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$\mathcal{Z}$-STRUCTURES AND SEMIDIRECT PRODUCTS WITH AN INFINITE CYCLIC GROUP

by

Brian Pietsch

A Dissertation Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

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at

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ABSTRACT

$\mathcal{Z}$-STRUCTURES AND SEMIDIRECT PRODUCTS WITH AN INFINITE CYCLIC GROUP

by

Brian Pietsch

The University of Wisconsin-Milwaukee, 2018
Under the Supervision of Professor Craig Guilbault

$\mathcal{Z}$-structures were originally formulated by Bestvina in order to axiomatize the properties that an ideal group boundary should have. In this dissertation, we prove that if a given group admits a $\mathcal{Z}$-structure, then any semidirect product of that group with an infinite cyclic group will also admit a $\mathcal{Z}$-structure. We then show how this can be applied to 3-manifold groups and strongly polycyclic groups.

I want to thank all of those who supported me and made this dissertation possible. First and foremost, I want to thank my advisor, Dr. Craig Guilbault, for all of the countless hours spent mentoring and lending words of encouragement. Without your valuable advice, both in mathematics and in life, I would not be where I am today.

I want to thank the topology group at UWM for the knowledge and feedback shared during my time in school. I was warned before the first seminar that the topology group runs “full-contact” seminars, but I would not have it any other way after experiencing the seminars firsthand. In particular, I want to thank Dr. Chris Hruska, Dr. Boris Okun, and Hoang Nguyen for their valuable input on this dissertation.

Lastly, I want to thank my family. I want thank my wife Nicole for all of the professional and personal sacrifices she made so that I could attend graduate school, as well as for being a great mother to Lucas. I also want to thank my parents, Max and Tina, and my second parents, Bob and Sue, for their support through the years, especially over the past year. And thanks to my brother-in-law Chris for lending the computer on which this dissertation was typed.

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1 Introduction

One can seek to understand the algebraic properties of a finitely generated group by instead studying the geometric properties of a topological space on which the group acts, and it is this theme that describes what geometric group theory is all about. Terms like “dimension” or “boundary” are used to describe algebraic objects such as groups, but these terms seem to have geometric connotations to them and so it makes sense to try to find a geometric way of defining them. This often becomes a matter of finding a space on which the group acts, analyzing its properties, and then asking: “is this a well-defined property of the group, or just a property of the space?” That is, if you find any other space on which that group acts, must that space also share the same properties? For example, it is a well known result in geometric group theory that δ-hyperbolic groups have well-defined boundaries that can be taken to be the boundary of any space on which the group acts geometrically. CAT(0) groups, on the other hand, do not have well-defined boundaries without adding additional hypotheses; there are examples of the same CAT(0) group acting geometrically on two different spaces with non-homeomorphic boundaries.

The challenge of geometric group theory then is to find the right balance between restricting with extra hypotheses, and generalizing to broader classes of objects, in order to discover just how strong the correspondence is between algebra and geometry. CAT(0) groups and δ-hyperbolic groups are two classes of groups that have received much focus in research, and they both have their own notions of a group boundary. Bestvina first introduced the notion of a $\mathcal{Z}$-structure [Bes96] as a way of capturing the idea of a group boundary as a set of axioms that reflects what happens in both the CAT(0) an δ-hyperbolic case. This then begs the question: besides CAT(0) and δ-hyperbolic groups, which other types of groups admit $\mathcal{Z}$-structures? A more complete listing of what is currently known is given in Section 2, but the goal of this paper is to prove the following main theorem:

Theorem A. *If a group G admits a $\mathcal{Z}$-structure with boundary Z, then any semidirect product of the form $G \rtimes_\phi \mathbb{Z}$ also admits a $\mathcal{Z}$-structure where the boundary is the suspension of Z.*

As a consequence of the above theorem, we are also able to prove the following two results in Section 9:

Theorem B. *Every strongly polycyclic group admits a $\mathcal{Z}$-structure where the boundary is a sphere of dimension $n-1$, where n is the Hirsch length of the group.*

Theorem C. *Every closed, orientable 3-manifold group admits a $\mathcal{Z}$-structure.*

Remark. *Theorem A was anticipated by Bestvina in [Bes96, Ex. 3.1]. The bulk of the work presented here involves providing a complete and detailed argument supporting the claim found there. In doing so, our methods diverged significantly from the hint provided there.*

2 Definitions, Examples, and Main Results

Every space in this paper will be assumed to be separable and metrizable.

Definition 2.1. *A locally compact space X is an* ***absolute neighborhood retract*** *(ANR) if, whenever X is embedded as a closed subset of any space Y, then some neighborhood of X retracts in Y. An ANR X is an* ***absolute retract*** *(AR) if, whenever X is embedded as a closed subset of Y, then all of Y retracts onto X.*

Definition 2.2. *A closed subset Z of a space X is a* ***$\mathcal{Z}$-set*** *if, there exists a homotopy $\alpha : X \times [0, 1] \to X$ such that $\alpha_0 = id_X$ and $\alpha_t(X) \subset X - Z$ for all $t > 0$. In this case, α will be referred to as a* ***$\mathcal{Z}$-set homotopy****.*

Definition 2.3. *A group G acting on a space X is said to act* ***properly*** *if for any compact set $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. The group is said to act* ***cocompactly*** *if there exists a compact set K such that the set of translates GK covers X.*

Remark. *Some authors may give an alternative definition for proper actions, but the definition given here is what will be used in this paper.*

Definition 2.4. *A collection of subsets $\mathcal{A}$ in a space X is a* ***null family*** *if, for any open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{B} \subset \mathcal{A}$ such that for all $A \in \mathcal{A} - \mathcal{B}$, there exists $U \in \mathcal{U}$ such that $A \subset U$.*

$\mathcal{Z}$-structures were introduced by Bestvina [Bes96] in order to provide axioms that an ideal group boundary should satisfy. Bestvina originally required finite dimensional spaces and free actions, and Dranishnikov [Dra06] later relaxed the definition to allow for groups with torsion. Work by Moran [Mor16] has shown that the finite dimensionality condition is also not necessary, and so we arrive at the following definition which appears in [GM18]:

Definition 2.5. *A* ***$\mathcal{Z}$-structure*** *on a group G is a pair of spaces $(\hat{X}, Z)$ satisfying:*

(1) *$\hat{X}$ is a compact AR.*

(2) *Z is a $\mathcal{Z}$-set in $\hat{X}$.*

(3) *$X = \hat{X} - Z$ admits a proper, cocompact action by G.*

(4) *(Nullity condition) For any compact set $K \subset X$, the collection of subsets $\{gK \mid g \in G\}$ is a null family in $\hat{X}$.*

Remark. *If the pair also satisfies:*

(5) *The action of G extends to $\hat{X}$,*

then it is called an ***$\mathcal{EZ}$-structure****. If only properties $(1) - (3)$ are satisfied, it is called a* ***weak $\mathcal{Z}$-structure****, or if properties $(1) - (3), (5)$ hold, it is a* ***weak $\mathcal{EZ}$-structure****.*

Because the action is proper and cocompact, in the case of infinite groups, the nullity condition can be interpreted as saying that translated sets get small near the boundary. There is no complete classification of which groups admit $\mathcal{Z}$-structures, but many special cases are known.

Example 2.1. (1) *Finite groups: Since a finite group acts properly and cocompactly on a space consisting of a single point via the trival action, we can* *say that* $(\{x\}, \emptyset)$ *is a* $\mathcal{EZ}$*-structure for any finite group. That is to say,* $\mathcal{Z}$*-structures don't tell us anything interesting about finite groups as is usually the case in geometric group theory.*

(2) *CAT(0) groups: For a group which acts geometrically on a proper CAT(0) space* X*, compactifying* X *with the visual boundary* ∂X *and giving* $X \cup \partial X$ *the cone topology creates a* $\mathcal{EZ}$*-structure on the group. [Bes96]*

(3) *Hyperbolic groups: For a* δ*-hyperbolic group* G*, one can create a* $\mathcal{EZ}$*-structure by compactifying a Rips complex* $P_d(G)$ *(where* d *depends on* δ*) with the Gromov boundary of the group. [BM91]*

(4) *Baumslag-Solitar groups: For integers* m *and* n*, the Baumslag-Solitar group is defined as* $BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$*. It is known that every such group admits a* $\mathcal{EZ}$*-structure. [GMT]*

(5) *Systolic groups: Systolic groups are defined as any group acting geometrically by simplicial automorphisms on a systolic complex, which is a type of contractible simplicial complex satisfying certain local combinatorial conditions. Every systolic group admits a* $\mathcal{Z}$*-structure. [OP09]*

(6) *Relatively hyperbolic groups: If a group* G *is hyperbolic relative to a set of peripheral subgroups* $\mathcal{H}$ *and if it is already known that each of the peripheral subgroups* $H \in \mathcal{H}$ *admits a* $\mathcal{Z}$*-structure, then the group* G *also admits a* $\mathcal{Z}$*-structure. [Dah03]*

(7) *Group extensions of type F groups: A group* G *is considered to be type F if it admits a finite* $K(G,1)$ *classifying space. One of the bigger open questions on the existence of* $\mathcal{Z}$*-structures is whether or not every type F group admits*

a $\mathcal{Z}$-structure. A partial result towards that end is that every extension of a nontrivial type F group by another nontrivial type F group will admit a weak $\mathcal{Z}$-structure. [Gui14]

(8) *Free products and direct products: Given two groups G and H which are already known to admit $\mathcal{Z}$-structures $(\hat{X}, Z_1)$ and $(\hat{Y}, Z_2)$ respectively, there are ways to construct $\mathcal{Z}$-structures for the groups $G * H$ and $G \times H$ that utilize the spaces X, Y and the boundaries Z_1, Z_2 in a natural way. [Tir11].*

The work of this paper builds heavily on the direct product construction to expand to including certain special cases of semidirect products.

Definition 2.6. *Let G and Q be groups, and let ϕ be a homomorphism $\phi : Q \to Aut(G)$ where $Aut(G)$ denotes the group of all automorphisms of G. The **semidirect product of** G **and** Q **with respect to** ϕ, denoted by $G \rtimes_\phi Q$, is defined as follows:*
As a set, $G \rtimes_\phi Q = G \times Q$, the ordinary Cartesian product. Multiplication of group elements is defined by the rule $(g_1, q_1) * (g_2, q_2) = \big(g_1\phi(q_1)(g_2), q_1q_2\big)$

In the special case of the above definition when the group Q is infinite cyclic, the semidirect product is easier to understand since the map $\phi : \mathbb{Z} \to G$ is completely determined by where ϕ sends 1. Suppose the group G has a finite presentation given by $\langle S|R\rangle$. Then $G \rtimes_\phi \mathbb{Z}$ has a presentation of the form $\langle S, t|R, t^{-1}st = \phi(s)$ for all $s \in S\rangle$.

The main goal of this paper is to prove the following result:

Theorem 2.7. *If a torsion-free group G admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$, then any semidirect product of the form $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure $(\hat{X}', Susp(Z))$.*

The theorem is easiest to state for the case of torsion-free groups as above, but with additional hypotheses, an analogous result can be stated for when the group may have torsion. The terms in the following theorem are defined more precisely in Section 8.

Theorem 2.8. *If G admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$ where X is an $\underline{E}G$ space, then any semidirect product of the form $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure $(\hat{X}', Susp(Z))$ where X' is an $\underline{E}(G \rtimes_\phi \mathbb{Z})$ space.*

Rough outline of proof

1. Since G is already assumed to admit a $\mathcal{Z}$-structure, we have a nice space X on which G acts. We then use X as a building block to construct an infinite mapping telescope on which $G \rtimes_\phi \mathbb{Z}$ acts.

2. Next, we build a carefully controlled homotopy equivalence v from Y to $X \times \mathbb{R}$.

3. Building upon methods developed by Tirel in [Tir11], we compactify $X \times \mathbb{R}$ by adding the suspension of X's boundary, $\mathrm{Susp}(\partial X)$, as the boundary. This requires topologizing the boundary in a way that v-images of a compact set whose translates cover Y form a null family in $\widehat{X \times \mathbb{R}}$.

4. Lastly, we use a "boundary swapping" technique developed in [GM18] to pull back the above boundary using v^{-1}. Extra control is built into the compactification done in Step 3 so that the final result is a $\mathcal{Z}$-structure for $G \rtimes_\phi \mathbb{Z}$.

3 An illustrative special case of the Main Theorem

We will start by giving a concrete example that demonstrates the steps laid out in the above overview. We will begin with the group $\mathbb{Z}^2$, which is already known to admit a $\mathcal{Z}$-structure, and then we will look at taking a semidirect product with $\mathbb{Z}$.

The discrete Heisenberg group The discrete Heisenberg group $H_3(\mathbb{Z})$ is a well-known group that has been studied in many contexts. It is one of the simplest examples of an infinite, non-abelian nilpotent group. There are several well-known presentations for the group, but I am most interested in thinking of the group as a semidirect product $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ where ϕ is the automorphism of $\mathbb{Z}^2$ given by the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. That is, we have a presentation given by $H_3(\mathbb{Z}) = \langle x, y, z | [x, y] = z, [x, z] = 1 = [y, z] \rangle$. This presentation can also be described with matrices by letting

$$x = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$H_3(\mathbb{Z})$ can be viewed as $\mathbb{Z}^3$ with a different multiplication rule: $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2 + z_1 y_2,\ y_1 + y_2,\ z_1 + z_2)$.
Since $H_3(\mathbb{Z})$ is a semidirect $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ and because $\mathbb{Z}^2$ is $CAT(0)$ (and hence admits a $\mathcal{Z}$-structure with $X = \mathbb{R}^2$), the Heisenberg group is the type of group that Theorem 2.7 can be applied to.

We will follow the method outlined in Section 1 for creating a $\mathcal{Z}$-structure on $H_3(\mathbb{Z})$ but with a warning. Things work out more simply in this case than than in arbitrary semidirect products of the form $G \rtimes_\phi \mathbb{Z}$. This is because in the case of the Heisenberg group, the automorphism ϕ can be realized as a homeomorphism of $\mathbb{R}^2$ to itself. In general, we will have to make do with a nicely controlled proper homotopy equivalence.

Let T^2 denote the standard torus $S^1 \times S^1$. One can construct a space on which $H_3(\mathbb{Z})$ acts by first taking a mapping torus of T^2, where the attaching map f is a homeomorphism that induces the group automorphism ϕ on the level of π_1. This space will be denoted $Tor_f(T^2)$. In the general proof found later in the paper, f is only guaranteed to be a homotopy equivalence and not necessarily a homeomorphism, so that is one of the features that makes this example simpler. The universal cover of this space then comes with a natural action of $H_3(\mathbb{Z})$.

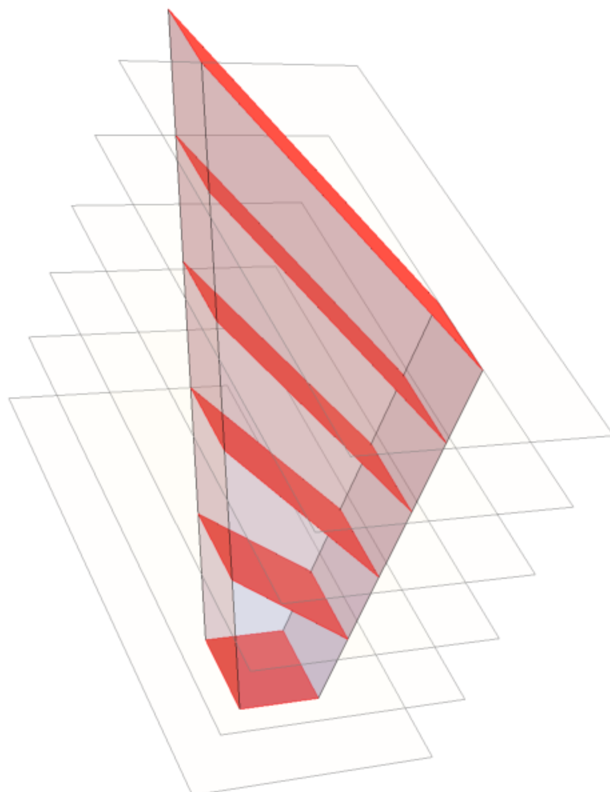

Figure 1: Several translates of the unit cube

This cover can be understood by creating it in two steps. First, consider at the cover corresponding to the $\mathbb{Z}$ quotient. That is to say, "unwrap" the mapping torus to create a bi-infinite mapping telescope consisting of mapping cylinders of the map $f : T^2 \to T^2$ glued end-to-end. Then take the universal cover of this space. It will be the bi-infinite mapping telescope of the map

$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which is the lift of f. Since f and $\tilde{f}$ are homeomorphisms, their mapping cylinders are homeomorphic to products. Therefore this new space created is topologically $\mathbb{R}^3$, where we have a countably infinite collection of planes representing the universal cover of T^2 stacked along the z-axis with intervals gluing them together in a skewed fashion. However, because of the way these planes are glued together, the geometry of this space is not that of standard Euclidean space; the geometry of this space is what defines Nil geometry.

The next step is to place a boundary on this space. Both $\mathbb{E}^3$ and $\mathbb{H}^3$ are equal to $\mathbb{R}^3$ as sets, so both can be $\mathcal{Z}$-compactified by adding a boundary sphere S^2 at infinity. The same can be done for this space on which $H_3(\mathbb{Z})$ acts, but a natural question is whether or not this produces a $\mathcal{Z}$-structure for the group $H_3(\mathbb{Z})$. In this case, the nullity condition is the most difficult to check. How does one topologize neighborhoods near the boundary sphere so that translates of compact sets are guaranteed to become small near the boundary? If we ignore the geometry of our bi-infinite mapping telescope and give $\mathbb{R}^3$ the standard compactification (add S^2 as a boundary by placing a point at infinity for each Euclidean ray emanating from $(0, 0, 0)$), the nullity condition fails since $H_3(\mathbb{Z})$ translates of the unit cube are distorted when translated in the z-direction (see Figure 1).

The remedy then is to change how that sphere at infinity is attached. This

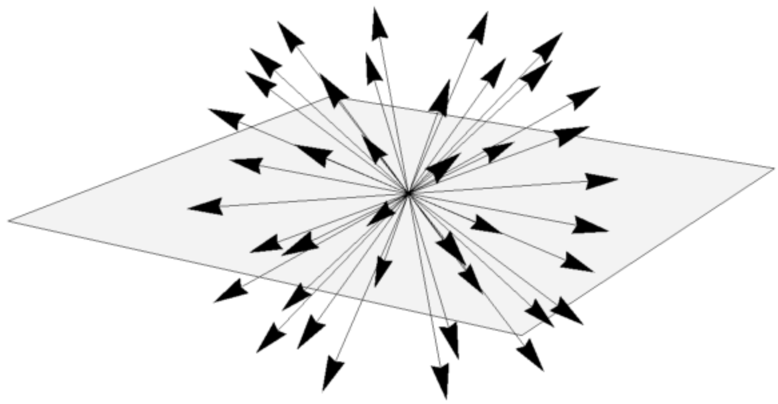

Figure 2: Standard Euclidean rays from the origin, corresponding to points on the sphere at infinity

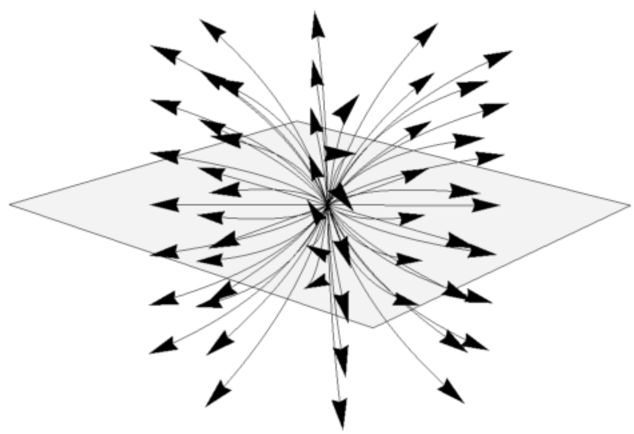

Figure 3: Modified rays corresponding to an alternate way of attaching the sphere at infinity

is done by defining a "slope function" that redefines which curves will be used in place of the standard Euclidean rays when deciding how to place points at infinity. In essence, these new rays of constant slope will bend away from the north and south poles so that those two points at infinity have "larger" neighborhoods that will swallow up translates of an initial compactum, which appear to be expanding as they are pushed in those directions. This must be done carefully so that translates in the other directions do not become larger as a result of the distortions we have introduced. This process is described more precisely in the general proof found in Section 6 below.

4 The Mapping Telescope

In order to create a $\mathcal{Z}$-structure for $G \rtimes_\phi \mathbb{Z}$, we must first come up with a candidate space on which that group acts, and then see if it can be compactified with all of the desired properties. The mapping telescope will be that space. The main goal is to prove Theorem 2.8, but we will begin by proving the torsion-free case stated as Theorem 2.7. After proving the torsion-free case, we will return to the case allowing for torsion in Section 8. If we are considering a group of the form $G \rtimes_\phi \mathbb{Z}$ where G is a torsion-free group that admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$, the following result of Bestvina says that it may as well be assumed that the space X is the universal cover of a $K(G, 1)$.

Lemma 4.1. *[Bes96] If a torsion-free group G admits a $\mathcal{Z}$-structure [resp. $\mathcal{EZ}$-structure] $(\hat{X}, Z)$, then G admits a finite $K(G, 1)$ complex K and there is a $\mathcal{Z}$-structure [resp. $\mathcal{EZ}$-structure] of the form $(\tilde{K} \cup Z, Z)$.*

Suppose we have a semidirect product of the form $G \rtimes_\phi \mathbb{Z}$ where G admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$. By Lemma 4.1, assume that $X := \hat{X} - Z$ is the universal cover of $K = K(G, 1)$. Throughout the rest of this paper, fix a base point $x_0 \in X$. Let $f : K \to K$ be a cellular map such that $f_* = \phi : G \to G$. Note that since ϕ is an automorphism, the above conditions and Whitehead's theorem tell us that f is a homotopy equivalence.

The mapping cylinder for the map $f : K \to K$, denoted $\mathcal{M}_{[a,b]}(f)$, is the quotient space $(K \times [a, b]) \sqcup K/ \sim$ where $\sim$ is the equivalence relation generated by the rule $(x, a) \sim f(x)$ (where $f(x)$ comes from the disjoint copy of K). Let $q_{[a,b]} : (K \times [a, b]) \sqcup K \to \mathcal{M}_{[a,b]}(f)$ be the quotient map. For each $t \in (a, b]$, $q_{[a,b]}$ restricts to an embedding of $K \times \{t\}$ into $\mathcal{M}_{[a,b]}(f)$ whose image will be denoted K_r. K_b will be referred to as the domain end of the mapping cylinder. The quotient map is also an embedding when restricted to the disjoint copy of K, and its image will be denoted K_a and called the range end of the mapping cylinder. The choice of $[a, b]$ used to label an interval is only a matter of convenience for what follows. The notation $Tel_f(K)$ will be used to denote the bi-infinite mapping telescope obtained by gluing together infinitely many mapping cylinders where the domain end of $\mathcal{M}_{[k-1,k]}(f)$ is attached to the range end of $\mathcal{M}_{[k,k+1]}(f)$. That is,

$$Tel_f(K) = \cdots \cup \mathcal{M}_{[-1,0]}(f) \cup \mathcal{M}_{[0,1]}(f) \cup \mathcal{M}_{[1,2]}(f) \cup \mathcal{M}_{[2,3]}(f) \ldots$$

The space we are really interested in using and the one that comes equipped with a proper, cocompact $G \rtimes_\phi \mathbb{Z}$ action is the universal cover, which is itself a mapping telescope for the map $\tilde{f} : X \to X$ where $X = \tilde{K}$. That is, we will eventually be compactifying the space $Tel_{\tilde{f}}(X)$ to construct a $\mathcal{Z}$-structure.

Understanding the mapping telescope: Recall that if the group G has a finite presentation given by $\langle S|R\rangle$, then $G \rtimes_\phi \mathbb{Z}$ has a presentation of the form $\langle S, t|R, t^{-1}st = \phi(s)$ for all $s \in S\rangle$. As one usually does with HNN extensions, take K and create the mapping torus for the map f. Without loss of generality, we will assume that K has been given a cell structure so that its 2-skeleton is the presentation 2-complex for $\langle S|R\rangle$, with a single vertex and one 1-cell for each generator in S. We will also assume that all of our mapping cylinders have been given the standard cell structure where the domain and range ends are subcomplexes, the only two vertices are the domain and range end copies of the single vertex from K, and all but one 1-cell lies in the domain or range end (with that one leftover 1-cell being the mapping cylinder line connecting the two vertices). We then wish to lift this cell structure to the telescope $Tel_{\tilde{f}}(X)$. By choosing K as described above, we can arrange that the 1-skeleton of $X = \tilde{K}$ is a Cayley graph and the 2-skeleton is a Cayley 2-complex for $G \rtimes_\phi \mathbb{Z}$. Because the mapping telescope consists of countably many mapping cylinders, each of which has a copy of X at their domain end, we end up with a copy of the Cayley graph for G at each integer level. As a set, the semidirect product $G \rtimes_\phi \mathbb{Z}$ is the same as $G \times \mathbb{Z}$, so it is no surprise that the 0-skeleton of the universal cover of this mapping torus is in $1-1$ correspondence with $G \times \mathbb{Z}$. We can visualize this 0-skeleton of the universal cover as being organized into horizontal strips corresponding to all of the cosets of G contained in the semidirect product. The vertices in the n^{th} level are labeled by the elements of G preceded by t^n. We can then proceed to fill in the 1-cells within each coset by obeying G's multiplication rules. It is the 1-cells that connect these cosets (i.e., multiplication by the group element t) that sets this semidirect product apart from an ordinary direct product, and this is what distorts the geometry in how distances are measured. These 1-cells are all of the lifts of the one exceptional 1-cell that connected the domain and range vertices in the mapping cylinders downstairs. From the identity vertex e, the t edges connect vertically in a straight line to t, t^2, t^3, etc., but from any other vertex for some $g \in G$, the t edge emanating from g needs to connect to the vertex labeled $t\phi(g)$. Depending on the word lengths of the elements $\phi(g)$ and g in G, it is possible that the shortest path between t and $t\phi(g)$ leaves the coset tG to take a shortcut through other cosets. The group action on this space is easy to understand, however. The group simply acts by isometries with t corresponding to a vertical shift of the vertices, and the action of any $g \in G$ can be seen as a horizontal shift that is understood by analyzing what it does on the eG coset and then making sure that all of the 1-cells connecting to other cosets are dragged along in the appropriate manner. This means that when focusing on the eG coset, the action of g looks just like multiplying by g, whereas the action of g on the t^nG coset will look like multiplication by $\phi^n(g)$. In the case that K is not 1-dimensional, the instructions for attaching higher dimension cells are encoded in the map $f : K \to K$. The mapping telescope $Tel_{\tilde{f}}(X)$ is built by just taking the universal cover of the mapping torus $Tor_f(K)$, and hence the attaching maps for higher dimension cells in the mapping telescope will just be lifts of whatever attaching maps were used in the mapping torus. To find a compact set whose translates cover $Tel_{\tilde{f}}(X)$, one can first identify a compact set $C' \subset X$ whose translates by G cover X and then consider its mapping cylinder $\mathcal{M}_{[0,1]}(\tilde{f}|_{C'}) =: C$. This set C is then a compact set whose translates under the action of $G \rtimes_\phi \mathbb{Z}$ will cover $Tel_{\tilde{f}}(X)$, and it is translated by isometries in the manner described above if the metric we use on $Tel_{\tilde{f}}(X)$ is a lift of a path metric on $Tor_f(K)$.

5 A homotopy equivalence

The goal of this section is to establish a controlled homotopy equivalence between the infinite mapping telescope $Tel_{\tilde{f}}(X)$ and the product $X \times \mathbb{R}$. The desired properties are that the homotopy is G-equivariant (where the G action on $Tel_{\tilde{f}}(X)$ is the restriction of the $G \rtimes_\phi \mathbb{Z}$ action, and the G action on $X \times \mathbb{R}$ is trivial on the second factor) and "nearly level-preserving," that is, $\mathcal{M}_{[k,k+1]}(\tilde{f})$ is mapped into $X \times [k, k+1]$. This begins with a close examination of [Gui14, Lemma 3.2].

Lemma 5.1. *[Gui14] If K is a compact connected ANR and $f : K \to K$ is a homotopy equivalence, then the canonical infinite cyclic cover, $Tel_f(K)$, of $Tor_f(K)$ is proper homotopy equivalent to $K \times \mathbb{R}$.*

To describe one of the maps involved in the preceding lemma, some notation will need to be established. Let $g : K \to K$ be a cellular homotopy inverse for f and $B : K \times [0,1] \to K$ with $B_0 = id_K, B_1 = fg$. Let $q_{[a,b]} : (K \times [a,b]) \sqcup K \to \mathcal{M}_{[a,b]}(f)$ be the quotient map that identifies $(x,a) \sim f(x)$ for all $x \in K$. Then the homotopy equivalence $u' : K \times \mathbb{R} \to Tel_f(K)$ from the preceding lemma can be described by piecing together the following functions defined for each integer n:

$$u'_n : K \times [n, n+1] \to \mathcal{M}_{[n,n+1]}(f) = \begin{cases} u'_n(x,r) = q_{[n,n+1]}(B_{r-n}(g^n(x)), r) & n \geq 0 \\ u'_n(x,r) = q_{[n,n+1]}(f^{-n}(x), r) & n < 0 \end{cases}$$

where g^0 is understood to be id_K.

Since our real interest is in developing a homotopy equivalence between $Tel_{\tilde{f}}(X)$ and $X \times \mathbb{R}$, we will take advantage of the fact that proper homotopy equivalences can be lifted to proper homotopy equivalences [Geo08, Section 10.1]. We will end up with a proper homotopy equivalence $u : X \times \mathbb{R} \to Tel_{\tilde{f}}(X)$ that consists of piecing together the following functions defined for each integer n:

$$u_n : X \times [n, n+1] \to \mathcal{M}_{[n,n+1]}(\tilde{f}) = \begin{cases} u_n(x,r) = q_{[n,n+1]}(\tilde{B}_{r-n}(\tilde{g}^n(x)), r) & n \geq 0 \\ u_n(x,r) = q_{[n,n+1]}(\tilde{f}^{-n}(x), r) & n < 0 \end{cases}$$

This lemma and its proof also give us extra control beyond simply being a homotopy equivalence. Because the lemma is first proven to create a proper homotopy equivalence downstairs, lifting from K to X results in G-equivariance for all of the maps. Because of the G-equivariance, the proper maps downstairs are guaranteed to lift to proper maps in the cover. That is, we end up with the following proper G-equivariant maps:

$u : X \times \mathbb{R} \to Tel_{\tilde{f}}(X)$

$v : Tel_{\tilde{f}}(X) \to X \times \mathbb{R}$

$H : Tel_{\tilde{f}}(X) \times [0,1] \to Tel_{\tilde{f}}(X)$ with $H_0 =$id and $H_1 = u \circ v$.

$J : X \times \mathbb{R} \to X \times \mathbb{R}$ with $J_0 =$id and $J_1 = v \circ u$.

Another important degree of control that these maps afford us is the property that they nearly preserve $\mathbb{R}$-levels, e.g., $\mathrm{Im}(u|_{X\times[n,n+1]}) \subset \mathcal{M}_{[n,n+1]}(\tilde{f})$. This is useful in placing bounds on the homotopies H and J in the sense that one only has to be concerned with how the homotopy track of a point wanders in the X direction because we have firm bounds in the $\mathbb{R}$ direction.

By inspection of the formula given for u, we see that mapping between $X \times \mathbb{R}$ and $Tel_{\tilde{f}}(X)$ results in applying the maps $\tilde{f}$ and $\tilde{g}$ to progressively higher powers as you move towards $\pm\infty$ in the $\mathbb{Z}$ direction. Since $\tilde{f}$ and $\tilde{g}$ are equivariant proper homotopy equivalences, they are quasi-isometries. Thus, there is some level of distortion occuring that gets progressively worse as you move towards $\pm\infty$ in the $\mathbb{Z}$ direction. To try to quantify this distortion, we will begin by considering a compact set C whose translates cover $Tel_{\tilde{f}}(X)$, and we will choose that set as described in Section 4 above by beginning with a compact set C' whose translates by G cover X and taking the mapping cylinder of $\tilde{f}$ restricted to C'. If we use the induced path metric on $Tel_{\tilde{f}}(X)$, we know that the diameter of C in this metric must be less than the diameter of $C' \times [0,1]$ under the usual taxicab metric (we know that this must be true since distances can only shrink from possible shortcuts added by taking mapping cylinder quotients $q_{[n,n+1]} : X \times [n, n+1] \to \mathcal{M}_{[n,n+1]}(\tilde{f})$ and by attaching neighboring mapping cylinders to form the bi-infinite mapping telescope). We know that translates of C within $Tel_{\tilde{f}}(X)$ do not change size since the action is by isometries, and we will use the taxicab estimate of $C' \times [0,1]$ for C's size. When we measure how the size of C compares to the size of its image once mapped into $X \times \mathbb{R}$ (where in $X \times \mathbb{R}$, we will use the taxicab metric), this boils down to comparing how much distortion occurs in the X direction because the maps u and v have the property that they nearly preserve $\mathbb{R}$-levels. The distortion in the X direction though is measured precisely by comparing the the sizes in X of C' with $\tilde{f}^n(C')$ or $\tilde{g}^n(C')$ (with the latter depending on which direction in the mapping telescope you are moving). The amount of stretching that occurs in the X direction is thus bounded above by an exponential, though knowing the exact formula is not required. In Section 6, a function η will be introduced to measure the upper bounds of distortion as required.

6 $\mathcal{Z}$-Compactifying the product

Our goal is a $\mathcal{Z}$-compactification of $Tel_{\tilde{f}}(X)$ which satisfies the nullity condition of Definition 2.5 with respect to the corresponding $G \rtimes_{\phi} \mathbb{Z}$ action.

For this construction, this is the most delicate task. The strategy is to first compactify the direct product with a suitable boundary and then use the homotopy equivalence established in Lemma 5.1 to pull back that boundary to the mapping telescope. The reason this is a delicate task is because as you translate a compact set in the $\mathbb{Z}$ direction of the mapping telescope and then look at the image of that set under the map $v : Tel_{\tilde{f}}(X) \to X \times \mathbb{R}$, you are forced to iterate the map f to higher and higher powers as you move further in the $\mathbb{Z}$ direction. This results in a worst-case scenario of the $G \rtimes_{\phi} \mathbb{Z}$ translates having images in $X \times \mathbb{R}$ that grow exponentially with respect to their $\mathbb{R}$-coordinate (the amount of distortion depends on the maximum word length of the image of any generator for G under the map ϕ). The goal then becomes to show that the product space can be compactified in a way so that even these exponentially distorted sets become small near the boundary, and moreover, that they become so small in the product that they still remain small when pulled back to the mapping telescope when we perform the boundary swap in the final step of the proof (where the homotopy used is potentially adding another degree of distortion). Proving this second statement relies on understanding the growth of homotopy tracks as you move further in the $\mathbb{Z}$ direction of the mapping telescope. For these reasons, we will define a function η to measure such growth.

For what follows, recall that the group G has a $\mathcal{Z}$-structure $(\hat{X}, \partial X)$ and assume a basepoint x_0 has been fixed. The metric on the space $X := \hat{X} - \partial X$

will be denoted by d, whereas the metric on $\hat{X}$ will be denoted by $\hat{d}$. The action by isometries of G is with respect to the space (X, d); even in the case of an $\mathcal{EZ}$-structure where the group action extends to the compactified boundary, the action of G on $(\hat{X}, \hat{d})$ is not by isometries. There is very little relationship between the metrics d and $\hat{d}$, and we will be primarily interested in the metric $\hat{d}$ when it comes to establishing the nullity condition. An undecorated x will be used to denote a point in X, $\bar{x}$ will denote a point in ∂X, and $\hat{x}$ will denote a point in $\hat{X}$ when the distinction between boundary or not is unneeded. Analogous notation will also be used for the group $\mathbb{Z}$: we have a $\mathcal{Z}$-structure $(\hat{\mathbb{R}}, \pm\infty)$, and $\hat{\sigma}$ will be used to denote any choice of metric on the extended real line (the exact choice will not matter). Ball notation, as in $B(x, r)$, will be used exclusively to refer to closed balls in the space X where the radius is measured by the metric d, and standard interval notation will be used for balls in $\mathbb{R}$.

As noted earlier, since $G \rtimes_\phi \mathbb{Z}$ acts cocompactly on $Tel_{\tilde{f}}(X)$, we can choose a compact set $C \subset \mathcal{M}_{[0,1]}(\tilde{f})$ whose translates cover $Tel_{\tilde{f}}(X)$. The letter t will be used to denote the generator of the $\mathbb{Z}$ factor. For the following definition, diameters in $X \times \mathbb{R}$ will be measured using the taxicab metric (with respect to the metric d on X and the standard metric on $\mathbb{R}$), and diameters in $Tel_{\tilde{f}}(X)$ will be measured with respect to the path metric previously described.

Definition 6.1. *Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying: For all $k \in \mathbb{N}$*

$$\eta(k) \geq max \begin{cases} diam\big(v(t^{\pm k}C)\big) \\ diam\big(H(t^{\pm k}C \times [0,1]\big) \\ diam\big(J(v(t^{\pm k}C) \times [0,1]\big) \end{cases}$$

Furthermore, choose η to be monotonic and such that $\lim_{r\to\infty} \eta(r) = \infty$.

The purpose of this definition is to establish upper bounds on how much distortion (of translates of our chosen compactum and of homotopy tracks contained in such translates) is happening when mapping between $\mathcal{M}_{[k,k+1]}(\tilde{f})$ and $X \times [k, k+1]$ as alluded to at the end of Section 5.

Theorem 6.2. $X \times \mathbb{R}$ *can be compactified to form a $\mathcal{Z}$-compactification $(\widehat{X \times \mathbb{R}}, Susp(\partial X))$ that satisfies the following version of the nullity condition:*

For any open cover $\mathcal{U}$ of $\widehat{X \times \mathbb{R}}$, there exists a compact $L \subset X \times \mathbb{R}$ such that any set of the form $B(x, \eta(k)) \times [k, k+1]$ that lies entirely outside of L is contained in some $U \in \mathcal{U}$ (where k is taken to be an integer here).

This is not quite the same as saying that the collection of sets $\{B(x, \eta(k)) \times [k, k+1] \mid x \in X,\ k \in \mathbb{Z}\}$ is a null family despite the action being proper, and that is because there are "too many" choices for x. But after we arrange for the above property to hold, we can take a subset of that family, and by applying properness of the action of G on X, we see that $\{B(gx_0, \eta(k)) \times [k, k+1] \mid g \in G,\ k \in \mathbb{Z}\}$ is a null family. As stated in Definition 6.1, η was defined to provide an upper bound on the diameter of images of our chosen compactum when translated only in the "vertical" direction of the mapping telescope (that is, when multiplying by powers of t). However, since the mapping telescope was constructed by taking lifts of the map $f : K \to K$, and since H and J are also lifts of homotopies downstairs, we have G-equivariance built-in and this means that η in fact provides an upper bound for the diameter of the image of any translate of our chosen compactum found in a given level of the mapping telescope. For this reason, this version of the nullity condition will suffice to establish that we do indeed have a $\mathcal{Z}$-structure for $G \rtimes_\phi \mathbb{Z}$ after the boundary is pulled back from the product into the telescope.

Much of the following construction follows as in [Tir11], but the "slope" function needs to be defined much more carefully. It is important to carefully select the slope function to compensate for the the potential distortion that occurs the case of a semidirect product, along with some additional control to allow for a boundary swap that was not required in [Tir11].

Definition 6.3. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying*
$\psi(k) \geq r_k$ where r_k is the radius at which balls of radius $\eta(k)$ in X have diameter less than $\frac{1}{k}$ in $\hat{X}$. That is, if $B(x, \eta(k)) \cap B(x_0, r_k) = \emptyset$, then $diam_{\hat{d}} B(x, \eta(k)) \leq \frac{1}{k}$. The existence of such a radius is always guaranteed by the fact that $(\hat{X}, G)$ is a $\mathcal{Z}$-structure.

Furthermore, choose ψ to have the following properties:

- *For some $R \in \mathbb{R}$, $\psi(s) \geq \eta(s)$ for all $s \geq R$* $\quad (\dagger)$

- *ψ is bijective*
- *$3\psi(s) \leq \psi(s+1)$ for all $s \geq R$*

Definition 6.4. *Let $p(x) : X \to \mathbb{R}^+$ be the function $p(x) := \ln(\psi^{-1}(d(x, x_0)) + 1)$. Let $\mu(x, r) : X \times \mathbb{R} \to \mathbb{R}$ be the function given by*

$$\mu(x,r) := \begin{cases} \frac{r}{p(x)} & \text{if } p(x) > 0 \\ \infty & \text{if } p(x) = 0 \text{ and } r \geq 0 \\ -\infty & \text{if } p(x) = 0 \text{ and } r < 0 \end{cases}$$

The definition of this slope function μ is one of the cruxes of this proof. While the reasons for all of the requirements placed on η will be referenced as needed in the coming lemmas, the basic goal is to ensure that p has arbitrarily small variation when measured on translates of compact sets that approach the boundary, even when these translates are in fact growing (at worst) exponentially as they near the boundary.

Definition 6.5. *The* suspension *of ∂X, denoted $Susp(\partial X)$, will be defined as $(\partial X \times [-\infty, \infty])/\sim$ where $\langle \bar{x}, -\infty \rangle \sim \langle \bar{x}', -\infty \rangle$ and $\langle \bar{x}, \infty \rangle \sim \langle \bar{x}', \infty \rangle$ for all $\bar{x}, \bar{x}' \in \partial X$*

Remark. *The choice of $[-\infty, \infty]$ as the interval used in the definition of the suspension was purely for notational convenience. The function μ will be used to parameterize arcs in the boundary that connect the two suspension points. It should also be noted that Tirel's construction for more general product spaces used the join of the two boundaries, and the suspension is a special case of a join.*

Remark. *Points in the suspension will continue to be denoted by* $\langle \bar{x}, \mu \rangle$ *so as to differentiate them from points* $(x, r) \in X \times \mathbb{R}$. *The equivalence classes of* $\langle \bar{x}, -\infty \rangle$ *and* $\langle \bar{x}, \infty \rangle$ *will be denoted* $\langle -\infty \rangle$ *and* $\langle \infty \rangle$ *respectively. This is done to reflect the independence of the choice of* $\bar{x}$ *for those two equivalence classes in* $Susp(\partial X)$.

Definition 6.6. *Define* $\widehat{X \times \mathbb{R}} := X \times \mathbb{R} \sqcup (Susp(\partial X))$ *where the topology is generated by all open subsets of* $X \times \mathbb{R}$ *together with open subsets along the boundary of the form:*

For $\langle \bar{x}, \mu \rangle \in \partial X \times (-\infty, \infty)$ *and* $\epsilon < \mu$,

$$U(\langle \bar{x}, \mu \rangle, \epsilon) := \{(x, r) \in X \times \mathbb{R} |\ \hat{d}(x, \bar{x}) < \epsilon,\ |\mu(x, r) - \mu| < \epsilon\} \\ \cup \{\langle \bar{x}', \mu' \rangle \in Susp(\partial X) |\ \hat{d}(\bar{x}', \bar{x}) < \epsilon,\ |\mu' - \mu| < \epsilon\}$$

For $\epsilon > 0$,

$$U(\langle -\infty \rangle, \epsilon) := \{(x, r) \in X \times \mathbb{R} |\ \hat{\sigma}(r, -\infty) < \epsilon,\ \mu(x, r) < \tfrac{-1}{\epsilon}\} \\ \cup \{\langle \bar{x}', \mu' \rangle \in Susp(\partial X) |\ \mu' < \tfrac{-1}{\epsilon}\}$$

For $\epsilon > 0$,

$$U(\langle \infty \rangle, \epsilon) := \{(x, r) \in X \times \mathbb{R} |\ \hat{\sigma}(r, \infty) < \epsilon,\ \mu(x, r) > \tfrac{1}{\epsilon}\} \\ \cup \{\langle \bar{x}', \mu' \rangle \in Susp(\partial X) |\ \mu' > \tfrac{1}{\epsilon}\}$$

Proposition 6.7. $\widehat{X \times \mathbb{R}}$ *is a compactification of* $X \times \mathbb{R}$

Proposition 6.8. *For any open cover $\mathcal{U}$ of Susp(∂X), there exists $\delta > 0$ such that for all $\langle \bar{x}, \mu \rangle$, there is an element of $\mathcal{U}$ containing $U(\langle \bar{x}, \mu \rangle, \delta)$*

Remark. *It should be noted that we allow for the possibility that $\mu = \pm\infty$ in Propostion 6.8, meaning the choice of $\bar{x}$ is unnecessary. That is, the claim is as true for neighborhoods of the type $U(\langle \bar{x}, \mu \rangle, \delta)$ as it is for the neighborhoods we denote by $U(\langle \infty \rangle, \delta)$. The proofs of the above two propositions are essentially the same as those in [Tir11] where they can be found as Propositions 3.10 and Claim 3.11, but the proofs are also included here for completeness.*

Proof. First, observe that the subspace topology $X \times \mathbb{R}$ inherits from $\widehat{X \times \mathbb{R}}$ is the same as the original topology on $X \times \mathbb{R}$, and that $X \times \mathbb{R}$ is open and dense in $\widehat{X \times \mathbb{R}}$. It remains to show that $\widehat{X \times \mathbb{R}}$ is compact to complete the claim that it is a compactification of $X \times \mathbb{R}$.

Let $\mathcal{U}$ be an open cover of $\widehat{X \times \mathbb{R}}$ by basic open sets. Since Susp(∂X) is compact (the suspension of a compact set is compact), we may choose a finite subset $\{U_i\}_{i=1}^{k}$ of $\mathcal{U}$ which covers Susp(∂X).

To prove Proposition 6.8, we define for each $i = 1, ..., k$ a function h_i : Susp$(\partial X) \to [0, \infty)$ by

$$h_i(\langle \bar{x}, \mu \rangle) := \begin{cases} 0 & \text{if } \langle \bar{x}, \mu \rangle \notin U_i \\ \sup\{r > 0 \mid U(\langle \bar{x}, \mu \rangle, r) \subset U_i\} & \text{if } \langle \bar{x}, \mu \rangle \in U_i \end{cases}$$

Each h_i is continuous, and for every $\langle \bar{x}, \mu \rangle \in$ Susp(∂X), there is some

$i \in \{1, ..., k\}$ for which $h_i(\langle \bar{x}, \mu \rangle) > 0$. Then, $h := \max\{h_i \mid i = 1, ..., k\}$ is a continuous, strictly positive function on the compact set $\mathrm{Susp}(\partial X)$ and so it has a minimum value δ which is the desired value. $\square$

For the rest of the paper, we will now fix an open cover $\mathcal{U}$ of $\widehat{X \times \mathbb{R}}$ and we will take δ to be the value promised by Proposition 6.8. The next three propositions work towards proving the modified nullity condition stated in Theorem 6.2. We can schematically view the product $X \times \mathbb{R}$ as a two-dimensional plane where the horizontal direction corresponds to the space X and the vertical direction corresponds to $\mathbb{R}$. The goal is then to find some large compact "box" or more specifically, a product of a compact set in X with a compact set in $\mathbb{R}$, so that sets of the form $B(x, \eta(k)) \times [k, k+1]$ are contained in some $U(\langle \bar{x}, \mu \rangle, \delta)$ if they lie entirely outside of the large box we choose. Proposition 6.9 shows that the box can be chosen in a way that if one of our sets lies above or below the box in our "plane," then our set is guaranteed to be in one of the neighborhoods $U(\langle \pm\infty \rangle, \delta)$. Proposition 6.10 shows that the box can be chosen so that sets lying to the left or right of it are ensured to be in a boundary neighborhood of the form $U(\langle \bar{x}, 0 \rangle, \delta)$. Then Proposition 6.11 describes what happens when looking at "diagonal" translates and how they end up in boundary neighborhoods of the form $U(\langle \bar{x}, \mu \rangle, \delta)$ where the slope lies somewhere between 0 and ∞.

Proposition 6.9. *For each compact set $J \subset X$, there exists a compact set $P_J \subset \mathbb{R}$ such that if $(B(x, \eta(k)) \times [k, k+1]) \cap (J \times P_J) = \emptyset$ and $J \cap B(x, \eta(k)) \neq$*

$\emptyset$, then for one of $-\infty$ or ∞, $(B(x,\eta(k))\times[k,k+1])\subset U(\langle\pm\infty\rangle,\delta)$

Proof. Without loss of generality, assume that $J = B(x_0, \eta(M))$ for $M \in \mathbb{R}$, and assume that $k > 0$.

Choose N large enough so that:

- $\hat{\mathbb{R}} - [-N, N] \subset N_\delta(\partial\mathbb{R})$
- $N \geq M$
- $N \geq R$ where R is as in $\big($Definition 6.3 † $\big)$
- $\dfrac{N}{\ln\Big(N+2\Big)} > \dfrac{1}{\delta}$

Let $P_J = [-N, N]$.

Assuming $J \cap B(x, \eta(k)) \neq \emptyset$, i.e., $k > N$. Then

$$\min\{\mu(x',r')\Big|(x',r')\in(B(x,\eta(k))\times[k,k+1])\} = \frac{\min\{r'|r'\in[k,k+1]\}}{\max\{p(x')|x'\in B(x,\eta(k))\}}$$

$$\geq \frac{k}{\ln\Big(\psi^{-1}\big(\eta(M)+2\eta(k)\big)+1\Big)}$$

$$\geq \frac{k}{\ln\Big(\psi^{-1}\big(3\eta(k)\big)+1\Big)}$$

$$\geq \frac{k}{\ln\Big(\psi^{-1}\big(3\psi(k)\big)+1\Big)}$$

$$\geq \frac{k}{\ln\Big(\psi^{-1}\big(\psi(k+1)\big)+1\Big)} > \frac{1}{\delta}$$

By the choice of N and the assumption that $k > 0$, $\hat{\sigma}(r', \infty) < \delta$ for all $r' \in [k, k+1]$.

Thus, $(B(x, \eta(k)) \times [k, k+1]) \subset U(\bar{r}, \infty)$. The case where $k < 0$ is completely analogous and the neighborhood at $\langle -\infty \rangle$ is used. $\square$

Proposition 6.10. *For all each compact set $K \subset \mathbb{R}$, there exists a compact set $Q_K \subset X$ such that if $(B(x, \eta(k)) \times [k, k+1]) \cap (Q_K \times K) = \emptyset$ and $K \cap [k, k+1] \neq \emptyset$, then there exists $\bar{x} \in \partial X$ such that $(B(x, \eta(k)) \times [k, k+1]) \subset U(\langle \bar{x}, 0 \rangle, \delta)$*

Proof. Without loss of generality, assume that $K = [-N, N]$, and again assume that $k \geq 0$.

Choose Q_K sufficiently large so that $\hat{d}(B(x, \eta(k)), \partial X) < \frac{\delta}{2}$ and $\mathrm{diam}_{\hat{d}}(B(x, \eta(k))) < \frac{\delta}{2}$ if $B(x, \eta(k)) \cap Q_K = \emptyset$. Note that the latter can be accomplished since $(\hat{X}, G)$ is a $\mathcal{Z}$-structure and since the assumption that $K \cap [k, k+1]$ ensures that k is bounded.

If $Q_K = B(x_0, \eta(M))$ for some M, then the largest possible slope occurs at a point $(x', N+1)$ where $d(x', x_0) = \eta(M)$, but $\dfrac{N+1}{\ln\Big(\psi^{-1}\big(\eta(M)\big) + 1\Big)} < \delta$ for M sufficiently large. That is, $\mu(x', r') < \delta$ for all $(x', r') \in B(x, \eta(k)) \times [k, k+1]$, and there exists $\bar{x} \in \partial X$ such that $\hat{d}(x', \bar{x}) < \delta$ for all $x' \in B(x, \eta(k))$. $\square$

We are now going to use the previous two propositions, along with some additional conditions concerning how large certain constants should be, to specify what we want our large "box" in $X \times \mathbb{R}$ to be.

Choose $J = B(x_0, \eta(S)) \subset X$ with $S \gg 0$ such that for all $x \in X - J$ (where ball notation still indicates closed balls, hence J is closed):

- $p(x) > \frac{2}{\delta}$
- $\hat{d}(x, \partial X) < \frac{\delta}{2}$
- $\psi^{-1}\big(\eta(S)\big) > R$ where R is as in $\big($Definition 6.3 † $\big)$
- $\ln\left(\dfrac{\psi^{-1}\big(3\eta(S)\big)+1}{\psi^{-1}\big(\eta(S)\big)+1}\right) < \delta$ (Note that this can only be done once the previous bullet is satisfied because ψ only behaves like an exponential for values greater than R)

Choose $K = [-N, N] \subset \mathbb{R}$ with $N \gg 0$ such that for all $r \in \mathbb{R} - K$ and for all $[k, k+1] \subset \mathbb{R} - K$:

- $\dfrac{1}{N} < \dfrac{\delta}{2}$
- $\bar{\sigma}(r, \partial\mathbb{R}) < \frac{\delta}{2}$
- $\mathrm{diam}_{\bar{\sigma}}([k, k+1]) < \frac{\delta}{2}$
- $N > R$ where R is as in $\big($Definition 6.3 † $\big)$
- $\dfrac{|r|}{\ln\big(|r|+2\big)} > \dfrac{1}{\delta}$

Let P_J and Q_K be as in Propositions 6.9 and 6.10. $\big((J \cup Q_K) \times (P_J \cup K)\big)$ is then our large compact "box" that has been referred to.

Proposition 6.11. *If* $\big(B(x, \eta(k)) \times [k, k+1]\big) \cap \big((J \cup Q_K) \times (P_J \cup K)\big) = \emptyset$, *then there exists* $\langle \bar{x}, \mu \rangle \in Susp(\partial X)$ *(where* μ *is possibly* $\pm\infty$*) such that* $\big(B(x, \eta(k)) \times [k, k+1]\big) \subset U(\langle \bar{x}, \mu \rangle, \delta)$.

Proof. If $B(x, \eta(k)) \cap J \neq \emptyset$, then Proposition 6.9 implies the result.

If $[k, k+1] \cap K \neq \emptyset$, then Proposition 6.10 implies the result.

Assume then that $B(x,\eta(k))\cap J=\emptyset=[k,k+1]\cap K$, and to simplify notation again, assume that $k>0$.

Let $M\in\mathbb{R}^+$ be such that $d(x,x_0)-\eta(k)=\eta(M)$. Note that such an M exists since $B(x,\eta(k))\cap J=\emptyset$.

Case 1: There exists $(x',r')\in\big(B(x,\eta(k))\times[k,k+1]\big)$ *such that* $\mu(x',r')\leq\frac{1}{\delta}$.

By the choice of J, there exists $\bar{x}\in\partial X$ such that $\hat{d}(x',\bar{x})<\frac{\delta}{2}$. By the choice of K and since we are dealing with the $k>0$ case, we know that $\bar{\sigma}(r',\infty)<\frac{\delta}{2}$

For any other $(x'',r'')\in\big(B(x,\eta(k))\times[k,k+1]\big)$,

$$
\begin{aligned}
|\mu(x'',r'')-\mu(x',r')| &= |\mu(x'',r'')-\mu(x'',r')+\mu(x'',r')-\mu(x',r')| \\
&= \left|\frac{r''}{p(x'')}-\frac{r'}{p(x'')}+\frac{r'}{p(x'')}-\frac{r'}{p(x')}\right| \\
&\leq \frac{1}{p(x'')}|r''-r'|+\mu(x',r')\frac{|p(x')-p(x'')|}{p(x'')} \\
&< \frac{\delta}{2}+\frac{1}{\delta}\cdot\frac{|p(x')-p(x'')|}{2/\delta}
\end{aligned}
$$

and so it remains to show that $|p(x')-p(x'')|<\delta$ to prove that the slopes of all points in $\big(B(x,\eta(k))\times[k,k+1]\big)$ vary no more than δ

Claim: $\eta(M)>\psi(k)$

The smallest possible slope in $\big(B(x,\eta(k)) \times [k,k+1]\big)$ is

$$\mu_{\min} = \frac{k}{\ln\Big(\psi^{-1}\big(\eta(M)+2\eta(k)\big)+1\Big)}$$

. By construction, $\psi(k) \geq \eta(k)$. Suppose that $\psi(k) \geq \eta(M)$.
Then, $\mu_{\min} \geq \dfrac{k}{\ln\Big(\psi^{-1}\big(3\psi(k)\big)+1\Big)} \geq \dfrac{k}{\ln\Big(\psi^{-1}\big(\psi(k+1)\big)+1\Big)} > \dfrac{1}{\delta}$

But this contradicts the existence of $(x', r') \in \big(B(x,\eta(k)) \times [k,k+1]\big)$ with $\mu(x',r') \leq \frac{1}{\delta}$ and thus the claim follows.

$$|p(x') - p(x'')| \leq \ln\Big(\psi^{-1}\big(\eta(M)+2\eta(k)\big)+1\Big) - \ln\Big(\psi^{-1}\big(\eta(M)\big)+1\Big)$$

$$< \ln\left(\frac{\psi^{-1}\big(3\eta(M)\big)+1}{\psi^{-1}\big(\eta(M)\big)+1}\right) < \delta$$

Note also that since $\eta(M) > \psi(k)$, we are guaranteed that $\operatorname{diam}_{\hat{d}}(B(x,\eta(k))) < \frac{1}{k} < \frac{\delta}{2}$. Thus by applying the triangle inequality, we see that $\big(B(x,\eta(k)) \times [k,k+1]\big) \subset U(\langle \bar{x}, \mu(x',r')\rangle, \delta)$.

Case 2: There does not exist $(x',r') \in \big(B(x,\eta(k)) \times [k,k+1]\big)$ *such that* $\mu(x',r') \leq \frac{1}{\delta}$.

Then $\mu(x',r') > \frac{1}{\delta}$ for all $(x',r') \in \big(B(x,\eta(k)) \times [k,k+1]\big)$. Since all of the slopes are greater than $\frac{1}{\delta}$, the choice of K guarantees that $\big(B(x,\eta(k)) \times [k,k+1]\big) \subset U(\langle\infty\rangle, \delta)$.

□

Proposition 6.12. *$\widehat{X \times \mathbb{R}}$ is an AR and Susp(∂X) is a $\mathcal{Z}$-set in $\widehat{X \times \mathbb{R}}$.*

To prove that $\widehat{X \times \mathbb{R}}$ is an AR and that Susp(∂X) is a $\mathcal{Z}$-set, a slight modification of Tirel's approach suffices (where the modifications are due to the fact that our boundary is a suspension of one boundary and hers is a more generalized join of two boundaries, and due to the fact that our slope function here is different). The idea is to construct "rays" from the base point $(x_0, 0)$ to the boundary and then retract along these rays. Note that these need not be geodesic rays. Since $\hat{X}$ and $\hat{\mathbb{R}}$ both already have $\mathcal{Z}$-structures, they come equipped with $\mathcal{Z}$-set homotopies α and β which we may assume are contractions to x_0 and 0 respectively (see [Tir11, Lemma 1.11]). This means that the homotopy tracks of boundary points in $\hat{X}$ and $\hat{\mathbb{R}}$ form "rays" to the base points x_0 and 0 using the maps α and β. We will then construct rays in $X \times \mathbb{R}$ by essentially taking a product of the rays defined by α in X and by β in $\mathbb{R}$, except that we want to have continuity when we extend to $\widehat{X \times \mathbb{R}}$. This means that our rays in the product $X \times \mathbb{R}$ need to trace out α and β at varying rates so that each ray has a constant slope near the boundary (where that slope μ corresponds to which point $\langle \hat{x}, \mu \rangle$ in the boundary the ray is approaching). A full proof can be found in [Tir11, Propositions 3.19, 3.20], but the key fact that allows her construction to work is that her slope function component p is constructed to have the following property (Lemma 6.13). This allows her to reparameterize the $\mathcal{Z}$-set homotopy α for $(\hat{X}, \partial X)$ in a way that the rays described above can be constructed from a product of the reparameterized homotopies for $(\hat{X}, \partial X)$ and $(\hat{\mathbb{R}}, \pm\infty)$. Since our function p was defined differently than Tirel's, we will first choose to reparameterize α and β such that the following lemma holds, and then from there Tirel's proof will follow.

Lemma 6.13. *There are reparameterizations $\hat{\alpha}$ and $\hat{\beta}$ of the $\mathcal{Z}$-set homotopies α and β so that $p(\hat{\alpha}(\bar{x}, t)) \in [\frac{1}{t}-1, \frac{1}{t}+2]$ and $|\hat{\beta}(\pm\infty, t)| \in [\frac{1}{t}-1, \frac{1}{t}+2]$ for all $t \in (0, 1]$ and for all $\bar{x} \in \partial X$.*

Sketch of proof of Lemma 6.13. That such reparameterizations can be done may not be obvious, but Tirel proves it as a consequence of the homotopies first being parameterized with a similar property. For the homotopy α, that property is: for some sequence $1 = t_0 > t_1 > t_2 > \cdots > 0$, we have $p(\alpha(\partial X \times [t_i, t_{i-1}))) \subset (i-1, i+1]$. The fact that α can be reparameterized to first meet this property before being reparameterized again is an easier observation. One can picture starting out at the basepoint $x_0 \in X$ and creating "bands" emanating outwards that divide X into countably many regions where the slope component function p has values lying in $[i, i+1]$. Then, control the speed at which the original α contracts to the basepoint x_0 and record the values t_i for when the boundary lies in the $[i, i+1]$ strip of p values. For more further details, refer to [Tir11, Lemma 3.8]. □

Now we will build the $\mathcal{Z}$-set homotopy on $\widehat{X \times \mathbb{R}}$ in a way that the slope function is respected near the boundary. This is the same proof as found in [Tir11] but included here for completeness. Define $\xi : [0,\infty) \to [0,1]$ by $\xi(t) = \frac{1}{1+t}$, and define $\alpha' : \hat{X} \times [0,\infty) \to X$ and $\beta' : \hat{\mathbb{R}} \times [0,\infty) \to \mathbb{R}$ by $\alpha'(\hat{x}, t) := \hat{\alpha}(\hat{x}, \xi(t))$ and $\beta'(\hat{r}, t) := \hat{\beta}(\hat{r}, \xi(t))$.

With these definitions and Lemma 6.13, we have that for any $t \in [0,\infty)$ and for any $\bar{x} \in \partial X$, $p(\alpha'(\bar{x}, t)) \in (t-1, t+3)$. We also have for any $t \in [0,\infty)$ that $|\beta'(\pm\infty, t) \in (t-1, t+3)$.

Let $\gamma' : \widehat{X \times \mathbb{R}} \times [0,\infty) \to X \times \mathbb{R}$ be given by:

- $\gamma'((x,r),t) := \Big(\alpha'(x, \frac{t}{\sqrt{(\mu(x,r))^2+1}}), \beta'(r, \frac{\mu(x,r)\cdot t}{\sqrt{(\mu(x,r))^2+1}})\Big)$ if $(x,r) \in X \times \mathbb{R}$.

- $\gamma'(\langle \bar{x}, \mu \rangle, t) := \Big(\alpha'(\bar{x}, \frac{t}{\sqrt{\mu^2+1}}), \beta'(\infty, \frac{\mu \cdot t}{\sqrt{\mu^2+1}})\Big)$ if $\langle \bar{x}, \mu \rangle \in \text{Susp}(\partial X)$ and $0 \leq \mu < \infty$.

- $\gamma'(\langle \bar{x}, \mu \rangle, t) := \Big(\alpha'(\bar{x}, \frac{t}{\sqrt{\mu^2+1}}), \beta'(-\infty, \frac{\mu \cdot t}{\sqrt{\mu^2+1}})\Big)$ if $\langle \bar{x}, \mu \rangle \in \text{Susp}(\partial X)$ and $-\infty < \mu < 0$.

- $\gamma'(\langle \infty \rangle, t) := \Big(x_0, \beta'(\infty, t)\Big)$

- $\gamma'(\langle -\infty \rangle, t) := \Big(x_0, \beta'(-\infty, t)\Big)$

Note that from the second bullet point, it follows that if $\langle \bar{x}, \mu \rangle$ is a boundary point where $\mu = 0$, then $\gamma'(\langle \bar{x}, 0 \rangle, t) = \big(\alpha'(\bar{x}, t), 0\big)$.

The map γ' applied to a boundary point in $\mathrm{Susp}(\partial X)$ traces out a ray in $X \times \mathbb{R}$ which converges in $\widehat{X \times \mathbb{R}}$ to that boundary point. We then build a homotopy γ which runs γ' in reverse, and this is our $\mathcal{Z}$-set homotopy for $\widehat{X \times \mathbb{R}}$. All that needs to be checked is that the map really is continuous, i.e., that the rays in $X \times \mathbb{R}$ really do converge on the boundary in $\widehat{X \times \mathbb{R}}$.

If the boundary point is one of the suspension points, say $\langle\infty\rangle$, then $\gamma'(\langle\infty\rangle, t) = (x_0, \beta'(\infty, t))$ and we see that $\beta'(\infty, t) \to \infty$ as $t \to \infty$. We also see that $\mu(\gamma'(\langle\infty\rangle, t)) = \mu((x_0, \beta'(\infty, t))) = \infty$ since $p(x_0) = 0$. Thus, $\gamma'(\langle\infty\rangle, t) \to \langle\infty\rangle$ as $t \to \infty$.

If the boundary point is of the form $\langle\bar{x}, \mu\rangle$ for $0 \leq \mu < \infty$, then for any t, we have:

$$\mu\big(\gamma'(\langle\bar{x},\mu\rangle, t)\big) = \frac{\beta'\Big(\infty, \frac{\mu \cdot t}{\sqrt{\mu^2+1}}\Big)}{p\Big(\alpha'\big(\bar{x}, \frac{t}{\sqrt{\mu^2+1}}\big)\Big)} \in \left(\frac{\frac{\mu\cdot t}{\sqrt{\mu^2+1}} - 2}{\frac{t}{\sqrt{\mu^2+1}} + 3}, \frac{\frac{\mu\cdot t}{\sqrt{\mu^2+1}} + 3}{\frac{t}{\sqrt{\mu^2+1}} - 2}\right)$$

$$= \left(\frac{\mu \cdot t - 2\sqrt{\mu^2+1}}{t + 3\sqrt{\mu^2+1}}, \frac{\mu \cdot t + 3\sqrt{\mu^2+1}}{t - 2\sqrt{\mu^2+1}}\right)$$

which implies that $\mu\big(\gamma'(\langle\bar{x},\mu\rangle, t)\big) \to \mu$ as $t \to \infty$, and thus $\gamma'(\langle\bar{x},\mu\rangle, t) \to \langle\bar{x},\mu\rangle$ as $t \to \infty$. This is all summarized by saying that the following map is continuous:

Let $\gamma : \widehat{X \times \mathbb{R}} \times [0,1] \to X \times \mathbb{R}$ be defined by:

$$\gamma(z,t) := \begin{cases} z & \text{if } t = 0 \\ \gamma'\big(z, \xi^{-1}(t)\big) & \text{if } t \in (0,1] \text{ and } r \geq 0 \end{cases}$$

Note that $\gamma(\widehat{X \times \mathbb{R}}, t) \subset X \times \mathbb{R}$ for $t > 0$.

Once one has the above $\mathcal{Z}$-set homotopy, an application of the following classical theorem in ANR theory tells you that the compactified space is in fact an ANR (and hence an AR since it is contractible):

Theorem 6.14 (Hanner's Criterion)**.** *[Han51] If for every open cover $\mathcal{U}$ of X there is an ANR which $\mathcal{U}$-dominates X, then X is an ANR.*

Since X is an ANR by hypothesis and $\mathbb{R}$ is an ANR, the well-known fact that a product of ANRs is an ANR gives us that $X \times \mathbb{R}$ is an ANR. $X \times \mathbb{R}$ is then used as the $\mathcal{U}$-dominating space for $\widehat{X \times \mathbb{R}}$ with the just-constructed $\mathcal{Z}$-set homotopy providing the necessary maps as described in [Tir11, Proposition 3.19]. Thus, $\widehat{X \times \mathbb{R}}$ is a contractible ANR (hence an AR) with $\mathcal{Z}$-set homotopy γ, and this completes the proof of Proposition 6.12.

7 Boundary Swap

Beginning with the $\mathcal{Z}$-compactification $\widehat{X \times \mathbb{R}} = X \times \mathbb{R} \sqcup \mathrm{Susp}(\partial X)$ just obtained, we will use the map $v : Tel_{\tilde{f}}(X) \to X \times \mathbb{R}$ constructed in Section 5 along with Proposition 7.1 to obtain a $\mathcal{Z}$-compactification $\widehat{Tel_{\tilde{f}}(X)} =$

$Tel_{\tilde{f}}(X) \sqcup \mathrm{Susp}(\partial X)$. Then we will show that due to the careful geometric controls built into the $\mathcal{Z}$-compactification of $X \times \mathbb{R}$, the resulting compactification of $Tel_{\tilde{f}}(X)$ satisfies the nullity condition with respect to the action of $G \rtimes_{\phi} \mathbb{Z}$.

Recall that the proper homotopy equivalence from Lemma 5.1 consists of the following proper, G-equivariant maps:

$u : X \times \mathbb{R} \to Tel_{\tilde{f}}(X)$

$v : Tel_{\tilde{f}}(X) \to X \times \mathbb{R}$

$H : Tel_{\tilde{f}}(X) \times [0,1] \to Tel_{\tilde{f}}(X)$ with $H_0 =$id and $H_1 = u \circ v$.

$J : X \times \mathbb{R} \to X \times \mathbb{R}$ with $J_0 =$id and $J_1 = v \circ u$.

These maps also have the property that they nearly preserve $\mathbb{R}$-coordinates, e.g., $\mathrm{Im}(u|_{X \times [n,n+1]}) \subset \mathcal{M}_{[n,n+1]}(\tilde{f})$

Proposition 7.1 (Boundary Swap)**.** *Given the $\mathcal{Z}$-compactification $\widehat{X \times \mathbb{R}}$ from Theorem 6.2, $\widehat{Tel_{\tilde{f}}(X)} := \Big(Tel_{\tilde{f}}(X) \sqcup Susp(\partial X), Susp(\partial X)\Big)$ can be topologized so that it too is a $\mathcal{Z}$-compactification.*

The topology on $\widehat{Tel_{\tilde{f}}(X)}$: Extend v to a function $\hat{v} : Tel_{\tilde{f}}(X) \sqcup \mathrm{Susp}(\partial X) \to \widehat{X \times \mathbb{R}}$ by letting $\hat{v}$ be the identity on $\mathrm{Susp}(\partial X)$. Then give $Tel_{\tilde{f}}(X) \sqcup \mathrm{Susp}(\partial X)$ the topology generated by the open subsets of $Tel_{\tilde{f}}(X)$ and sets of the form $\hat{v}^{-1}(U)$ where $U \subset \widehat{X \times \mathbb{R}}$ is open, and let $\widehat{Tel_{\tilde{f}}(X)}$ denote the resulting topological space. Clearly, $\hat{v}$ is continuous and $\widehat{Tel_{\tilde{f}}(X)}$ is compact, Hausdorff, and second countable. It follows that $\widehat{Tel_{\tilde{f}}(X)}$ is metrizable and separable. The following proposition is originally due to Ferry [Fer00], and Guilbault-Moran provided an alternate proof with slightly relaxed hypotheses [GM18]. It will be what allows us to claim that $\text{Susp}(\partial X)$ is a $\mathcal{Z}$-set in $\widehat{Tel_{\tilde{f}}(X)}$.

Proposition 7.2. *[Fer00],[GM18] Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ be continuous maps with $f(X - A) \subset Y - B$, $g(Y - B) \subset X - A$, and $g \circ f|_A = id_A$. Suppose further that there is a homotopy $H : X \times [0, 1] \to X$ which is fixed on A and satisfies $H_0 = id_X$, $H_1 = g \circ f$, and $H((X - A) \times [0, 1]) \subset X - A$. If B is a $\mathcal{Z}$-set in Y, then A is a $\mathcal{Z}$-set in X.*

Proof of Proposition 7.1. Before proceeding, we establish some notation. Let $\hat{u} : \widehat{X \times \mathbb{R}} \to \widehat{Tel_{\tilde{f}}(X)}$ and $\hat{H} : \widehat{Tel_{\tilde{f}}(X)} \times [0, 1] \to \widehat{Tel_{\tilde{f}}(X)}$ be the obvious extensions which are the identity on $\text{Susp}(\partial X)$. Whenever $\hat{U}$ denotes a subset of $\widehat{Tel_{\tilde{f}}(X)}$ [resp., $\widehat{X \times \mathbb{R}}$], U will denote $\hat{U} \cap Tel_{\tilde{f}}(X)$ [resp., $\hat{U} \cap (X \times \mathbb{R})$]. To satisfy the hypotheses of Proposition 7.2, it only needs to be shown that these maps are still continuous after we have extended them to the boundary.

Claim 1: $\hat{u}$ is continuous.

Let $z = \langle \bar{x}, \mu \rangle \in \text{Susp}(\partial X)$. Suppose that $\hat{v}^{-1}(\hat{U})$ is a basic open neighborhood of z in $\widehat{Tel_{\tilde{f}}(X)}$, where $\hat{U} = U(\langle \bar{x}, \mu \rangle, \epsilon)$. The goal is to pick a smaller open set $\hat{V} \subset \hat{U}$ such that $\hat{u}(\hat{V}) \subset \hat{v}^{-1}(\hat{U})$. It is clear that $\hat{v} \circ \hat{u}(z') \in \hat{U}$ for any $z' \in (\text{Susp}(\partial X)) \cap \hat{V}$ since both maps are the identity on the boundary. It remains to be checked that the same holds for all $y \in V$.

Since we know that $v \circ u$ is homotopic to the identity, if we can show that V can be chosen small enough so that the homotopy tracks under J of points in V do not wander outside of U, then we are done. We know though that a point (x, k) has a homotopy track bounded in the X direction by $\eta(k)$ (this was built into the definition of η, and we know that the homotopy track is contained in the $\mathbb{R}$ direction within the set $[k, k+1]$. Therefore, it suffices to show that we can choose the set V small enough so that for any $x \in V$, $(B(x, \eta(k)) \times [k, k+1]) \subset U$. As observed earlier, sets of the form $(B(x, \eta(k)) \times [k, k+1])$ do not necessarily form a null family in $\widehat{X \times \mathbb{R}}$, but if we restrict to sets of the form $(B(gx_0, \eta(k)) \times [k, k+1])$ then we do have a null family that still covers $Tel_{\tilde{f}}(X)$. Any arbitrary set $(B(x, \eta(k)) \times [k, k+1])$ is contained in the star of one of the sets of the form $(B(gx_0, \eta(k)) \times [k, k+1])$, and since the family of stars of a null family is itself a null family, the following proposition from Hruska-Ruane accomplishes what we are after:

Proposition 7.3. *[HR17] Let $\mathcal{A}$ be a null family of compact sets in a metric space M. Suppose $z \in M$ is not contained in any member of the family $\mathcal{A}$. Then each neighborhood U of z contains a smaller neighborhood V of z such that each $A \in \mathcal{A}$ intersecting V is contained in U.*

Claim 2: $\hat{H}$ is continuous.

Let $z = \langle \bar{x}, \mu \rangle \in \mathrm{Susp}(\partial X)$. Suppose that $\hat{U}$ is a basic open neighborhood of z in $\widehat{Tel_{\tilde{f}}(X)}$. The goal is then to choose a smaller open neighborhood $\hat{V}$ with the property that for all $y \in V$, $H(y \times [0,1]) \subset U$ and hence $\hat{H}(V \hat{\times} [0,1]) \subset \hat{U}$. Because of how $\widehat{Tel_{\tilde{f}}(X)}$ was topologized, $\hat{U}$ is actually the inverse image of an open set $\hat{U}' \subset \widehat{X \times \mathbb{R}}$. Thus, we want to show that we can choose V

small enough so that for all $y \in V$, $H(y \times [0,1]) \subset v^{-1}(U)$. That is, we want for all $y \in V$ $v(H(y \times [0,1])) \subset U$. Because η was defined to bound the growth of homotopy tracks of H, $v(H(y \times [0,1]))$ is contained in some subset of the form $(B(x, \eta(k)) \times [k, k+1])$. Thus, the same Hruska-Ruane null family argument can be applied to tell us that we can select V small enough so that if the homotopy tracks of any point of V has nonempty intersection with U, then in fact the entire track is contained in U. $\square$

The boundary swap was the final step in proving that $G \rtimes_\phi \mathbb{Z}$ admits a $\mathcal{Z}$-structure in the case that G is torsion-free, and so we arrive at the following proposition:

Proposition 7.4. *$(\widehat{Tel_{\tilde{f}}(X)}, Susp(\partial X))$ is a $\mathcal{Z}$-structure for $G \rtimes_\phi \mathbb{Z}$.*

Proof. Assume that we fix open covers $\mathcal{U}$ and $\mathcal{V}$ for $\widehat{Tel_{\tilde{f}}(X)}$ and $\widehat{X \times \mathbb{R}}$ respectively, and choose them (by refining $\mathcal{U}$ if necessary) so that the following property is satisfied: for each $V \in \mathcal{V}$ with $V \cap \mathrm{Susp}(\partial X) \neq \emptyset$, there is a $U \in \mathcal{U}$ such that $U = \hat{v}^{-1}(V)$. We will then choose the number δ corresponding to this cover $\mathcal{V}$ as in Proposition 6.8. We then need to check that the four conditions of Definition 2.5 are satisfied.

We begin by observing that $Tel_{\tilde{f}}(X)$ is an ANR due to a classical theorem by Borsuk-Whitehead-Hanner [Hu65] that says that the mapping cylinder of a proper map between ANRs is itself an ANR. We already know that $\widehat{X \times \mathbb{R}}$ is an AR by Proposition 6.12. We can then apply Hanner's Theorem (Theorem 6.14) to deduce that $\widehat{Tel_{\tilde{f}}(X)}$ is an ANR since it is dominated by $\widehat{X \times \mathbb{R}}$

via the maps $\hat{v}$, $\hat{u}$, and $\hat{H}$. Lastly, since $Tel_{\tilde{f}}(X)$ is homotopy equivalent to $\widehat{X \times \mathbb{R}}$, it is contractible and hence an AR.

Proposition 7.1 just showed that $(\widehat{Tel_{\tilde{f}}(X)}, \text{Susp}(\partial X))$ is a $\mathcal{Z}$-compactification, i.e., $\text{Susp}(\partial X)$ is a $\mathcal{Z}$-set in $\widehat{Tel_{\tilde{f}}(X)}$.

The proper, cocompact group action is already given on $Tel_{\tilde{f}}(X)$, and so only the nullity condition remains to be verified.

Suppose we are given the compact set $C \subset \widehat{Tel_{\tilde{f}}(X)}$ for which we want to show the nullity condition is satisfied. By the choice of δ, Theorem 6.2 says that a large enough compact set K can be found in $X \times \mathbb{R}$ so that any image of a $G \rtimes_\phi \mathbb{Z}$-translate of C, which is disjoint from K, is contained in some $V \in \mathcal{V}$ where $V \cap \text{Susp}(\partial X) \neq \emptyset$. Because v is a proper map, $v^{-1}(K)$ is still a compact set in $Tel_{\tilde{f}}(X)$. Since open neighborhoods on the boundary in $\mathcal{U}$ where chosen to be in correspondence with open sets on the boundary in $\mathcal{V}$, that is, open sets on the boundary in $\mathcal{U}$ are of the form $\hat{v}^{-1}(V)$ for some $V \in \mathcal{V}$, any $G \rtimes_\phi \mathbb{Z}$-translate of C that is disjoint from $v^{-1}(K)$ will be contained in some neighborhood $U \in \mathcal{U}$. Because the group action is proper, this is enough to prove the nullity condition. This also completes the proof of Theorem 2.7. □

8 In the case of groups with torsion

Returning to the question of when the group G has torsion and hence we cannot expect the G action to be free, we see that the correct analog to finite

$K(G,1)$'s is to now consider $\underline{E}G$ spaces. Note that some authors require an $\underline{E}G$ to be a CW complex, but we will use a broader definition that allows us to work in the category of ANRs.

Definition 8.1. *A space X on which a group G acts properly is an $\underline{E}G$ space if for all finite subgroups $H \subset G$, the fixed point set of H is contractible.*

In the initial case of torsion-free groups, the main tool that kickstarted everything was Lemma 5.1 which provided a proper, G-equivariant homotopy equivalence between the infinite mapping telescope and the direct product space, where the mapping telescope was the universal cover of a mapping torus of a finite $K(G,1)$. An analogous result can be proven for when we do not have finite $K(G,1)$'s at our disposal, after which the rest of the construction for a $\mathcal{Z}$-structure on $G \rtimes_\phi \mathbb{Z}$ follows directly. That is, in place of a map from a $K(G,1)$ to itself that induces the automorphism ϕ, we have the following:

Proposition 8.2. *If an ANR X is an $\underline{E}G$ space on which G acts cocompactly, and ϕ is any automorphism of G, there exists a proper homotopy equivalence $f : X \to X$ satisfying the following "ϕ-variance" property: for all $g \in G$ and for all $x \in X$, $f(gx) = \phi(g)f(x)$.*

Farrell-Jones proved the following result which is close to what we need:

Theorem 8.3. *[FJ93, Theorem A.2] Let X and Y be spaces on which G acts such that X has a cellular G action with finite stabilizers and Y has the* *property that the fixed point sets of all finite subgroups of G are contractible. Then there is a G-equivariant map from X to Y, and any two G-maps are homotopic through G-maps.*

Remark. *Their theorem is stated with a bit more generality, but they do require that X has a cellular action by G, i.e., X is a CW complex and the stabilizer of any cell acts trivially on that cell. We wish to remove this hypothesis so as to allow for a broader class of spaces. However, given a semidirect product $G \rtimes_\phi \mathbb{Z}$ and an $\underline{E}G$ CW complex X, the Farrell-Jones theorem could be applied to get a ϕ-variant homotopy equivalence from X to X. This is accomplished by letting X denote the space X with it's given action by G, and then letting X' be the space X but with a different action, namely, an element g acts on X' in the way that $\phi(g)$ acts on X. The theorem then produces a ϕ-variant map $f : X \to X'$ (which as a topological function is really just a map $f : X \to X$ since X and X' are the same space). By also applying the theorem in the reverse direction with the map ϕ^{-1}, one gets a ϕ^{-1}-variant map $g : X' \to X$ (which again will be considered as a map $g : X \to X$. The compositions $f \circ g : X \to X$ and $g \circ f : X \to X$ are both G-equivariant, as is the map $id : X \to X$, and hence according to the second part of Theorem 8.3, f and g are actually homotopy equivalences. If one does not assume X to be a CW complex, additional work is required.*

Proof of Proposition 8.2. Ontaneda [Ont05] proved a similar result in the case of CAT(0) groups and our proof is based on his. We will assume that our action is by isometries which, while not required in the definition of a

$\mathcal{Z}$-structure, causes us no loss of generality [GM18, Proposition 6.3]. We begin by building a suitable open cover of our space X, and then map into the nerve of that open cover. This will produce a simplicial complex to which we can apply the Farrell-Jones theorem.

Since the orbit Gx of any $x \in X$ is discrete, there is a radius r_x such that the closed ball $B(x, r_x) \cap (Gx) = \{x\}$. This implies that for all $g \in G$, either $B(x, \frac{r_x}{2}) \cap gB(x, \frac{r_x}{2}) = \emptyset$ or $gx = x$, with the latter implying that $B(x, \frac{r_x}{2}) = gB(x, \frac{r_x}{2})$. Since the action is cocompact, there is a finite collection $\mathcal{V}$ of balls $B(x, r_x)$ such that $\mathcal{U} := \{gV |\ g \in G,\ V \in \mathcal{V}\}$ is an open cover of X.

Next, we show that every ball $U \in \mathcal{U}$ intersects only finitely many elements in $\mathcal{U}$. If not, then there would be some $B(x, \frac{r_x}{4})$ and $B(y, \frac{r_y}{4})$ along with a sequence $\{g_i\} \subset G$ such that infinitely many distinct $g_iB(x, \frac{r_x}{4})$ all have nonempty intersection with $B(y, \frac{r_y}{4})$. It can be assumed that $r_x > r_y$. Thus, if $g_iB(x, \frac{r_x}{4})$ and $g_jB(x, \frac{r_x}{4})$ both intersect $B(y, \frac{r_y}{4})$, then we know that $y \in g_iB(x, \frac{r_x}{2}) \cap g_jB(x, \frac{r_x}{2})$. By the previous observation that either $B(x, \frac{r_x}{2}) \cap gB(x, \frac{r_x}{2}) = \emptyset$ or $B(x, \frac{r_x}{2}) = gB(x, \frac{r_x}{2})$, this contradicts the $g_iB(x, \frac{r_x}{4})$ being distinct. Thus every $U \in \mathcal{U}$ intersects only finitely many other elements.

We now construct the nerve of this open cover, denoted by $N(\mathcal{U})$, which is the simplicial complex consisting of one vertex for every $U \in \mathcal{U}$ and simplices of the form $[U_0, ..., U_n]$ whenever $U_0 \cap ... \cap U_n \neq \emptyset$. Because of the previous paragraph's observation, we see that $N(\mathcal{U})$ is locally finite and finite dimensional. $N(\mathcal{U})$ also comes equipped with a natural, simplicial action by G

since $\mathcal{U}$ consists of G translates of open sets. Moreover, the stabilizer of any simplex fixes that simplex pointwise. This follows from the fact that for any $U \in \mathcal{U}$ and any $g \in G$, $gU \cap U \neq \emptyset$ implies that $gU = U$.

We now let X' be the space X but where an element g's action on X' is given by how $\phi(g)$ acts on X. Since we have proven that the stabilizers act trivially on $N(\mathcal{U})$ and since it is assumed that X' has contractible fixed point sets, we may apply the Farrell-Jones theorem to get a G-equivariant map h from $N(\mathcal{U})$ to X', which is equivalent to having a ϕ-variant map from $N(\mathcal{U})$ to X. To get the ϕ-variant map from X to X, we consider $h \circ \beta$ where β is the barycentric map $\beta : X \to N(\mathcal{U})$. To describe the barycentric map, we first create a partition of unity $\{\lambda_U\}_{U \in \mathcal{U}}$ where for a given $U_0 \in \mathcal{U}$, $\lambda_{U_0} : X \to [0, 1]$ is defined by $\lambda_{U_0}(x) = d(x, X - U_0)/(\sum_{U \in \mathcal{U}} d(x, X - U))$. Since our cover is locally finite, these sums are finite and continuous. The barycentric map is then defined by $\beta(x) = \sum_{U \in \mathcal{U}} \lambda_U(x) v_U$ where v_U denotes the vertex represented by U in the nerve. Because the cover $\mathcal{U}$ is generated by G translates of open sets, the map β is G-equivariant, meaning that the map $h \circ \beta : X \to X'$ is G-equivariant, and by switching the range to X we get the ϕ-variant map we want.

All that remains is to prove that this map is also a proper homotopy equivalence. To show this, we first observe that G acts cocompactly on all of the spaces involved $(X, N(\mathcal{U}), X')$ and hence any two G-equivariant maps between the spaces will be boundedly close, large-scale uniform maps. By the same method as above, we can construct a ϕ^{-1}-variant map from X to

X (or a G-equivariant map from X' to X). By composing our ϕ-variant and ϕ^{-1}-variant maps in either direction, we get G-equivariant maps that must be boundedly close to the identity map. By the following coarse geometry result found in [GM18], we can conclude that we $h \circ \beta$ is indeed a proper homotopy equivalence:

Lemma 8.4. *[GM18, Corollary 5.3] Suppose $f, g : X \to Y$ are continuous, boundedly close, large-scale uniform maps, where X has finite macroscopic dimension and Y is a uniformly contractible ANR. Then f and g are boundedly (hence properly) homotopic.*

The fact that our space X satisfies these hypotheses (finite macroscopic dimension, uniformly contractible) follows from the fact that we have an ANR with a proper, cocompact action by our discrete group G. □

Once we have our ϕ-variant proper homotopy equivalence, the rest of the construction for a $\mathcal{Z}$-structure on $G \rtimes_\phi \mathbb{Z}$ goes through in the exact same way, except that we are now focused on a mapping telescope using the map from Proposition 8.2 instead of a lifted map from a $K(G, 1)$ to itself. Inspection of the proof of Lemma 5.1 shows that the assumptions of the map in Proposition 8.2 are enough to get the G-equivariant proper homotopy equivalence between $X \times \mathbb{R}$ and $Tel_f(X)$. Because it would be convenient to be able to inductively apply this construction to repeated semidirect products with infinite cyclic factors, we also need to know that if our construction begins with an $\underline{E}G$ space X that admits a $\mathcal{Z}$-structure, then our constructed $\mathcal{Z}$-structure for $G \rtimes_\phi \mathbb{Z}$ is in fact an $\underline{E}(G \rtimes_\phi \mathbb{Z})$ space. To prove this, it needs to be shown that every finite subgroup in $G \rtimes_\phi \mathbb{Z}$ has a contractible fixed point set in $Tel_f(X)$, but this is true since every finite subgroup's fixed point set is a mapping telescope of the map f restricted to the fixed point set of a finite subgroup $H \subset G$. By the assumption that X is an $\underline{E}G$ space, we know that the fixed point sets of finite subgroups are contractible in X, and the mapping telescope of a contractible subspace is itself a contractible subspace. With these observations, the proof of Theorem 2.8 is complete.

9 Applications

In this section, we look at two applications of Theorem 2.7. The first concerns strongly polycyclic groups, and the second involves 3-manifold groups.

Definition 9.1. *A group is polycyclic if it admits a subnormal series with cyclic factors. A group is strongly polycyclic if each of these factors is infinite cyclic. The Hirsch length of a polycylic group is the number of infinite cyclic factors in its subnormal series.*

Theorem 9.2. *Every strongly polycyclic group with Hirsch length n admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$ where $Z = S^{n-1}$.*

Proof. If G has a subnormal series $G = G_0 \triangleleft G_1 ... \triangleleft G_{n-1} \triangleleft G_n = 1$ where each $G_i/G_{i+1} = \mathbb{Z}$, then one uses the fact that $G_{n-1} = \mathbb{Z}$ has a $\mathcal{Z}$-structure

$(\hat{\mathbb{R}}, \pm\infty)$ as the base case of an induction proof, and repeated applications of Theorem 2.7 to the extensions $1 \to G_{i+1} \to G_i \to G_i/G_{i+1} \to 1$ to get the rest. Because the extension is by $\mathbb{Z}$ every time, the boundary becomes $(n-1)$-fold join of S^0. □

We do not know whether every polycyclic group (and hence every nilpotent group) admits a $\mathcal{Z}$-structure. It is known that every polycylic group can be expressed as a finite extension of a strongly polycyclic group, so the above argument could also be used to prove that polycyclic groups admit $\mathcal{Z}$-structures if it could be taken one step further when extended by a finite group at the end. Since finite groups tend not to be interesting in the eyes of geometric group theory (they act properly and cocompactly on a point), it seems plausible that one could develop a boundary swapping argument where the same spherical boundary is used. This could involve developing a proof showing that the finite index subgroup's action could be extended to a proper, cocompact action of the full group on the same space, analogous to Bieberbach's theorems for groups that are virtually $\mathbb{Z}^n$, but it would also suffice to prove that you could construct any AR on which the larger group acts properly and cocompactly, at which point one could then apply the following boundary swapping theorem of [GM18].

Theorem 9.3. *[GM18] Suppose quasi-isometric groups G and H act geometrically on proper metric ARs X and Y, respectively, and Y can be compactified to a $\mathcal{Z}$-structure $(\hat{Y}, Z)$ for H. Then X can be compactified, by the* *addition of the same boundary, to a $\mathcal{Z}$-structure $(\hat{X}, Z)$ for G.*

The next application of Theorem 2.7 relates to the fundamental groups of 3-manifolds. For further reference on the background material involving 3-manifold groups, see [AFW15]. Many closed 3-manifold groups were already known to admit $\mathcal{Z}$-structures by the work of others, and those results are collected here. It is our Theorem 2.7 that gives us the tool required to place $\mathcal{Z}$-structures on the fundamental group of manifolds admitting Sol or Nil geometry, and these were the final pieces in the puzzle for completing the following theorem:

Theorem 9.4. *Every closed, orientable 3-manifold group admits a $\mathcal{Z}$-structure.*

Proof. Because of the Prime Decomposition Theorem due to Milnor [Mil62], we know that every 3-manifold has a unique decomposition as the connect sum of prime 3-manifolds. Therefore, it suffices to prove the claim for prime manifolds because if a manifold splits as a connect sum, then it's fundamental group splits as a free product, and Tirel and Dahmani both have constructions that tell us how to build $\mathcal{Z}$-structures on free products when the individual factors admit $\mathcal{Z}$-structures [Tir11],[Dah03]. Since there is only one closed 3-manifold that is prime but not irreducible, and it's fundamental group is $\mathbb{Z}$ (which is known to admit a $\mathcal{Z}$-structure), it further reduces to proving the claim for irreducible manifolds. For irreducible manifolds, there are three essential cases to consider: geometric manifolds, mixed manifolds, and graph manifolds. For our purposes, we will only consider mixed and graph manifolds with at least one JSJ torus in their decomposition, and those without any JSJ tori will be viewed only as geometric manifolds. This framework also relies on Perelman's resolution of the Geometrization Theorem [Per03].

A mixed manifold is one whose prime JSJ decomposition includes at least one hyperbolic block, and Leeb proved that Haken mixed manifolds admit nonpositively curved Riemannian metrics [Lee94]. If the mixed manifold has at least one JSJ torus, then the manifold is guaranteed to be Haken and so Leeb's theorem applies. This provides a proof that a mixed 3-manifold's group admits a $\mathcal{Z}$-structure, namely the CAT(0) boundary on the universal cover.

A graph manifold is one whose prime JSJ decomposition does not include any hyperbolic blocks, and Kapovich-Leeb proved that for a Haken graph manifold M (where we again focus only on the case of graph manifolds with at least one JSJ torus), one can find a nonpositively curved 3-manifold N such that there is a bi-Lipschitz homeomorphism between the universal covers of M and N [KL98]. As a result, $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric, where it is then known that $\pi_1(N)$ admits a $\mathcal{Z}$-structure in the form of $\tilde{N}$'s CAT(0) boundary. One can then apply Theorem 9.3 to swap boundaries and get a $\mathcal{Z}$-structure for $\pi_1(M)$.

In the case that the 3-manifold admits a geometric structure, it was already known how to apply a $\mathcal{Z}$-structure to groups for a majority of the geometries. For S^3 geometry, the groups are all finite, and these all admit trivial $\mathcal{Z}$-structures where we have the groups act on a point. For $S^2 \times \mathbb{R}$ geometry, the only group that arises in the orientable case is $\mathbb{Z}$, which is known to admit a $\mathcal{Z}$-structure. Groups modeled on $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ are all CAT(0). It is also a well-known result in 3-manifold theory that $\widetilde{SL_2(\mathbb{R})}$ is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$ which is CAT(0), so boundary swapping can be used again in that case. Last is the question of Sol and Nil manifold groups, but these groups are precisely of the form to which Theorem 2.7 applies. The fundamental group of a closed 3-manifold that admits Sol or Nil geometry is going to be a semidirect of the form $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ [AFW15], and since $\mathbb{Z}^2$ is a CAT(0) group and hence admits a $\mathcal{Z}$-structure, we can apply Theorem 2.7 (and in these two cases, the boundary provided by our construction will be a 2-sphere). $\square$

We are hopeful that the above theorem can also be extended to include non-orientable closed 3-manifold groups, but there is additional work to be done.

10 Group Extensions

In this section, we briefly discuss an open question in the study of $\mathcal{Z}$-structures.

Definition 10.1. *Let N, G, and Q be groups. If there exists a short exact* *sequence of the form*

$$1 \to N \to G \to Q \to 1$$

then we say that G is a ***extension*** *of N by Q(or Q by N; the literature is inconsistent).*

Recall that a group G is considered to be type F if it admits a finite $K(G, 1)$. It is known that an extension of a nontrivial type F group by another nontrivial type F group will admit a weak $\mathcal{Z}$-structure [Gui14], meaning that all of the conditions required for a $\mathcal{Z}$-structure except the nullity condition are satisfied. Direct products are also a special case of group extensions, and it is known how to create $\mathcal{Z}$-structures out of direct products. Theorem 2.7 looks at the special case of extending a group G by $\mathbb{Z}$. This leads to the following:

Open Question: If N and Q are assumed to admit $\mathcal{Z}$-structures, must any extension of N by Q also must admit a $\mathcal{Z}$-structure?

We are hopeful that a resolution of the above question could lend insight into resolving the more classical group extension problem, for which no complete classification is yet known.

Group Extension Problem: Classify all possible extensions of N by Q.

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