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Optimal Deductibles: A Theoretical Analysis from an Insured's Perspective

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ABSTRACT

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A stop-loss policy as a tool for protection against a large loss is one of the most common insurance forms. For fixed premiums and therefore a uniquely determined insurance deductible, it has been well-established that the stop-loss form is superior to all other common insurance forms (Arrow, 1963). Using the expected premium principal, one can relax the assumption of a fixed premium and allow the insured to choose an arbitrary deductible that fits their needs.

This thesis presents a stop-loss insurance policy model from an insured's perspective for a flexible premium. It shows the existence and uniqueness of an optimal deductible for a single risk model and derives several properties of the optimal deductibles in a bivariate excess-of-loss risk model where the insured faces two risks. The theoretical analysis is exemplified by several utility concepts which do not only illustrate the overall results but also give insights in the necessity of insurance and the influence of the risk structure on the findings.

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Introduction

Stop-loss insurance as protection against large losses is one of the most common policies for insurance contracts. This intuitive form, which takes effect after a prearranged retention level, the so-called deductible, prevents the insured from extreme losses by transferring this part of the risk to the insurance carrier. Naturally, this insurance form is favored if there are large claims possible. For example, individual risk takers use it for car insurance and householder's comprehensive policies. Furthermore, companies providing health insurance often insure themselves with stop-loss policies to avoid catastrophic claims.

Once a risk undertaker takes a stop-loss insurance contract into account, he has to decide for a deductible. Since the insurance premium he has to pay naturally increases with the portion of the risk he transfers, the risk undertaker wants to find an optimal deductible in this trade-off situation. The optimal deductible might be influenced by the personal risk preference of the risk undertaker as well as the structure of the risk.

These considerations can be extended if there is an insured that faces several different risks he wants to insure against. For instance, a car owner that holds multiple cars has to think about how much money he is willing to spend to insure his property. But since there are multiple risks, the question arises how to spread the financial resources over the risks, i.e. how to choose the deductibles. Apart from the individual's personal risk preference, the answer to this question could be influenced by the relative value of the cars, for instance if one car is higher-priced than another car.

In this thesis, we start by introducing two common utility concepts, the exponential utility and the quadratic utility. Furthermore, we derive some basic properties of risks and their modifications. In the third chapter, we motivate and introduce the single risk model for stop-loss policies by placing it in context with other research results. We formally state the optimization problem of finding an optimal deductible, prove that there exists a unique optimal solution and derive some stronger results by specifying the utility function or the distribution. In the forth chapter, we expand the model by including a second risk. In this so-called excess-of-loss model, we derive properties of an optimal solution of the optimization problem. By specifying the utility function, we can prove the existence and uniqueness of an optimal solution if the risks are independent. In the final part of the thesis, we discuss the implications of the results and introduce potential further ideas of interest in this research area.

Preliminaries

In this chapter, we examine necessary results for the thesis.

2.1 Utility Functions

In our mathematical framework, we will use the utility function concept to compare the goodness of insurance contracts for the insured. We introduce the common principle of a risk-averse individual who buys the insurance policy. In particular, we want a utility function $u(.)$ that fulfills $u'(.) > 0$ as well as $u''(.) < 0$, i.e. it is a strictly increasing concave function. We will discuss results for two explicit utility functions we are going to introduce in the next subsection.

2.1.1 Exponential Utility

We introduce the exponential utility function $u(x) = 1 - e^{-\gamma x}$, where $\gamma > 0$ is a parameter that measures the risk-aversion of the decision-maker. Consequently, it is $u'(x) = \gamma e^{-\gamma x} > 0$ and $u''(x) = -\gamma^2 e^{-\gamma x} < 0$ and the postulated properties of a strictly increasing concave utility function are fulfilled.

Moreover, the exponential utility concept has an interesting feature because it implies a constant absolute risk aversion, that is

$$-\frac{u''(x)}{u'(x)} = \gamma.$$

This property makes decisions independent of the initial wealth a risk undertaker has.

2.1.2 Quadratic Utility

As a special form of the class of power utility functions, we denote with $u(x) = -(\eta - x)^2$ the quadratic utility function, where $\eta > 0$ is a positive parameter and we restrict the domain of $u(.)$ to numbers $x < \eta$. We obtain $u'(x) = 2(\eta - x) > 0$ and $u''(x) = -2 < 0$, i.e. the quadratic utility function is strictly increasing concave.

Furthermore, the absolute risk aversion is

$$-\frac{u''(x)}{u'(x)} = \frac{1}{\eta - x}$$

which is increasing in x. Therefore, decisions under this utility function are influenced by the initial wealth.

In general, there are two disadvantages of the quadratic utility function as we introduced it. First of all, the restricted domain allows us only considerations where the maximum final wealth is limited. In our setup, the insured owns an initial wealth and buys an insurance contract to avoid large losses from his risks. In particular, the final wealth is restricted to the initial wealth and there are no problems as long as we choose the parameter η high enough. Secondly, the increasing risk aversion is in conflict with empirical studies from the reality. Nevertheless, we decided to use this utility function as an alternative to the exponential utility principle.

2.2 Properties of Risks and their modifications

In this thesis, we model risks with a random variable X. We assume $X \geq 0$ to be positive and continuous with density function $f_X(x)$. We denote with $S_X(x) = \mathbb{P}(X > x)$ the survival function of X.

Furthermore, for a given $d \geq 0$, we define the following expressions:

1. $(X \wedge d) \coloneqq \min\{X, d\}$

2. $(X - d)_+ \coloneqq \max\{X - d, 0\}$

We use the following observations for calculations in this thesis:

Lemma 2.1. *For a risk $X \geq 0$ and a constant $d \geq 0$ it holds:*

1. $\mathbb{E}[X] = \mathbb{E}[(X \wedge d)] + \mathbb{E}[(X - d)_+]$

2. $\frac{\partial}{\partial d}\mathbb{E}[(X \wedge d)] = S_X(d)$

3. $\frac{\partial}{\partial d}\mathbb{E}[(X - d)_+] = -S_X(d)$

Proof:

1. It is

$$
\begin{aligned}
\mathbb{E}[(X \wedge d)] + \mathbb{E}[(X - d)_+] &= \int_0^\infty \min\{x, d\} f_X(x)dx + \int_0^\infty \max\{x - d, 0\} f_X(x)dx \\
&= \int_0^d x \; f_X(x)dx + \int_d^\infty d \; f_X(x)dx \\
&\quad + \int_d^\infty x \; f_X(x)dx - \int_d^\infty d \; f_X(x)dx \\
&= \int_0^\infty x \; f_X(x)dx \\
&= \mathbb{E}[X]
\end{aligned}
$$

2. Since $f(x, d) \coloneqq x f_X(x)$ as well as $\frac{\partial}{\partial d} f(x, d) = 0$ are both continuous in x and d, we can apply Leibniz rule to calculate the term $\frac{\partial}{\partial d}\int_0^d x f_X(x)dx$:

$$\begin{aligned}
\frac{\partial}{\partial d}\mathbb{E}[(X \wedge d)] =& \frac{\partial}{\partial d}\int_0^d x\ f_X(x)dx + \frac{\partial}{\partial d}\int_d^\infty d\ f_X(x)dx \\
=& d\ f_X(d)\cdot 1 - d\ f_X(d)\cdot 0 + \int_0^d \frac{\partial}{\partial d}x\ f_X(x)dx + \frac{\partial}{\partial d}\int_d^\infty f_X(x)dx \\
=& d\ f_X(d) + 1\cdot S_X(d) - d\ f_X(d) \\
=& S_X(d)
\end{aligned}$$

3. Follows from the two previous parts together with the fact that $\frac{\partial}{\partial d}\mathbb{E}[X] = 0$.

□

In the later part of the thesis, we are going to compare deductible for different risks. Therefore, we introduce the following notation:

Definition 2.2. *Let X_1 and X_2 be two risks.*
We say X_1 is stochastically greater than X_2 if $F_{X_1}(x) \leq F_{X_2}(x)$ for all $x \in \mathbb{R}$ and there exists at least one $x_0 \in \mathbb{R}$ where $F_{X_1}(x_0) < F_{X_2}(x_0)$.
Denote $X_1 \succeq X_2$.

Single Risk Model

3.1 Model Introduction

We start by considering a model with a univariate risk X we want to insure against. Our goal is to find an optimal insurance strategy for the insured. We use the following general mathematical model:

We denote with $I : \mathbb{R} \to \mathbb{R}$ an insurance strategy. With this strategy, the risk undertaker retains $I(X)$ to himself while he cedes the remaining part $\tilde{I}(X) = X - I(X)$. In order to transfer parts of his risk, the risk undertaker pays a risk-specific premium $\pi(X)$. In our framework, we use the common expected premium principle, i.e. the charged premium equals the expected value of the risk plus a safety loading θ. Applying this principle to the ceded risk $\tilde{I}(X)$, we obtain a premium of $\pi(\tilde{I}(X)) = (1+\theta)\mathbb{E}[\tilde{I}(X)] = (1+\theta)\mathbb{E}[X - I(X)]$.

Our goal is now to optimize the insurance contract I for the insured in this framework. There are different interpretations of the term optimization. Cai and Tan (2007) analyzed a similar model to the one we will introduce where the insured seeks a minimization of the value-at-risk and the conditional tail expectation. In this thesis, we will focus on another approach following the expected utility concept. This approach has been well-established and is commonly used (Huang et al., 2013). We assume that the insured has a monotone strictly increasing and concave utility function u and he wants to maximize his expected utility function.

Mathematically, the optimization problem is formulated as

$$\max_{I \in \mathcal{D}} \; \mathbb{E}\big[u(w - I(X) - (1+\theta)\mathbb{E}[\tilde{I}(X)])\big]$$

where w is the initial wealth of the insured before entering the contract and $\mathcal{D}$ is the set of all possible insurance strategies.

If we restrict $\mathcal{D}$ to a common strategy class, we obtain a framework that has been well-established. One main result of the research is that the stop-loss insurance is optimal in this setup (Arrow, 1963). Therefore, the optimal strategy has the form $I(X) = X \wedge d$ where d is the deductible or retention level. Consequently, we obtain $\mathbb{E}[\tilde{I}(X)] = \mathbb{E}[X - X \wedge d] = \mathbb{E}[(X-d)_+]$ and the optimization problem reduces to finding an optimal d. Consequently, the optimization problem is

$$\max_{d \geq 0} \; \mathbb{E}\big[u(w - X \wedge d - (1+\theta)\mathbb{E}[(X-d)_+])\big] = \max_{d \geq 0} L(d) \tag{3.1}$$

where $L(d) \coloneqq \mathbb{E}\big[u(w - X \wedge d - (1+\theta)\mathbb{E}[(X-d)_+])\big]$ is the expected utility of the insured if he buys the stop-loss insurance policy with deductible d.

3.2 Model Analysis

In this section, we are interested in finding an optimal solution to (3.1). We will find out that there exists a unique optimal retention level d^* that maximizes $L(d)$.

Denote with $W_d(X) = w - X \wedge d - (1+\theta)\mathbb{E}[(X-d)_+]$ the wealth of the insured after the realization of the risk X. For optimization, we consider the derivative with respect to d. It might be that the functions $u(.)$ and $W_d(X)$ are only right-side differentiable. In that case, we consider the right-side derivative instead.

Cai and Wei (2012) show that under appropriate conditions for $u()$ and X, it holds

$$L'(d) = \mathbb{E}\big[u'(W_d(X))\big] = \mathbb{E}\big[u'(W_d(X))W_d'(X)\big]$$

where the (right-side) derivatives $L'(d)$ and $W_d'(X)$ are with respect to d.

Observing that

$$W_d'(x) = 0 - \mathbb{1}_{\{X>d\}} - (1+\theta)\big(\mathbb{E}[X] - \mathbb{E}[X \wedge d]\big)' = -\mathbb{1}_{\{X>d\}} + (1+\theta)S_X(d)$$

allows us to calculate

$$\begin{aligned} L'(d) &= \mathbb{E}\big[u'(W_d(X))((1+\theta)S_X(d) - \mathbb{1}_{\{X>d\}})\big] \\ &= S_X(d)\mathbb{E}\big[u'(W_d(X))\big]\left((1+\theta) - \frac{\mathbb{E}\big[u'(W_d(X))|X>d\big]}{\mathbb{E}\big[u'(W_d(X))\big]}\right) \\ &= S_X(d)\mathbb{E}\big[u'(W_d(X))\big]\big(1+\theta-\Phi(d)\big) \end{aligned}$$

where $\Phi(d) = \dfrac{\mathbb{E}\big[u'(W_d(X))|X>d\big]}{\mathbb{E}\big[u'(W_d(X))\big]}$.

Using this explicit form of $L'(d)$, we find a lower limiting value for the optimal deductible:

Lemma 3.1. *Let d^* be an optimal solution to* (3.1)*. Then it holds $d^* \geq S_X^{-1}(\frac{1}{1+\theta})$.*

Proof: Let $d < S_X^{-1}(\frac{1}{1+\theta})$ be a smaller retention level. Then

$$L'(d) = \mathbb{E}\big[u'(W_d(X))((1+\theta)S_X^{-1}(d) - \mathbb{1}_{\{X>d\}})\big] > \mathbb{E}\big[u'(W_d(X))((1+\theta)\frac{1}{1+\theta} - \mathbb{1}_{\{X>d\}})\big] \geq 0.$$

The first inequality holds because of $d < S_X^{-1}(\frac{1}{1+\theta})$ while the second inequality holds due to $u'(.) > 0$.

Consequently, $L(d)$ is a strictly increasing function in d for $d < d^*$. Therefore, the function $L(d)$ reaches its maximum for a $d^* \geq S_X^{-1}(\frac{1}{1+\theta})$. □

We now analyze the behavior of the function $\Phi(d)$. The proof of the following lemma is similar to Chi and Wei (2018, pp.12–13) who discuss a more general framework.

Lemma 3.2. *As a function of d, $\Phi(d)$ is strictly increasing in d for $d > S_X^{-1}(\frac{1}{1+\theta})$.*

Proof: We define $g_1(d) = \mathbb{E}[u'(W_d(X))|X > d]$ and $g_2(d) = \mathbb{E}[u'(W_d(X))]$.
Then $\Phi(d) = \frac{g_1(d)}{g_2(d)}$.

We want to show that $\Phi(d)'(g_2(d))^2 = g_1'g_2 - g_1g_2' > 0$.

Now

$$\begin{aligned}
g_2'(d) =& \big(\mathbb{E}[u'(W_d(x))]\big)' = \mathbb{E}[u''(W_d(X))W_d'(X)] \\
=& \mathbb{E}\big[u''(W_d(X))((1+\theta)S_X(d) - \mathbb{1}_{\{X>d\}})\big] \\
=& S_X(d)(1+\theta)\mathbb{E}[u''(W_d(X))] - \mathbb{E}[u''(W_d(X))\mathbb{1}_{\{X>d\}}] \\
\leq& S_X(d)(1+\theta)\mathbb{E}[u''(W_d(X))\mathbb{1}_{\{X>d\}}] - \mathbb{E}[u''(W_d(X))\mathbb{1}_{\{X>d\}}] \\
=& \big((1+\theta)S_X(d) - 1\big)\mathbb{E}\big[u''(W_d(X))\mathbb{1}_{\{X>d\}}\big].
\end{aligned}$$

The inequality holds because of $u''(.) < 0$.

To analyze the derivative of $g_1(d)$, we define the function

$$l(x,y) = \mathbb{E}\big[u'(w - x - (1+\theta)\mathbb{E}[(X-d)_+])|X > y\big].$$

One can easily see that $g_1(d) = l(d,d)$.

We observe that $l(x,y)$ is increasing in y, i.e. $\frac{\partial}{\partial y}l(x,y) \geq 0$.

On the other hand, it is

$$\begin{aligned}
\frac{\partial}{\partial x}l(x,y) =& \mathbb{E}\big[u''(w - x - (1+\theta)\mathbb{E}[(X-d)_+])|X > d\big]\frac{\partial}{\partial x}\big(w - x - (1+\theta)\mathbb{E}[(X-d)_+]\big) \\
=& \mathbb{E}\big[u''(w - x - (1+\theta)\mathbb{E}[(X-d)_+])|X > d\big]\big(0 - 1 + (1+\theta)S_X(d)\big) \\
=& \mathbb{E}\big[u''(w - x - (1+\theta)\mathbb{E}[(X-d)_+])|X > d\big]\big((1+\theta)S_X(d) - 1\big).
\end{aligned}$$

We obtain

$$
\begin{aligned}
g_1'(d) =& \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) l(x,y)|_{(x,y)=(d,d)} \\
\geq& \mathbb{E}\big[u''(W_d(X))|X > d\big]\big((1+\theta)S_X(d) - 1\big).
\end{aligned}
$$

Finally

$$
\begin{aligned}
\Phi(d)'(g_2(d))^2 =& g_1' \cdot g_2 - g_1 \cdot g_2' \\
\geq& \mathbb{E}\big[u''(W_d(X))|X > d\big]\big((1+\theta)S_X(d) - 1\big) \cdot \mathbb{E}\big[u'(W_d(X))\big] \\
& - \mathbb{E}\big[u'(W_d(X))|X > d\big] \cdot \big((1+\theta)S_X(d) - 1\big)(1+\theta)\mathbb{E}\big[u''(W_d(X))\mathbb{1}_{\{X>d\}}\big] \\
=& \big((1+\theta)S_X(d) - 1\big) \Big\{\mathbb{E}\big[u'(W_d(X))\big] \cdot \mathbb{E}\big[u''(W_d(X))|X > d\big] \\
& - \mathbb{E}\big[u'(W_d(X))|X > d\big] \cdot \mathbb{E}\big[u''(W_d(X))\mathbb{1}_{\{X>d\}}\big]\Big\} \\
=& \big((1+\theta)S_X(d) - 1\big)\ \mathbb{E}\big[u''(W_d(X))|X > d\big] \\
& \Big\{\mathbb{E}\big[u'(W_d(X))\big] - \mathbb{E}\big[u'(W_d(X))\mathbb{1}_{\{X>d\}}\big]\Big\} \\
=& \big((1+\theta)S_X(d) - 1\big)\ \mathbb{E}\big[u''(W_d(X))|X > d\big]\mathbb{E}\big[u'(W_d(X))\mathbb{1}_{\{X\leq d\}}\big] \\
>& 0.
\end{aligned}
$$

The last inequality holds because of $d \geq S_X^{-1}(\frac{1}{1+\theta})$ and $u''() < 0$.

Therefore, $\Phi'(d) > 0$. □

Theorem 3.3. *There exists a unique solution d^* to the optimization problem* (3.1)*.*

Proof: From Lemma 3.1 we know that an optimal solution d^* that maximizes $L(d)$ satisfies $d^* \geq S_X^{-1}(\frac{1}{1+p})$. According to Lemma 3.2, it is $L'(d) = S_X(d)\mathbb{E}\big[u'(W_d(X))\big]\big(1+\theta-\Phi(d)\big)$ where $\Phi(d)$ is a strictly increasing function in d for $d > S_X^{-1}(\frac{1}{1+p})$. Therefore, the function $L(d)$ reaches its global maximum for a unique $d^* \geq S_X^{-1}(\frac{1}{1+\theta})$. □

Remark 3.4. *The optimal solution might be $d^* = \infty$. In practice, this means that the risk owner rejects any insurance contract as optimal solution.*

3.3 Examples

In this section, we specify the utility function $u(.)$ to derive more precise results about the optimal solution d^*.

Example 3.5. *Let $u(x) = 1 - e^{-\gamma x}$ be the exponential utility function and the risk $X \sim Exp(\lambda)$ be exponentially distributed.*

1. *The optimal deductible d^* satisfies $d^* < \infty$.*

2. *For $\gamma > \frac{1}{\lambda}$, there is a negative relationship between λ and d^*.*

Proof:

1. We calculate

$$\begin{aligned}\Phi(d) =&\frac{\mathbb{E}[-\gamma e^{-\gamma W_d(X)}|X > d]}{\mathbb{E}[\gamma e^{-\gamma W_d(X)}]}\\ =&\frac{\mathbb{E}[e^{-\gamma(w-d-(1+\theta)\mathbb{E}[(X-d)_+])}]}{\mathbb{E}[e^{-\gamma(w-X\wedge d-(1+\theta)\mathbb{E}[(X-d)_+])}]}\\ =&\frac{e^{\gamma d}}{\mathbb{E}[e^{\gamma(X\wedge d)}]}.\end{aligned}$$

   Since the risk X is exponentially distributed, one now can show (see Appendix) that if $\lambda \neq \frac{1}{\gamma}$, it is

$$\mathbb{E}[e^{\gamma(X\wedge d)}] = \frac{1}{\lambda\gamma - 1}(e^{d(\gamma-1/\lambda)} - 1) \tag{3.2}$$

and therefore

$$\Phi(d) = \frac{\lambda\gamma - 1}{\lambda\gamma e^{-d/\lambda} - e^{-d\gamma}}.$$

Now the optimal deductible d^* is infinity if and only if it holds $\Phi(d) < 1 + \theta$ for all d. But since it is

$$\lim_{d\to\infty} \Phi(d) = \infty$$

we can conclude that $d^* < \infty$.

If $\lambda = \frac{1}{\gamma}$, the same procedure yields (see Appendix) $\mathbb{E}[e^{\gamma(X \wedge d)}] = d\gamma + 1$ and therefore $\Phi(d) = \frac{e^{\gamma d}}{\gamma d + 1}$. Using l'Hospital rule, one can conclude that $\lim_{d\to\infty} \Phi(d) = \infty$. Therefore, $d^* < \infty$ holds.

2. We differentiate $\Phi(d) = \frac{\lambda\gamma - 1}{\lambda\gamma e^{-d/\lambda} - e^{-d\gamma}}$ with the quotient rule:

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\Phi(d) \ \ (\lambda\gamma e^{-d/\lambda} - e^{-d\gamma})^2 =& (\lambda\gamma e^{-d/\lambda} - e^{-d\gamma})\gamma - (\lambda\gamma - 1)(\gamma e^{-d/\lambda} - \frac{\gamma}{\lambda} e^{-d/\lambda}) \\
=& \lambda\gamma^2 e^{-d/\lambda} - \gamma e^{-d\gamma} + \gamma e^{-d/\lambda} - d\frac{\gamma}{\lambda} e^{-d/\lambda} - \gamma^2\lambda e^{-d/\lambda} + d\gamma^2 e^{-d/\lambda} \\
=& (d\gamma - \frac{d}{\lambda}) e^{-d/\lambda} + e^{-d/\lambda} - e^{-\gamma d}. \qquad (3.3)
\end{aligned}$$

The first term of (3.3) is positive for $\gamma > 1/\lambda$. The second and the third term add to a positive number for $\gamma > 1/\lambda$. Consequently, $\Phi(d)$ is increasing in λ for $\gamma > 1/\lambda$. Regarding the fact that an increase in $\Phi(d)$ coincides with a smaller optimal retention level d^*, this proves the second part of the theorem.

□

Example 3.6. *Let $u(x) = -(x - \eta)^2$ be the quadratic utility function. Then the optimal deductible d^* satisfies $d^* < \infty$.*

Proof: We calculate

$$\begin{aligned}\Phi(d) =& \frac{2\big(\eta - \mathbb{E}[W_d(X)|X > d]\big)}{2\big(\eta - \mathbb{E}[W_d(X)]\big)}\\ =& \frac{\eta - w + d + (1+\theta)\mathbb{E}[(X-d)_+]}{\eta - w + E[(X \wedge d)_+] + (1+\theta)\mathbb{E}[(X-d)_+]}.\end{aligned}$$

Again, the optimal deductible d^* equals ∞ if and only if it holds $\Phi(d) < 1+\theta$ for all d. But this is not the case: Since $\mathbb{E}[(X-d)_+]$ approaches 0 for high d and $\mathbb{E}[X \wedge d] \leq \mathbb{E}[X]$ is limited, it follows that $\Phi(d)$ is turning to ∞ for large d. The same argument as for the prior example utility function completes the proof. □

Bivariate Risk Model

4.1 Model Introduction

We now want to examine equivalent results in a model with multiple risks instead of only one. In particular, we will study two independent risks X_1 and X_2. These risks are insured with two different contracts, i.e. the insured chooses two strategies $I_1, I_2 \in D$ from the set of allowed insurance strategies. The optimal solution to the corresponding optimization problem depends on the relationship between the two risk.

If X_1 and X_2 are independent or positively dependent, Cai and Wei (2012) show that an insurance policy with two separate stop-loss insurance contracts with the deductibles d_1 and d_2 is optimal, a so-called excess-of-loss strategy.

Following the ideas and notation from the previous section, we can state the optimization problem for the bivariate risk model:

$$\begin{aligned} &\max_{d_1,d_2\geq 0} \; \mathbb{E}\big[u(w - X_1 \wedge d_1 - X_2 \wedge d_2 - (1+\theta)\mathbb{E}[(X_1-d_1)_+] - (1+\theta)\mathbb{E}[(X_2-d_2)_+])\big] \\ &= \max_{d_1,d_2\geq 0} L(d_1,d_2) \qquad (4.1) \end{aligned}$$

where $L(d_1,d_2) := \mathbb{E}\big[u(w - X_1 \wedge d_1 - X_2 \wedge d_2 - (1+\theta)\mathbb{E}[(X_1-d_1)_+] - (1+\theta)\mathbb{E}[(X_2-d_2)_+])\big]$ is the expected utility of the insured if he enters an excess-of-loss insurance policy with deductibles d_1 and d_2.

Equivalently to the univariate model, we denote with

$$W_{d_1,d_2}(X_1,X_2) = w - X_1 \wedge d_1 - X_2 \wedge d_2 - (1+\theta)\mathbb{E}[(X_1-d_1)_+] - (1+\theta)\mathbb{E}[(X_2-d_2)_+] \tag{4.2}$$

the wealth of the insured for realizations of the risks X_1 and X_2.

4.2 Model Analysis

Using the results from the preliminaries, we can calculate the partial derivatives for the function $L_{d_1,d_2}(X_1,X_2)$:

$$\begin{aligned}
\frac{\partial}{\partial d_1}L(d_1,d_2) &= \mathbb{E}\big[u'(W_{d_1,d_2}(X_1,X_2))\frac{\partial}{\partial d_1}W_{d_1,d_2}(X_1,X_2)\big] \\
&= \mathbb{E}\big[u'(W_{d_1,d_2}(X_1,X_2))(-\mathbb{1}_{\{X_1>d_1\}} + (1+\theta)S_{X_1}(d_1))\big] \\
&= S_{X_1}(d_1)\mathbb{E}\big[u'(W_{d_1,d_2}(X_1,X_2))\big]\left((1+\theta) - \frac{\mathbb{E}[u'(W_{d_1,d_2}(X_1,X_2))|X_1>d_1]}{\mathbb{E}[u'(W_{d_1,d_2}(X_1,X_2))]}\right) \\
&= S_{X_1}(d_1)\mathbb{E}\big[u'(W_{d_1,d_2}(X_1,X_2))\big]\big(1+\theta-\Phi_{d_2}(d_1)\big)
\end{aligned}$$

where $\Phi_{d_2}(d_1) = \frac{\mathbb{E}[u'(W_{d_1,d_2}(X_1,X_2))|X_1>d_1]}{\mathbb{E}[u'(W_{d_1,d_2}(X_1,X_2))]}$.

Computing $\frac{\partial}{\partial d_2}L(d_1,d_2)$, we can derive a symmetric result with analogous definition of $\Phi_{d_1}(d_2)$.

We now investigate results for the partial derivatives $\Phi_{d_1}(d_2)$ and $\Phi_{d_1}(d_2)$ that are in some way equivalent to Lemma 3.1 and Lemma 3.2 from the univariate case:

Lemma 4.1. *Let (d_1^*, d_2^*) be an optimal solution to the optimization problem in the independent case. Then it fulfills $d_i^* \geq S_{X_i}^{-1}(\frac{1}{1+\theta})$ where $i \in \{1,2\}$.*

Lemma 4.2. *Let $i,j \in \{1,2\},\ i \neq j$. As a function of d_i, $\Phi_{d_j}(d_i)$ is strictly increasing in d_i for $d_i > S_{X_i}^{-1}(\frac{1}{1+\theta})$ and fixed d_j.*

The proofs of these two lemmas work equivalently to the ones we gave in the univariate model. Using the fact that

$$\frac{\partial}{\partial d_i} W(d_1, d_2) = -\mathbb{1}_{\{X_i > d_i\}} + (1+\theta) S_{X_i}(d_i)$$

is independent of d_j, we can use the same argument as in the proof of Lemma 3.1 to prove Lemma 4.1.

For Lemma 4.2, when considering $\Phi_{d_2}(d_1)$, the independence of X_1 and X_2 allows us to treat the terms X_2 and $X_2 \wedge d_2$ as constants even if they are conditioned on X_1. In particular, when computing the partial derivative with respect to d_1, all ideas from the proof of Lemma 3.2 can be applied where we differentiated with respect to d. One can show that for $g_1(d_1) := \mathbb{E}\big[u'(W_{d_1,d_2}(X_1, X_2))\big]$, it holds

$$\frac{\partial}{\partial d_1} g_1(d_1) \leq \big((1+\theta) S_{X_1}(d_1) - 1\big) \mathbb{E}\big[u''(W_{d_1,d_2}(X_1, X_2)) \mathbb{1}_{X_1 > d_1}\big]$$

and on the other hand, for $g_2(d_1) := \mathbb{E}\big[u'(W_{d_1,d_2}(X_1, X_2)) | X_1 > d_1\big]$, it holds

$$\frac{\partial}{\partial d_1} g_2(d_1) \geq \mathbb{E}\big[u''(W_{d_1,d_2}(X))\big] \big((1+\theta) S_X(d) - 1\big).$$

Since it is $\Phi_{d_2}(d_1) = \frac{g_1(d_1)}{g_2(d_2)}$, an equivalent estimate to the one in the final part of Lemma 4.2 proves the result after repeating this procedure with changed roles of d_1 and d_2.

As contrasted with the single risk model, we cannot prove the existence of a unique solution in the bivariate case with these two lemmas. Instead, we get the following equation system a potential finite optimal solution has to satisfy:

Remark 4.3. *If an optimal solution* (d_1^*, d_2^*) *to the optimization problem* (4.1) *fulfills* $d_1^* < \infty, d_2^* < \infty$*, then it holds*

$$\Phi_{d_2}(d_1^*) = \Phi_{d_1}(d_2^*) = 1 + \theta. \tag{4.3}$$

Proof: $L(d_1, d_2)$ is a function of two variables $d_1, d_2 \geq 0$ and reaches its maximum somewhere in the open interval $(0, \infty) \times (0, \infty)$. Now since (d_1^*, d_2^*) is a global maximum (and therefore a local maximum) and $L(d_1, d_2)$ is defined on a surrounding of (d_1^*, d_2^*), the partial derivatives of the function $L(d_1, d_2)$ have to equal zero at (d_1^*, d_2^*). According to prior calculations, this is the case if and only if $\Phi_{d_1}(d_2^*) = \Phi_{d_2}(d_1^*) = 1 + \theta$. □

In the bivariate risk model, there arises naturally the question about the relationship between the deductibles. We find the following result:

Theorem 4.4. *Let* X_1 *and* X_2 *be independent risks, where* X_1 *is stochastically greater than* X_2*, i.e.* $X_1 \succeq X_2$ *. If there exists a finite optimal solution* (d_1^*, d_2^*)*, then it is* $d_1^* \geq d_2^*$*.*

Proof: According to Remark 4.3, the solution (d_1^*, d_2^*) satisfies

$$\Phi_{d_2}(d_1^*) = \frac{\mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))|X_1 > d_1^*\big]}{\mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))\big]} = \frac{\mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))|X_2 > d_2^*\big]}{\mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))\big]} = \Phi_{d_1}(d_2^*).$$

Consequently, it holds

$$\mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))|X_1 > d_1^*\big] = \mathbb{E}\big[u'(W_{d_1^*,d_2^*}(X_1, X_2))|X_2 > d_2^*\big]. \tag{4.4}$$

We now define the two random variables $Y_1 = d_1^* + X_2 \wedge d_2^*$ and $Y_2 = d_2^* + X_1 \wedge d_1^*$. Together with the constant $c = w - (1 + \theta)\mathbb{E}[(X_1 - d_1^*)_+] - (1 + \theta)\mathbb{E}[(X_1 - d_1^*)_+]$ and the definition of $W_{d_1^*,d_2^*}(X_1, X_2)$ we gave at the beginning of this chapter, we can rewrite (4.4) as

$$\mathbb{E}[u'(Y_1 + c)] = \mathbb{E}[u'(Y_2 + c)]. \tag{4.5}$$

Now we assume that $d_1^* < d_2^*$.

Then because of $X_1 \succeq X_2$, we can conclude that $(X_1 - d_1^*)_+ \succeq (X_2 - d_2^*)_+$.

It follows

$$Y_1 = d_1^* + X_2 \wedge d_2^* = d_1^* + d_2^* - (d_2^* - X_2)_+ \succeq d_1^* + d_2^* - (d_1^* - X_1)_+ = d_2^* + X_1 \wedge d_1^* = Y_2.$$

Since c is constant, it also holds $Y_1 + c \succeq Y_2 + c$.

Furthermore, we assumed that $u'()$ is strictly decreasing, which implies $u'(Y_1+c) \preceq u'(Y_2+c)$.

But then it follows

$$\mathbb{E}[u'(Y_1 + c)] < \mathbb{E}[u'(Y_2 + c)]$$

which is a contradiction to (4.5).

Therefore, it is $d_1^* \geq d_2^*$. □

4.3 Examples

Example 4.5. *Let $u(x) = 1 - e^{-\gamma x}$ be the exponential utility function. For independent risks X_1 and X_2 there exists a unique optimal solution (d_1^*, d_2^*) to the optimization problem* (4.1).

Proof: Using $u'(x) = \gamma e^{-\gamma x}$, we can compute

$$\begin{aligned}
\Phi_{d_2}(d_1) =& \frac{\mathbb{E}[\gamma e^{-\gamma(W_{d_1,d_2}(X_1,X_2))} | X_1 > d_1]}{\mathbb{E}\big[\gamma e^{-\gamma(W_{d_1,d_2}(X_1,X_2))}\big]} \\
=& \frac{\mathbb{E}[e^{-\gamma(w - X_1\wedge d_1 - X_2\wedge d_2 - (1+\theta)\mathbb{E}[\gamma(X_1-d_1)_+ - (1+\theta)\mathbb{E}[(X_2-d_2)_+])} | X_1 > d_1]}{\mathbb{E}\big[e^{-\gamma(w - X_1\wedge d_1 - X_2\wedge d_2 - (1+\theta)\mathbb{E}[(X_1-d_1)_+ - (1+\theta)\mathbb{E}[(X_2-d_2)_+]))}\big]} \\
=& \frac{\mathbb{E}[e^{-\gamma(-X_2\wedge d_2 - (1+\theta)]\mathbb{E}[(X_2-d_2)_+])}\big]\mathbb{E}[e^{-\gamma(w - X_1\wedge d_1 - (1+\theta)\mathbb{E}[(X_1-d_1)_+])} | X_1 > d_1]}{\mathbb{E}\big[e^{-\gamma(-X_2\wedge d_2 - (1+\theta)\mathbb{E}[(X_2-d_2)_+])}\big]\mathbb{E}\big[e^{-\gamma(w - X_1\wedge d_1 - (1+\theta)\mathbb{E}[(X_1-d_1)_+])}\big]} \\
=& \frac{\mathbb{E}[u'(W_{d_1}(X_1)) | X_1 > d_1]}{\mathbb{E}[u'(W_{d_1}(X_1))]} = \Phi(d_1).
\end{aligned}$$

We used that X_1 and X_2 are independent while w, $\mathbb{E}[(X_1 - d_1)_+]$ and $\mathbb{E}[(X_2 - d_2)_+]$ are constants.

As we can see, $\Phi_{d_2}(d_1) = \Phi(d_1)$ is independent of d_2 and equivalent to $\Phi(d)$ from the single risk model. We know that an optimal solution (d_1^*, d_2^*) has to satisfy $\Phi_{d_2}(d_1^*) = \Phi_{d_1}(d_2^*) = 1 + \theta$. But since $\Phi_{d_j}(d_i) = \Phi(d_i)$ is independent of d_j, the bivariate optimization problem reduces here to two univariate optimization problems. Using the results from the previous section, in particular Theorem 3.3, there is a unique solution. $\square$

This heritage of the independence for the exponential utility function allows us to use Example 3.5 to derive the following result:

Remark 4.6. *Let $u(x) = 1 - e^{-\gamma x}$ be the exponential utility function and the independent risks $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$ be exponentially distributed.*

1. *There exists a unique optimal solution (d_1^*, d_2^*) to the optimization problem (4.1) which is finite.*
2. *For each $i \in \{1, 2\}$, there is a negative relationship between λ_i and d_i^* if $\gamma > \frac{1}{\lambda_i}$.*

Example 4.7. *Let $u(x) = -(\eta - x)^2$ be the quadratic utility function. Let the risks X_1 and X_2 be independent. Then the optimization problem (4.1) has a unique optimal finite solution (d_1^*, d_2^*).*

Proof: We start by calculating the explicit expression $\Phi_{d_2}(d_1)$:

$$\begin{aligned}
&\Phi_{d_2}(d_1) \\
&= \frac{2\big(\eta - w + \mathbb{E}[X_1 \wedge d_1 + X_2 \wedge d_2 + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_2 - d_2)_+] \,|X_1 > d_1]\big)}{2\big(\eta - w + \mathbb{E}[X_1 \wedge d_1 + X_2 \wedge d_2 + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_2 - d_2)_+]]\big)} \\
&= \frac{\eta - w + d_1 + \mathbb{E}[X_2 \wedge d_2] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+]}{\eta - w + \mathbb{E}[X_1 \wedge d_1] + \mathbb{E}[X_2 \wedge d_2] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_2 - d_2)_+]}.
\end{aligned}$$

We now consider the following equation system that is equivalent to the one postulated in Remark 4.3:

$$\Phi_{d_2}(d_1) = \Phi_{d_1}(d_2) \tag{4.6}$$

$$\Phi_{d_2}(d_1) + \Phi_{d_1}(d_2) = 2 + 2\theta \tag{4.7}$$

Taking into account the prior calculations about $\Phi_{d_2}(d_1)$ together with the symmetry of $\Phi_{d_2}(d_1)$, we obtain from (4.6):

$$\begin{aligned} &\eta - w + d_1 + \mathbb{E}[X_2 \wedge d_2] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] \\ =\ &\eta - w + d_2 + \mathbb{E}[X_2 \wedge d_1] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] \end{aligned}$$

This simplifies to

$$d_1 - \mathbb{E}[X_1 \wedge d_1] = d_2 - \mathbb{E}[X_2 \wedge d_2]. \tag{4.8}$$

Furthermore, (4.7) together with prior calculations and subtracting one on both sides yields

$$\frac{\eta - w + d_1 + d_2 + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_2 - d_2)_+]}{\eta - w + \mathbb{E}[X_1 \wedge d_1] + \mathbb{E}[X_2 \wedge d_2] + (1+\theta)\mathbb{E}[(X_1 - d_1)_+] + (1+\theta)\mathbb{E}[(X_2 - d_2)_+]} = 1 + 2\theta.$$

After simplifying, this yields to the equation

$$d_1 - (2\theta+1)\mathbb{E}[X_1 \wedge d_1] - 2\theta(1+\theta)\mathbb{E}[(X_1 - d_1)_+] = -d_2 + (2\theta+1)\mathbb{E}[X_2 \wedge d_2] + 2\theta(1+\theta)\mathbb{E}[(X_2 - d_2)_+]$$

or equivalently

$$\begin{aligned} &d_1 - 2\theta\mathbb{E}[X_1] - \mathbb{E}[X_1 \wedge d_1] - 2\theta^2\mathbb{E}[(X_1 - d_1)_+] = \\ &-d_2 + 2\theta\mathbb{E}[X_2] + \mathbb{E}[X_2 \wedge d_2] + 2\theta^2\mathbb{E}[(X_2 - d_2)_+]. \end{aligned} \tag{4.9}$$

Now both (4.8) and (4.9) describe relationships between the deductibles d_1 and d_2. If we denote the left side of (4.8) with $f_1(d_1) = d_1 - \mathbb{E}[X_1 \wedge d_1]$ and the right side with $f_2(d_2) = d_2 - \mathbb{E}[X_2 \wedge d_2]$, we can calculate $\frac{\partial}{\partial d_1} f_1(d_1) = 1 - S_{X_1}(d_1) > 0$ and $\frac{\partial}{\partial d_2} f_2(d_2) = 1 - S_{X_2}(d_2) > 0$.

Therefore, equation (4.8) describes a positive relationship between d_1 and d_2. Moreover, f_1 and f_2 are strictly increasing and continuous. Observing that $f_1(0) = 0 \leq f_2(d_2)$ for all $d_2 \geq 0$ and $\lim_{d_1 \to \infty} f_1(d_1) = \infty$, we conclude that there is for each $d_1 \geq 0$ exactly one $d_2 \geq 0$ such that the equation is fulfilled. Denote with f the function that finds to a deductible $d_1 > 0$ the corresponding $d_2 = f(d_1)$ such that (4.8) is fulfilled.

We proceed equivalently for (4.9):
Denoting $g_1(d_1) = d_1 - 2\theta\mathbb{E}[X_1] - \mathbb{E}[X_1 \wedge d_1] - 2\theta^2\mathbb{E}[(X_1 - d_1)_+]$ yields $\frac{\partial}{\partial d_1} g_1(d_1) = 1 - S_{X_1}(d_1) + 2\theta^2 S_{X_1}(d_1) > 0$.
On the other hand, for $g_2(d_2) = -d_2 + 2\theta\mathbb{E}[X_2] + \mathbb{E}[X_2 \wedge d_2] + 2\theta^2\mathbb{E}[(X_2 - d_2)_+]$ we have $\frac{\partial}{\partial d_2} g_2(d_2) = -1 + S_{X_2}(d_2) - 2\theta^2 S_{X_2}(d_2) < 0$.
We do have a strictly negative relationship between d_1 and d_2 in (4.9). Since $g_1(0) < g_2(0)$, there exists for every $d_1 \geq 0$ a unique $d_2 \geq 0$ such that $g_1(d_1) = g_2(d_2)$. We denote with g the function that returns to a $d_1 \geq 0$ the unique $d_2 = g(d_1) \geq 0$ such that (4.9) is fulfilled.

Due to Remark 4.3, an optimal finite solution has to satisfy $f(d_1^*) = g(d_1^*) = d_2^*$.

We observe that

$$\lim_{d_1 \to \infty} f(d_1) = \infty \quad \text{and} \quad \lim_{d_1 \to 0} f(d_1) = 0$$

and because $f, g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ are both continuous, f strictly increasing and g strictly decreasing, it follows that there is a unique critical point $(d_1^*, d_2^*) \in \mathbb{R}^2_{>0}$ with $f(d_1^*) = g(d_1^*) =$

d_2^*, i.e. where the partial derivatives are 0.

Now look at the behavior of the function $L(d_1, d_2)$ at the boundaries of the domain, i.e. if one deductible d_i approaches zero or infinity. We use the explicit form of the function $\Phi_{d_2}(d_1)$ from the beginning of this proof together with the explicit expression of $\frac{\partial}{\partial d_i} L(d_1, d_2)$ and obtain:

- For any $d_2 \geq 0$, it is $\Phi_{d_2}(0) = 1 < 1 + \theta$, hence $\frac{\partial}{\partial d_1} L(0, d_2) > 0$.
- For any $d_2 \geq 0$, it is $\lim\limits_{d_1 \to \infty} L(d_1, d_2) = \infty$, hence $\frac{\partial}{\partial d_1} \lim\limits_{d_1 \to \infty} L(d_1, d_2) < 0$.

Using the symmetry of d_1 and d_2, we can conclude that the function $L(d_1, d_2)$ has a global maximum and reaches this global maximum only at points in the open interval $\mathbb{R}^2_{>0}$. But then every global maximum is also a local maximum, that means all partial derivatives equal zero. Since we found only one point (d_1^*, d_2^*) that fulfills this, we do have a unique finite solution to the optimization problem. $\square$

Summary

Based on the expected-utility approach, we set up an optimal deductible problem for stop-loss insurance policies from an insured's perspective who faces a single risk. We showed that there exists a unique solution and verified that there is the necessity of an insurance contract both for the exponential utility concept and for the quadratic utility concept. Including a second risk into our analysis which was independent to the first one, we expanded the optimization problem to finding two deductibles for a so-called excess-of-loss insurance policy. We found out that if one risk is relatively larger than the other one, then the corresponding deductible in the insurance contract should also be larger. Specifying the utility function again to exponential or quadratic utility allowed us to prove the existence of a unique optimal solution in this extended model.

We also showed that there might be a lot of parameters that take influence on the optimal deductible, such as the dimension of the risk or the personal risk preference of the individual. These variations get even more complex if there are multiple risks. Consequently, an insurance company should offer a variety of insurance contracts an individual can choose from. In an ideal case where welfare is maximized, every policyholder can choose arbitrary deductibles that fit their personal needs.

An interesting idea for further research is to relax the assumption of independence in the multivariate risk model. For instance when considering the initial car insurance example, there might be positive correlations in the risks due to the danger of damage by hail. In order to include such considerations into the model, one might introduce positively dependent risks. A mathematical risk model with positive dependencies that might fit our needs could be the one Cai and Wei (2012) introduced. Again by specifying the utility concepts, it could be possible to analyze the behavior of the corresponding deductibles once positive dependencies occur.

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Appendix

Proof of Example 3.7

Let $X \sim Exp(\lambda)$. Then for $\lambda \neq \frac{1}{\gamma}$, it is

$$\begin{aligned}
\mathbb{E}[e^{\gamma(X\wedge d)}] &= \int_0^\infty e^{\gamma(x\wedge d)} f_X(x)\,\mathrm{d}x \\
&= \int_0^d e^{\gamma x}\frac{1}{\lambda}e^{-x/\lambda}\,\mathrm{d}x + \int_d^\infty e^{\gamma d} f_X(x)\,\mathrm{d}x \\
&= \frac{1}{\lambda}\int_0^d e^{(\gamma-1/\lambda)x}\,\mathrm{d}x + e^{\gamma d}\mathbb{P}(X > d) \\
&= \frac{1}{\lambda}\left(\frac{1}{\gamma - 1/\lambda}e^{(\gamma-1/\lambda)x)})\right|_0^d + e^{\gamma d}e^{-d/\lambda} \\
&= \frac{1}{\lambda\gamma - 1}(e^{(\gamma-1/\lambda)d} - 1)
\end{aligned}$$

On the other hand, if $\lambda = \frac{1}{\gamma}$, we obtain

$$\begin{aligned}
\mathbb{E}[e^{\gamma(X\wedge d)}] &= \frac{1}{\lambda}\int_0^d e^{(\gamma-1/\lambda)x}\,\mathrm{d}x + e^{\gamma d}\mathbb{P}(X > d) \\
&= \frac{1}{\lambda}\int_0^d 1\,\mathrm{d}x + e^{\gamma d}\mathbb{P}(X > d) \\
&= \frac{d}{\lambda} + e^{\gamma d}e^{-\gamma d} \\
&= d\gamma + 1
\end{aligned}$$