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Optimal Insurance with Background Risk: An Analysis in the Presence of Moderate Negative Dependence

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OPTIMAL INSURANCE WITH BACKGROUND RISK: AN ANALYSIS IN THE PRESENCE OF MODERATE NEGATIVE DEPENDENCE

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ABSTRACT

OPTIMAL INSURANCE WITH BACKGROUND RISK: AN ANALYSIS IN THE PRESENCE OF MODERATE NEGATIVE DEPENDENCE

by

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As an individual or a corporation, there are various types of risks one faces. For many of these risks, there are insurance policies available for purchase that provide some protection against potential losses. However, there are also risks that are not insurable. These risks remain present as a background factor and affect the insured's final wealth. Consequentially, they have an impact on the optimal insurance for the insurable risk through the dependence structure between the insurable and uninsurable risk.

In this thesis, we take a look at the optimal insurance problem given an insurable risk X and a background risk Y that are partly moderately negative dependent. We will investigate the implications of this dependence structure for the optimal solution to the optimal insurance problem that uses an approach based on [Chi and Wei, 2018]. First, focusing on whether coverage is demanded or not, we later on make assumptions about the utility function of the insured and further specify the form of the dependence structure. These analytic results are followed up by a numerical analysis that has the goal to illustrate the previously obtained results of this thesis, and [Chi and Wei, 2018], for an exponentially distributed risk X, and a Pareto distributed risk X respectively.

Table of Contents

List of Tables

Introduction

The optimal insurance problem can be formulated for various situations, and finding a solution to this problem is of great interest for insurance companies as well as for individuals or corporations. In this thesis, we want to consider a setting that commonly arises in the field of non-life insurance where an individual or a corporation faces two different risks: One being insurable by obtaining an insurable policy from an insurance provider, and the second being uninsurable. Examples for the second risk include the volatility of share returns, inflation, and general economic conditions, as demonstrated by [Huang et al., 2013] and [Doherty and Schlesinger, 1983]. These types of risks are usually not insurable, and can therefore be seen as background risks in situations where the optimal insurance for an insurable risk is sought. Other authors regard this setting as an optimal insurance problem with random initial wealth, which is another approach that we will not further develop though.

When discussing approaches and methods to determine solutions to the optimal insurance problem, one major characteristic of modeling is the dependence structure between the insurable risk and the background risk. As both risks are random variables that represent random losses or gains for the insured party, the questions of whether and how these two pay-offs relate to one another arise. Assuming independence of these two risks might be a tempting approach. However, this often does not reflect the reality of the insured. As an example, consider the following scenario: The owner of a car is legally required to obtain liability car insurance for their vehicle. In addition, comprehensive coverage is available to the insured as an optional feature. In situations where the car owner decides to only purchase liability coverage but no comprehensive coverage, the risk due to claims that are covered by the liability policy is the insured risk, and the risk due to claims that are not covered by the liability coverage is the background risk. In this setting, rather than there being independence between the two risks, there exists a dependence structure that needs to be specified.

With a plethora of possible dependence structures, we want to focus on a scenario where moderate negative dependence prevails for parts of the claim size range of the risk X. In a working paper by Chi and Wei, [Chi and Wei, 2018], the two authors give an optimal solution to the optimal insurance problem in the scenario above. The solution requires some conditions and is of a multi-layer structure, similar to the structure of a stop-loss policy, in which there are two critical values, d_1^* and d_2^* - one, d_1^*, denoting the beginning of the interval of the claim sizes for which there is a linearly increasing payment from the insurance provider to the insured, and the other, d_2^*, denoting the beginning of the interval of the claim sizes for which there is a constant payment made by the insurance provider to the insured.

Based on the results by Chi and Wei, the thesis develops as follows: After introducing the model with its main assumptions, we specify the dependence structure between the insured risk X and the background risk Y. Using these results by Chi and Wei, we are then able to show that if there exists a solution with certain values for d_1^* and d_2^*, these need to be above a certain lower boundary. When it comes to interpreting the values for d_1^* and d_2^*, the question of insurance coverage versus no insurance coverage arises. As we will discuss in chapter three, the first possible solution to think of is that there is no demand for insurance coverage. By the end of this chapter, we will have developed a criterion that helps us to identify those case where there is no demand for insurance coverage. Therefore, in chapter four, we assume a quadratic utility function for the insured as well as the dependence structure to follow a piece-wise linear law. Applying these assumptions yields more specified conditions for d_1^* and d_2^* that can later be used to determine these quantities. In chapters five and six, we furthermore assume the risk X to be exponentially distributed, and Pareto distributed, respectively. For these distribution types, we can obtain analytic results about the conditions, which is followed up by a numerical analysis of d_1^* and d_2^* for various choices for the parameter values. Finally, we are able to connect the question about whether insurance is demanded with the numerical results, as well as use these results to better understand the impact of the individual parameters on the values of d_1^* and d_2^*, the critical numbers in the optimal solution to the optimal insurance problem we consider for this thesis.

A Model for the Background Risk Y

2.1 Description

First, we need to describe the setting that we want to investigate. As an individual or corporation who faces two sources of risks, given some initial wealth w, they would like to reduce their risk exposure by obtaining an insurance policy: The two sources of risks are denoted by X and Y, where the risk X is assumed to be insurable and positive, $\mathbb{P}(X > 0) = 1$, representing a loss for the insured. The background risk Y, however, is not insurable and may be negative, representing a loss or a gain for the insured.

When obtaining the insurance policy, the insured's wealth changes as claims for the risks X and Y occur, due to payments received from the insurance company that are related to the insured risk X, and due to the premium payment made by the insured when concluding the contract. Therefore, we consider the following: The insured's ceded loss function $f(X)$ is the amount that is ceded to an insurer, which yields the residual risk $I(X) = X - f(X)$, the insured's retained loss function, to be the amount the insured retains. To avoid the phenomenon of moral hazard, we assume that one should pay more for a larger realization of the loss, i.e., $f(x)$ and $I(x)$ are both increasing functions. This yields $0 \leq f'(x) \leq 1$ holds almost everywhere, and $f(0) = 0$. Thus, the set of admissible ceded loss functions is given by

$$\mathfrak{A} = \{0 \leq f(x) \leq x : I(x) \text{ and } f(x) \text{ are increasing functions}\}.$$

For the premium payment made by the insured to the insurer, we assume that the insurer is risk-neutral as the premium $\pi(f(X))$ charged for the insurance coverage is determined in accordance with the expected value principle. Hence, $\pi(f(X)) = (1+\rho)\mathbb{E}[f(X)]$ holds for some positive safety loading coefficient ρ.

With this said, the insured's final wealth $W_f(X,Y)$ is of the form

$$W_f(X,Y) = w - Y - X + f(X) - (1+\rho)\mathbb{E}[f(X)]. \tag{1.1}$$

Since it is our objective to maximize the expected utility of the insured's final wealth, we obtain the following optimization problem:

$$\max_{f \in \mathfrak{A}} \mathbb{E}[u(W_f(X,Y))] \tag{1.2}$$

for some utility function of a risk-averse insured for which it holds: $u' > 0$ and $u'' < 0$.

Additionally, we want to state the notation used for a frequently encountered insurance form, the stop-loss insurance:

$$f_d^{sl}(x) = (x-d)_+ = \max\{x-d, 0\} \tag{1.3}$$

Furthermore, we define $\Phi_f(x)$, an expression in the expected marginal utility function, as follows:

$$\Phi_f(x) = \frac{\mathbb{E}[u'(W_f(X,Y))|X > x]}{\mathbb{E}[u'(W_f(X,Y))]}, \quad \text{for} \quad 0 \leq x \leq \text{ess sup } X$$

In this setting, we are now able to infer properties of the optimal solution as well as specify the dependence structure that we want to consider for this thesis.

2.2 Properties of the Optimal Solution

At this point, we want to refer to [Chi and Wei, 2018] and mention two results that have been proven in their working paper.

The first result is in regard to the ceded loss function. Their result suggests that the optimal insurance strategy f^* usually admits a multi-layer structure. The theorem states that the ceded loss function being an optimal solution to problem 1.2 is equivalent to the derivative of the ceded loss function obtaining certain values for certain values of $\Phi_f(x)$. Specifically, the marginal indemnity f'^* takes value of either 0 or 1 except at some critical points.

They further establish the uniqueness of the optimal solution to problem 1.2. That is, once a strategy is verified to be optimal, then it is unique in the sense that any other optimal strategy would produce the same utility.

2.3 Dependence Structure between X and Y

One major feature of modeling the risks is the dependence structure that is assumed to hold between the insurable risk X and the background risk Y. We want to investigate a special class of dependence structures which are represented by $X + Y = m(X)$ where, in the following, X is a continuous random variable and $m(x)$ is continuous and differentiable. This yields the insured's final wealth to be $W_f(X) = w - m(X) + f(X) - (1 + \rho)\mathbb{E}[f(X)]$ and problem 1.2 can be rewritten as

$$\max_{f \in \mathfrak{A}} \mathbb{E}[u(W_f(X))] \tag{3.1}$$

The three major relations, apart from independence, that can hold for X and Y are described in the following:

Positive Dependence: $\mathbf{m'(x) \geq 1}$

An increase in x leads to a greater increase in $m(x)$, meaning when X increases, Y increases as well. Therefore, we have a positive dependence between the insurable risk X and the background risk Y. As the loss caused by the insurable risk X increases, the loss caused by the background risk Y increases as well, which results in a greater overall loss $X+Y = m(X)$. With this said, the stop-loss insurance strategy appears to be a reasonable choice in this scenario as it eliminates the tail risk of X.

Strong Negative Dependence: $\mathbf{m'(x) \leq 0}$

An increase in x leads to a decrease in $m(x)$ meaning that when X increases, Y decreases greater than the increase of X. As a result, we have a strong negative dependence structure between the insurable risk X and the background risk Y. For example, as the loss caused by the insurable risk X increases, the loss caused by the background risk Y decreases greater, and consequentially does not only absorb the additional loss, but it also leads to a decrease in the combined loss. Overall, this means that a greater loss caused by X yields a smaller overall loss $X + Y = m(X)$. One might refer to this as "X becoming completely hedged by Y", and does not requite insurance coverage therefore.

Moderate Negative Dependence: $\mathbf{0 \leq m'(x) \leq 1}$

An increase in x leads to a smaller increase in $m(x)$, meaning that when X increases, Y decreases, but the decrease is smaller than the increase of X. Therefore, we have a moderately negative dependence structure between the insurable risk X and the background risk Y because as the loss caused by the insurable risk X increases, the loss caused by the background risk Y decreases in contrast. However, it is not able to completely absorb the additional loss caused by X, and the combined loss still increases. Overall, this means that a greater loss caused by X yields a greater overall loss $X + Y = m(X)$. This is referred to as "X is partly hedged by Y". This relation turns out to cause some complications when attempting to find an optimal solution for a given risk structure where the two risks display a moderate negative dependence on a certain interval.

In the following analysis, we want to focus on one specific scenario where there is a moderate negative dependence structure present between X and Y.

2.3.1 Special Case

The special case for which we try to solve the optimal insurance problem, given the structure $X+Y=m(X)$ between the two risks, is a mixture of two of the structures we have discussed so far, and can be described as follows: Assume that there exists $x_0 \geq 0$ such that it holds for $X+Y=m(X)$:

- $0 \leq m'(x) \leq 1$ for $0 \leq x \leq x_0$
- $m'(x) \leq 0$ for $x > x_0$

Hence, on the one hand, m is increasing, with slope smaller than 1, and we have a moderate negative dependence for $0 \leq x \leq x_0$. On the other hand, m is decreasing, without any further information about the slope, and we have strong negative dependence for $x > x_0$. In this special case, the optimal solution to problem 1.2 is given by a proposition in [Chi and Wei, 2018]:

Proposition 3.2. *With the dependence structure stated in 2.3.1, the optimal solution to problem 1.2 is*

$$f_3^*(x) = \begin{cases} (m(x) - m(d_1^*))_+ & \text{for } x \leq d_2^* \\ m(d_2^*) - m(d_1^*) & \text{for } x > d_2^* \end{cases} \tag{3.3}$$

if there exist d_1^*, d_2^*, *such that* $0 \leq d_1^* \leq d_2^* \leq x_0$ *and*

$$\begin{cases} \mathbb{E}[u'(W_{f_3^*}(X))|X > d_1^*] = (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] & (1) \\ \mathbb{E}[u'(W_{f_3^*}(X))|X > d_2^*] = (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] & (2) \end{cases} \tag{3.4}$$

By defining the function $E(d) := \mathbb{E}[u'(W_{f_3^*}(X))|X > d]$, the two conditions (1) and (2) in Proposition 3.2 can been seen as $E(d)$ attaining the same value, that is $(1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))]$, for certain d's that are d_1^* and d_2^*. Therefore, the notation $E(d)$ will come up later again in our discussion where we will consider the function $E(d)$ for further analysis.

The following table illustrates the consequences of the assumptions about m and proposition 3.2 for the optimal solution $f_3^*(x)$, the final wealth of the insured $W_{f_3^*}(x)$, and the marginal utility $u'(W_{f_3^*}(x))$.

	$0 \leq x \leq d_1^*$	$d_1^* \leq x \leq d_2^*$	$d_2^* \leq x \leq x_0$	$x \geq x_0$
$f_3^*(x) =$	0	$m(x) - m(d_1^*)$	$m(d_2^*) - m(d_1^*)$	$m(d_2^*) - m(d_1^*)$
monotonicity	constant	increasing with $0 \leq m'(x) \leq 1$	constant	constant
$W_{f_3^*}(x) =$	$w - m(x) - \pi$	$w - m(d_1^*) - \pi$	$w - m(x) - \pi$	$w - m(x) - \pi$
monotonicity	decreasing with $-1 \leq -m'(x) \leq 0$	constant	decreasing with $-1 \leq -m'(x) \leq 0$	increasing with $-m'(x) \geq 0$
$u'(W_{f_3^*}(x)) =$	-	-	-	-
monotonicity	increasing	constant	increasing	decreasing

Table 2.1: Functions involved in the optimal insurance problem

2.3.2 Lower Boundary for d

Assuming that there exists a d^* that fulfills the conditions in Proposition 3.2, we can show an additional property of d^*: There exists a lower boundary for d^*, meaning d^* is not only greater than zero, but also greater than $S_X^{-1}(\frac{1}{1+\rho})$.

Proposition 3.5. *Given the dependence structure stated in 2.3.1: If there exists* d^* *such that* $0 \leq d^* \leq x_0$ *and* $\mathbb{E}[u'(W_{f_3^*}(X))|X > d^*] = (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))]$ *holds, then* $d^* \geq S_X^{-1}\left(\frac{1}{1+\rho}\right)$ *holds.*

Proof:

Using conditional expectation, it holds:

$$E(d^*) \;=\; \mathbb{E}[u'(W_{f_3^*}(X))|X > d^*] = \frac{1}{\mathbb{P}(X > d^*)}\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}]$$

and

$$\begin{aligned}
& (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] \\
= \;& (1+\rho)\Big(\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{d^* \geq X > 0\}] + \mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}]\Big) \\
= \;& (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{d^* \geq X > 0\}] + (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}].
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}[u'(W_{f_3^*}(X))|X > d^*] &= (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] \\
\frac{1}{\mathbb{P}(X > d^*)}\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}] &= (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{d^* \geq X > 0\}] \\
&\quad + (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}] \\
\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}] &= \mathbb{P}(X > d^*)(1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{d^* \geq X > 0\}] \\
&\quad + \mathbb{P}(X > d^*)(1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}]
\end{aligned}$$

$$
\begin{aligned}
0 &= \mathbb{P}(X > d^*)(1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{d^* \geq X > 0\}] \\
&\quad + \Big(\mathbb{P}(X > d^*)(1+\rho) - 1\Big)\mathbb{E}[u'(W_{f_3^*}(X))\mathbb{1}\{X > d^*\}]
\end{aligned}
$$

Since u' is positive, both expectations are greater than or equal to 0. Furthermore, with $\mathbb{P}(X > d^*) \geq 0$ and $\rho \geq 0$, we can infer that the first summand in the last equation is non-negative. Therefore, the second summand needs to be non-positive. Since the expectation is non-negative, the factor needs to be non-positive. Thus, it needs to hold $\mathbb{P}(X > d^*)(1+\rho) \leq 1$. With $S_X^{-1}(x)$ being decreasing, we obtain the following inequality:

$$
\begin{aligned}
\mathbb{P}(X > d^*)(1+\rho) &\leq 1 \\
\mathbb{P}(X > d^*) &\leq \frac{1}{1+\rho} \\
S_X(d^*) &\leq \frac{1}{1+\rho} \\
d^* &\geq S_X^{-1}\Big(\frac{1}{1+\rho}\Big)
\end{aligned}
$$

This completes the proof. Therefore, if there exists a d^* that fulfills the equations in Proposition 3.2, we know that $d^* \geq S_X^{-1}\left(\frac{1}{1+\rho}\right)$ needs to hold.

In the next chapter, we investigate the case where the optimal solution to problem 1.2 is no coverage, i.e., the insured does not demand any insurance coverage.

No Insurance Demand

The first question that arises when consider the optimal insurance problem is whether the insured will demand coverage for the risk X, or not. If we can show that the optimal solution to problem II.1.2 is no coverage, we are finished and we have found f_3^* as $f_3^* \equiv 0$. According to [Chi and Wei, 2018], we need to check whether the following holds for all d:

$$\Phi_{f_3^*}(d) = \frac{\mathbb{E}[u'(W_{f_3^*}(X))|X > d]}{\mathbb{E}[u'(W_{f_3^*}(X))]} \leq 1 + \rho$$

$$\Leftrightarrow \quad \mathbb{E}[u'(W_{f_3^*}(X))|X > d] \leq (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))]$$

On the other hand, if we can show that this inequality doesn't hold, i.e., there exist d such that $\mathbb{E}[u'(W_{f_3^*}(X))|X > d] > (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))]$, we know that "no coverage" is not the optimal solution. Hence, purchasing an insurance policy with coverage is recommended and we need to conduct further research about the nature of this insurance coverage. The investigation will deal with the condition $0 \leq d_1^* \leq d_2^* \leq x_0$ for the d's as stated in Proposition II.3.2.

3.1 Setting

Since there is no coverage, $f_3^* \equiv 0$, there is no premium payment, and the final wealth of the insured $W_{f_3^*}$ becomes $W_{f_3^*}(X) = w - m(X)$, which is free of d.

The following table illustrates the consequences of these assumptions for the optimal solution $f_3^*(x)$, the final wealth of the insured $W_{f_3^*}(x)$, and the marginal utility $u'(W_{f_3^*}(x))$, in a shortened version. It is followed by a more detailed version of the table that gives insight on d_1^* and d_2^*, results we want to return to when analyzing the numerical results obtained in chapters 4.2.2 and 5.2.4.

	$0 \leq x \leq x_0$	$x > x_0$
$f_3^*(x) =$	0	0
monotonicity	constant	constant
$W_{f_3^*}(x) =$	$w - m(x)$	$w - m(x)$
monotonicity	decreasing with $-1 \leq -m'(x) \leq 0$	increasing with $-m'(x) \geq 0$
$u'(W_{f_3^*}(x)) =$	-	-
monotonicity	increasing	decreasing

Table 3.2: Simplified version of the functions involved in the optimal insurance problem when there is no insurance demand

	$0 \leq x \leq d_1^*$	$d_1^* < x \leq d_2^*$	$d_2^* < x \leq x_0$	$x > x_0$
$f_3^*(x) =$	0	0	0	0
monotonicity	constant	constant	constant	constant
$W_{f_3^*}(x) =$	$w - m(x)$	$w - m(x)$	$w - m(x)$	$w - m(x)$
monotonicity	decreasing with $-1 \leq -m'(x) \leq 0$	decreasing with $-1 \leq -m'(x) \leq 0$	decreasing with $-1 \leq -m'(x) \leq 0$	increasing with $-m'(x) \geq 0$
$u'(W_{f_3^*}(x)) =$	-	-	-	-
monotonicity	increasing	increasing	increasing	decreasing

Table 3.3: Functions involved in the optimal insurance problem when there is no insurance demand

Recalling $E(d) = \mathbb{E}[u'(W_{f_3^*}(X))|X > d]$, we now want to investigate whether the inequality below holds for all d.

$$\Phi_{f_3^*}(d) = \frac{\mathbb{E}[u'(W_{f_3^*}(X))|X > d]}{\mathbb{E}[u'(W_{f_3^*}(X))]} \leq 1 + \rho$$

$$\Leftrightarrow \quad \mathbb{E}[u'(W_{f_3^*}(X))|X > d] \leq (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))]$$

3.2 Investigation of $E(d)$

Let $g(X) := u'(W_{f_3^*}(X))$, and with the above said, it follows $g(X) = u'(w - m(X))$.
In this case, $E(d)$ becomes $E(d) = \mathbb{E}[u'(W_{f_3^*}(X))|X > d] = \mathbb{E}[g(X)|X > d]$. Observe that the variable d only appears in the condition. For analyzing the monotonicity and extreme points of $E(d)$, one approach is to consider the first derivative $E'(d) = \frac{\partial}{\partial d}E(d)$

3.2.1 General Results

Lemma 2.1. *The function $E(d) = \mathbb{E}[u'(W_{f_3^*}(X))|X > d]$ has the derivative*

$$E'(d) = \frac{\partial}{\partial d}E(d) = \frac{f_X(d)}{\mathbb{P}(X > d)}\Big[\mathbb{E}[g(X)|X > d] - g(d)\Big]$$

with $g(X) = u'(W_{f_3^*}(X))$.

Proof:

Finding the first derivative yields:

$$\begin{aligned} E'(d) &= \frac{\partial}{\partial d}\mathbb{E}[g(X)|X > d] \\ &= \frac{\partial}{\partial d}\Big[\frac{1}{\mathbb{P}(X > d)}\mathbb{E}[g(X)\mathbb{1}\{X > d\}]\Big] \end{aligned}$$

$$= \frac{\partial}{\partial d}\Big[\frac{1}{\mathbb{P}(X > d)} \int\limits_d^\infty g(x) f_X(x) dx\Big]$$

$$= \frac{\partial}{\partial d}\Big[\Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1} \int\limits_d^\infty g(x) f_X(x) dx\Big]$$

Using the product rule and the rules for differentiation for parameter integrals, we obtain:

$$\begin{aligned}
E'(d) &= \frac{\partial}{\partial d}\Big[\Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1}\Big] \int\limits_d^\infty g(x) f_X(x) dx \\
&\quad + \frac{\partial}{\partial d}\Big[\int\limits_d^\infty g(x) f_X(x) dx\Big]\Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1} \\
&= -\Big(\int\limits_d^\infty f_X(x)dx\Big)^{-2} \frac{\partial}{\partial d}\Big[\int\limits_d^\infty f_X(x)dx\Big] \int\limits_d^\infty g(x) f_X(x) dx \\
&\quad - g(d) f_X(d) \Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1} \\
&= -\Big(\int\limits_d^\infty f_X(x)dx\Big)^{-2} (-f_X(d)) \int\limits_d^\infty g(x) f_X(x) dx \\
&\quad - g(d) f_X(d) \Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1} \\
&= \Big(\int\limits_d^\infty f_X(x)dx\Big)^{-2} f_X(d) \int\limits_d^\infty g(x) f_X(x) dx \\
&\quad - g(d) f_X(d) \Big(\int\limits_d^\infty f_X(x)dx\Big)^{-1}
\end{aligned}$$

$$= \Big(\int_d^\infty f_X(x)dx\Big)^{-1} f_X(d)\Big[\Big(\int_d^\infty f_X(x)dx\Big)^{-1}\int_d^\infty g(x)f_X(x)dx - g(d)\Big]$$

$$= \frac{f_X(d)}{\mathbb{P}(X>d)}\Big[\frac{1}{\mathbb{P}(X>d)}\int_d^\infty g(x)f_X(x)dx - g(d)\Big]$$

$$= \frac{f_X(d)}{\mathbb{P}(X>d)}\Big[\frac{1}{\mathbb{P}(X>d)}\mathbb{E}[g(X)\mathbb{1}\{X>d\}] - g(d)\Big]$$

$$= \frac{f_X(d)}{\mathbb{P}(X>d)}\Big[\mathbb{E}[g(X)|X>d] - g(d)\Big]$$

This finishes the proof.

Since we are interested in the monotonicity behavior of $E(d)$, observe that $f_X(d)$ and $\mathbb{P}(X > d)$ are both non-negative, and therefore, the further analysis focuses on the third factor. Due to the change in monotonicity of $m(X)$ at x_0, we want to consider the following two separate cases.

3.2.2 Results for $d > x_0$

Lemma 2.2. *Given the dependence structure stated in 2.3.1: $E(d)$ is decreasing in d for $d > x_0$.*

Proof:

With Lemma 2.1, we can infer for $d \geq x_0$: From the fact that $g(x) = u'(W_{f_3^*}(x))$ is decreasing for $x \geq x_0$, see table 3.1, we obtain the following inequality for the third factor in the derivative $E'(d)$:

$$
\begin{aligned}
&\mathbb{E}[g(X)|X > d] - g(d) \\
&= \frac{1}{\mathbb{P}(X > d)} \int_d^\infty g(x) f_X(x) dx - g(d) \\
&\leq \frac{1}{\mathbb{P}(X > d)} \int_d^\infty g(d) f_X(x) dx - g(d) \\
&= \frac{1}{\mathbb{P}(X > d)} g(d) \int_d^\infty f_X(x) dx - g(d) \\
&= \frac{1}{\mathbb{P}(X > d)} g(d) \mathbb{P}(X > d) - g(d) \\
&= g(d) - g(d) = 0
\end{aligned}
$$

In short, $\mathbb{E}[g(X)|X > d] - g(d) \leq 0$. Since the other two factors in the product are non-negative, we know that $E'(d)$ is non-positive for $d \geq x_0$, which means that $E(d)$ is decreasing in d on $[x_0, \infty)$. We can further infer that the maximum of $E(d)$ needs to be to the left of x_0. This completes the proof.

3.2.3 Results for $d \leq x_0$

Starting off with $E(0) = \mathbb{E}[g(X)|X > 0] = \mathbb{E}[g(X)]$, assuming that $\mathbb{P}(X > 0) = 1$, we encounter two subcases that need to be considered separately:

Subcase: $\mathbf{E}[\mathbf{g}(\mathbf{X})] \leq \mathbf{g}(\mathbf{0})$

Lemma 2.3. *Given the dependence structure stated in 2.3.1: If $E[g(X)] \leq g(0)$ $(*)$ holds, this implies $E(d) = \mathbb{E}[g(X)|X > d] \leq \mathbb{E}[g(X)]$ for all $d \leq x_0$.*

Proof:

For the proof, we need the fact that $g(0) \leq g(x)$ holds for all $x \in [0, x_0]$. This is due to the monotonicity of m(x): Since $0 \leq m'(x) \leq 1$ holds for $x \in [0, x_0]$ (see table), we know that $g(x) = u'(w - m(x))$ is increasing on $[0, x_0]$. Hence, $g(0) \leq g(x)$ holds for all $x \in [0, x_0]$ $(**)$

First, we use some general properties of the expected value and conditional expectation in order to express $E(d) = \mathbb{E}[g(X)|X > d]$ in another form:

$$\begin{aligned}
\mathbb{E}[g(X)] &= \mathbb{E}[g(X)(\mathbb{1}\{X \leq d\} + \mathbb{1}\{X > d\})] \\
&= \mathbb{E}[g(X)\mathbb{1}\{X \leq d\}] + \mathbb{E}[g(X)\mathbb{1}\{X > d\}] \\
&= \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d] + \mathbb{P}(X > d)\mathbb{E}[g(X)|X > d]
\end{aligned}$$

This yields

$$\mathbb{E}[g(X)|X > d] = \frac{\mathbb{E}[g(X)] - \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d]}{\mathbb{P}(X > d)},$$

and we can apply the previously mentioned properties to obtain:

$$\begin{aligned}
\mathbb{E}[g(X)|X > d] &= \frac{\mathbb{E}[g(X)] - \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d]}{\mathbb{P}(X > d)} \\
&= \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(X)\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)} \\
&\overset{(**)}{\leq} \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(0)\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{E}[g(X)] - g(0)\mathbb{E}[\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)} \\
&= \frac{\mathbb{E}[g(X)] - g(0)\mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&\overset{(*)}{\leq} \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(X)]\mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]\frac{1 - \mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]\frac{\mathbb{P}(X > d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]
\end{aligned}
$$

Hence, $E(d) = \mathbb{E}[g(X)|X > d] \leq \mathbb{E}[g(X)]$ for all $d \leq x_0$. This completes the proof.

Subcase: $\mathbf{E[g(X)] > g(0)}$

Lemma 2.4. *Given the dependence structure stated in 2.3.1: If $E[g(X)] > g(0)$ $(*)$ holds, this implies that there exists $d \in [0, x_0]$ such that $E(d) = \mathbb{E}[g(X)|X > d] > \mathbb{E}[g(X)]$ holds.*

Proof:

Now, consider the following: In order for $\mathbb{E}[g(X)] > g(0)$ to hold, the continuity of $g(x)$ implies that there exists $z \in [0, x_0]$ such that $g(z) = \mathbb{E}[g(X)]$. We can state the interval for z due to the following reasoning: Since $g(x)$ is increasing on $[0, x_0]$ and decreasing on $[x_0, \infty)$, the maximum of $g(x)$ is attained on $[0, x_0]$, and thus, there exists $z \in [0, x_0]$ with $g(z) = \mathbb{E}[g(X)]$. Due to the monotonicity of $g(x)$, it holds $g(x) \leq g(z)$ for all $x \in [0, z]$ $(**)$

Taking these observations into consideration, we obtain the following for all $d \in [0, z]$: First, we use some general properties of the expected value and conditional expectation in order to express $E(d) = \mathbb{E}[g(X)|X > d]$ in another form.

$$\begin{aligned}
\mathbb{E}[g(X)] &= \mathbb{E}[g(X)(\mathbb{1}\{X \leq d\} + \mathbb{1}\{X > d\})] \\
&= \mathbb{E}[g(X)\mathbb{1}\{X \leq d\}] + \mathbb{E}[g(X)\mathbb{1}\{X > d\}] \\
&= \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d] + \mathbb{P}(X > d)\mathbb{E}[g(X)|X > d]
\end{aligned}$$

This yields

$$\mathbb{E}[g(X)|X > d] = \frac{\mathbb{E}[g(X)] - \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d]}{\mathbb{P}(X > d)},$$

and we can apply the previously mentioned observations to obtain:

$$\begin{aligned}
\mathbb{E}[g(X)|X > d] &= \frac{\mathbb{E}[g(X)] - \mathbb{P}(X \leq d)\mathbb{E}[g(X)|X \leq d]}{\mathbb{P}(X > d)} \\
&= \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(X)\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)} \\
&\overset{(**)}{\geq} \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(z)\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)} \\
&= \frac{\mathbb{E}[g(X)] - g(z)\mathbb{E}[\mathbb{1}\{X \leq d\}]}{\mathbb{P}(X > d)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{E}[g(X)] - g(z)\mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&\overset{(*)}{>} \frac{\mathbb{E}[g(X)] - \mathbb{E}[g(X)]\mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]\frac{1 - \mathbb{P}(X \leq d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]\frac{\mathbb{P}(X > d)}{\mathbb{P}(X > d)} \\
&= \mathbb{E}[g(X)]
\end{aligned}
$$

Hence, $E(d) = \mathbb{E}[g(X)|X > d] > \mathbb{E}[g(X)]$ for all $d \in [0, z]$. Since $z \in [0, x_0]$, this proves the existence of some $d \in [0, x_0]$ such that $E(d) = \mathbb{E}[g(X)|X > d] > \mathbb{E}[g(X)]$ holds, and the proof is complete.

3.3 Inference about the Optimal Solution

With the lemmas from the previous sections, we are able to draw conclusions about whether insurance is demanded.

3.3.1 No Insurance Demand

The following theorem states a condition that implies the optimal solution to be no insurance coverage.

Theorem 3.1. *Given the dependence structure stated in 2.3.1: If* $E[g(X)] \leq g(0)$*, the optimal solution to problem II.1.2 is no insurance coverage, i.e.,* $f_3^* \equiv 0$.

Proof:

By lemma 2.3, $E(d) = \mathbb{E}[g(X)|X > d] \leq \mathbb{E}[g(X)] \leq (1+\rho)\mathbb{E}[g(X)]$ holds for all $d \in [0, x_0]$. By lemma 2.2, $E(d)$ is decreasing on $d \in [x_0, \infty)$. Hence, $E(d) = \mathbb{E}[g(X)|X > d] \leq \mathbb{E}[g(X)] \leq (1+\rho)\mathbb{E}[g(X)]$ for all $d \geq 0$. This is exactly what we wanted to show. Therefore, it holds

$$\Phi_{f_3^*}(d) = \frac{\mathbb{E}[u'(W_{f_3^*}(X))|X > d]}{\mathbb{E}[u'(W_{f_3^*}(X))]} \leq 1 + \rho$$

for all $d \geq 0$, and [Chi and Wei, 2018] implies that the optimal solution to problem II.1.2 is no insurance demand, i.e., $f_3^* \equiv 0$. This finishes the proof.

3.3.2 Insurance Demand

The following theorem states a condition that implies the optimal solution to be insurance coverage.

Theorem 3.2. *Given the dependence structure stated in 2.3.1: If $E[g(X)] > g(0)$, the optimal solution to problem II.1.2 is insurance coverage.*

Proof:

Assuming there is no insurance contract concluded, we can set $\pi(f(X)) = 0$.
By lemma 2.4, there exists some d for which $\mathbb{E}[g(X)|X > d] > \mathbb{E}[g(X)] > (1+0)\mathbb{E}[g(X)]$ holds. Therefore, it holds

$$\Phi_{f_3^*}(d) = \frac{\mathbb{E}[u'(W_{f_3^*}(X))|X > d]}{\mathbb{E}[u'(W_{f_3^*}(X))]} > 1 + \rho$$

for some d, which contradicts the properties that need to hold in order for "no coverage" to be the optimal solution to problem II.1.2. Hence, "no coverage" $f_3^* \equiv 0$ is not the optimal solution, and the optimal insurance needs to be insurance coverage. This finishes the proof.

Insurance Demand

In this chapter, we want to investigate the case where there is insurance coverage for the risk X. Furthermore, we want certain assumptions to hold, for the utility function of the insured, as well as for the function type of $m(X)$ which describes the dependence structure between the insurable risk X and the background risk Y.

4.1 Quadratic Utility Function

To begin with, we want to specify the type of utility function. We assume the insured has assessed their final wealth according to a quadratic utility function. For this type of a utility function, we take a look at the following preliminaries first.

4.1.1 Preliminaries

The quadratic utility function used should have the parametric representation $u(\xi)$ with $= -(\eta - \xi)^2$ holding for $\xi \leq \eta$, with an appropriate choice of η. Since we need $u' > 0$ to hold for all ξ, and we only consider the half of the parabola that is to the left of the vertex, η needs to be chosen large enough. This becomes especially relevant when we consider a certain distribution for the risk X. We might need to adjust the distribution to keep the final wealth of the insured bounded allowing for the value of η to be finite. In general, this assumption is reasonable as in application, the insured's final wealth is bounded from above due to natural economic restrictions.

For the derivative $u'(\xi)$, it holds $u'(\xi) = -2(\eta-\xi)(-1) = 2(\eta-\xi)$, and therefore, plugging in $W_{f_3^*}(x)$ as the argument, we obtain

$$
\begin{aligned}
& u'(W_{f_3^*}(x)) = 2\Big(\eta - W_{f_3^*}(x)\Big) \\
= \;& 2\Big(\eta - (w - m(x) + f_3^*(x) - (1+\rho)\mathbb{E}[f_3^*(X)])\Big) \\
= \;& 2\Big(\eta - w + m(x) - f_3^*(x) + (1+\rho)\mathbb{E}[f_3^*(X)]\Big)
\end{aligned}
$$

The quadratic utility function is convenient in this case as its linear structure allows the marginal utility to be split up into several parts that can then be analyzed individually.

4.1.2 Application

Lemma 1.1. *Given the dependence structure stated in 2.3.1, assuming a quadratic utility function of the form $u(\xi) = -(\eta-\xi)^2$, the two equations stated in Proposition II.3.2 are equivalent to*

$$
\begin{cases}
& (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] - \mathbb{E}[f_{d_1,d_2}(X)|X > d_1] \\
= & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \\
\\
& (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\
= & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)]
\end{cases}
$$

Proof:

The equations in proposition II.3.2 are as follows:

$$
\begin{cases}
\mathbb{E}[u'(W_{f_3^*}(X))|X > d_1^*] = (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] & (1) \\
\mathbb{E}[u'(W_{f_3^*}(X))|X > d_2^*] = (1+\rho)\mathbb{E}[u'(W_{f_3^*}(X))] & (2)
\end{cases}
$$

Assuming the existence of d_1^* and d_2^*, we can try to solve this system of equations to determine the values of d_1^* and d_2^*. The optimal solution f_3^* then depends on the two variables d_1 and d_2, which is why we denote the ceded function as f_{d_1,d_2} in the following. With the assumption of a quadratic utility function, this yields

$$\begin{cases} \begin{aligned} & \mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)|X > d_1] \\ = \; & (1+\rho)\mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)] \quad (1) \\ \\ & \mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)|X > d_2] \\ = \; & (1+\rho)\mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)] \quad (2) \end{aligned} \end{cases}$$

If we furthermore assume the optimal solution to be given in the form of

$$f_{d_1,d_2}(x) = \begin{cases} (m(x) - m(d_1))_+ & \text{for } x \le d_2 \\ m(d_2) - m(d_1) & \text{for } x > d_2 \end{cases}$$

the two conditions can be simplified analyzing each part of the equation individually.

Starting off with $\mathbb{E}[f_{d_1,d_2}(X)]$, we observe that $f_{d_1,d_2}(x) = 0$ for $x \in (0, d_1]$ since m is increasing on $[0, x_0]$. Using conditional expectation, the first expectation becomes:

$$\begin{aligned} & \mathbb{E}[f_{d_1,d_2}(X)] \\ = \; & \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{d_1 \ge X > 0\}] + \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{d_2 \ge X > d_1\}] + \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_2\}] \\ = \; & \mathbb{E}[(m(X) - m(d_1))\mathbb{1}\{d_2 \ge X > d_1\}] + \mathbb{E}[(m(d_2) - m(d_1))\mathbb{1}\{X > d_2\}] \\ = \; & \mathbb{E}[(m(X) - m(d_1))\mathbb{1}\{d_2 \ge X > d_1\}] + (m(d_2) - m(d_1))\mathbb{E}[\mathbb{1}\{X > d_2\}] \\ = \; & \mathbb{E}[(m(X) - m(d_1))\mathbb{1}\{d_2 \ge X > d_1\}] + (m(d_2) - m(d_1))\mathbb{P}(X > d_2) \end{aligned}$$

for which it further holds:

$$
\begin{aligned}
& \mathbb{E}[f_{d_1,d_2}(X)] \\
= \; & \mathbb{E}[(m(X) - m(d_1))\mathbb{1}\{d_2 \geq X > d_1\}] + (m(d_2) - m(d_1))\mathbb{P}(X > d_2) \\
= \; & \mathbb{E}[m(X)\mathbb{1}\{d_2 \geq X > d_1\}] - \mathbb{E}[m(d_1)\mathbb{1}\{d_2 \geq X > d_1\}] \\
& + m(d_2)\mathbb{P}(X > d_2) - m(d_1))\mathbb{P}(X > d_2) \\
= \; & \mathbb{E}[m(X)\mathbb{1}\{d_2 \geq X > d_1\}] - m(d_1)\mathbb{E}[\mathbb{1}\{d_2 \geq X > d_1\}] \\
& + m(d_2)\mathbb{P}(X > d_2) - m(d_1))\mathbb{P}(X > d_2) \\
= \; & \mathbb{E}[m(X)\mathbb{1}\{d_2 \geq X > d_1\}] - m(d_1)\mathbb{P}(d_2 \geq X > d_1) \\
& + m(d_2)\mathbb{P}(X > d_2) - m(d_1))\mathbb{P}(X > d_2) \\
= \; & \mathbb{E}[m(X)\mathbb{1}\{d_2 \geq X > d_1\}] - m(d_1)\mathbb{P}(X > d_1) + m(d_2)\mathbb{P}(X > d_2)
\end{aligned}
$$

The two conditional expectations become:

$$
\begin{aligned}
\mathbb{E}[f_{d_1,d_2}(X)|X > d_1] &= \frac{1}{\mathbb{P}(X > d_1)}\mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_1\}] \\
&= \frac{1}{\mathbb{P}(X > d_1)}\Big(0 + \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_1\}]\Big) \\
&= \frac{1}{\mathbb{P}(X > d_1)}\Big(\mathbb{E}[0\mathbb{1}\{d_1 \geq X > 0\}] + \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_1\}]\Big) \\
&= \frac{1}{\mathbb{P}(X > d_1)}\Big(\mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{d_1 \geq X > 0\}] + \mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_1\}]\Big) \\
&= \frac{1}{\mathbb{P}(X > d_1)}\mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > 0\}] \\
&= \frac{1}{\mathbb{P}(X > d_1)}\mathbb{E}[f_{d_1,d_2}(X)]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[f_{d_1,d_2}(X)|X > d_2] &= \frac{1}{\mathbb{P}(X > d_2)}\mathbb{E}[f_{d_1,d_2}(X)\mathbb{1}\{X > d_2\}] \\
&= \frac{1}{\mathbb{P}(X > d_2)}\mathbb{E}[(m(d_2) - m(d_1))\mathbb{1}\{X > d_2\}] \\
&= \frac{1}{\mathbb{P}(X > d_2)}(m(d_2) - m(d_1))\mathbb{E}[\mathbb{1}\{X > d_2\}] \\
&= \frac{1}{\mathbb{P}(X > d_2)}(m(d_2) - m(d_1))\mathbb{P}(X > d_2) \\
&= m(d_2) - m(d_1)
\end{aligned}
$$

Returning to the two equations, it holds for the right-hand side:

$$
\begin{aligned}
&\mathbb{E}\Big[2(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)])\Big] \\
= \;& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}\Big[m(X) - f_{d_1,d_2}(X)\Big] \\
= \;& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)] - 2\mathbb{E}[f_{d_1,d_2}(X)] \\
= \;& 2(\eta - w) + 2(1+\rho-1)\mathbb{E}[f_{d_1,d_2}(X)] + 2\mathbb{E}[m(X)] \\
= \;& 2(\eta - w) + 2\rho\mathbb{E}[f_{d_1,d_2}(X)] + 2\mathbb{E}[m(X)] \\
= \;& 2(\eta - w + \mathbb{E}[m(X)]) + 2\rho\mathbb{E}[f_{d_1,d_2}(X)]
\end{aligned}
$$

Similarly, we obtain for the left-hand side with d_1:

$$
\begin{aligned}
&\mathbb{E}\Big[2(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)])|X > d_1\Big] \\
= \;& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}\Big[m(X) - f_{d_1,d_2}(X)|X > d_1\Big] \\
= \;& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_1] - 2\mathbb{E}[f_{d_1,d_2}(X)|X > d_1]
\end{aligned}
$$

which becomes the following, by using the results above:

$$
\begin{aligned}
& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_1] - 2\mathbb{E}[f_{d_1,d_2}(X)|X > d_1] \\
= \; & 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_1] - 2\frac{1}{\mathbb{P}(X > d_1)}\mathbb{E}[f_{d_1,d_2}(X)] \\
= \; & 2(\eta - w) + 2\Big((1+\rho - \frac{1}{\mathbb{P}(X > d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1]\Big)
\end{aligned}
$$

Similarly, we obtain for the left-hand side with d_2:

$$
\begin{aligned}
& \mathbb{E}\Big[2(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)])|X > d_2\Big] \\
= \; & 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}\Big[m(X) - f_{d_1,d_2}(X)|X > d_2\Big] \\
= \; & 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_2] - 2\mathbb{E}[f_{d_1,d_2}(X)|X > d_2]
\end{aligned}
$$

which becomes the following, by using the results above:

$$
\begin{aligned}
& 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_2] - 2\mathbb{E}[f_{d_1,d_2}(X)|X > d_2] \\
= \; & 2(\eta - w + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]) + 2\mathbb{E}[m(X)|X > d_2] - 2(m(d_2) - m(d_1)) \\
= \; & 2(\eta - w) + 2\Big((1+\rho)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1))\Big)
\end{aligned}
$$

This finishes the individual analysis. Putting these identities together turns the equations

(1) and (2)

$$\begin{cases} & \mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)|X > d_1] \\ = & (1+\rho)\mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)] \quad (1) \\ \\ & \mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)|X > d_2] \\ = & (1+\rho)\mathbb{E}[2\Big(\eta - w + m(X) - f_{d_1,d_2}(X) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)]\Big)] \quad (2) \end{cases}$$

into the following conditions:

$$\begin{cases} & 2(\eta - w) + 2\Big((1+\rho - \frac{1}{\mathbb{P}(X>d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1]\Big) \\ = & (1+\rho)\Big(2(\eta - w + \mathbb{E}[m(X)]) + 2\rho\mathbb{E}[f_{d_1,d_2}(X)]\Big) \\ \\ & 2(\eta - w) + 2\Big((1+\rho)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1))\Big) \\ = & (1+\rho)\Big(2(\eta - w + \mathbb{E}[m(X)]) + 2\rho\mathbb{E}[f_{d_1,d_2}(X)]\Big) \end{cases}$$

These can be further simplified by dividing by 2, and rearranging terms, to:

$$\begin{cases} & (\eta - w) + (1+\rho - \frac{1}{\mathbb{P}(X>d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] \\ = & (1+\rho)\Big((\eta - w + \mathbb{E}[m(X)]) + \rho\mathbb{E}[f_{d_1,d_2}(X)]\Big) \\ \\ & (\eta - w) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\ = & (1+\rho)\Big((\eta - w + \mathbb{E}[m(X)]) + \rho\mathbb{E}[f_{d_1,d_2}(X)]\Big) \end{cases}$$

$$\begin{cases} & (\eta - w) + (1 + \rho - \frac{1}{\mathbb{P}(X>d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] \\ = & (1+\rho)(\eta - w) + (1+\rho)\mathbb{E}[m(X)]) + (1+\rho)\rho\mathbb{E}[f_{d_1,d_2}(X)] \\ & \\ & (\eta - w) + (1+\rho)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\ = & (1+\rho)(\eta - w) + (1+\rho)\mathbb{E}[m(X)]) + (1+\rho)\rho\mathbb{E}[f_{d_1,d_2}(X)] \end{cases}$$

$$\begin{cases} & (1 + \rho - (1+\rho)\rho - \frac{1}{\mathbb{P}(X>d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \\ & \\ & (1 + \rho - (1+\rho)\rho)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \end{cases}$$

$$\begin{cases} & (1 - \rho^2 - \frac{1}{\mathbb{P}(X>d_1)})\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \\ & \\ & (1 - \rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \end{cases}$$

which can also be written as:

$$\begin{cases} & (1 - \rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] - \mathbb{E}[f_{d_1,d_2}(X)|X > d_1] \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \\ & \\ & (1 - \rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\ = & \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \end{cases}$$

4.2 Piece-wise Linear Function m

We assume m to be a piece-wise linear function in order to simplify the discussion and allow for more inference.

4.2.1 Preliminaries

The function m should display the following structure:

$$m(x) = \begin{cases} m_1 x & \text{for } x \leq x_0 \\ m_1 x_0 + m_2(x - x_0) = (m_1 - m_2)x_0 + m_2 x & \text{for } x > x_0 \end{cases}$$

with $0 \leq m_1 \leq 1$, since $0 \leq m'(x) \leq 1$ should hold for $0 \leq x \leq x_0$, and $m_2 \leq 0$, since $m'(x) \leq 0$ should hold for $x > x_0$.

4.2.2 Application

Lemma 2.1. *Given the dependence structure stated in 2.3.1, assuming a quadratic utility function of the form $u(\xi) = -(\eta - \xi)^2$, and a linear structure of m with*

$$m(x) = \begin{cases} m_1 x & \textit{for } x \leq x_0 \\ m_1 x_0 + m_2(x - x_0) = (m_1 - m_2)x_0 + m_2 x & \textit{for } x > x_0, \end{cases}$$

the two equations stated in Lemma 1.1 are equivalent to

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \\ \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ & -m_1(d_2-d_1) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]. \end{cases}$$

Proof:

With this additional assumption about the structure of m, we want to simplify the equations below:

$$\begin{cases} & (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)]+\mathbb{E}[m(X)|X>d_1]-\mathbb{E}[f_{d_1,d_2}(X)|X>d_1] \\ = & \rho(\eta-w)+(1+\rho)\mathbb{E}[m(X)] \\ \\ & (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)]+\mathbb{E}[m(X)|X>d_2]-(m(d_2)-m(d_1)) \\ = & \rho(\eta-w)+(1+\rho)\mathbb{E}[m(X)] \end{cases}$$

Considering each expectation individually, we obtain:

For $\mathbb{E}[f_{d_1,d_2}(X)]$, it follows:

$$\begin{aligned}
\mathbb{E}[f_{d_1,d_2}(X)] &= \mathbb{E}[(m(X)-m(d_1))\mathbb{1}\{d_2 \geq X > d_1\}] + (m(d_2)-m(d_1))\mathbb{P}(X > d_2) \\
&= \mathbb{E}[m(X)\mathbb{1}\{d_2 \geq X > d_1\}] - m(d_1)\mathbb{P}(X > d_1) + m(d_2)\mathbb{P}(X > d_2) \\
&= \mathbb{E}[m_1 X \mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2) \\
&= m_1 \mathbb{E}[X \mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)
\end{aligned}$$

For $\mathbb{E}[f_{d_1,d_2}(X)|X > d_1]$, we obtain:

$$\begin{aligned}
&\mathbb{E}[f_{d_1,d_2}(X)|X > d_1] \\
= \ &\frac{1}{\mathbb{P}(X > d_1)}\mathbb{E}[f_{d_1,d_2}(X)] \\
= \ &\frac{1}{\mathbb{P}(X > d_1)}\Big(m_1 \mathbb{E}[X \mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big)
\end{aligned}$$

Furthermore, it holds:

$$\begin{aligned}
\mathbb{E}[m(X)] &= \mathbb{E}[m_1 X \mathbb{1}\{x_0 \geq X > 0\}] + \mathbb{E}[(m_1 x_0 + m_2 X)\mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2) x_0 \mathbb{E}[\mathbb{1}\{X > x_0\}] \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2) x_0 \mathbb{P}(X > x_0) \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[m(X)|X > d_1] &= \mathbb{E}[m_1 X \mathbb{1}\{x_0 \geq X > d_1\}] + \mathbb{E}[(m_1 x_0 + m_2 X)\mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > d_1\}] + (m_1 - m_2) x_0 \mathbb{E}[\mathbb{1}\{X > x_0\}] \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > d_1\}] + (m_1 - m_2) x_0 \mathbb{P}(X > x_0) \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[m(X)|X > d_2] &= \mathbb{E}[m_1 X \mathbb{1}\{x_0 \geq X > d_2\}] + \mathbb{E}[(m_1 x_0 + m_2 X)\mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > d_2\}] + (m_1 - m_2) x_0 \mathbb{E}[\mathbb{1}\{X > x_0\}] \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}] \\
&= m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > d_2\}] + (m_1 - m_2) x_0 \mathbb{P}(X > x_0) \\
&\quad + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]
\end{aligned}
$$

With these identities, it yields:

$$
\begin{cases}
\quad (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_1] - \mathbb{E}[f_{d_1,d_2}(X)|X > d_1] \\
= \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)] \\
\\
\quad (1-\rho^2)\mathbb{E}[f_{d_1,d_2}(X)] + \mathbb{E}[m(X)|X > d_2] - (m(d_2) - m(d_1)) \\
= \rho(\eta - w) + (1+\rho)\mathbb{E}[m(X)]
\end{cases}
$$

$$
\begin{cases}
\begin{aligned}
& (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big) \\
& + m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_1\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}] \\
& - \tfrac{1}{\mathbb{P}(X>d_1)}\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big) \\
= \;& \rho(\eta - w) + (1+\rho) \\
& \Big(m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}]\Big)
\end{aligned} \\
\\
\begin{aligned}
& (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big) \\
& + m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_2\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}] \\
& - m_1(d_2 - d_1) \\
= \;& \rho(\eta - w) + (1+\rho) \\
& \Big(m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}]\Big)
\end{aligned}
\end{cases}
$$

$$
\begin{cases}
\begin{aligned}
& (1-\rho^2 - \tfrac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big) \\
& + m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_1\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}] \\
= \;& \rho(\eta - w) + (1+\rho) \\
& \Big(m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}]\Big)
\end{aligned} \\
\\
\begin{aligned}
& (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X > d_1) + m_1 d_2 \mathbb{P}(X > d_2)\Big) \\
& + m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_2\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}] \\
& - m_1(d_2 - d_1) \\
= \;& \rho(\eta - w) + (1+\rho) \\
& \Big(m_1\mathbb{E}[X\mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0\mathbb{P}(X > x_0) + m_2\mathbb{E}[X\mathbb{1}\{X > x_0\}]\Big)
\end{aligned}
\end{cases}
$$

Rearranging the terms yields:

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big)\\ & +m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(m_1-m_2)x_0\mathbb{P}(X>x_0)+m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ = & \rho(\eta-w)\\ & +(1+\rho)m_1\mathbb{E}[X(\mathbb{1}\{x_0\geq X>d_1\}+\mathbb{1}\{d_1\geq X>0\})]\\ & +(1+\rho)(m_1-m_2)x_0\mathbb{P}(X>x_0)+(1+\rho)m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big)\\ & +m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(m_1-m_2)x_0\mathbb{P}(X>x_0)+m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ & -m_1(d_2-d_1)\\ = & \rho(\eta-w)\\ & +(1+\rho)m_1\mathbb{E}[X(\mathbb{1}\{x_0\geq X>d_2\}+\mathbb{1}\{d_2\geq X>0\})]\\ & +(1+\rho)(m_1-m_2)x_0\mathbb{P}(X>x_0)+(1+\rho)m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \end{cases}$$

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big)\\ & +m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(m_1-m_2)x_0\mathbb{P}(X>x_0)+m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ & =\rho(\eta-w)\\ & +(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}]\\ & +(1+\rho)(m_1-m_2)x_0\mathbb{P}(X>x_0)+(1+\rho)m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big)\\ & +m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(m_1-m_2)x_0\mathbb{P}(X>x_0)+m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]\\ & -m_1(d_2-d_1)\\ = & \rho(\eta-w)\\ & +(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}]\\ & +(1+\rho)(m_1-m_2)x_0\mathbb{P}(X>x_0)+(1+\rho)m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}], \end{cases}$$

which then simplifies to

$$\begin{cases} (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ =\ \rho(\eta-w) \\ \quad +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}] \\ \quad +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \\ \\ (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ \quad -m_1(d_2-d_1) \\ =\ \rho(\eta-w) \\ \quad +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}] \\ \quad +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}]. \end{cases}$$

At this point, we want to continue assuming a distribution for the risk X.

Exponentially-distributed Risk $\boldsymbol{X}$

In the following analysis, we want to consider two different distributions for the risk X, Exponential and Pareto distribution. These two distributions are from different categories of distributions: The Exponential distribution is light-tailed, whereas the Pareto distribution is heavy-tailed.

5.1 Analytic Results

5.1.1 Preliminaries

Since we need $m(x)$ to be bounded from below, the assumed linear structure of m requires us to restrict the random variable $\widetilde{X}$ to the interval $[0, D]$, $D \geq 0$. A lower boundary for the choice of D can be given by the following reasoning: To preserve the risk structure, which is the change in behavior of the function m, or more precisely, the change of the slope from m_1 for $x \in [0, x_0]$ to m_2 for $x > x_0$, while not cut off the second part of the function, the choice $D > x_0$ seems to reasonable. One interesting point can be obtained from the fact that m is decreasing linearly from x_0 on, with slope m_2. For $m(D) \geq 0$ to hold for $D > x_0$, the linear structure of m yields that $m(x_0) + (D - x_0)m_2 \geq 0$ needs to hold. With $m(x_0) = m_1 x_0$, solving the equation for D, we obtain:

$$
\begin{aligned}
m(x_0) + (D - x_0)m_2 &\geq 0 \\
\Leftrightarrow (D - x_0)m_2 &\geq -m_1 x_0 \\
\overset{m_2 \leq 0}{\Leftrightarrow} D - x_0 &\leq -\frac{m_1}{m_2} x_0 \\
\Leftrightarrow D &\leq -\frac{m_1}{m_2} x_0 + x_0 \\
\Leftrightarrow D &\leq \left(1 - \frac{m_1}{m_2}\right) x_0 =: \widetilde{D}
\end{aligned}
$$

Therefore, we can choose D to be $D = \left(1 - \frac{m_1}{m_2}\right) x_0$.

5.1.2 Distribution of $\boldsymbol{X}$

Our goal is to have X distributed similarly to the exponential distribution. However, with what we have just discussed above, we need to make some adjustments to obtain a bounded random variable. Otherwise, the structure of m would lead to the final wealth being unbounded which would cause complications with the quadratic utility function, or more specifically, as choice of the value for the parameter η.

Nevertheless, we want to start off with an exponentially distributed random variable $\widetilde{X}$:

$\widetilde{\boldsymbol{X}} \sim \boldsymbol{Exp(\theta)}$

Let $\widetilde{X}$ be exponentially distributed with:

- Parameter $\theta > 0$
- Probability density function $f_{\widetilde{X}}(x) = \frac{1}{\theta} e^{-\frac{1}{\theta}x} \mathbb{1}\{x \geq 0\}$
- Cumulative distribution function $F_{\widetilde{X}}(x) = \mathbb{P}(\widetilde{X} \leq x) = 1 - e^{-\frac{1}{\theta}x}$
- Survival function $S_{\widetilde{X}}(x) = 1 - \mathbb{P}(\widetilde{X} \leq x) = e^{-\frac{1}{\theta}x}$
- Inverse survival function $S_{\widetilde{X}}^{-1}(x) = -\theta ln(x)$

- $\mathbb{E}[\widetilde{X}] = \theta$

Now, we choose D in accordance with the boundaries mentioned above. This yields the truncated random variable $X = \widetilde{X}|\widetilde{X} \leq D$ that now describes the insurable risk X. The risk X then is a truncated version of the exponentially distributed random variable $\widetilde{X}$.

$\boldsymbol{X \sim Exp_D(\theta)}$

Let $X = \widetilde{X}|\widetilde{X} \leq D$ with the following characteristics:

- Parameter $\theta > 0$
- Probability density function

$$f_X(x) = \frac{f_{\widetilde{X}}(x)}{F_{\widetilde{X}}(D)} \mathbb{1}\{D \geq x\} = \frac{\frac{1}{\theta} e^{-\frac{1}{\theta}x}}{1 - e^{-\frac{1}{\theta}D}} \mathbb{1}\{D \geq x \geq 0\}$$

- Cumulative distribution function

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x) = \frac{F_{\widetilde{X}}(x)}{F_{\widetilde{X}}(D)} \mathbb{1}\{x \leq D\} + \mathbb{1}\{x > D\} \\ &= \frac{1 - e^{-\frac{1}{\theta}x}}{1 - e^{-\frac{1}{\theta}D}} \mathbb{1}\{x \leq D\} + \mathbb{1}\{x > D\} \end{aligned}$$

- Survival function

$$\begin{aligned} S_X(x) &= \mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x) \\ &= 1 - \Big(\frac{1 - e^{-\frac{1}{\theta}x}}{1 - e^{-\frac{1}{\theta}D}} \mathbb{1}\{x \leq D\} + \mathbb{1}\{x > D\} \Big) \\ &= \Big(1 - \frac{1 - e^{-\frac{1}{\theta}x}}{1 - e^{-\frac{1}{\theta}D}} \Big) \mathbb{1}\{x \leq D\} = \Big(1 - \frac{1 - e^{-\frac{1}{\theta}x}}{F_{\widetilde{X}}(D)} \Big) \mathbb{1}\{x \leq D\} \end{aligned}$$

- Inverse survival function

$$S_X^{-1}(x) = -\theta ln\Big(1-(1-x)(1-e^{-\frac{1}{\theta}D})\Big) = -\theta ln\Big(1-(1-x)F_{\widetilde{X}}(D)\Big)$$

- Expected value:

$$\begin{aligned}\mathbb{E}[X] &= \int\limits_{-\infty}^{\infty} x f_X(x)dx = \frac{1}{F_{\widetilde{X}}(D)}\int\limits_{0}^{D} x f_{\widetilde{X}}(x)dx \\ &= \frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D \geq \widetilde{X} > 0\}]\end{aligned}$$

5.1.3 Application

Lemma 1.1. *Given the dependence structure stated in 2.3.1, assuming a quadratic utility function of the form* $u(\xi) = -(\eta - \xi)^2$*, a linear structure of* m *with*

$$m(x) = \begin{cases} m_1 x & \text{for } x \leq x_0 \\ m_1 x_0 + m_2(x - x_0) = (m_1 - m_2)x_0 + m_2 x & \text{for } x > x_0, \end{cases}$$

and the risk being a truncated exponential random variable $X = \widetilde{X} | \widetilde{X} \leq D$*, which is derived from* $\widetilde{X} \sim Exp(\theta)$*, the two equations stated in Lemma IV.2.1 are equivalent to*

$$\begin{cases} \begin{aligned} & m_1(1 - \rho^2 - \Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)^{-1})\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1) - (d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\ & -d_1\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big) + d_2\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\ & + m_1 \tfrac{1}{F_{\widetilde{X}}(D)}(d_1 + \theta)exp(-\tfrac{1}{\theta}d_1) \\ = \; & \rho(\eta - w) \\ & +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}\theta + \rho(m_1 - m_2)x_0\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\ & +\rho(m_2 - m_1)\tfrac{1}{F_{\widetilde{X}}(D)}(x_0+\theta)exp(-\tfrac{1}{\theta}x_0) \\ & -\rho m_2 \tfrac{1}{F_{\widetilde{X}}(D)}(D+\theta)exp(-\tfrac{1}{\theta}D) \\ \\ & m_1(1-\rho^2)\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1) - (d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\ & -d_1\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big) + d_2\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\ & -m_1(d_2 - d_1) \\ & + m_1 \tfrac{1}{F_{\widetilde{X}}(D)}(d_2 + \theta)exp(-\tfrac{1}{\theta}d_2) \\ = \; & \rho(\eta - w) \\ & +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}\theta + \rho(m_1 - m_2)x_0\Big(1 - \tfrac{1 - e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\ & +\rho(m_2 - m_1)\tfrac{1}{F_{\widetilde{X}}(D)}(x_0+\theta)exp(-\tfrac{1}{\theta}x_0) \\ & -\rho m_2 \tfrac{1}{F_{\widetilde{X}}(D)}(D+\theta)exp(-\tfrac{1}{\theta}D) \end{aligned} \end{cases}$$

Proof:

For the two equations in Lemma IV.2.1

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \\ & \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ & -m_1(d_2-d_1) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}], \end{cases}$$

we need to determine the following quantities:

- $\mathbb{P}(X>d_1)$, $\mathbb{P}(X>d_2)$, $\mathbb{P}(X>x_0)$
- $\mathbb{E}[X]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_2\geq\widetilde{X}>d_1\}]$
- $\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_1\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_2\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{x_0\geq\widetilde{X}>d_1\}]$
- $\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{x_0\geq\widetilde{X}>d_2\}]$
- $\mathbb{E}[X\mathbb{1}\{X>x_0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D\geq\widetilde{X}>x_0\}]$

For the first two probabilities, we take a look at the survival function S_X, and obtain:

- $\mathbb{P}(X > d_1) = \left(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\right)\mathbb{1}\{d_1 \leq D\}$
- $\mathbb{P}(X > d_2) = \left(1 - \frac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\right)\mathbb{1}\{d_2 \leq D\}$
- $\mathbb{P}(X > x_0) = \left(1 - \frac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\right)\mathbb{1}\{x_0 \leq D\}$

For the expected values, we recall the following, using partial integration:

$$
\begin{aligned}
\mathbb{E}[\widetilde{X}\mathbb{1}\{b \geq \widetilde{X} > a\}] &= \int_a^b x f_{\widetilde{X}}(x)dx = \int_a^b x\frac{1}{\theta}exp(-\frac{1}{\theta}x)dx \\
&= \Big[-xexp(-\frac{1}{\theta}x)\Big]_a^b + \int_a^b exp(-\frac{1}{\theta}x)dx \\
&= -bexp(-\frac{1}{\theta}b) + aexp(-\frac{1}{\theta}a) + \Big[-\theta exp(-\frac{1}{\theta}x)\Big]_a^b \\
&= -bexp(-\frac{1}{\theta}b) + aexp(-\frac{1}{\theta}a) - \theta exp(-\frac{1}{\theta}b) + \theta exp(-\frac{1}{\theta}a) \\
&= (a+\theta)exp(-\frac{1}{\theta}a) - (b+\theta)exp(-\frac{1}{\theta}b)
\end{aligned}
$$

Putting everything together, this yields the following system of equations:

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \\ \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ & -m_1(d_2-d_1) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \end{cases}$$

$$
\begin{cases}
& (1-\rho^2 - \Big(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)^{-1})\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\frac{1}{\theta}d_1) - (d_2+\theta)exp(-\frac{1}{\theta}d_2)] \\
& -m_1 d_1\Big(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big) + m_1 d_2\Big(1 - \frac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
= & \rho(\eta - w) \\
& +\rho m_1 \frac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\frac{1}{\theta}d_1) - (x_0+\theta)exp(-\frac{1}{\theta}x_0)] \\
& +(1+\rho)m_1 \frac{1}{F_{\widetilde{X}}(D)}[(0+\theta)exp(-\frac{1}{\theta}0) - (d_1+\theta)exp(-\frac{1}{\theta}d_1)] \\
& +\rho(m_1-m_2)x_0\Big(1 - \frac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho m_2 \frac{1}{F_{\widetilde{X}}(D)}[(x_0+\theta)exp(-\frac{1}{\theta}x_0) - (D+\theta)exp(-\frac{1}{\theta}D)] \\
\\
& (1-\rho^2)\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\frac{1}{\theta}d_1) - (d_2+\theta)exp(-\frac{1}{\theta}d_2)] \\
& -m_1 d_1\Big(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big) + m_1 d_2\Big(1 - \frac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
& -m_1(d_2-d_1) \\
= & \rho(\eta - w) \\
& +\rho m_1 \frac{1}{F_{\widetilde{X}}(D)}[(d_2+\theta)exp(-\frac{1}{\theta}d_2) - (x_0+\theta)exp(-\frac{1}{\theta}x_0)] \\
& +(1+\rho)m_1 \frac{1}{F_{\widetilde{X}}(D)}[(0+\theta)exp(-\frac{1}{\theta}0) - (d_2+\theta)exp(-\frac{1}{\theta}d_2)] \\
& +\rho(m_1-m_2)x_0\Big(1 - \frac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho m_2 \frac{1}{F_{\widetilde{X}}(D)}[(x_0+\theta)exp(-\frac{1}{\theta}x_0) - (D+\theta)exp(-\frac{1}{\theta}D)]
\end{cases}
$$

Evaluating the expressions containing zero, and factoring m_1 out yields:

$$\begin{cases}
\begin{aligned}
& m_1(1-\rho^2-\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)^{-1})\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)-(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\
& -d_1\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
= \; & \rho(\eta-w) \\
& +\rho m_1\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)-(x_0+\theta)exp(-\tfrac{1}{\theta}x_0)] \\
& +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}[\theta-(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)] \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[(x_0+\theta)exp(-\tfrac{1}{\theta}x_0)-(D+\theta)exp(-\tfrac{1}{\theta}D)] \\
\\
& m_1(1-\rho^2)\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)-(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\
& -d_1\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
& -m_1(d_2-d_1) \\
= \; & \rho(\eta-w) \\
& +\rho m_1\tfrac{1}{F_{\widetilde{X}}(D)}[(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)-(x_0+\theta)exp(-\tfrac{1}{\theta}x_0)] \\
& +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}[\theta-(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[(x_0+\theta)exp(-\tfrac{1}{\theta}x_0)-(D+\theta)exp(-\tfrac{1}{\theta}D)]
\end{aligned}
\end{cases}$$

Some of the terms cancel out, which yields:

$$
\begin{cases}
\begin{aligned}
& m_1(1-\rho^2 - \Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)^{-1})\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)-(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\
& -d_1\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
= \; & \rho(\eta-w) \\
& +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}\theta \\
& -m_1\tfrac{1}{F_{\widetilde{X}}(D)}(d_1+\theta)exp(-\tfrac{1}{\theta}d_1) \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho(m_2-m_1)\tfrac{1}{F_{\widetilde{X}}(D)}(x_0+\theta)exp(-\tfrac{1}{\theta}x_0) \\
& -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}(D+\theta)exp(-\tfrac{1}{\theta}D) \\
\\
& m_1(1-\rho^2)\Big(\tfrac{1}{F_{\widetilde{X}}(D)}[(d_1+\theta)exp(-\tfrac{1}{\theta}d_1)-(d_2+\theta)exp(-\tfrac{1}{\theta}d_2)] \\
& -d_1\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_1}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-e^{-\frac{1}{\theta}d_2}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
& -m_1(d_2-d_1) \\
= \; & \rho(\eta-w) \\
& +(1+\rho)m_1\tfrac{1}{F_{\widetilde{X}}(D)}\theta \\
& -m_1\tfrac{1}{F_{\widetilde{X}}(D)}(d_2+\theta)exp(-\tfrac{1}{\theta}d_2) \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho(m_2-m_1)\tfrac{1}{F_{\widetilde{X}}(D)}(x_0+\theta)exp(-\tfrac{1}{\theta}x_0) \\
& -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}(D+\theta)exp(-\tfrac{1}{\theta}D)
\end{aligned}
\end{cases}
$$

$$
\begin{cases}
\begin{aligned}
& m_1(1-\rho^2 - \left(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\tilde{X}}(D)}\right)^{-1})\left(\frac{1}{F_{\tilde{X}}(D)}[(d_1+\theta)exp(-\frac{1}{\theta}d_1) - (d_2+\theta)exp(-\frac{1}{\theta}d_2)]\right. \\
& \left. -d_1\left(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\tilde{X}}(D)}\right) + d_2\left(1 - \frac{1-e^{-\frac{1}{\theta}d_2}}{F_{\tilde{X}}(D)}\right)\right) \\
& +m_1\frac{1}{F_{\tilde{X}}(D)}(d_1+\theta)exp(-\frac{1}{\theta}d_1) \\
= \; & \rho(\eta - w) \\
& +(1+\rho)m_1\frac{1}{F_{\tilde{X}}(D)}\theta + \rho(m_1 - m_2)x_0\left(1 - \frac{1-e^{-\frac{1}{\theta}x_0}}{F_{\tilde{X}}(D)}\right) \\
& +\rho(m_2 - m_1)\frac{1}{F_{\tilde{X}}(D)}(x_0+\theta)exp(-\frac{1}{\theta}x_0) \\
& -\rho m_2\frac{1}{F_{\tilde{X}}(D)}(D+\theta)exp(-\frac{1}{\theta}D) \\
\\
& m_1(1-\rho^2)\left(\frac{1}{F_{\tilde{X}}(D)}[(d_1+\theta)exp(-\frac{1}{\theta}d_1) - (d_2+\theta)exp(-\frac{1}{\theta}d_2)]\right. \\
& \left. -d_1\left(1 - \frac{1-e^{-\frac{1}{\theta}d_1}}{F_{\tilde{X}}(D)}\right) + d_2\left(1 - \frac{1-e^{-\frac{1}{\theta}d_2}}{F_{\tilde{X}}(D)}\right)\right) \\
& -m_1(d_2 - d_1) \\
& +m_1\frac{1}{F_{\tilde{X}}(D)}(d_2+\theta)exp(-\frac{1}{\theta}d_2) \\
= \; & \rho(\eta - w) \\
& +(1+\rho)m_1\frac{1}{F_{\tilde{X}}(D)}\theta + \rho(m_1 - m_2)x_0\left(1 - \frac{1-e^{-\frac{1}{\theta}x_0}}{F_{\tilde{X}}(D)}\right) \\
& +\rho(m_2 - m_1)\frac{1}{F_{\tilde{X}}(D)}(x_0+\theta)exp(-\frac{1}{\theta}x_0) \\
& -\rho m_2\frac{1}{F_{\tilde{X}}(D)}(D+\theta)exp(-\frac{1}{\theta}D)
\end{aligned}
\end{cases}
$$

In the last step, we arranged the terms in a way that yields the same right hand side of both equations. The variables d_1 and d_2 are now both on the left hand side and the right hand side is constant. Since it appears difficult to solve this system of equations analytically, we use the software package "R"" to compute solutions for certain parameter values of the model to obtain some numerical solutions. These solutions can also be used to illustrate concepts and results obtained without assuming any distribution for the risk X, such as presented in section 2.3.2 where we have shown that if d_1 and d_2 exist, they need to be greater than or equal to $S_X^{-1}(\frac{1}{1+\rho})$.

5.2 Numerical Results

For the numerical analysis, we need to choose values for a number of parameters. Our objective is to investigate the impact of the dependence structure between X and Y, represented by the function m, on the optimal solution, or more specifically, d_1 and d_1 as part of f_3^*. Therefore, we let the parameters x_0, m_1, and m_2 vary while we keep the remaining parameters of the model constant. Since we want to compare the results for varying slopes and constant x_0, we choose D to be the maximum of all $\widetilde{D}$ for all combinations m_1 and m_2. Hence, D is constant for the same x_0, and varies for different x_0.

5.2.1 Parameters

Constant Parameters:

- θ: Parameter of Exponential distribution $\widetilde{X} \sim Exp(\theta)$ for $X = \widetilde{X}|\widetilde{X} \leq D$, here $\theta = 1$
- ρ: Safety loading coefficient, here $\rho = 0.1$
- η and w: Since we choose η such that u represents a certain risk aversion which is unspecified here, we choose $\eta - w = 0$ for simplicity here.

Varying Parameters:

- x_0: Point where the behavior of m changes, here $x_0 \in \{0.5, 1, 1.5, 2.5\}$
- m_1 and m_2: Slope parameters of m,
  here $m_1 \in M_1 = \{0.25,\ 0.5,\ 0.75,\ 1\}$ and $m_2 \in M_2 = \{-0.5, -0.75, -1, -1.5, -2\}$
- D: Cut-off value, here $D = \max\limits_{(m_1,m_2)\in M_1\times M_2} \{\widetilde{D} : \widetilde{D} = \left(1 - \frac{m_1}{m_2}\right)x_0\}$

5.2.2 Objectives

Using the software package " R", we solve the system of equations for d_1 and d_2. The following tables display our findings, see 5.2.3. The code used for this analysis can be found in the appendix.

We want to illustrate the theoretical result in theorem III.3.1, that is, if $\mathbb{E}[g(x)] \le g(0)$ holds, the optimal solution to problem II.1.2 is "no coverage". With the assumptions above, it holds:

$$
\begin{aligned}
\mathbb{E}[g(X)] &= \mathbb{E}[u'(w - m(X))] = \mathbb{E}[2(\eta - w + m(X))] = 2(\eta - w) + 2\mathbb{E}[m(X)] \\
&= 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \ge X > 0\}] + \mathbb{E}[((m_1 - m_2)x_0 + m_2 X)\mathbb{1}\{X > x_0\}]\Big) \\
&= 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \ge X > 0\}] + (m_1 - m_2)x_0 \mathbb{E}[\mathbb{1}\{X > x_0\}] + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]\Big) \\
&= 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \ge X > 0\}] + (m_1 - m_2)x_0 \mathbb{P}(X > x_0) + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]\Big) \\
&= 2\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)} \mathbb{E}[\widetilde{X} \mathbb{1}\{x_0 \ge \widetilde{X} > 0\}] + (m_1 - m_2)x_0 \Big(1 - \frac{1 - e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
&\quad + m_2 \frac{1}{F_{\widetilde{X}}(D)} \mathbb{E}[\widetilde{X} \mathbb{1}\{D \ge \widetilde{X} > x_0\}]\Big) \\
&= 2\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)} [\theta - (x_0 + \theta) exp(-\frac{1}{\theta}x_0)] + (m_1 - m_2)x_0 \Big(1 - \frac{1 - e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big) \\
&\quad + m_2 \frac{1}{F_{\widetilde{X}}(D)} [(x_0 + \theta) exp(-\frac{1}{\theta}x_0) - (D + \theta) exp(-\frac{1}{\theta}D)]\Big)
\end{aligned}
$$

and

$$g(0) \;=\; \mathbb{E}[u'(w - m(0))] = \mathbb{E}[2(\eta - w + m(0))] \stackrel{m(0)=m_1\cdot 0}{=} 2(\eta - w) = 0$$

Hence, we check, after dividing both sides by 2, whether it holds

$$m_1 \frac{1}{F_{\widetilde{X}}(D)}[\theta - (x_0+\theta)exp(-\frac{1}{\theta}x_0)] + (m_1 - m_2)x_0\Big(1 - \frac{1 - e^{-\frac{1}{\theta}x_0}}{F_{\widetilde{X}}(D)}\Big)$$

$$+m_2 \frac{1}{F_{\widetilde{X}}(D)}[(x_0+\theta)exp(-\frac{1}{\theta}x_0) - (D+\theta)exp(-\frac{1}{\theta}D)]$$
$$\leq \; 0,$$

and by theorem III.3.1, this implies that "no coverage" is optimal. In the tables, *yes* represents the inequality is satisfied, *no* represents the inequality doesn't hold. This means, *yes* represents the cases where "no coverage" is optimal, and *no* represents the cases where there is some form of coverage for the risk X.

5.2.3 Findings

For the inequality, our choice of the minimum for truncation parameter D yields that the inequality is never satisfied. There is always need for insurance coverage for the considered parameter combination.

For the lower boundary $S_X^{-1}(\frac{1}{1+\rho})$, and the values d_1 and d_2 the results are displayed in the tables below:

$m_1 \setminus m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.1518	-0.1510	-0.1502	-0.1486	-0.1469
	d_2	0.0251	0.0249	0.0246	0.0241	0.0236
0.5	d_1	-0.1526	-0.1522	-0.1518	-0.1510	-0.1502
	d_2	0.0254	0.0253	0.0251	0.0249	0.0246
0.75	d_1	-0.1529	-0.1526	-0.1524	-0.1518	-0.1513
	d_2	0.0255	0.0254	0.0253	0.0251	0.0250
1	d_1	-0.1530	-0.1528	-0.1526	-0.1522	-0.1518
	d_2	0.0255	0.0255	0.0254	0.0253	0.0251

Table 5.4: $X \sim Exp_D(\theta)$: Numerical Results for $x_0 = 0.5$, $D = 0.5625$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0399$

$m_1 \setminus m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.2524	-0.2513	-0.2502	-0.2480	-0.2458
	d_2	0.0469	0.0465	0.0461	0.0452	0.0444
0.5	d_1	-0.2535	-0.2530	-0.2524	-0.2513	-0.2502
	d_2	0.0473	0.0471	0.0469	0.0465	0.0461
0.75	d_1	-0.2539	-0.2535	-0.2532	-0.2524	-0.2517
	d_2	0.0474	0.0473	0.0471	0.0469	0.0466
1	d_1	-0.2541	-0.2538	-0.2535	-0.2530	-0.2524
	d_2	0.0475	0.0474	0.0473	0.0471	0.0469

Table 5.5: $X \sim Exp_D(\theta)$: Numerical Results for $x_0 = 1$, $D = 1.125$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0634$

$m_1 \ \backslash \ m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.3196	-0.3184	-0.3173	-0.3151	-0.3128
	d_2	0.0646	0.0641	0.0636	0.0627	0.0618
0.5	d_1	-0.3207	-0.3201	-0.3196	-0.3184	-0.3173
	d_2	0.0650	0.0648	0.0646	0.0641	0.0636
0.75	d_1	-0.3211	-0.3207	-0.3203	-0.3196	-0.3188
	d_2	0.0652	0.0650	0.0649	0.0646	0.0643
1	d_1	-0.3212	-0.3210	-0.3207	-0.3201	-0.3196
	d_2	0.0653	0.0651	0.0650	0.0648	0.0646

Table 5.6: $X \sim Exp_D(\theta)$: Numerical Results for $x_0 = 1.5$, $D = 1.6875$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.077$

$m_1 \setminus m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.3942	-0.3934	-0.3926	-0.3909	-0.3892
	d_2	0.0885	0.0881	0.0877	0.0870	0.0862
0.5	d_1	-0.3951	-0.3947	-0.3942	-0.3934	-0.3926
	d_2	0.0889	0.0887	0.0885	0.0881	0.0877
0.75	d_1	-0.3954	-0.3951	-0.3948	-0.3942	-0.3937
	d_2	0.0891	0.0889	0.0888	0.0885	0.0883
1	d_1	-0.3955	-0.3953	-0.3951	-0.3947	-0.3942
	d_2	0.0891	0.0890	0.0889	0.0887	0.0885

Table 5.7: $X \sim Exp_D(\theta)$: Numerical Results for $x_0 = 2.5$, $D = 2.8125$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0893$

5.2.4 Interpretation

Overall, it seems that there are two different types of results obtained from running the code and solving the two equations for d_1 and d_2: The first type of results are negative values for d_1, the second type of results are positive values for d_2. Hence, the inequality $d_1 \leq d_2$ holds. However, with $d_1 < 0 < d_2 \leq x_0$, we are not able to provide values for d_1 and d_2 that fulfill the conditions stated in theorem II.3.2. This is rather unsatisfying, and demands further investigation. Since these are numerical results that have been obtained using a certain software package; a certain code; a certain method for determining the solutions; and certain input parameters for these methods, such as an initial guess for the solutions, there are various potential sources that can cause the numerical analysis to produce these undesired results. Another potential source for these results that needs to be considered are the assumptions that have been made. Maybe some of the assumptions need to be revised and adjustments need to made in order to obtain values for d_1 and d_2 that can be used such that theorem II.3.2 may provide the optimal solution.

In regard to the impact of m_1, we can observe that as m_1 increases, d_1 becomes smaller, and d_2 becomes greater. This implies that as m_1 increases, the difference between d_1 and d_2 increases as well. Hence, for greater m_1, the claim size for which the insurance coverage becomes effective decreases, i.e., the insurance company already provides a payment for smaller claim sizes - the “deductible” decreases in a way. In addition, the claim size causing the insurance coverage to become capped begins to increase, and the insurance company provides an increasing payment for even larger claim sizes. This can be seen following the individual columns from top to bottom, since m_1 is increased, top to bottom, taking the values 0.25, 0.5, 0.75, and 1.

In regard to the impact of m_2, we can observe that as m_2 decreases, d_1 becomes greater, and d_2 becomes smaller. This implies that as m_2 decreases, the difference between d_1 and d_2 decreases as well. Hence, for smaller m_2, meaning more negative m_2, the claim size for which the insurance coverage becomes effective increases, i.e., the insurance company provides a payment for larger claim sizes than before - the "deductible" increases in a way. In addition, the claim size that causes the insurance coverage being capped decreases, i.e., the insurance company provides an increasing payment for smaller claim sizes than before. This can be seen following the individual rows from right to left, since m_2 is decreased going from left to right in the table, taking the values -0.5, -0.75, -1, -1.5, and -2.

In regard to the impact of x_0, a greater value for x_0 results into a bigger gap between the two levels d_1 and d_2, which is reasonable as a greater x_0 means that overall loss $m(X) = X+Y$ increases on a longer interval, and also decreases on a longer interval, yielding a scaling effect.

Pareto-distributed Risk X

In the following analysis, we want to consider the Pareto distribution for the risk X.

6.1 Analytic Results

6.1.1 Preliminaries

With the same reasoning as in 5.1.1, we choose D to be $D = \left(1 - \frac{m_1}{m_2}\right)x_0$.

6.1.2 Distribution of X

Our goal is to have X distributed similarly to the exponential distribution. However, with what we have just discussed above, we need to make some adjustments to obtain a bounded random variable. Otherwise, the structure of m would lead to the final wealth being unbounded which would cause complications with the quadratic utility function, more specifically, for choice of the value for the parameter η.

Nevertheless, we want to start off with a Pareto-distributed random variable $\widetilde{X}$:

$\widetilde{\mathbf{X}} \sim \mathbf{Pareto}(\alpha, \lambda)$

Let $\widetilde{X}$ be Pareto distributed with:

- Parameters $\alpha > 0$ and $\lambda > 0$
- Probability density function $f_{\widetilde{X}}(x) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}$

- Cumulative distribution function $F_{\widetilde{X}}(x) = \mathbb{P}(\widetilde{X} \leq x) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}$

- Survival function $S_{\widetilde{X}}(x) = 1 - \mathbb{P}(\widetilde{X} \leq x) = (1 + \frac{x}{\lambda})^{-\alpha}$

- Inverse survival function $S_{\widetilde{X}}^{-1}(x) = \lambda(x^{-\frac{1}{\alpha}} - 1)$

- $\mathbb{E}[\widetilde{X}] = \frac{\lambda}{\alpha-1}$ holds for $\alpha > 1$

Now, we choose D in accordance with the boundaries. This yields the truncated random variable $X = \widetilde{X}|\widetilde{X} \leq D$ that now describes the insurable risk X. The risk X then is a truncated version of the Pareto distributed random variable $\widetilde{X}$.

$\mathbf{X} \sim \mathbf{Pareto_D}(\alpha, \lambda)$

Let $X = \widetilde{X}|\widetilde{X} \leq D$ with the following characteristics:

- Parameters $\alpha > 0$ and $\lambda > 0$

- Probability density function

$$f_X(x) = \frac{f_{\widetilde{X}}(x)}{F_{\widetilde{X}}(D)} \mathbb{1}\{D \geq x\} = \frac{\frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}}{1 - (1 + \frac{D}{\lambda})^{-\alpha}} \mathbb{1}\{D \geq x \geq 0\}$$

- Cumulative distribution function

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x) = \frac{F_{\widetilde{X}}(x)}{F_{\widetilde{X}}(D)} \mathbb{1}\{x \leq D\} + \mathbb{1}\{x > D\} \\ &= \frac{1 - (1 + \frac{x}{\lambda})^{-\alpha}}{1 - (1 + \frac{D}{\lambda})^{-\alpha}} \mathbb{1}\{x \leq D\} + \mathbb{1}\{x > D\} \end{aligned}$$

- Survival function

$$\begin{aligned}
S_X(x) &= \mathbb{P}(X > x) = 1 - \mathbb{P}(X \le x) \\
&= 1 - \Big(\frac{1-(1+\frac{x}{\lambda})^{-\alpha}}{1-(1+\frac{D}{\lambda})^{-\alpha}}\mathbb{1}\{x \le D\} + \mathbb{1}\{x > D\}\Big) \\
&= \Big(1 - \frac{1-(1+\frac{x}{\lambda})^{-\alpha}}{1-(1+\frac{D}{\lambda})^{-\alpha}}\Big)\mathbb{1}\{x \le D\} = \Big(1 - \frac{1-(1+\frac{x}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\mathbb{1}\{x \le D\}
\end{aligned}$$

- Inverse survival function

$$S_X^{-1}(x) = \lambda\Big[\Big(1-(1-x)F_{\widetilde{X}}(D)\Big)^{-\frac{1}{\alpha}} - 1\Big]$$

- Expected value:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx = \frac{1}{F_{\widetilde{X}}(D)}\int_0^D x f_{\widetilde{X}}(x)dx \\
&= \frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D \ge \widetilde{X} > 0\}]
\end{aligned}$$

6.1.3 Application

Lemma 1.1. *Given the dependence structure stated in 2.3.1, assuming a quadratic utility function of the form* $u(\xi) = -(\eta - \xi)^2$*, a linear structure of* m *with*

$$m(x) = \begin{cases} m_1 x & \text{for } x \leq x_0 \\ m_1 x_0 + m_2(x - x_0) = (m_1 - m_2)x_0 + m_2 x & \text{for } x > x_0, \end{cases}$$

and the risk being a truncated Pareto random variable $X = \widetilde{X}|\widetilde{X} \leq D$*, which is derived from* $\widetilde{X} \sim Pareto(\alpha, \lambda)$*, the two equations stated in Lemma IV.2.1 are equivalent to*

$$\begin{cases} \begin{aligned} & m_1(1 - \rho^2 - \left(1 - \tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)^{-1}) \\ & \left(\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha+1} - d_2\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha} - \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha+1}] \right. \\ & \left. -d_1\left(1 - \tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) + d_2\left(1 - \tfrac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\ & +m_1\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha+1}] \\ = \; & \rho(\eta - w) + m_1\tfrac{(1+\rho)}{F_{\widetilde{X}}(D)}\tfrac{\lambda}{\alpha-1} + \rho(m_1 - m_2)x_0\left(1 - \tfrac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & +\rho(m_2 - m_1)\tfrac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\tfrac{x_0}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{x_0}{\lambda}\right)^{-\alpha+1}] \\ & -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[D\left(1+\tfrac{D}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{D}{\lambda}\right)^{-\alpha+1}] \end{aligned} \\ \\ \begin{aligned} & m_1(1 - \rho^2) \\ & \left(\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_1}{\lambda}\right)^{-\alpha+1} - d_2\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha} - \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha+1}] \right. \\ & \left. -d_1\left(1 - \tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) + d_2\left(1 - \tfrac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\ & -m_1(d_2 - d_1) \\ & +m_1\tfrac{1}{F_{\widetilde{X}}(D)}[d_2\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{d_2}{\lambda}\right)^{-\alpha+1}] \\ = \; & \rho(\eta - w) + m_1\tfrac{(1+\rho)}{F_{\widetilde{X}}(D)}\tfrac{\lambda}{\alpha-1} + \rho(m_1 - m_2)x_0\left(1 - \tfrac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & +\rho(m_2 - m_1)\tfrac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\tfrac{x_0}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{x_0}{\lambda}\right)^{-\alpha+1}] \\ & -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[D\left(1+\tfrac{D}{\lambda}\right)^{-\alpha} + \tfrac{\lambda}{\alpha-1}\left(1+\tfrac{D}{\lambda}\right)^{-\alpha+1}]. \end{aligned} \end{cases}$$

Proof:

For the two equations in Lemma IV.2.1

$$\begin{cases} & (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}] \\ & \\ & (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]-m_1d_1\mathbb{P}(X>d_1)+m_1d_2\mathbb{P}(X>d_2)\Big) \\ & -m_1(d_2-d_1) \\ = & \rho(\eta-w) \\ & +\rho m_1\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]+(1+\rho)m_1\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}] \\ & +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0)+\rho m_2\mathbb{E}[X\mathbb{1}\{X>x_0\}], \end{cases}$$

we need to determine the following quantities:

- $\mathbb{P}(X>d_1)$, $\mathbb{P}(X>d_2)$, $\mathbb{P}(X>x_0)$
- $\mathbb{E}[X]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{d_2\geq X>d_1\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_2\geq\widetilde{X}>d_1\}]$
- $\mathbb{E}[X\mathbb{1}\{d_1\geq X>0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_1\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{d_2\geq X>0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{d_2\geq\widetilde{X}>0\}]$
- $\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_1\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{x_0\geq\widetilde{X}>d_1\}]$
- $\mathbb{E}[X\mathbb{1}\{x_0\geq X>d_2\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{x_0\geq\widetilde{X}>d_2\}]$
- $\mathbb{E}[X\mathbb{1}\{X>x_0\}]=\frac{1}{F_{\widetilde{X}}(D)}\mathbb{E}[\widetilde{X}\mathbb{1}\{D\geq\widetilde{X}>x_0\}]$

For the first two probabilities, we take a look at the survival function S_X, and obtain:

- $\mathbb{P}(X > d_1) = \Big(1 - \frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\mathbb{1}\{d_1 \leq D\}$
- $\mathbb{P}(X > d_2) = \Big(1 - \frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\mathbb{1}\{d_2 \leq D\}$
- $\mathbb{P}(X > x_0) = \Big(1 - \frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\mathbb{1}\{x_0 \leq D\}$

For the expected values, we recall the following, using partial integration:

$$
\begin{aligned}
& \mathbb{E}[\widetilde{X}\mathbb{1}\{b \geq \widetilde{X} > a\}] \\
= \; & \int_a^b x f_{\widetilde{X}}(x)dx = \int_a^b x\frac{\alpha}{\lambda}\Big(1+\frac{x}{\lambda}\Big)^{-(\alpha+1)}dx \\
= \; & \Big[-x\Big(1+\frac{x}{\lambda}\Big)^{-\alpha}\Big]_a^b + \int_a^b \Big(1+\frac{x}{\lambda}\Big)^{-\alpha}dx \\
= \; & -b\Big(1+\frac{b}{\lambda}\Big)^{-\alpha} + a\Big(1+\frac{a}{\lambda}\Big)^{-\alpha} + \Big[\frac{\lambda}{-\alpha+1}\Big(1+\frac{x}{\lambda}\Big)^{-\alpha+1}\Big]_a^b \\
= \; & -b\Big(1+\frac{b}{\lambda}\Big)^{-\alpha} + a\Big(1+\frac{a}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{-\alpha+1}\Big(1+\frac{b}{\lambda}\Big)^{-\alpha+1} - \frac{\lambda}{-\alpha+1}\Big(1+\frac{a}{\lambda}\Big)^{-\alpha+1} \\
= \; & a\Big(1+\frac{a}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{\alpha-1}\Big(1+\frac{a}{\lambda}\Big)^{-\alpha+1} - b\Big(1+\frac{b}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{\alpha-1}\Big(1+\frac{b}{\lambda}\Big)^{-\alpha+1}
\end{aligned}
$$

Putting everything together, this yields the following system of equations:

$$
\begin{cases}
\quad (1-\rho^2-\frac{1}{\mathbb{P}(X>d_1)})\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X>d_1) + m_1 d_2 \mathbb{P}(X>d_2)\Big) \\
= \rho(\eta - w) \\
\quad +\rho m_1 \mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_1\}] + (1+\rho) m_1 \mathbb{E}[X\mathbb{1}\{d_1 \geq X > 0\}] \\
\quad +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0) + \rho m_2 \mathbb{E}[X\mathbb{1}\{X>x_0\}] \\
\\
\quad (1-\rho^2)\Big(m_1\mathbb{E}[X\mathbb{1}\{d_2 \geq X > d_1\}] - m_1 d_1 \mathbb{P}(X>d_1) + m_1 d_2 \mathbb{P}(X>d_2)\Big) \\
\quad -m_1(d_2-d_1) \\
= \rho(\eta - w) \\
\quad +\rho m_1 \mathbb{E}[X\mathbb{1}\{x_0 \geq X > d_2\}] + (1+\rho) m_1 \mathbb{E}[X\mathbb{1}\{d_2 \geq X > 0\}] \\
\quad +\rho(m_1-m_2)x_0\mathbb{P}(X>x_0) + \rho m_2 \mathbb{E}[X\mathbb{1}\{X>x_0\}]
\end{cases}
$$

$$
\begin{cases}
& (1-\rho^2-\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)^{-1}) \\
& \left(m_1\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\
& -m_1d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
& \left.+m_1d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\
= & \rho(\eta-w) \\
& +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}[0\left(1+\frac{0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{0}{\lambda}\right)^{-\alpha+1}-d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}] \\
& +\rho m_1\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\
& +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
& +\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}-D\left(1+\frac{D}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{D}{\lambda}\right)^{-\alpha+1}] \\
\\
& (1-\rho^2) \\
& \left(m_1\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\
& -m_1d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
& \left.+m_1d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\
& -m_1(d_2-d_1) \\
= & \rho(\eta-w) \\
& +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}[0\left(1+\frac{0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{0}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}] \\
& +\rho m_1\frac{1}{F_{\widetilde{X}}(D)}[d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}-x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\
& +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
& +\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}-D\left(1+\frac{D}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{D}{\lambda}\right)^{-\alpha+1}]
\end{cases}
$$

Evaluating the expressions containing zero, and factoring m_1 out yields:

$$\begin{cases} & m_1(1-\rho^2-\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)^{-1}) \\ & \left(\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\ & -d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & \left.+d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\ = & \rho(\eta-w) \\ & +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}[\frac{\lambda}{\alpha-1}-d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}] \\ & +\rho m_1\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\ & +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & +\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}-D\left(1+\frac{D}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{D}{\lambda}\right)^{-\alpha+1}] \\ \\ & m_1(1-\rho^2) \\ & \left(\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\ & -d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & \left.+d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\ & -m_1(d_2-d_1) \\ = & \rho(\eta-w) \\ & +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}[\frac{\lambda}{\alpha-1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}] \\ & +\rho m_1\frac{1}{F_{\widetilde{X}}(D)}[d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}-x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\ & +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\ & +\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}-D\left(1+\frac{D}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{D}{\lambda}\right)^{-\alpha+1}] \end{cases}$$

Some of the terms cancel out, which yields:

$$
\left\{
\begin{array}{rl}
 & m_1\left(1-\rho^2-\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)^{-1}\right) \\
 & \left(\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\
 & -d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
 & \left.+d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\
= & \rho(\eta-w) \\
 & +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}\frac{\lambda}{\alpha-1} \\
 & -m_1\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}] \\
 & +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
 & +\rho(m_2-m_1)\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\
 & -\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[D\left(1+\frac{D}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1-\frac{D}{\lambda}\right)^{-\alpha+1}] \\
 & \\
 & m_1(1-\rho^2) \\
 & \left(\frac{1}{F_{\widetilde{X}}(D)}[d_1\left(1+\frac{d_1}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_1}{\lambda}\right)^{-\alpha+1}-d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}-\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}]\right. \\
 & -d_1\left(1-\frac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
 & \left.+d_2\left(1-\frac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right)\right) \\
 & -m_1(d_2-d_1) \\
= & \rho(\eta-w) \\
 & +m_1\frac{(1+\rho)}{F_{\widetilde{X}}(D)}\frac{\lambda}{\alpha-1} \\
 & -m_1\frac{1}{F_{\widetilde{X}}(D)}[d_2\left(1+\frac{d_2}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{d_2}{\lambda}\right)^{-\alpha+1}] \\
 & +\rho(m_1-m_2)x_0\left(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\right) \\
 & +\rho(m_2-m_1)\frac{1}{F_{\widetilde{X}}(D)}[x_0\left(1+\frac{x_0}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{x_0}{\lambda}\right)^{-\alpha+1}] \\
 & -\rho m_2\frac{1}{F_{\widetilde{X}}(D)}[D\left(1+\frac{D}{\lambda}\right)^{-\alpha}+\frac{\lambda}{\alpha-1}\left(1+\frac{D}{\lambda}\right)^{-\alpha+1}]
\end{array}
\right.
$$

$$
\begin{cases}
\begin{aligned}
& m_1(1-\rho^2-\Big(1-\tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)^{-1}) \\
& \Big(\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha+1}-d_2\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha}-\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha+1}] \\
& -d_1\Big(1-\tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
& +m_1\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha+1}] \\
= \;& \rho(\eta-w) \\
& +m_1\tfrac{(1+\rho)}{F_{\widetilde{X}}(D)}\tfrac{\lambda}{\alpha-1} \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho(m_2-m_1)\tfrac{1}{F_{\widetilde{X}}(D)}[x_0\Big(1+\tfrac{x_0}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{x_0}{\lambda}\Big)^{-\alpha+1}] \\
& -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[D\Big(1+\tfrac{D}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{D}{\lambda}\Big)^{-\alpha+1}] \\
\\
& m_1(1-\rho^2) \\
& \Big(\tfrac{1}{F_{\widetilde{X}}(D)}[d_1\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_1}{\lambda}\Big)^{-\alpha+1}-d_2\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha}-\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha+1}] \\
& -d_1\Big(1-\tfrac{1-(1+\frac{d_1}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)+d_2\Big(1-\tfrac{1-(1+\frac{d_2}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big)\Big) \\
& -m_1(d_2-d_1) \\
& +m_1\tfrac{1}{F_{\widetilde{X}}(D)}[d_2\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{d_2}{\lambda}\Big)^{-\alpha+1}] \\
= \;& \rho(\eta-w) \\
& +m_1\tfrac{(1+\rho)}{F_{\widetilde{X}}(D)}\tfrac{\lambda}{\alpha-1} \\
& +\rho(m_1-m_2)x_0\Big(1-\tfrac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big) \\
& +\rho(m_2-m_1)\tfrac{1}{F_{\widetilde{X}}(D)}[x_0\Big(1+\tfrac{x_0}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{x_0}{\lambda}\Big)^{-\alpha+1}] \\
& -\rho m_2\tfrac{1}{F_{\widetilde{X}}(D)}[D\Big(1+\tfrac{D}{\lambda}\Big)^{-\alpha}+\tfrac{\lambda}{\alpha-1}\Big(1+\tfrac{D}{\lambda}\Big)^{-\alpha+1}]
\end{aligned}
\end{cases}
$$

In the last step, we arranged the terms in a way that gives us the same right hand side of both equations. The variables d_1 and d_2 are now both on the left hand side and the right hand side is constant. Since it appears difficult to solve this system of equations analytically, we use the software package "R" to compute solutions for certain parameter values of the model to obtain some numerical solutions. These solutions can also be used to illustrate results such as presented in section 2.3.2 where we have shown that if d_1 and d_2 exist, they need to be greater than or equal to $S_X^{-1}(\frac{1}{1+\rho})$.

6.2 Numerical Results

For the numerical analysis, we need to choose values for a number of parameters. Our objective is to investigate the impact of the dependence structure between X and Y, represented by the function m, on the optimal solution, or more specifically, d_1 and d_1 as part of f_3^*. Therefore, we let the parameters x_0, m_1, and m_2 vary while we keep the remaining parameters of the model constant. Since we want to compare the results for varying slopes and constant x_0, we choose D to be the maximum of all $\widetilde{D}$ for all combinations m_1 and m_2. Hence, D is constant for the same x_0, and varies for different x_0.

6.2.1 Parameters

Constant Parameters:

- θ: Parameter of Pareto distribution $\widetilde{X} \sim Pareto(\alpha, \lambda)$ for $X = \widetilde{X}|\widetilde{X} \leq D$, here $\alpha = 2$ and $\lambda = 1$
- ρ: Safety loading coefficient, here $\rho = 0.1$
- η and w: Since we choose η such that u represents a certain risk aversion which is unspecified here, we choose $\eta - w = 0$ for simplicity here.

Varying Parameters:

- x_0: Point where the behavior of m changes, here $x_0 \in \{1, 2, 4, 10\}$
- m_1 and m_2: Slope parameters of m,

  here $m_1 \in M_1 = \{0.25,\ 0.5,\ 0.75,\ 1\}$ and $m_2 \in M_2 = \{-0.5, -0.75, -1, -1.5, -2\}$
- D: Cut-off value, here $D = \max_{(m_1,m_2)\in M_1\times M_2} \{\widetilde{D} : \widetilde{D} = \left(1 - \frac{m_1}{m_2}\right)x_0\}$

6.2.2 Objectives

Using the software package "R", we solve the system of equations for d_1 and d_2. The following tables display our findings (see 6.2.3). The code used for this analysis can be found in the appendix.

We want to illustrate the theoretical result in theorem III.3.1: If $\mathbb{E}[g(x)] \leq g(0)$ holds, the optimal solution to problem II.1.2 is "no coverage". With the assumptions above, it holds:

$$
\begin{aligned}
& \mathbb{E}[g(X)] \\
= \;& \mathbb{E}[u'(w - m(X))] = \mathbb{E}[2(\eta - w + m(X))] = 2(\eta - w) + 2\mathbb{E}[m(X)] \\
= \;& 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > 0\}] + \mathbb{E}[((m_1 - m_2)x_0 + m_2 X)\mathbb{1}\{X > x_0\}]\Big) \\
= \;& 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0 \mathbb{E}[\mathbb{1}\{X > x_0\}] + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]\Big) \\
= \;& 2\Big(m_1 \mathbb{E}[X \mathbb{1}\{x_0 \geq X > 0\}] + (m_1 - m_2)x_0 \mathbb{P}(X > x_0) + m_2 \mathbb{E}[X \mathbb{1}\{X > x_0\}]\Big) \\
= \;& 2\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)} \mathbb{E}[\widetilde{X} \mathbb{1}\{x_0 \geq \widetilde{X} > 0\}] + (m_1 - m_2)x_0 \Big(1 - \frac{1 - (1 + \frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big) \\
& + m_2 \frac{1}{F_{\widetilde{X}}(D)} \mathbb{E}[\widetilde{X} \mathbb{1}\{D \geq \widetilde{X} > x_0\}]\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2\Big(m_1 \frac{1}{F_{\widetilde{X}}(D)}[\frac{\lambda}{\alpha-1} - x_0\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{\alpha-1}\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha+1}] \\
& +(m_1-m_2)x_0\Big(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big) \\
& +m_2\frac{1}{F_{\widetilde{X}}(D)} \\
& [x_0\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{\alpha-1}\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha+1} - D\Big(1+\frac{D}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{\alpha-1}\Big(1+\frac{D}{\lambda}\Big)^{-\alpha+1}]\Big)
\end{aligned}
$$

and

$$
g(0) \;=\; \mathbb{E}[u'(w-m(0))] = \mathbb{E}[2(\eta-w+m(0))] \overset{m(0)=m_1\cdot 0}{=} 2(\eta-w)=0
$$

Hence, we check, after dividing both sides by 2, whether it holds

$$
\begin{aligned}
& m_1 \frac{1}{F_{\widetilde{X}}(D)}[\frac{\lambda}{\alpha-1} - x_0\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{\alpha-1}\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha+1}] \\
& +(m_1-m_2)x_0\Big(1-\frac{1-(1+\frac{x_0}{\lambda})^{-\alpha}}{F_{\widetilde{X}}(D)}\Big) + m_2\frac{1}{F_{\widetilde{X}}(D)} \\
& [x_0\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{\alpha-1}\Big(1+\frac{x_0}{\lambda}\Big)^{-\alpha+1} - D\Big(1+\frac{D}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{\alpha-1}\Big(1+\frac{D}{\lambda}\Big)^{-\alpha+1}] \\
\leq \;& 0,
\end{aligned}
$$

and by theorem III.3.1, this implies that "no insurance" is optimal. In the tables, *yes* represents the inequality is satisfied, *no* represents the inequality doesn't hold. This means, *yes* represents the cases where "no coverage" is optimal, and *no* represents the cases where there is some form of coverage for the risk X.

6.2.3 Findings

For the inequality, our choice of the minimum for truncation parameter D yields that the inequality is never satisfied. There is always need for insurance coverage for the considered parameter combination.

For the lower boundary $S_X^{-1}(\frac{1}{1+\rho})$, and the values d_1 and d_2 the results are displayed in the tables below:

$m_1 \setminus m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.1649	-0.1644	-0.1639	-0.1628	-0.1618
	d_2	0.0409	0.0406	0.0403	0.0397	0.0392
0.5	d_1	-0.1654	-0.1652	-0.1649	-0.1644	-0.1639
	d_2	0.0411	0.0410	0.0409	0.0406	0.0403
0.75	d_1	-0.1656	-0.1654	-0.1653	-0.1649	-0.1646
	d_2	0.0412	0.0411	0.0410	0.0409	0.0407
1	d_1	-0.1657	-0.1656	-0.1654	-0.1652	-0.1649
	d_2	0.0413	0.0412	0.0411	0.0410	0.0409

Table 6.8: $X \sim Pareto_D(\alpha, \lambda)$: Numerical Results for $x_0 = 1$,$D = 1.125$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0374$

$m_1 \ \backslash \ m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.2137	-0.2133	-0.2128	-0.2119	-0.2110
	d_2	0.0650	0.0647	0.0644	0.0638	0.0631
0.5	d_1	-0.2142	-0.2140	-0.2137	-0.2133	-0.2128
	d_2	0.0654	0.0652	0.0650	0.0647	0.0644
0.75	d_1	-0.2143	-0.2142	-0.2140	-0.2137	-0.2134
	d_2	0.0655	0.0654	0.0653	0.0650	0.0648
1	d_1	-0.2144	-0.2143	-0.2142	-0.2140	-0.2137
	d_2	0.0655	0.0654	0.0654	0.0652	0.0650

Table 6.9: $X \sim Pareto_D(\alpha, \lambda)$: Numerical Results for $x_0 = 2$, $D = 2.25$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0439$

$m_1 \ \backslash \ m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.2527	-0.2523	-0.2520	-0.2513	-0.2507
	d_2	0.0921	0.0918	0.0915	0.0909	0.0903
0.5	d_1	-0.2530	-0.2528	-0.2527	-0.2523	-0.2520
	d_2	0.0924	0.0922	0.0921	0.0918	0.0915
0.75	d_1	-0.2531	-0.2530	-0.2529	-0.2527	-0.2524
	d_2	0.0925	0.0924	0.0923	0.0921	0.0919
1	d_1	-0.2532	-0.2531	-0.2530	-0.2528	-0.2527
	d_2	0.0925	0.0924	0.0924	0.0922	0.0921

Table 6.10: $X \sim Pareto_D(\alpha, \lambda)$: Numerical Results for $x_0 = 4$, $D = 4.5$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0471$

$m_1 \ \backslash \ m_2$		-0.5	-0.75	-1	-1.5	-2
0.25	d_1	-0.2861	-0.2859	-0.2857	-0.2853	-0.2849
	d_2	0.1230	0.1228	0.1226	0.1222	0.1218
0.5	d_1	-0.2862	-0.2861	-0.2861	-0.2859	-0.2857
	d_2	0.1232	0.1231	0.1230	0.1228	0.1226
0.75	d_1	-0.2863	-0.2862	-0.2862	-0.2861	-0.2859
	d_2	0.1233	0.1232	0.1232	0.1230	0.1229
1	d_1	-0.2863	-0.2863	-0.2862	-0.2861	-0.2861
	d_2	0.1233	0.1233	0.1232	0.1231	0.1230

Table 6.11: $X \sim Pareto_D(\alpha, \lambda)$: Numerical Results for $x_0 = 10$, $D = 11.25$, and $S_X^{-1}(\frac{1}{1+\rho}) = 0.0485$

6.2.4 Interpretation

The interpretation of results for the Pareto-distributed risk X is very similar to the interpretation of the results for the exponentially-distributed risk X:

Overall, it seems that there are two different types of results obtained from running the code and solving the two equations for d_1 and d_2: The first type of results are found to be negative values for d_1, the second type of results are found to be positive values for d_2. Hence, the inequality $d_1 \leq d_2$ holds. However, with $d_1 < 0 < d_2 \leq x_0$, we are not able to provide values for d_1 and d_2 which fulfill the conditions stated in theorem II.3.2. This is rather unsatisfying, and demands further investigation. Since these are numerical results that have been obtained using a certain software package; a certain code; a certain method for determining the solutions; and certain input parameters for these methods, such as an initial guess for the solutions, there are various potential sources that can cause the numerical analysis to produce these undesired results. Another potential source for these results that needs to be considered are the assumptions that have been made. Maybe some of these assumptions need to be revised and adjustments need to made in order to obtain values for d_1 and d_2 that can be used such that theorem II.3.2 may provide the optimal solution.

In regard to the impact of m_1, we can observe that as m_1 increases, d_1 becomes smaller, and d_2 becomes greater. This implies that as m_1 increases, the difference between d_1 and d_2 increases as well. Hence, for greater m_1, the claim size for which the insurance coverage becomes effective decreases, i.e., the insurance company already provides a payment for smaller claim sizes - the "deductible" decreases in a way. In addition, the claim size causing the insurance coverage to become capped begins to increase, the insurance company provides an increasing payment for even larger claim sizes. This can be seen following the individual columns from top to bottom, since m_1 is increased, top to bottom, taking the values 0.25, 0.5, 0.75, and 1.

In regard to the impact of m_2, we can observe that as m_2 decreases, d_1 becomes greater, and d_2 becomes smaller. This implies that as m_2 decreases, the difference between d_1 and d_2 decreases as well. Hence, for smaller m_2, meaning more negative m_2, the claim size for which the insurance coverage becomes effective increases, i.e., the insurance company provides a payment for larger claim sizes than before - the "deductible" increases in a way. In addition, the claim size that causes the insurance coverage being capped decreases, i.e., the insurance company provides an increasing payment for smaller claim sizes than before. This can be seen following the individual rows, going from left to right in the table, since m_2 is being decreased going from left to right, taking the values -0.5, -0.75, -1, -1.5, and -2.

In regard to the impact of x_0, a greater value for x_0 results into a bigger gap between the two levels d_1 and d_2, which is reasonable as a greater x_0 resulting in the overall loss $m(X) = X + Y$ increases on a longer interval, and also decreases on a longer interval, yielding a scaling effect.

Summary

Considering a special case for the dependence structure, where $X + Y = m(X)$ holds for the two risks and m represents a mixture of moderate negative dependence and strong negative dependence between X and the background risk Y, we were able to establish a lower boundary for the d's as part of the optimal solution to the optimal insurance problem that is provided by a theorem by Chi and Wei, [Chi and Wei, 2018]. Without making any further assumptions, we were also able to develop a criterion that implies "no insurance coverage" is the optimal solution. In the numerical analysis, the examples we considered turned out to display the exact same behavior as predicted by the criterion.

Adding the assumptions about the utility function and the linear structure of m, led to a system of equations that we simplified as much as possible. Afterwards, assuming the risk X to be exponentially distributed, or Pareto distributed respectively, we applied our previous results to these two distribution types. Due to the assumed quadratic utility function, some amendments were necessary, and after adjusting the distribution type to a truncated version that represents the risk X for the further analysis, we obtained a more complex system of two equations. Therefore, using a built-in solver of the software package ,"R", we solved this system numerically yielding results that require cautious interpretation. On the one hand, the analytic result regarding "coverage" versus "no coverage" were perfectly mirrored in the numerical results. On the other hand, the values for d_1^* and d_2^* provided by the solver for the case where the optimal solution is a certain coverage of the risk X, didn't completely follow the conditions required for the theorem in [Chi and Wei, 2018] to hold. However, the overall picture of how the dependence structure, or more specifically, how strongly negative dependent, and how moderately negative dependent the two risks are, affects the optimal solution became clear by the numerical analysis. With X and Y being dependent in a way that Y is not able to balance out the overall loss on a great part of the domain, insurance in form of a certain coverage will be demanded by the insured. The layers of the coverage depend on the parameters that determine the strong negative dependence and the moderate negative dependence. This allows insight on the behavior of the optimal solution.

Outlook

For this thesis, we mainly focused on one special case for the dependence structure, which was a combination of moderate negative dependence and strong negative dependence between the insurable risk X and the background risk Y. As mentioned in [Chi and Wei, 2018], there are various other combinations that can be investigated as well.

Regarding the special case we considered, one of the first assumptions we made was in regards to the utility function that represents the insured's behavior. We assumed a quadratic utility function, but a exponential utility function could also hold. So, one interesting topic to investigate could be how the choice of the utility function affects the question whether the insured decided to obtain a certain coverage, or whether they decide to not have any coverage for the risk X. Since we could try choose the parameters of the two utility functions in a way they deviate only very little from each other, they could both represent the behavior of the insured as both are assumed models for the behavior. These boundary cases could be very interesting.

Another assumption made was the linear structure of m. If we loosen this assumption, other notable scenarios occur. For example, allowing m to exponentially increase for one part of the domain, and exponentially decreasing on the other part. Also, combinations of linear behavior and exponential behavior are possible, such as m increasing linearly from 0 up to x_0, and decreasing exponentially from x_0 on.

This also has an impact on the later assumptions we had regarding the distribution function of the insurable risk X. If we choose m to be exponentially decreasing from x_0 on, such that there exists a lower boundary that $m(X)$ doesn't fall below, we can allow X

to be the non-truncated version of the random variable. This, of course, depends on the application and is very much dependent on the risk X.

In regards to X being a truncated random variable, the choice of D has an impact on the optimal solution as well. Different choices of D might lead to different values of d_1^* and d_2^*, or even “no coverage” as the optimal solution. Given a different reasoning, other D’s than the ones used in the numerical analysis are valid as well.

Taking a look at the numerical analysis, we quickly observe the following: Since the solver used to determine the values of d_1^* and d_2^* requires an initial guess for the solution, the output might depend on this choice. As it turns out, this is the case in our analysis, and therefore, there is some variation in the output values as there seem to be several solutions for the system of equations we have investigated. Furthermore, there are several tools available for solving the system of equations, and other methods might yield different results. With the above mentioned, the numerical analysis part might be adjustable in a way that it provides values for d_1^* and d_2^* that fulfill the requirements in [Chi and Wei, 2018] allowing for a statement about the optimal solution in the cases where the optimal solution is “insurance coverage”.

Overall, we see that there are many more aspects to consider, various methods to implement, and different assumptions possible that demand further research in this area.

Bibliography

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[Doherty and Schlesinger, 1983] Doherty, N. A. and Schlesinger, H. (1983). The optimal deductible for an insurance policy when initial wealth is random. *The Journal of Business*, 56(4):555–565.

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Appendix

A Code for Exponentially-distributed Risk X

```
# Numerical Analysis: Exponential Distribution

# package used for solving
library(rootSolve)

# constant parameters
theta <- 1
rho <- 0.1

# varying parameters
xzero <- c(0.5, 1, 1.5, 2.5)
m1smaller <- c(0.25, 0.5, 0.75, 1)
m2greater <- c(-0.5, -0.75, -1, -1.5, -2)

# set constant parameters
r <- rho
t <- theta
x <- xzero[1]

# find maximum among D's
Dall <- 1000
for(b in 1:4){
  for(c in 1:5){
m1 <- m1smaller[b]
m2 <- m2greater[c]
Dnew <- (1 - m1/m2)*x
if(Dnew < Dall){
Dall <- Dnew
    }
  }
}

# set D
D <- Dall

# initialize tables
table1 = matrix(0, nrow=4,ncol=5)
table2 = matrix(0, nrow=4,ncol=5)
tablelower = matrix(0, nrow=4,ncol=5)
tabletest = matrix(0, nrow=4,ncol=5)

# loop over m1 and m2
for(b in 1:4){
  for(c in 1:5){

# reset d
d <- c(0,0)

# set varying parameters
m1 <- m1smaller[b]
```

```
m2 <- m2greater[c]

# compute quantities from parameter values
FXD <- 1 - exp(-1/t*D)
Sinv <- -t*log(1-(1-(1/(1+r)))*FXD)

# lower boundary for d's
tablelower[b,c] <- Sinv

# right-hand side of the equations
right <- ((1+r)*m1/FXD*t
          +r*(m2-m1)/FXD*(x+t)*exp(-1/t*x)
          +r*(m1-m2)*x*(1-(1-exp(-1/t*x))/FXD)
          -r*m2/FXD*(D+t)*exp(-1/t*D))

# lower boundary for d's: S_{X}^{-1}(1/(1+rho))
lower <- Sinv

# upper boundary for d's: x_{0}
upper <- x

# initial value for solver
initial <- c(lower, upper)

# define function with parameter values
model <- function(d) {
  F1 <- (m1*(1-r^2-1/(1-(1-exp(-1/t*d[1]))/FXD))
          *(   1/FXD*((d[1]+t)*exp(-1/t*d[1])-(d[2]+t)*exp(-1/t*d[2]))
               -d[1]*((1-(1-exp(-1/t*d[1]))/FXD))
               +d[2]*((1-(1-exp(-1/t*d[2]))/FXD))    )
         +m1/FXD*(d[1]+t)*exp(-1/t*d[1])-right)
  F2 <- (m1*(1-r^2)
          *(   1/FXD*((d[1]+t)*exp(-1/t*d[1])-(d[2]+t)*exp(-1/t*d[2]))
               -d[1]*((1-(1-exp(-1/t*d[1]))/FXD))
               +d[2]*((1-(1-exp(-1/t*d[2]))/FXD)))
               -m1*(d[2]-d[1])
         +m1/FXD*(d[2]+t)*exp(-1/t*d[2])-right)
  c(F1 = F1, F2 = F2)
}

# find solutions
d1d2object <- multiroot(f = model, start = initial)
ds <- d1d2object$root

# save values in table
table1[b,c] <- ds[1]
table2[b,c] <- ds[2]

# expectation
e <- (  m1/FXD*(t-(x+t)*exp(-1/t*x))
        +(m1-m2)*x*(1-(1-exp(-1/t*x))/FXD)
        +m2/FXD*((x+t)*exp(-1/t*x)-(D+t)*exp(-1/t*D))
)

# test for E(g(X)) <= g(0)
if(e<=0){
  tabletest[b,c] <- 1
}

  }
}

# display tables
round(table1, digits = 4)
round(table2, digits = 4)
round(tablelower, digits = 4)
tabletest
```

B Code for Pareto-distributed Risk X

```
# Numerical Analysis: Pareto Distribution

# package used for solving
library(rootSolve)

# constant parameters
alpha <- 2
lambda <- 1
rho <- 0.1

# varying parameters
xzero <- c(1, 2, 4, 10)
m1smaller <- c(0.25, 0.5, 0.75, 1)
m2greater <- c(-0.5, -0.75, -1, -1.5, -2)

# set constant parameters
r <- rho
a <- alpha
l <- lambda
x <- xzero[1]

# find maximum among D's
Dall <- 1000
for(b in 1:4){
  for(c in 1:5){
    m1 <- m1smaller[b]
    m2 <- m2greater[c]
    Dnew <- (1 - m1/m2)*x
    if(Dnew < Dall){
      Dall <- Dnew
    }
  }
}

# set D
D <- Dall

# initialize tables
table1 = matrix(0, nrow=4,ncol=5)
table2 = matrix(0, nrow=4,ncol=5)
tablelower = matrix(0, nrow=4,ncol=5)
tabletest = matrix(0, nrow=4,ncol=5)

# loop over m1 and m2
for(b in 1:4){
  for(c in 1:5){

# reset d
d <- c(0,0)

# set varying parameters
m1 <- m1smaller[b]
m2 <- m2greater[c]

# compute quantities from parameter values
FXD <- 1 - (1+D/l)^(-a)
Sinv <- l*((1-(1-1/(1+r))*FXD)^(-1/a)-1)

# lower boundary for d's
tablelower[b,c] <- Sinv

# right-hand side of the equations
right1 <- 0+m1*(1+r)/FXD*l/(a-1)
right2 <- r*(m1-m2)*x*(1-(1-(1+x/l)^(-a))/FXD)
```

```
right3 <- r*(m2-m1)/FXD*(x*(1+x/l)^(-a)+l/(a-1)*(1+x/l)^(-a+1))
right4 <- -r*m2/FXD*(D*(1+D/l)^(-a)+l/(a-1)*(1+D/l)^(-a+1))
right <- right1 + right2 + right3 + right4

# lower boundary for d's: S_{X}^{-1}(1/(1+rho))
lower <- Sinv

# upper boundary for d's: x_{0}
upper <- x

# initial value for solver
initial <- c(lower, upper)

# define function with parameter values
model <- function(d) {
  F1 <- (  m1*(1-r^2-(1-(1-(1+d[1]/l)^(-a))/FXD)^(-1))*
              (
                1/FXD*(   d[1]*(1+d[1]/l)^(-a)
                         +l/(a-1)*(1+d[1]/l)^(-a+1)
                          -d[2]*(1+d[2]/l)^(-a)
                         -l/(a-1)*(1+d[2]/l)^(-a+1)
                       )
                -d[1]*(1-(1-(1+d[1]/l)^(-a))/FXD)
                +d[2]*(1-(1-(1+d[2]/l)^(-a))/FXD)
              )
            +m1/FXD*(d[1]*(1+d[1]/l)^(-a)+l/(a-1)*(1+d[1]/l)^(-a+1))
            - right)

  F2 <-  ( m1*(1-r^2)*
              (
                1/FXD*(    d[1]*(1+d[1]/l)^(-a)
                          +l/(a-1)*(1+d[1]/l)^(-a+1)
                          -d[2]*(1+d[2]/l)^(-a)
                          -l/(a-1)*(1+d[2]/l)^(-a+1)
                   )
                -d[1]*(1-(1-(1+d[1]/l)^(-a))/FXD)
                +d[2]*(1-(1-(1+d[2]/l)^(-a))/FXD)
              )
            -m1*(d[2]-d[1])
            +m1/FXD*(d[2]*(1+d[2]/l)^(-a)+l/(a-1)*(1+d[2]/l)^(-a+1))
            -right)
  c(F1 = F1, F2 = F2)
}

# find solutions
d1d2object <- multiroot(f = model, start = initial)
ds <- d1d2object$root

# save values in table
table1[b,c] <- ds[1]
table2[b,c] <- ds[2]

# expectation
e <- (
      m1/FXD*(l/(a-1)-x*(1+x/l)^(-a)-l/(a-1)*(1+x/l)^(-a+1))
     +r*(m1-m2)*x*(1-(1-(1+x/l)^(-a))/FXD)
     +m2/FXD*(  x*(1+x/l)^(-a)
                +l/(a-1)*(1+x/l)^(-a+1)
                -D*(1+D/l)^(-a)
                -l/(a-1)*(1+D/l)^(-a+1)
              )
     )

# test for E(g(X)) <= g(0)
if(e<=0){
  tabletest[b,c] <- 1
}

```

```
  }
}

# display tables
round(table1, digits = 4)
round(table2, digits = 4)
round(tablelower, digits = 4)
tabletest
```