve the same result.

Theorem 4.3.3. *Suppose H is a connected Hopf algebra with*

$$\textit{r.gl.dim}(H) = \dim_F P(H) < \infty.$$

If $U(P(H))$ is a normal Hopf subalgebra of H, then $H = U(P(H))$.

Proof. Assume the contrary, $H \neq U(P(H))$. Then $P_2(H) \neq P(H)$ by [29, Lemma 2.4], thus by Corollary 4.2.11, $\mathfrak{h} = P(H) \oplus F\{t\}$ is an ALE which implies that if A is the Hopf subalgebra of H generated by $\mathfrak{h}$, then $A \cong U(\mathfrak{h})$ as algebras, hence $\text{gl.dim}(A) = \dim_F P(H)+1$. Because A is a Noetherian Hopf subalgebra of H with finite global dimension, we have that

$$\text{r.gl.dim}(H) \geq \text{gl.dim}(A) > \dim_F P(H),$$

thus a contradiction. Therefore $H = U(P(H))$. $\square$

Additionally we may also apply the same technique for Krull dimension. However to mimic Theorem 4.3.2 we need to improve the structure of the Lie algebra.

We denote the right Krull dimension of an algebra A by $\text{K.dim}(A_A)$.

Lemma 4.3.4. *Suppose H is a right Noetherian connected Hopf algebra and A is a Hopf subalgebra of H. Then $\textit{K.dim}(A_A) \leq \textit{K.dim}(H_H)$.*

Proof. Since H is right Noetherian then so is A by [15, Theorem 1.3] and [11, Exercise 17T]. Apply [11, Exercise 15U]. $\square$

Theorem 4.3.5. *If H is any right Noetherian connected Hopf algebra such that*

$$K.dim(H_H) = \dim_F P(H) < \infty,$$

and $P(H)$ is nilpotent, then $H = U(P(H))$.

Proof. (Similar to the proof of Theorem 4.3.2.) Assume the contrary; $H \neq U(P(H))$. Let A be subalgebra of A generated by the coalgebra $P_2(H)$. Obviously $A \in \mathcal{A}(P(H))$, so by Proposition 4.1.5, there exists $t \in P_2(A) = P_2(H)$ such that $P(H) \oplus F\{t\}$ is an ALE. If A' is a the subalgebra of A generated by the coalgebra $P(H) \oplus F\{t\}$, then $A' \cong U(\mathfrak{g})$ as algebras for some finite dimensional Lie algebra $\mathfrak{g}$ with $\dim_F \mathfrak{g} > \dim_F P(H)$. Additionally $P(H) \oplus F\{t\}$ is a finite dimensional completely solvable Lie algebra and so is $\mathfrak{g}$ since the algebra-isomorphism is the identity restricted on $P(H) \oplus F\{t\}$. By [11], $\text{K.dim}(A'_{A'}) = \dim_F \mathfrak{g}$. Applying Lemma 4.3.4 shows that

$$\text{K.dim}(H_H) \geq \text{K.dim}(A'_{A'}) > \dim_F P(H),$$

which is a contradiction. Therefore $H = U(P(H))$. □

Note that both Theorem 4.3.2, Theorem 4.3.3, and Theorem 4.3.5 are analogous to [30, Lemma 7.2] with additional conditions. We have the following result about low dimensional connected Hopf algebras with finite dimensional Lie algebras that is also analogous to [30, Lemma 7.4].

Corollary 4.3.6. *If H is a connected Hopf algebra with r.gl.dim$(H) \leq 2$ and $P(H)$ is finite dimensional, then $H = U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$.*

Proof. There are two cases to consider: $d = 0$ and $0 < d \leq 2$, where $d = \text{r.gl.dim}(H)$. If $d = 0$, then H is a semisimple algebra, hence Artinian, and so by the main theorem of [14] and [10, Proposition 3.5.19], $H = F$.

Now let $d \leq 2$. Applying Lemma 4.3.1 shows that $\dim_F P(H) \leq 2$. If both $\dim_F P(H)$ and d are 2, then by Theorem 4.3.2, $H = U(\mathfrak{g})$ where $\mathfrak{g} = P(H)$. If $\dim_F P(H) = 1$, then by [29, Lemma 1.3], $H = F[x]$ where $x \in P(H)$ whence $H = U(P(H))$. □

4.4 Extra: The Antipode

In this section we focus on the antipode of $A = A(\mathfrak{g}) \in \mathcal{A}(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$. We will see that the antipode of A has only two outcomes in regards to its order. And if the invariant subalgebra $A^{\langle S^2 \rangle}$ is exactly the universal enveloping algebra, then we retrieve some information about the Lie algebra.

Recall that the antipode of any pointed, whence connected, Hopf algebra is bijective due to [18, Corollary 5.2.11]. We will be using this result later but first a simple fact about the antipode of pointed Hopf algebras.

Lemma 4.4.1. *If $H \neq F$ is a connected Hopf algebra with antipode S and $P(H) \neq 0$, then either $S^m = id_H$ for some even number m, or $S^m \neq id_H$ for any $m \in \mathbb{N}$. In other words, S has either even order or infinite order.*

Proof. Suppose that the order of S is finite but $S^k = \mathrm{id}_H$ for some odd number k. Then for any $x \in P(H) - 0$ we have $S^k(x) = S(x)$, since $S^2|_{P(H)} = \mathrm{id}_{P(H)}$. Thus $x = -x$ and given the characteristic of F is not 2 then $x = 0$, a contradiction. Therefore k must be an even number. □

The next proposition states that given a finite dimensional Lie algebra $\mathfrak{g}$, the antipode of $A \in \mathcal{A}(\mathfrak{g})$ has only two options.

Proposition 4.4.2. *Let $\mathfrak{g}$ be any Lie algebra, and consider $A \in \mathcal{A}(\mathfrak{g})$. If S is the antipode of A, then either $S^2 = id_A$, or $S^k \neq id_A$ for any $k \in \mathbb{Z} - 0$. In other words, either A is involutive or S has infinite order.*

Proof. First notice that for any $t \in P_2(A)$ with $\delta(t) = x \otimes y - y \otimes x$, where $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} 0 = \varepsilon(t) &= (\mathrm{id}_A * S)(t) = t + S(t) + xS(y) - yS(x) \\ &= t + S(t) + [y, x]. \end{aligned}$$

Therefore $S(t) = -t + [x, y]$.

Let's assume that $S^k = \mathrm{id}_A$ with $2 \leq k < \infty$. Since A is generated by the coalgebra $P_2(A)$, then we only need to consider $t \in P_2(A)/\mathfrak{g}$. Set $\delta(t) = x \otimes y - y \otimes x$ for some $x, y \in \mathfrak{g}$ with $x \neq y$. Then we have $S(t) = -t + [x, y]$. If $[x, y] = 0$ then $S^2(t) = t$, and thus $[\mathfrak{g}, \mathfrak{g}] = 0$ if and only if $S^2|_{P_2(A)} = \mathrm{id}_{P_2(A)}$.

Let's assume that $[x, y] \neq 0$, whence $k > 2$. Then $S^2(t) = -S(t) + S([x, y]) = t - 2[x, y]$, and so by induction, $S^n(t) = (-1)^n(t - n[x, y])$ for all $n \in \mathbb{N}$. By our assumption, $t = (-1)^k(t - k[x, y])$. If k is even, we have $0 = -k[x, y]$, and since the characteristic of F is zero, $[x, y] = 0$ which contradicts our previous assumption. If k is odd, we have $2t = k[x, y]$ which shows that t is cocommutative which is absurd. (A similar argument can be applied for S^m where m is a negative integer.) Therefore either $S^2 = \mathrm{id}_A$, or $S^k \neq \mathrm{id}_A$ for any $k \in \mathbb{Z} - 0$. □

Additionally, the antipode of $A(\mathfrak{g})$ tells us more about the Lie algebra $\mathfrak{g}$.

Corollary 4.4.3. *If H is a connected Hopf algebra such that $S^2 = id_H$, $H \neq U(P(H))$, and $\dim_F P(H) = 2$, then $P(H)$ is Abelian.*

Proof. Consider $P(H) = F\{x, y\}$, Since $H \neq U(P(H))$, then by [29, Lemma 2.4], $P_2(H) \neq P(H)$. So let $t \in P_2(H)$ be non-primitive with $\delta(t) = x \otimes y - y \otimes x$. As F is characteristic zero, $S^2(t) = t - 2[x, y] = t$ which forces $[x, y] = 0$. □

Corollary 4.4.4. *Assume H is a connected Hopf algebra such that*

$$\dim_F[P_2(H)/P(H)] = \binom{\dim_F P(H)}{2}.$$

If $S^2 = id_H$ then $P(H)$ is Abelian.

Proof. Since $S|_A^2 = \mathrm{id}_A$ where A is the Hopf subalgebra of H generated by $P_2(H)$, then by Proposition 4.4.2, $P(H)$ is Abelian. $\square$

Example 4.4.5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. If $H \subsetneq A$ is a Hopf subalgebra, then $S|_H^2 = \mathrm{id}_H$ does not imply that $\mathfrak{g}$ is Abelian. For example, let $\mathfrak{h} = F\{x, y, z\}$ be the 3-dimensional Heisenberg algebra over F with $z \in Z(\mathfrak{h})$, and let $t = s_{yz} \in A(\mathfrak{h})$. We know that $\mathfrak{g} = \mathfrak{h} \oplus F\{t\}$ is an ALE of $\mathfrak{h}$, and $S(t) = -t + zy - yz = -t$, hence $S|_H^2 = \mathrm{id}_H$ where $H = U(\mathfrak{g})$. However, $\mathfrak{h}$ is not Abelian since $[x, y] = z$.

Proposition 4.4.6. *Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra and $A \in \mathcal{A}(\mathfrak{g})$ with $\dim_F \mathfrak{g} \geq 3$. If the invariant subalgebra $A^{\langle S^2 \rangle} = U(\mathfrak{g})$ then $\mathfrak{g}$ is a semisimple Lie algebra.*

Proof. Assuming $\dim_F \mathfrak{g} \geq 3$, hence $\dim_F(P_2(A)/\mathfrak{g}) \geq 3$, and suppose that $A^{\langle S^2 \rangle} = U$. Let $\mathfrak{j}$ be an Abelian ideal of $\mathfrak{g}$; so showing that $\mathfrak{j} = 0$ implies that $\mathfrak{g}$ is semisimple. Consider $a \in \mathfrak{j}$, then for any $x \in \mathfrak{g} - 0$, $z = [x, a] \in \mathfrak{j}$ and thus $[a, z] = 0$. Since $s_{az} \in A$ with $\delta(s_{az}) = a \otimes z - z \otimes a$, then it follows that $S(s_{az}) = -s_{az} + [z, a] = -s_{az}$. Hence $s_{az} \in X$ which is impossible since s_{az} is not cocommutative. This shows that $a = 0$ or $[x, a] = 0$. If $[x, a] = 0$ then again we have that $s_{ax} \in X$ where $\delta(s_{ax}) = a \otimes x - x \otimes a$, which forces $a = 0$ since $x \neq 0$. Hence $\mathfrak{j} = 0$, as desired. $\square$

Remark 4.4.7. The reason why we need $\dim_F P(H) \geq 3$ in Proposition 4.4.6 is that if $\mathfrak{g} = F\{x, y\}$ is the 2-dimensional non-Abelian Lie algebra, then $S(s_{xy}) = -s_{xy} + x$, where $s_{xy} \in P_2(A(\mathfrak{g}))$. This shows that $X = U$ but $\mathfrak{g}$ is obviously not semisimple.

Additionally we have the following property about the linear map $S^2 - \mathrm{id}_A$.

Lemma 4.4.8. *For any Lie algebra $\mathfrak{g}$ with $A \in \mathcal{A}(\mathfrak{g})$, the linear map $D = S^2 - id_A$ is a locally nilpotent skew-derivation on A.*

Proof. It is clear that S^2 is a Hopf automorphism; S is a bijective anit-homomorphism on A by [18, Corollary 5.2.11], and

$$\Delta \circ S^2 = \tau \circ (S \otimes S) \circ \tau \circ (S \otimes S) \circ \Delta = (S^2 \otimes S^2) \circ \Delta,$$

hence S^2 is also a coalgebra homomorphism, hence S^2 is a Hopf automorphism. If G is the group of all Hopf automorphisms on A, then in the pointed Hopf algebra FG, $D = S^2 - \mathrm{id}_A$ is a skew-primitve element, therefore D is a skew derivation on A.

To see that D is locally nilpotent, we see that $D(\mathfrak{g}) = 0$ while $D(P_2(A)) \subseteq \mathfrak{g}$, whence $D^2(P_2(A)) = 0$. Using the fact that D is a linear map then for any word $t = t_1, ..., t_k \in A$, where $t_1, ..., t_k$ are elements of the basis of $P_2(A)$, we have by induction

$$D^n(t) = \sum_{O}^{n} D^{e_1}(t_1)D^{e_2}(t_2)\cdots D^{e_k}(t_k),$$

where $O = \{(e_1, ..., e_k) \in \mathbb{N}^k : \sum_{i=1}^{k} e_i = n\}$. Since every $t_i \in P_2(A)$, setting $n = k+1$ implies that some $e_i \geq 2$ therefore $D(t) = 0$, as desired. □

4.5 Minor Result & Further Questions

Lastly one of the useful facts about ALE is that it can describe the algebra $A(\mathfrak{g})$. So the next proposition uses and generalizes one of Passman's result on universal enveloping algebras.

Proposition 4.5.1. *Given a Lie algebra $\mathfrak{g}$, if $A \in \mathcal{A}(\mathfrak{g})$ is PI, then A is commutative.*

Proof. By Passman's result, the subalgebra $U(\mathfrak{g})$ is PI, hence it's commutative. But this implies that $P_2(A)$ is an ALE of $\mathfrak{g}$, hence $A \cong U(\mathfrak{h})$ as algebras, where $\mathfrak{h} = P_2(A)$. Therefore $U(\mathfrak{h})$ is PI, hence commutative, hence A is commutative. □

This begs the following question.

Question 4.5.2. If a connected Hopf algebra is affine PI, is it commutative?

We have seen that global dimension can be just as effective as the Gelfand-Kirillov dimension given the right conditions, which leads to the following question.

Question 4.5.3. Does there exist a connected Hopf algebra H with infinite GK-dimension but both $\dim_F P(H)$ and r.gl.dim(H) are finite?

Another question we can ask is if there are any free subalgebras. So the next question is not only the main motivation for this research, but can answer the previous question.

Question 4.5.4. Given any finite dimensional Lie algebra $\mathfrak{g}$, does some $A \in \mathcal{A}(\mathfrak{g})$ have a free subalgebra?

Analogous to classifying via GK-dimension, we ask to same using global dimension.

Question 4.5.5. If H is a connected Hopf algebra of global dimesnion up to 4, what are the possible algebra structures on H?

We end with asking the obligatory Noetherian condition.

Question 4.5.6. If H is an affine connected Hopf algebra with finite dimensional $P(H)$, is H Noetherian?

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Daniel Yee

Curriculum Vitae

Education

- **University of Wisconsin-Milwaukee (UWM)** Milwaukee, WI
  PhD in Mathematics (Expected) *2012 - 2017*
  – Thesis Title: Extending Enveloping Algebras via Anti-Cocommutative Elements.
  Advisor: Professor Allen Bell.
- **UWM**
  MS in Mathematics *2010 - 2012*
- **UWM**
  BA in Mathematics *2005 - 2009*
  – Undergraduate Thesis: A Population vs. Environment Model.
  Advisor: Eric Key.

Teaching Experience

- **Graduate Teaching Assistant** UWM
  Vocational *2010 - 2017*
  – Math 240 - Matrices and Applications [Spring 2017]
  – Math 233 - Calculus and Analytical Geometry III [Summer 2016]
  – Math 233 (co-taught) - Calculus and Analytical Geometry III [Summer 2014]
  – Math 232 - Calculus and Analytical Geometry II [Fall 2016, Spring 2016]
  – Math 211- Survey in Calculus and Analytical Geometry [Fall 2013, Spring 2014]
  – Math 116 (with ALEKS) - College Algebra [Fall 2015, Summer 2017]
  – Math 108 (with ALEKS) - Algebraic Literacy II [Spring 2015]
  – Math 105 (with MyMathLab) - Intermediate Algebra [Fall 2012]
  – Math 105 (Online Course) - Intermediate Algebra [Summer 2011, Fall 2011, Spring 2012]
  – Math 103 - Contemporary Applications of Mathematics [Fall 2009, Spring 2010]
  – Math 98 (with ALEKS) - Algebraic Literacy I [Fall 2014]

  Within the last academic year, my students have rated my overall teaching performance as ”excellent” or ”very good.” Cumulative evaluation GPA is 4.1 out of 5.
- **Graduate Student Colloquium Coordinator** UWM
  Program Coordinator *2015 - 2016*
- **Mathematics Tutor** UWM
  Tutor *2009 - 2010*

Languages & Computer Proficiency

- **English**
  Main

- **French** Passed the PhD language requirement at UWM in French
  Secondary
- **Programming** Taken courses at UWM regarding said programming languages
  C++, Java
- **Math Specific Programs**
  Maple, Mathematica, Magma
- **Typesetting**
  TeX, LaTeX

Research Interests

- **Algebra**
  Noncommutating Ring Theory, Hopf Algebras, Quantum Groups, Hopf Actions, Invariant Theory

Selected Talks

- **Generalizations of Universal Enveloping Algebras** April 2017
  Algebra Seminar *UWM*
- **Global Dimension on Connected Hopf Algebras** January 2017
  AMS Contributed Paper Session, Joint Mathematics Meeting *Atlanta, GA*
- **Coalgebras: Another Side of Algebras** October 2016
  Graduate Collloquium *UWM*
- **Examples of Connected Hopf Algebras satisfying the Noetherian or Ore Condition** September 2016
  AMS Sectional Meeting, Bowdoin College *Burnswick, ME*
- **Plücker Coordinates via Stasheff Polytopes** April 2016
  Algebra Seminar *UWM*
- **A Basic Introduction to Locally Nilpotent Derivations** March 2016
  Graduate Colloquium *UWM*
- **The Gelfand-Kirillov Dimension on Algebras and Groups** October 2015
  Graduate Colloquium *UWM*
- **Geometry of Syzygies: Monomial Ideals and Simplicial Complexes** October 2015
  Algebra Seminar *UWM*
- **Tilting Modules** February 2015
  Algebra Seminar *UWM*
- **The Fourier Transform and its Properties** September 2013
  Analysis Seminar *UWM*
- **Introduction to Polynomial Invariance under Finite Group Actions** September 2013
  Algebra Seminar *UWM*

Awards, Grants & Honors

Travel Funds [Algebra Extravaganza Conference] . . . . Summer 2017
Mark Lawrence Teply Award . . . . Spring 2017
AMS Travel Grant [Sectional Meeting, Bowdoin College] . . . . Fall 2016
GAANN Fellowship [UWM Math] . . . . Summer 2015
UWM Graduate Travel Fund [AMS Meeting in Chicago] . . . . Fall 2015
UWM Graduate Travel Fund [Hopf Algebra Workshop, Seattle] . . . . Fall 2014
GAANN Fellowship [UWM Math] . . . . Summer 2013
GAANN Fellowship [UWM Math] . . . . Spring 2013
GAANN Fellowship [UWM Math] . . . . Summer 2012

Publications & Pre-Prints

- A list of publications and pre-prints is available upon request.

References

- **Allen Bell**
  Advisor, Professor *UWM*
  - adbell@uwm.edu
  - (414) 229-4233
- **Bruce Wade**
  Department Chair *UWM*
  - wade@uwm.edu
  - (414) 229-3371
- **Yi Ming Zou**
  Associate Chair, Graduate Program Chair *UWM*
  - ymzou@uwm.edu
  - (414) 229-5110
- **Craig Guilbault**
  Professor *UWM*
  - craigg@uwm.edu
  - (414) 229-4568
- **Ian Musson**
  Professor *UWM*
  - musson@uwm.edu
  - (414) 229-5953
- **Jeb Willenbring**
  Professor *UWM*
  - jw@uwm.edu
  - (414) 229-3371