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Numerical Solution of Stochastic Control Problems Using the Finite Element Method

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ABSTRACT

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Based on linear programming formulations for infinite horizon stochastic control problems, a numerical technique in fashion of the finite element method is developed. The convergence of the approximate scheme is shown and its performance is illustrated on multiple examples. This thesis begins with an introduction of stochastic optimal control and a review of the theory of the linear programming approach. The analysis of existence and uniqueness of solutions to the linear programming formulation for fixed controls represents the first contribution of this work. Then, an approximate scheme for the linear programming formulations is established. To this end, a novel discretization of the involved measures and constraints using finite dimensional function subspaces is introduced. Its convergence is proven using weak convergence of measures, and a detailed analysis of the approximate relaxed controls. The applicability of the established method is shown through a collection of examples from stochastic control. The considered examples include models with bounded or unbounded state space, models featuring continuous and singular control as well as discounted or long-term average cost criteria. Analyses of various model parameters are given, and in selected examples, the approximate solutions are compared to available analytic solutions. A summary and an outlook on possible research directions is given.

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Introduction

I.1 On Mathematical Control

The area of mathematical control considers dynamic systems and investigates how they can be influenced, or controlled, in a desired way. Inherently, this requires that one is able to describe the evolution of the dynamic system with appropriate mathematical models. Furthermore, a terminology has to be established to specify what a desired influence on the given dynamic system is.

The description of the dynamics can have various forms depending on the object under consideration. Classically, one distinguishes between models that describe the state of the system at discrete time points, and models that describe it in continuous time. In the same manner, one can either consider models that assume that the dynamic system can take finite, or countably infinite different states on the one hand, or consider models that allow the states to lie in a continuous, uncountable infinite set on the other hand. Evidently, the illustrated approaches for the time points and the states can be combined to most suitably model a given dynamical system. The type of model is referred to as a discrete, or continuous time and discrete, or continuous state space model, respectively. Across all possible four types of models, one can further distinguish between models that show deterministic behavior, and such models that include stochastic behavior.

Depending on the type of the model, the mathematical formulation of the dynamics takes different forms. Deterministic models in continuous time with a continuous state space can be modeled using ordinary differential equations. If the considered dynamical system also shows random behavior, stochastic differential equations can be chosen as a suitable model. In a discrete state space setting including randomness, one can use stochastic processes given by Markov chains, while in the deterministic case, an automaton model could be chosen as a model.

Key to all these choices is that the respective mathematical formulation must include, to some extent, a way in which the system can be influenced. For example, using an automaton model, it is possible that the transition from one state to another depends on an input given by an entity referred to as the control. In a stochastic model, that transition might not completely underlie the control's discretion, but the transition probabilities can be influenced in accordance to the control's preference. Using differential equation models, such influence is given through the coefficient functions that are found in the differential equation, such as drift and diffusion coefficients.

Given an appropriate model for the dynamics and the influence the control is able to enact, the objective which the control seeks to fulfill has to be formulated. For example, it might be desirable that at a given point in time, the dynamical system is in a specific state, or one might simply decide to stop the system as soon as it reaches a specific state. Further, the control might be interested in minimizing a running cost that accrues as the system evolves over time, depending on the state and also, on the influence enacted by the control.

It is also necessary to distinguish between systems which are running for a finite amount of time, systems which can be stopped at a desired time, and systems that are assumed to be running an infinite amount of time. The first case is called a finite time horizon, the second case is referred to as a problem of optimal stopping while the last case is called an infinite time horizon problem. A measure that specifies how well any of the described objectives is met is called an optimality criterion. It can, for example, take the form of a functional on the paths that the dynamical system takes, or simply a function on the state space which is evaluated as soon as the system stops evolving.

The choice of an optimal control which fulfills a given objective is a complex mathematical problem. In all of the cases we have introduced, analytic solution methods as well as approximate numerical methods are available.

The contribution of this thesis is to investigate an approximate numerical method for models that feature a continuous time domain, and a continuous state space and include random behavior. The considerations are limited to infinite time horizon models. Generally, the considered problems lie in the area of stochastic optimal control, or simply stochastic control. The next section gives a more precise description of the models which are considered in this thesis, introduces the relevant literature and elaborates on the scope of this work.

I.2 Motivation and Overview

Connecting to the preceded introduction to mathematical control, in this section we specify the type of stochastic control problem under investigation in the present thesis. This is followed by an overview of scientific problems that can be solved with such stochastic control problems. The existing literature, especially on approximate techniques for the formulated problems, is reviewed and it is illustrated in which way this work contributes to the current state of research on these techniques. Finally, an outline of this thesis is given. Although mathematical language is used to a certain extent in this section, we partially omit references to the literature and hereby point to the following chapters for a rigorous treatment of the mathematical ideas pertinent to this thesis.

The basis for the control problems considered in this thesis is given by a description of the dynamics by stochastic differential equations (SDEs) given by their integral form

$$X_t = x_0 + \int_0^t b(X_s, u_s)ds + \int_0^t \sigma(X_s, u_s)dW_s + \xi_t \tag{2.1}$$

where X_t describes the state of the system at time t, and u_t specifies the control at time t. At time $t = 0$, X_0 is assumed to be in the state x_0, called the starting value. The choice of u_t influences the coefficient functions b and σ, called drift and diffusion, respectively. Obviously, drift and diffusion are crucial to the evolution of the system. The process ξ can be used to model behavior that does not evolve continuously with time, like instantaneous jumps, or reflections of the process. The term dW_s refers to an increment of a Brownian motion process, which is a classic model used in stochastic analysis. As soon as ξ is non-zero in a given model, we speak of a singular stochastic control problem, as opposed to a stochastic control problem where $\xi \equiv 0$.

We assume that the system runs for an infinite amount of time. An example for an optimality criterion in this case is given by the so-called infinite horizon discounted criterion given by two positive cost functions c_0 and c_1, a discounting rate α and the expression

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s} c_0(X_s, u_s)ds + \int_0^\infty e^{-\alpha s} c(X_s, u_s)d\xi_s\right].$$

A precise and in-depth treatment of the stochastic differential equation models and cost criteria can be found in Section II.1. Note that it is totally arbitrary whether one tries to minimize costs or maximize reward. Unless stated differently, this thesis deals with the minimization of costs.

Problems of the aforementioned type have their origins in defensive and strategic analysis. In order to intercept hostile missiles one tries to imitate the flight path of an object, modeled by a Brownian motion process W, with a deterministic process given by integrating over the drift b. This classical 'bounded follower' problem is dealt with in Benes et al. (1980). Another classic reference is Bather and Chernoff (1967), which considers the control of a spaceship trying to reach a certain target. This setting can easily be transferred to a scientific problem where random movements of particles have to be countered by a deterministic drift, for example to keep these particles within certain bounds.

A large area of research concerns itself with evaluating ecologic or economic systems. Optimal harvesting problems, which consider the growth of a population under both deterministic and random influence, with some entity reducing the population by harvesting, give an example. The growth behavior of the population is often described by a stochastic logistic growth model, as seen in Lungu and Øksendal (1997) and Framstad (2003). A more general growth model is considered in Stockbridge and Zhu (2013).

Another application is inventory control, where unpredictability of demand and possible returns in conjunction with cost for holding items on stock pose an involved control problem. A specific control policy was analyzed in Helmes et al. (2017), and the references therein provide a good overview on this field of study.

The rise of quantitative analysis in finance and business has posed a line of interesting stochastic control problems. An optimal investment model is considered in Guo and Pham (2005). With asset prices frequently described by geometric Brownian motion processes or mean-reverting models like the Ornstein-Uhlenbeck processes, derivative pricing or asset allocation problems can also be expressed in terms of stochastic control problems. See Davis and Norman (1990) for a consideration of the classic Merton problem. A paper from Lu et al. (2017) presents a stock allocation problem, and the references therein give an overview on financial trading rules based on mathematical models. More applications of stochastic of stochastic control can as well be found in Yong and Zhou (1999) or Pham (2009).

As in many areas of applied mathematics, approximate numerical methods are needed to solve involved problems. This is due to several reasons. First and foremost, many models pose problems whose analytic solution is not attainable, and thus has to be approximated. This might be the case where the structure of the underlying dynamics is too complicated. Note that there is only a small class of SDEs of the type (2.1) for which solutions are known. The same is true when analytic expressions for the involved coefficient functions are unavailable - that is if b and σ in (2.1) can only be evaluated pointwise, perhaps by a complex numerical procedure itself. On the other hand, analytic investigations of stochastic control problems are time-consuming. Even if a solution to the underlying dynamics can be found, finding the optimal control is another challenging task which usually can only be solved for fairly ‘obvious’ problems, or with a considerable number of simplifying assumptions. The use of approximate numerical techniques can help circumvent this time intensive process.

The classic solution approach for stochastic control problems is based on the dynamic programming principle. It derives a differential equation, called the Hamilton-Jacobi-Bellman (HJB) equation, that characterizes the so-called value function. The value function describes the value of a cost criterion under the optimal control. A fundamental treatment of this approach, for both deterministic and stochastic control problems, can be found in Fleming and Rishel (1975). A more recent text, focusing on stochastic control, is given by Pham (2009). An overview of analytically solvable stochastic control problems can be found in Benes et al. (1980), which is mainly concerned with tracking a Brownian Motion process under certain restriction on the control. Another analytic example, featuring optimal harvesting, can be found in Lande et al. (1995).

As the HJB equation is usually fairly irregular (given by a second-order non-linear partial differential equation), the most successful solving techniques are based on viscosity solutions, as introduced in Crandall et al. (1992). The application of the notion of viscosity solution to stochastic control is presented in Lions (1983a) and Lions (1983b).

Numerical methods based on the dynamic programming principle have been state-of-the-art in deterministic and stochastic control, and can generally be split up in two branches. The first branch seeks to discretize the dynamics in such a way that a discrete time, discrete state space model is obtained. An overview of such techniques is presented in Kushner (1990), and an extensive treatment of finite-difference based approaches is given by Kushner and Dupuis (2001). The second branch considers fully analytic HJB equations and seeks to solve them using solvers for partial differential equations. In their paper, Barles and Souganidis (1991) present a general framework to approximate viscosity solutions to the HJB equations, and an implementation of this framework, using finite element approximations, is described in Jensen and Smears (2013). In both branches, the techniques of value iteration and policy iteration (see Puterman (1994) for an introduction in discrete time and space) can be used to solve the discrete optimal control problem as soon as the problem is discretized. A different numerical technique using dynamic programming was analyzed in Anselmi et al. (2016).

An alternative approach to stochastic optimal control is given by the so-called linear programming approach, which considers infinite-dimensional linear programs for measures that describe the 'average' behavior of the dynamics. This approach has been used to control the running maximum of a diffusion in Heinricher and Stockbridge (1993), solve optimal stopping problems in Cho and Stockbridge (2002) and to consider regime-switching diffusions in Helmes and Stockbridge (2008). Frequently, authors have considered models with singular dynamics, like reflection or jump processes or even singularly controlled processes. The fundamental theory of the linear programming approach in stochastic control is presented in Kurtz and Stockbridge (2017).

The linear programming approach relies on a relaxed formulation of the dynamics in the form of martingale problems. This relaxation allows for a mathematically more suitable treatment of stochastic control problems. An example can be found in Stockbridge and Zhu (2013), which deals with a harvesting model for which the dynamic programming approach indicated the use of a 'chattering' control, harvesting at an infinite rate for small portions of time, see Alvarez (2000).

While being instrumental in providing analytic solutions to a line of control problems, the linear programming approach has led the way for the introduction of novel approximation techniques in stochastic control. A very general setting is presented in Mendiondo and Stockbridge (1998). Moment-based approaches were extensively studied in a line of publications, as can be seen in Helmes et al. (2001), Helmes and Stockbridge (2000) and Lasserre and

Prieto-Rumeau (2004). These methods rely on a computation of the moments of the involved expected occupation measures in order to approximate cost criteria.

Recent research has investigated a different numerical approach to infinite-dimensional linear programs. The structure of the linear constraints in form of an operator-integral equation, similar to variational equalities encountered in solving partial differential equations has suggested the use of finite element type approximations. The idea of the finite element method (see Solin (2006) for an example) is to introduce finite dimensional subspaces of the involved function spaces, and solve the discrete problem that results from this discretization. The initial work on this approach was conducted in Kaczmarek et al. (2007), which features using an approximation of both constraints and measure densities by continuous piecewise linear functions. Furthermore, the performance of this method is compared to state-of-the-art methods presented by Kushner and Dupuis (2001), indicating a better performance of the finite element approach on a selected numerical example. The thesis of Rus (2009) conducted a more thorough and theoretical investigation of this idea by using a least squares finite element approach with cubic Hermite polynomial basis functions for both constraints and measures. A comparable performance to Kaczmarek et al. (2007) could be observed, however, the underlying linear structure of the problem was not exploited for the optimization. Finally, Lutz (2007) applied similar ideas to price American lookback options, again using continuous piecewise linear basis functions.

While indicating strong numerical performance of finite element-type approximate methods, all three aforementioned papers (Kaczmarek et al. (2007), Rus (2009), Lutz (2007)) fail to provide a complete convergence proof of the suggested numerical scheme. Furthermore, the techniques presented were only used on a limited selection of problems. The broad applicability of these methods remains in question.

The contribution of the present work is to provide an adjusted approximation scheme for which the convergence can be shown analytically, and apply it to a broad range of stochastic control problems. In contrast to previous work, the constraints and measures are discretized using distinct function spaces tailored towards the analysis conducted in the convergence proof, which imparts detailed insight on the approximation properties of the proposed method. The arguments presented are general enough to be applied to models with or without singular behavior, models with singular control, and models with both a bounded or an unbounded state space. Furthermore, the cases of optimization and plain evaluation of a given control policy are distinguished. As a side product of the latter, existence and uniqueness of analytic solutions to the linear constraint given by the linear program are shown. To demonstrate the method's flexibility, it is tested on a variety of stochastic control problems. These include the example calculated in Kaczmarek et al. (2007) and Rus (2009), for which an analytic solution can be found and the convergence can be investigated. Several new examples are presented, featuring models with costs of control, singular control, unbounded state space and additional constraints on the use of the control. In many examples, an analysis of the influence of the model parameters is provided, supporting the relevance of the present method.

This thesis is structured as follows. The second chapter, 'Stochastic Control and Mathematical Background', thoroughly introduces the stochastic control problems of interest and discusses the linear programming approach. A first contribution is given by the analysis of existence and uniqueness of solution to certain linear constraints given by a fixed control. Mathematical concepts pertinent to the understanding of this thesis are also introduced. The third chapter, 'Approximation' introduces the numerical scheme used for the proposed method. On the one hand, we consider how a finite-dimensional linear program can be obtained from the infinite-dimensional formulation. On the other hand, we describe how similar techniques can be used to evaluate the optimality criterion for a fixed, not necessarily optimal, control. Further attention is directed to the computational adaption of the approximation. Chapter IV, 'Convergence Analysis' provides the theory proving that the proposed numerical scheme produces solutions converging towards the analytic optimal solution. Again, some attention is brought towards the case of evaluating a fixed, not necessarily optimal control. The fifth chapter, 'Numerical Examples' shows the performance of the developed numerical method when solving a collection of example problems from various scientific fields. Finally, an outlook is given indicating future research directions and extensions to this thesis. An appendix providing material omitted in the main document for the sake of readability and including a list of frequently used abbreviations is to follow. The bibliography and a curriculum vitae conclude this thesis.

We use a continuous numbering scheme for equations, lemmas, propositions, theorems, corollaries, remarks and examples in the following. These objects will be referenced by a roman numeral indicating the chapter, followed by an arabic number indicating the section and a second arabic number representing the consecutive number of the object in that section, separated by a period, respectively. To give an example, '(III.2.1)' is the first object, an equation, appearing in the second section of Chapter III. The next object in this section is a definition - and is thus referenced by 'Definition III.2.2'. However, for the sake of readability, the roman numerals are omitted when the object lies in the same chapter from where it is referred to. So, the aforementioned 'Definition III.2.2' will appear as 'Definition 2.2' in Chapter III, and as 'Definition III.2.2' in any other chapter.
Figures and tables follow separate continuous numbering schemes, both consisting of a roman numeral to indicate the chapter, and an arabic number representing the consecutive number of the figure or table, respectively, within the given chapter. Throughout the text, they are referenced using both the roman numeral and the arabic number. For example, 'Figure V.27' is the 27th figure in Chapter V, and 'Table V.35' is the 35th table in Chapter V.

Stochastic Control and Mathematical Background

This chapter rigorously introduces the topic of stochastic control and the mathematical background needed for this thesis. The first section introduces the mathematics of stochastic control, in particular the linear programming approach. The analysis of existence and uniqueness in the linear programming setting, conducted in the second section of this chapter, represents a first contribution of the present thesis. Sections reviewing weak convergence of measures and cubic spline interpolation close this chapter.

II.1 Stochastic Control Problems

This section thoroughly introduces the mathematical formalism used to describe the stochastic control problems of interest in this thesis. We start by considering a formulation involving stochastic differential equations. Then we show how this formulation can be transformed into an infinite-dimensional linear program, which is the basis for the method we will propose in Section II.4.

II.1.1 Models Using Stochastic Differential Equations

In this subsection, we will introduce specific stochastic differential equations (SDEs) modeling the dynamical system that are of interest in this thesis. For a thorough introduction to stochastic differential equations, the interested reader is referred to Karatzas and Shreve

(1991). A condensed exposition suited for the application of stochastic control can be found in Pham (2009).

Let the state space E and the control space U be subsets of the real line. The specific form of E depends on the model of consideration, which will be elaborated. In the same way, U can take various forms, but this thesis focuses on the case where U is a closed interval. We equip E and U with their Borel σ-algebras $\mathscr{B}(E)$ and $\mathscr{B}(U)$, respectively. Let $b, \sigma : E \times U \mapsto E$ be continuous and hence measurable functions. Consider a Brownian motion W on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and a progressively measurable process u, called the control, taking values in the control space U. The set of progressively measurable processes taking values in U is referred to as $\mathcal{A}$, the set of admissible controls. We consider dynamic systems modeled by the SDE in integral form

$$\begin{aligned} X_t &= x_0 + \int_0^t b(X_s, u_s)ds + \int_0^t \sigma(X_s, u_s)dW_s + \xi_t \\ X_0 &= x_0. \end{aligned} \tag{1.1}$$

Formally, we have $E = \{x \in \mathbb{R} | x = X_t(\omega) \text{ for some } (t, \omega)\} \subset \mathbb{R}$. To better describe the situation, we split it into two parts. Set $Y_t = x_0 + \int_0^t b(X_s, u_s)ds + \int_0^t \sigma(X_s, u_s)dW_s$. Then, $X_t = Y_t + \xi_t$. Since the coefficients b and σ do not depend on the time t, Y_t is a diffusion process. It will be referred to as the continuous part of X_t, as it evolves continuously with time. The process ξ_t is a right-continuous process of bounded variation, which can be used as an integrator in a Lebesgue-Stieltjes integral. It is used to model behavior that happens instantaneously and is therefore called the singular part of X_t. A classic SDE model is obtained when $\xi \equiv 0$. In this case, the form of the coefficient functions determines E. For example, a simple Brownian motion process is modeled by the SDE $X_t = \int_0^t dW_t = W_t$, and as the paths of W are unbounded, $E = (-\infty, \infty)$. The introduction of an aptly chosen process ξ could however force $X_t = W_t + \xi_t$ to remain within a bounded interval $E = [e_l, e_r]$. For example, ξ could model a jump from e_l and e_r to the origin, ensuring that the process remains with $[e_l, e_r]$. Finally, specific SDEs have coefficient functions b and σ which guarantee that X remains in a bounded interval, as the following example shows.

Example 1.2. *A solution to the SDE*

$$X_t = x_0 + \int_0^t X_s(K - X_s)ds + \int_0^t X_s(K - X_s)dW_s$$

for $K, x_0 \geq 0$ has the state space $E = [0, K]$ which is closed and bounded, see Lungu and Øksendal (1997).

It is easy to imagine that the described cases can be combined such that we encounter state spaces of the form $E = (-\infty, \infty)$, $E = (-\infty, e_r]$, $E = [e_l, \infty)$ and $E = [e_l, e_r]$. If one of the boundaries is infinite, we speak of an unbounded state space model, if both boundaries are finite, we speak of a bounded state space model.

From now on, assume that Y_t and ξ_t are processes such that $X_t \in E$ for all $t \geq 0$. A pair of processes (X_t, ξ_t), with ξ possibly being zero, is called a solution to the SDE if it fulfills (1.1). If ξ is non-zero, (1.1) is called a singular SDE, due to the existence of singular behavior and control problems for such SDEs are referred to as singular control problems.

Existence and uniqueness results for SDE models without a singular part are extensively discussed in the literature, see Karatzas and Shreve (1991) or Pham (2009). They can be generalized to the case of singular SDEs by considering piecewise solutions between two consecutive increments or decrements of the singular process. Considering reflected processes in particular, Section 3.6 C of Karatzas and Shreve (1991) provides another framework to show existence and uniqueness by addressing the so-called Skorohod problem. Nevertheless, as the main focus of this work lies on a relaxed formulation of the dynamics we will not dwell on existence and uniqueness results for (1.1). Still, it is worth mentioning that typically one has to assume stronger conditions on b and σ than sheer continuity as we did here. For the linear programming formulations that are considered later on, continuity is sufficient for a well-posed problem.

In order to fully describe the sets of problems we are considering, we need to specify possible form of the singular process ξ. It will be used to model two types of singular behavior. The first one is jump behavior. Upon entering a certain point $x \in E$, the process immediately moves to a second distinct point $s \in E$. Jumps are modeled using an increasing sequence of stopping times $\{\tau_k\}_{k \in \mathbb{N}}$, which describes the times a jump is triggered, and a Borel measurable function $h : [0, \infty) \times E \times U \mapsto \mathbb{R}$. The value $h(t, x, v)$ describes the size of the jump happening at time t, given that the process approached the state x, with the control process approaching the point v. This allows the jumps to be time-dependent and controlled. Throughout this thesis, we will assume that $u \mapsto h(t, x, u)$ is continuous. The actual jump process is given by

$$\xi_t = \int_0^t h(s, X_{s-}, u_{s-})\, d\hat{\xi}_s, \qquad \hat{\xi}_s = \sum_{k=1}^{\infty} I_{\tau_k \leq s}.$$

A detailed discussion of jump processes, including a derivation of Itô's formula for processes featuring jumps, is provided in Appendix A.1. In the scope of this thesis, we will consider jumps that happen as soon as the process enters a given point x in the state space. This means that in particular, the jump size h will not depend on time, and the sequence of jump times is given by

$$\begin{aligned} \tau_0 &= \inf\{t \geq 0 \,|\, X_t = x\} \\ \tau_k &= \inf\{t \geq \tau_{k-1} \,|\, X_t = x\}. \end{aligned}$$

The second type of singular process is a reflection. This means that if the process hits a specific point $r \in E$, it is reflected directly back into the opposite direction. Such a reflection is modeled by a local time process which only increases at the times t when $X_t = r$. Local time process are denoted by $L_r^X(t)$. A discussion of local time processes, including a derivation of Itô's formula for processes featuring reflections is provided in Appendix A.2. A reflection at r to the right is modeled by $\xi_t = L_r^X(t)$, a reflection to the left by $\xi_t = -L_r^X(t)$. The following is an example of a singular SDE.

Example 1.3. *If ξ is given by two local time process $L_{\{0\}}^X$ and $L_{\{1\}}^X$, which model reflection, a solution to the SDE*

$$X_t = x_0 + \int_0^t b(X_s, u_s)\,ds + \int_0^t \sigma(X_s, u_s) dW_s + L_{\{0\}}^X(t) - L_{\{1\}}^X(t)$$

with $x_0 \in [0, 1]$ has the state space $E = [0, 1]$.

In this example, as well as the general case, the control process u formally depends on the time t and the random element $\omega \in \Omega$. Note that this means that the control can depend on the full history of the process X. If a control only depends on the current state of the process, say by a measurable function $v : E \mapsto U$ such that $u_t = v(X_t)$, u is called a feedback control.

As previously mentioned, the optimality of a given control u is measured using cost criteria. Let (X^u, ξ^u) be a solution to the dynamics (1.1) when the control u is being used. The following two cost criteria are of interest in this thesis.

Definition 1.4. *Let c_0, c_1 be continuous functions from $E \times U$ into $R_{\geq 0}$. The long-term average cost criterion is given by*

$$J : \mathcal{A} \ni u \mapsto J(u) := \limsup_{t\to\infty} \frac{1}{t} \mathbb{E}\left[\int_0^t c_0(X_s^u, u_s)\,ds + \int_0^t c_1(X_s^u, u_s)\,d\xi_s^u\right].$$

Definition 1.5. *Let c_0, c_1 be continuous functions from $E \times U$ into $R_{\geq 0}$. For a discounting rate $\alpha > 0$, the infinite horizon discounted cost criterion is given by*

$$J : \mathcal{A} \ni u \mapsto J(u) := \mathbb{E}\left[\int_0^\infty e^{-\alpha s} c_0(X_s^u, u_s)\,ds + \int_0^\infty e^{-\alpha s} c_1(X_s^u, u_s)\,d\xi_s^u\right].$$

Because we are considering the process X over the whole time line, we say these cost criteria have an infinite time horizon. The function c_0 is referred to as the cost function of the continuous behavior, c_1 is referred to as the cost function of the singular behavior. If we have a non-singular model with $\xi \equiv 0$, it is obvious that the second summand in the expectations can simply be dropped.

Having introduced the dynamics, and a choice of possible cost criteria, we can now define singular stochastic control problems in SDE form.

Definition 1.6. *A singular stochastic optimal control problem in SDE form is the task to find a control $u \in \mathcal{A}$ for the process X given by (1.1) such that one of the cost criteria (1.4) and (1.5) is minimal over all of $\mathcal{A}$.*

Remark 1.7. *For a stochastic optimal control problem to be well-posed, we need to ensure that for a given control u, there is -in some sense- only one solution X^u to the SDE, since otherwise the cost criteria are not well-defined. The presented cost criteria do only depend on the law of X, and hence are identical for two weakly unique solutions. Hence it suffices to require weak existence and uniqueness of solutions.*

II.1.2 Martingale Problems and Relaxed Formulations

We proceed to reformulate stochastic control problems in SDE form into linear programs. This will be conducted in three steps. First, we will motivate the martingale problem formulation of the SDE given by (1.1). In terms of weak solutions, martingale problems are equivalent to (1.1), and thus pose an equivalent optimal control problem when combined with one of the cost criteria, defined in Definitions 1.4 and 1.5. In a second step we consider relaxed martingale problems. This in particular yields to the introduction of relaxed controls, on the cost of losing the equivalence to weak solutions of the considered stochastic differential equation. However, the relaxation allows for a mathematically more complete consideration of control problems. Third, the optimal control problem defined by the relaxed martingale problem will be expressed as an infinite-dimensional linear program with the help of so-called expected occupation measures.

To begin with, we consider the following situation. Let $B(E)$ denote the set of Borel measurable functions from E to $\mathbb{R}$, $C_c(E)$ the set of continuous functions from E to $\mathbb{R}$ with compact support, and $C_c^2(E)$ the subset thereof of functions which are twice continuously differentiable.

Definition 1.8. *Let $A : C_c^2(E) \mapsto B(E \times U)$ be a linear operator, and let u be an admissible control. A stochastic process X is called a solution to the controlled martingale problem for (A, x_0) if for all $f \in C_c^2(E)$*

$$f(X_t) - f(x_0) - \int_0^t Af(X_s, u_s)\, ds \tag{1.9}$$

is a martingale.

Definition 1.10. *Let $A : C_c^2(E) \mapsto B(E \times U)$ and $B : C_c^2(E) \mapsto B(E \times U)$ be linear operators, and let u be an admissible control. A pair of stochastic processes (X, ξ) is called a solution to the singular controlled martingale problem for (A, B, x_0) if for all $f \in C_c^2(E)$*

$$f(X_t) - f(x_0) - \int_0^t Af(X_s, u_s)\, ds - \int_0^t Bf(X_s, u_s)\, d\xi_s \tag{1.11}$$

is a martingale.

Remark 1.12. *The operator A is called the generator of the continuous behavior of X, B is called the generator of the singular behavior of X. It is easy to see that with $B \equiv 0$, the controlled martingale problem is a special case of the singular controlled martingale problem.*

The specific form of the generators will determine the dynamic behavior of X. While the formulations of Definitions 1.8 and 1.10 are fairly general, with A and B allowed to take various forms, we will only consider such generators that stem from processes governed by (singular) SDEs, as seen in Section II.1.1.

Definition 1.13. *The generator of the continuous behavior is given by*

$$Af(x,u) = b(x,u)f'(x) + \frac{1}{2}\sigma^2(x,u)f''(x)$$

where b and σ are the coefficients from (1.1). Also, let $\alpha \geq 0$ and define

$$A_\alpha(x,u) = A(x,u) - \alpha f(x).$$

The operator A characterizes the dynamics specified in the SDE. The operator A_α will later be used to express an equivalent reformulation of the martingale problem needed when considering the infinite horizon discounted criterion. The generator of the singular behavior can take different forms depending on the specified singular behavior of the process ξ. The generators for jump processes and reflection processes are defined as follows.

Definition 1.14. *Let ξ model the singular behavior of a jump from x to $x+u$ with $x+u \in E$. The generator of ξ is given by*

$$Bf(x,u) = f(x+u) - f(x),$$

where u can either be constant or be determined by a control. In the latter case, we have to assert that $u \in U$.

Definition 1.15. *Let ξ model the singular behavior of a reflection to the right at x with $x \in E$. The generator of ξ is given by*

$$Bf(x,u) = f'(x).$$

If ξ models a reflection to the left at x, the generator is given by

$$Bf(x,u) = -f'(x).$$

Remark 1.16. *Note that we can indeed encounter different cases of singular behavior, as seen in Example 1.3. In this case, we have to introduce additional terms into (1.11). We need to replace*

$$\int_0^t Bf(X_s,u_s)d\xi_s \equiv \int_0^t f'(x)d\xi_s^{(1)} + \int_0^t -f'(x)d\xi_s^{(2)},$$

where $\xi^{(1)}$ and $\xi^{(2)}$ are local time processes. For the sake of generality, we will use the notation $\int_0^t Bf(X_s,u_s)d\xi_s$ and will keep in mind that the actual expression might be a combination of the expressions seen in Definition 1.14 and Definition 1.15.

Some remarks on the regularity of these operators are in order. First, note that for any $f \in C_c^2(E)$, Af, $A_\alpha f$ and Bf are well-defined and are indeed bounded. For the subsequent analysis, we need to consider their domain $C_c^2(E)$ as a normed space in the following sense. From here on $\|\cdot\|_\infty$ denotes the uniform norm of functions that is well-defined for any bounded function.

Definition 1.17. *For $f \in C_c^2(E)$, define the norm*

$$\|f\|_{\mathscr{D}} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$$

The space $\mathscr{D}_\infty$ is the normed space $(C_c^2(E), \|\cdot\|_{\mathscr{D}})$.

For the following result, we consider A_α and treat A as a special case of the former, with $\alpha = 0$. Since $C_c(E \times U) \subset C_b(E \times U)$, the space of continuous bounded functions on $E \times U$, we can consider the operators A_α and B as mappings between normed spaces in the following way.

$$A_\alpha, B : \mathscr{D}_\infty \mapsto (C_b(E \times U), \|\cdot\|_\infty)$$

When E and U are compact, the following result holds.

Proposition 1.18. *Let E and U be compact. Then A_α and B are continuous operators.*

Proof. First, for $f \in \mathscr{D}_\infty$ we have that

$$\begin{aligned}
\|bf' + \frac{1}{2}\sigma f''\|_\infty &= \sup_{(x,u)\in E\times U} |b(x,u)f'(x) + \frac{1}{2}\sigma(x,u)f''(x)| \\
&\leq \sup_{(x,u)\in E\times U} \left(|b(x,u)||f'(x)| + \frac{1}{2}|\sigma(x,u)||f''(x)|\right) \\
&\leq \sup_{(x,u)\in E\times U} |b(x,u)| \sup_{(x,u)\in E\times U} |f'(x)| + \frac{1}{2} \sup_{(x,u)\in E\times U} |\sigma(x,u)| \sup_{(x,u)\in E\times U} |f''(x)| \\
&\leq \|b\|_\infty \|f'\|_\infty + \frac{1}{2}\|\sigma\|_\infty \|f''\|_\infty
\end{aligned}$$

where by continuity of b and σ and the compactness of E and U, $\|b\|_\infty$ and $\|\sigma\|_\infty$ are finite. Now consider $g, h \in \mathscr{D}_\infty$. With $f = g - h$, we deduce from the above equation that

$$\begin{aligned}
\|A_\alpha g - A_\alpha h\|_\infty &= \|A_\alpha(g-h)\|_\infty \\
&\leq \|b\|_\infty \|(g-h)'\|_\infty + \frac{1}{2}\|\sigma\|_\infty \|(g-h)''\|_\infty + \alpha\|(g-h)\|_\infty \\
&\leq \max\left\{\|b\|_\infty, \frac{1}{2}\|\sigma\|_\infty\|, \alpha\right\}\left(\|(g-h)\|_\infty + \|(g-h)'\|_\infty + \|(g-h)''\|_\infty\right)
\end{aligned}$$

For $\epsilon > 0$, if $\|g - h\|_{\mathscr{D}} \leq \frac{\epsilon}{\max\left\{\|b\|_\infty, \frac{1}{2}\|\sigma\|_\infty\|, \alpha\right\}}$, we see that $\|A_\alpha g - A_\alpha h\|_\infty \leq \epsilon$. If B models a jump from x to s, if $\|g - h\|_{\mathscr{D}} \leq \frac{1}{2}\epsilon$, we have that

$$\|Bg - Bh\|_\infty \leq \sup_{(x,u)\in E\times U} \{|(g-h)(s)| + |(g-h)(x)|\} \leq 2\|g-h\|_\infty \leq 2\|g-h\|_{\mathscr{D}} \leq \epsilon$$

or if B models reflection at a point x and $\|g - h\|_{\mathscr{D}} \leq \epsilon$,

$$\|Bg - Bh\|_\infty \leq \sup_{(x,u)\in E\times U} \{|(g-h)'(x)|\} \leq \|(g-h)'\|_\infty \leq \|g-h\|_{\mathscr{D}} \leq \epsilon,$$

which proves the claim. □

Remark 1.19. *Denote $C_b^u(E \times U)$ the space of bounded, uniformly continuous functions on $E \times U$. If $E \times U$ is compact, $C_b(E \times U) \subset C_b^u(E \times U)$. This means that the range of A and B as described in Proposition 1.18 lies in $C_b^u(E \times U)$.*

The relation between solutions of the martingale problem (1.11) and solutions to the SDE (1.1) is expressed in the following result.

Theorem 1.20. *A pair of processes* (X, ξ) *is a solution to the martingale problem as defined in Definition 1.10 with* A *as in Definition 1.13 and* B *according to Remark 1.16 if and only if* X *is a weak solution to the SDE*

$$
\begin{aligned}
X_t &= x_0 + \int_0^t b(X_s, u_s)ds + \int_0^t \sigma(X_s, u_s)dW_s + \xi_t \\
X_0 &= x_0.
\end{aligned}
\tag{1.1}
$$

Proof. We will provide the if-implication for the long-term average case. For the remainder of the proof, we refer to the literature, in particular Ethier and Kurtz (1986) and Lamperti (1977). We begin with a weak solution to the SDE (1.1). For any $f \in C_c^2(E)$, Itô's formula for singular processes (compare Appendix A) reveals that

$$
f(X_t) = f(X_0) + \int_0^t Af(X_s, u_s)ds + \int_0^t Bf(X_s, u_s)d\xi_s + \int_0^t \sigma(X_s, u_s)f''(X_s)dW_s \tag{1.21}
$$

where $\int_0^t Bf(X_s, u_s)d\xi_s$ is a generic term modeling various types of singular behavior (compare Remark 1.16). Equation (1.21) is equivalent to

$$
f(X_t) - f(X_0) - \int_0^t Af(X_s, u_s)ds - \int_0^t Bf(X_s, u_s)d\xi_s = \int_0^t \sigma(X_s, u_s)f''(X_s)dW_s,
$$

but since f is bounded with compact support and hence in L^2, the right hand side is a martingale. $\square$

As we are considering a solution (X, ξ), we can use the same cost criteria (Definitions 1.4 and 1.5) for solutions to the martingale problem. Two different weak solutions have the same law, and hence will give the same values for the cost criteria. From this standpoint, minimizing the cost criteria over the set of solutions to the martingale problem is equivalent to minimizing the cost criteria over the set of weak solutions to the SDE.

As previously mentioned, this set up might lead to situations in which an optimal control process u does not exist. A more general formulation of the martingale problem is thus better suited for the minimization of the cost criteria, although the equivalence to the SDE-based definition of a stochastic control problem is lost. In the following, denote the space of probability measures on U by $\mathcal{P}(U)$.

Definition 1.22. *Let A be the generator of the continuous behavior. Let X be a stochastic process taking values in E and Λ a stochastic process taking values in $\mathcal{P}(U)$. Then (X, Λ) is called a solution to the relaxed controlled martingale problem for (A, x_0) if there is a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that X and Λ are $\mathscr{F}_t$-adapted, X and Λ are $\mathscr{F}_t$-progressively measurable and for all $f \in C_c^2(E)$*

$$f(X_t) - f(x_0) - \int_0^t \int_U Af(X_s, u)\Lambda_s(du)\, ds \tag{1.23}$$

is a martingale.

Definition 1.24. *Let A and B be the generators of the continuous and singular behavior, respectively. Let X be a stochastic process taking values in E, Λ a stochastic process taking values in $\mathcal{P}(U)$, and Γ a random measure on $[0, \infty) \times E \times U$. The triplet (X, Λ, Γ) is called a solution to the relaxed singular controlled martingale problem for (A, B, x_0) if there is a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that X and Λ are $\mathscr{F}_t$-adapted, X, Λ and Γ_t, the restriction of Γ to $[0, t] \times E \times U$, are $\mathscr{F}_t$-progressively measurable and for all $f \in C_c^2(E)$*

$$f(X_t) - f(x_0) - \int_0^t \int_U Af(X_s, u)\Lambda_s(du)\, ds - \int_{[0,t]\times E\times U} Bf(x, u)\, \Gamma(ds \times dx \times du) \tag{1.25}$$

is a martingale.

The 'relaxed' form of the martingale problem is given by the fact that the continuous control is now a stochastic process Λ which takes values in the space of probability measures on U, and the singular control is included in the random measure Γ. The expression $\int_U Af(X_s, u)\Lambda_s(du)$ can be understood in a way that we are averaging the drift b and diffusion

σ that appear in A by integrating against Λ_s. The random measure Γ is a random variable taking values in the space of Borel measures on $[0, \infty) \times E \times U$, denoted by $\mathcal{M}([0, \infty) \times E \times U)$. It is assumed that for every $t \geq 0$, $\Gamma([0, t] \times E \times U)$ is finite. A line of additional technical constraints have to be considered concerning this relaxed formulation. We refer the interested reader to Kurtz and Stockbridge (2017) for a complete analysis. Here, we proceed to describe a stochastic control problem in terms of the relaxed martingale formulation.

To model feedback controls in the relaxed setting, we need the following concept. Note that $\mathscr{B}(E \times U) = \mathscr{B}(E) \times \mathscr{B}(U)$ as both E and U are separable.

Definition 1.26. *Let $(E \times U, \mathscr{B}(E \times U), \mu)$ be a measure space, and let $X : E \times U \ni (x, u) \mapsto x \in E$ be the projection map onto E. Let μ_E be the distribution of X. A map $\eta : \mathscr{B}(U) \times E \mapsto [0, 1]$ is called a regular conditional probability (rcp) if*

i) for each $x \in E$, $\eta(\cdot, x) : \mathscr{B}(U) \mapsto [0, 1]$ is a probability measure,

ii) for each $V \in \mathscr{B}(U)$, $\eta(V, \cdot) : E \mapsto [0, 1]$ is a measurable function, and

iii) for all $V \in \mathscr{B}(U)$ and all $F \in \mathscr{B}(E)$ we have

$$\mu(F \times V) = \int_F \eta(V, x) \mu_E(dx).$$

Remark 1.27. *This definition is tailored towards our purposes. We refer to Ethier and Kurtz (1986), Appendix 8, for the general definition and a short theoretic treatment. In particular, results from this source reveal that if $E \times U$ is complete and separable, an rcp does exist for the measure μ. A feedback control in the relaxed sense is given by an rcp η by setting $\Lambda_t(\cdot) = \eta(\cdot, X_t)$.*

The introduction of relaxed controls requires an adaption of the cost criteria. From here on, the set $\mathcal{A}$ of admissible controls is the set of all pairs (Λ, Γ) for which there exists a process X such that (X, Λ, Γ) is a solution to the relaxed singular controlled martingale problem.

Definition 1.28. *Let c_0, c_1 be continuous functions from $E \times U$ into $R_{\geq 0}$. The long-term average cost criterion in the relaxed setting is given by $J : \mathcal{A} \ni (\Lambda, \Gamma) \mapsto J(\Lambda, \Gamma)$ with*

$$J(\Lambda, \Gamma) := \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[\int_0^t \int_U c_0(X_s, u_s) \, \Lambda_s(du) \, ds + \int_{[0,\infty) \times E \times U} c_1(X_s, u_s) \, \Gamma(ds \times dx \times du) \right].$$

Definition 1.29. *Let c_0, c_1 be continuous functions from $E \times U$ into $R_{\geq 0}$. For a discounting rate $\alpha > 0$, the infinite horizon discounted cost criterion in the relaxed setting is given by $J : \mathcal{A} \ni (\Lambda, \Gamma) \mapsto J(\Lambda, \Gamma) \in \mathbb{R}$ with*

$$J(\Lambda, \Gamma) = \mathbb{E} \left[\int_0^\infty \int_U e^{-\alpha s} c_0(X_s, u_s) \, \Lambda_s(du) \, ds + \int_{[0,\infty) \times E \times U} e^{-\alpha s} c_1(X_s, u_s) \, \Gamma(ds \times dx \times du) \right].$$

From here on, we will always consider stochastic control problems in the relaxed sense according to the following definition.

Definition 1.30. *A stochastic optimal control problem in the relaxed sense is given by the dynamics in the relaxed martingale form of Definition 1.24 with one of the cost criteria given by Definitions 1.28 and 1.29.*

II.1.3 Linear Programming Formulations

Based on stochastic control problems in the relaxed sense we derive a linear programming formulation that considers these problems as linear optimization problems over a space of measures. The definitions of the cost criteria in the previous subsection feature a cost structure that considers, to some extent, only the average behavior of the process. It should hence suffice to have information about the average behavior of the process rather than full information about each possible path. This reduction of information is achieved by the introduction of so-called expected occupation measures. In the following, I_F is used to denote the indicator function of a set F.

Definition 1.31. *Let (X, Λ, Γ) be a solution to the relaxed martingale problem. The continuous and singular expected occupation measures $\mu_0 \in \mathcal{P}(E \times U)$ and $\mu_1 \in \mathcal{M}(E \times U)$ for the long-term average cost criterion are defined by*

$$\mu_0(F) = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t \int_U I_F(X_s, u)\Lambda_s(du)ds\right], \quad F \in \mathscr{B}(E \times U)$$

and

$$\mu_1(F) = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_{[0,t]\times E\times U} I_F(X_s, u)\Gamma(ds \times dx \times du)\right], \quad F \in \mathscr{B}(E \times U)$$

respectively.

Remark 1.32. *The phrases 'continuous' and 'singular' refer to the fact that the behavior μ_0 models occurs continuously in time, and the behavior μ_1 models occurs only on a set of points with a Lebesgue measure of zero.*

Definition 1.33. *Let (X, Λ, Γ) be a solution to the singular relaxed martingale problem. The continuous and singular expected occupation measures $\mu_0 \in \mathcal{P}(E \times U)$ and $\mu_1 \in \mathcal{M}(E \times U)$ for the discounted infinite horizon cost criterion are defined by*

$$\mu_0(F) = \alpha\,\mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} I_F(X_s, u)\Lambda_s(du)ds\right], \quad F \in \mathscr{B}(E \times U)$$

and

$$\mu_1(F) = \alpha\,\mathbb{E}\left[\int_{[0,\infty)\times E\times U} e^{-\alpha s} I_F(X_s, u)\Gamma(ds \times dx \times du)\right], \quad F \in \mathscr{B}(E \times U)$$

respectively.

Remark 1.34. *In the case of a (non-singular) relaxed martingale problem, the only expected occupation measure of interest is μ_0.*

Note that the definition of the respective expected occupation measures strongly resembles the cost criteria defined by Definitions 1.29 and 1.29 . This allows for a reformulation of the cost criteria as presented in the following proposition.

Proposition 1.35. *The long-term average cost criterion in the relaxed setting is given by*

$$\int_{E\times U} c_0(x,u)\,\mu_0(dx\times du) + \int_{E\times U} c_1(x,u)\,\mu_1(dx\times du).$$

The infinite horizon discounted criterion is given by

$$\int_{E\times U} \frac{1}{\alpha} c_0(x,u)\,\mu_0(dx\times du) + \int_{E\times U} \frac{1}{\alpha} c_1(x,u)\,\mu_1(dx\times du).$$

Proof. We present the proof for the case of the discounted infinite horizon criterion. The case of the long-term average cost criterion, the proof is more cumbersome. The interested reader is referred to Kurtz and Stockbridge (2017).
Approximate c_0 and c_1 by two sequence of elementary functions $\{\varphi_n^{(0)}\}_{n\in N}$ and $\{\varphi_n^{(1)}\}_{n\in N}$ with $\varphi_n^{(i)} = \sum_{k=1}^n \alpha_{k,n}^{(i)} I_{F_{k,n}^{(i)}}$, $i = 0,1$ for aptly chosen real numbers $\alpha_{k,n}$ and Borel sets $F_{k,n} \in \mathscr{B}(E\times U)$. By the monotone convergence theorem (note that $c_0 > 0$ and $c_1 > 0$)

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} c_0(X_s,u_s)\,\Lambda_s(du)\,ds\right] &= \lim_{n\to\infty} \sum_{k=1}^n \alpha_{k,n}^{(0)} \frac{1}{\alpha}\,\alpha\,\mathbb{E}\left[\int_0^\infty \int_U e^{-\alpha s} I_{F_{k,n}^{(0)}}\,\Lambda_s(du)\,ds\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^n \alpha_{k,n}^{(0)} \frac{1}{\alpha}\,\mu_0(F_{k,n}^{(0)}) \\
&= \lim_{n\to\infty} \sum_{k=1}^n \alpha_{k,n}^{(0)} \frac{1}{\alpha} \int_{E\times U} I_{F_{k,n}^{(0)}}\,\mu_0(dx\times du) \\
&= \int_{E\times U} \frac{1}{\alpha} c_0(x,u)\,\mu_0(dx\times du)
\end{aligned}$$

holds. A similar approach can be taken for the expression

$$\mathbb{E}\left[\int_{[0,\infty)\times E\times U} e^{-\alpha s} c_1(X_s,u_s)\,\Gamma(ds\times dx\times du)\right].$$

$\square$

We now present a way of characterizing the expected occupation measures in such a way that they indeed are expected occupation measures of a solution to the relaxed martingale problem. We confine ourselves to motivating this reformulation in the case of the discounted infinite horizon criterion, and again refer to the literature, see Kurtz and Stockbridge (2017), for a detailed analysis. Also, we present only the singular case, as again the non-singular case comes as a special case of the former. Consider a solution X to the singular relaxed martingale problem. This means that

$$f(X_t) - f(x_0) - \int_0^t \int_U Af(X_s, u)\Lambda_s(du)\, ds - \int_{[0,t]\times E\times U} Bf(x,u)\,\Gamma(ds \times dx \times du)$$

is a martingale for all $f \in C_c^2(E)$. According to Ethier and Kurtz (1986), Lemma 4.3.2, this is equivalent to

$$\begin{aligned} e^{-\alpha t} f(X_t) - f(x_0) - \int_0^t \int_U e^{-\alpha s} \left[Af(X_s, u) - \alpha f(X_s)\right] \Lambda_s(du)\, ds \\ - \int_{[0,t]\times E\times U} e^{-\alpha s} Bf(x,u)\,\Gamma(ds \times dx \times du) \end{aligned}$$

being a martingale for all $f \in C_c^2(E)$. In particular, for fixed $f \in C_c^2(E)$ we have that the above expression equals 0 for $t = 0$, hence it is true that for all $t \geq 0$

$$\begin{aligned} \mathbb{E}\Bigg[e^{-\alpha t} f(X_t) - f(x_0) - \int_0^t \int_U e^{-\alpha s} \left[A_\alpha f(X_s, u)\right] \Lambda_s(du)\, ds \\ - \int_{[0,t]\times E\times U} e^{-\alpha s} Bf(x,u)\,\Gamma(ds \times dx \times du) \Bigg] = 0, \end{aligned}$$

recalling that we defined $Af(x,u) - \alpha f(x) = A_\alpha f(x,u)$. Using the boundedness of the involved quantities we can send $t \to \infty$ to derive that

$$
\begin{aligned}
-f(x_0) &= \mathbb{E}\Bigg[\int_0^\infty \int_U e^{-\alpha s} A_\alpha f(X_s,u)\Lambda_s(du)\,ds \\
&\qquad\qquad + \int_{[0,\infty]\times E\times U} e^{-\alpha s} Bf(x,u)\,\Gamma(ds\times dx\times du)\Bigg]
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
-\alpha f(x_0) &= \alpha\,\mathbb{E}\Bigg[\int_0^\infty \int_U e^{-\alpha s} A_\alpha f(X_s,u)\Lambda_s(du)\,ds \\
&\qquad\qquad + \int_{[0,\infty]\times E\times U} e^{-\alpha s} Bf(x,u)\,\Gamma(ds\times dx\times du)\Bigg] \\
&= \int_{E\times U} A_\alpha(x,u)\,\mu_0(dx\times du) + \int_{E\times U} Bf(x,u)\,\mu_1(dx\times du).
\end{aligned}
$$

The relation $\int_{E\times U} A_\alpha(x,u)\,\mu_0(dx\times du) + \int_{E\times U} Bf(x,u)\,\mu_1(dx\times du) = -\alpha f(x_0)$, which has to hold for all $f \in C_c^2(E)$, will be referred to as the linear constraints since it poses a linear relationship for the two measures μ_0 and μ_1. A similar argument can be used to derive the linear constraints for the long-term average problem, which reads, given $f \in C_c^2(E)$,

$$
\int_{E\times U} Af(x,u)\mu_0(dx\times du) + \int_{E\times U} Bf(x,u)\mu_1(dx\times du) = 0.
$$

Naturally, the question arises if a pair of measures (μ_0,μ_1) which fulfills either of these constraints can somehow be related to a solution of the relaxed martingale problem. The answer to this question is positive, if we consider these constraints jointly with the respective cost criteria as in the following definitions.

Definition 1.36. *The linear program for the long-term average cost criterion is the following optimization problem. Minimize*

$$
\int_{E\times U} c_0(x,u)\mu_0(dx\times du) + \int_{E\times U} c_1(x,u)\mu_1(dx\times du).
$$

such that $\mu_0 \in \mathcal{P}(E \times U)$ *and* $\mu_1 \in \mathcal{M}(E \times U)$ *and furthermore*

$$\int_{E\times U} Af(x,u)\mu_0(dx \times du) + \int_{E\times U} Bf(x,u)\mu_1(dx \times du) = 0, \quad \forall\, f \in C_c^2(E),$$

holds.

Definition 1.37. *The linear program for the infinite horizon discounted cost criterion is the following optimization problem. Minimize*

$$\frac{1}{\alpha}\int_{E\times U} \frac{1}{\alpha} c_0(x,u)\mu_0(dx \times du) + \frac{1}{\alpha}\int_{E\times U} \frac{1}{\alpha} c_1(x,u)\mu_1(dx \times du).$$

such that $\mu_0 \in \mathcal{P}(E \times U)$ *and* $\mu_1 \in \mathcal{M}(E \times U)$ *and furthermore*

$$\int_{E\times U} A_\alpha f(x,u)\mu_0(dx \times du) + \int_{E\times U} Bf(x,u)\mu_1(dx \times du) = -\alpha f(x_0), \quad \forall\, f \in C_c^2(E),$$

holds.

Remark 1.38. *Again, it is easy to see that with* $B \equiv 0$ *and* $c_1 \equiv 0$*, we obtain a non-singular problem. The formal definition of these linear programs is omitted to avoid unnecessary repetitions.*

Remark 1.39. *The considerations that follow will deal with a generic linear program with constraints*

$$\int_{E\times U} Af(x,u)\mu_0(dx \times du) + \int_{E\times U} Bf(x,u)\mu_1(dx \times du) = Rf \tag{1.40}$$

where A *can be any of the two presented operators* A *and* A_α *(this slight abuse of notation will ease the exposition going forward) and* $R : \mathscr{D}_\infty \mapsto \mathbb{R}$ *is a suitably chosen right hand side functional, either* $Rf = 0$ *or* $Rf = -\alpha f(x_0)$*. The cost will be expressed by*

$$\int_{E\times U} c_0(x,u)\mu_0(dx \times du) + \int_{E\times U} c_1(x,u)\mu_1(dx \times du).$$

It is obvious that both the long-term average problem as well as the infinite horizon discounted problem can be expressed in this form, as can singular and non-singular problems by the choice of B.

Proposition 1.41. *$R : \mathscr{D}_\infty \mapsto \mathbb{R}$ is a continuous functional.*

Proof. The case $Rf = 0$ is obvious. For the case $Rf = -\alpha f(x_0)$, take $\epsilon > 0$. Take $0 < \delta \leq \frac{\epsilon}{\alpha}$ and consider two functions f and g such that $\|f - g\|_{\mathscr{D}} < \delta$. Then, in particular, $\|f - g\|_\infty < \delta \leq \frac{\epsilon}{\alpha}$, which implies that $\|Rf - Rg\| = \alpha\|f(x_0 - g(x_0)\| < \epsilon$. □

Linear programs of this type are referred to as infinite-dimensional linear programs to distinguish them from 'classic' linear programs in matrix-vector form. They display some type of infinite-dimensionality in two ways. First, the variables come in the form of measures. Some analysis that is presented will reveal that for certain solutions to the linear programs, μ has a density and thus can be identified with the infinite-dimensional function space $L^1(E)$. Second, the constraints are represented by functions in $C_c^2(E)$, which is also an infinite-dimensional function space.

We now present the two main theorems that relate linear programs of the above kind with the stochastic control problems in the relaxed setting which come in the form of a relaxed martingale problem. These results are the centerpiece of the so-called linear programming approach to stochastic control and are extensively discussed in Kurtz and Stockbridge (2017). In particular, these results assert the existence of an optimal control in feedback form, this means that for a given optimal solution (μ_0^*, μ_1^*) to the linear program, we can consider the regular conditional probabilities (η_0, η_1) with respect to the state space marginals

$$\mu_{0,E}(dx) = \mu_0(dx \times U) \qquad \text{and} \qquad \mu_{1,E}(dx) = \mu_1(dx \times U),$$

such that, in accordance with Definition 1.26, we have for $V \in \mathscr{B}(E)$ and $F \in \mathscr{B}(U)$

$$\mu_0(V \times F) = \int_F \eta_0(V, x)\mu_{0,E}(dx) \qquad \text{and} \qquad \mu_1(V \times F) = \int_F \eta_1(V, x)\mu_{1,E}(dx).$$

The optimal relaxed feedback controls will then be given by $\eta_0(\cdot, X_t)$ and $\eta_1(\cdot, x)\tilde{\Gamma}_t$, where $\tilde{\Gamma}$ is the marginal of Γ with respect to $[0, \infty) \times E$. As before, we present the theory only for the singular case, and regard the non-singular case as a special case thereof. The following results are analogous to Theorem 2.1 and Theorem 3.3 in Kurtz and Stockbridge (2017).

Theorem 1.42. *The problem of minimizing the long-term average cost criterion (1.28) over the set of all solutions (X, Λ, Γ) to the relaxed martingale problem given by (Definition 1.24) is equivalent to the linear program in Definition 1.36. Moreover, there exists an optimal solution (μ_0^*, μ_1^*). Let η_0^* and η_1^* be the regular conditional probabilities of μ_0^* and μ_1^* with respect to their state space marginals. Then an optimal relaxed control is given in feedback form by $\Lambda_t^* = \eta_0^*(\cdot, X_t^*)$ and $\Gamma^*(dt \times dx \times du) = \eta_1^*(du, x)\tilde{\Gamma}^*(dt \times dx)$ for a random measure $\tilde{\Gamma}^*$ on $[0, \infty) \times E$, where $(X^*, \Lambda^*, \Gamma^*)$ solves the relaxed singular controlled martingale problem, having occupation measures $(\mu_0^* \mu_1^*)$.*

Theorem 1.43. *The problem of minimizing the discounted infinite horizon cost criterion (1.29) over the set of all solutions (X, Λ, Γ) to the relaxed martingale problem given by (Definition 1.24) is equivalent to the linear program in Definition 1.37. Moreover, there exists an optimal solution (μ_0^*, μ_1^*). Let η_0^* and η_1^* be the regular conditional probabilities of μ_0^* and μ_1^* with respect to their state space marginals. Then an optimal relaxed control is given in feedback form by $\Lambda_t^* = \eta^*(\cdot, X_t^*)$ and $\Gamma^*(dt \times dx \times du) = \eta_1^*(du, x)\tilde{\Gamma}^*(dt \times dx)$ for a random measure $\tilde{\Gamma}^*$ on $[0, \infty) \times E$, where $(X^*, \Lambda^*, \Gamma^*)$ solves the relaxed singular controlled martingale problem, having occupation measures $(\mu_0^* \mu_1^*)$.*

The second part of each of these two theorems sets the stage for a solution approach to the linear programs. It is sufficient to consider feedback controls represented by regular conditional probabilities, which are to some extent computationally tractable. It is the main goal of this thesis to develop a numerical algorithm that borrows ideas from the finite element method, mainly used to solve partial differential equations, in order to approximately solve the presented linear programs and thereby find approximate solutions to stochastic control problems in the relaxed setting.

A careful examination of the statements of the two preceding theorems reveals that they indeed guarantee the existence of an optimal solution. However, it remains unclear if this solution yields a finite value for the cost criterion. Furthermore, uniqueness of the optimal solution is not guaranteed. At last, the question remains open if for any fixed relaxed control given by a regular conditional probability, there is a solution to the linear program that has a regular conditional probability. To some extent, these question are addressed in the next section, where we consider existence and uniqueness of solutions to the linear program in the singular case, given a fixed control.

II.2 Existence and Uniqueness under a Fixed Control

This section provides an extension to the existing theory of the linear programming approach to stochastic control. It investigates the solvability of the linear constraints given by (1.40) in the presence of singular behavior, which means that B is non-zero. For a specific class of controls, which comprises the controls the later-on proposed numerical method uses, existence and uniqueness of solutions to these constraints will be shown. The long-term average problem and the infinite horizon discounted problem have to be treated separately due to the different form of the generator A. In both cases, different boundary behavior of the process demands a slightly different analysis, however, the main ideas remain identical. To avoid unnecessary repetition in this section, the general method is outlined in the following paragraph, and examples for each of the long-term average and the infinite horizon discounted problem are presented. The lemmas treating any other combination of cost criterion and boundary behavior can then be found in Appendix B.

Throughout this section, we assume that the boundary behavior is uncontrolled, in other words, we have that $B(x, u) \equiv B(x)$. Three possible forms of the integral term including B

are displayed in Table II.1. Case 1 is the case of a reflection at both the left endpoint e_l and the right endpoint e_r of the state space. Case 2 is that of a reflection at e_l of the state space, with a jump from e_r into E, while case 3 represents the 'opposite' case of a reflection at e_r and a jump into E at e_l.

Case	$\int_E Bf\, d\bar{\mu}_1$
1	$f'(e_l)\bar{\mu}_1(\{e_l\}) - f'(e_r)\bar{\mu}_1(\{e_r\})$
2	$f'(e_l)\bar{\mu}_1(\{e_l\}) + \big(f(s) - f(e_r)\big)\bar{\mu}_1(\{e_r\})$
3	$\big(f(s) - f(e_l)\big)\bar{\mu}_1(\{e_l\}) - f'(e_r)\bar{\mu}_1(\{e_r\})$

Table II.1: Form of the singular integral term for different boundary behavior

For the considerations in this section, the following assumption has to be placed on the relaxed control η_0. Let η_0 be a fixed relaxed control such that the functions $\bar{b}$ and $\bar{\sigma}$ defined by

$$\bar{b} : E \ni x \mapsto \bar{b}(x) = \int_U b(x,u)\,\eta_0(du,x), \qquad \bar{\sigma} : E \ni x \mapsto \bar{\sigma}(x) = \sqrt{\int_U \sigma^2(x,u)\,\eta_0(du,x)}$$

are bounded, and continuous in E except for finitely many points. At these finitely many points, they are assumed to be either left or right continuous.

Remark 2.1. *The above assumption is satisfied when η_0 is constant on a fixed number of intervals, and on these intervals, $\eta_0(\cdot,x)$ is given by a discrete probability measure on finitely many points in the control space U. This class of controls embraces so-called 'bang-bang' controls, where η_0 puts full mass on only one point in the control space, and switches only finitely many times, and the type of controls used in the approximation to be introduced in Section III.1.1.*

Remark 2.2. *The above assumption is satisfied if there is a function g that is continuous everywhere except on finitely many points such that $1 = \eta_0(\{g(x)\},x)$ for all x. Indeed, since in this case, $\int_U b(x,u)\,\eta_0(du,x) = b(x,g(x))$.*

To some extent, $\bar{b}$ and $\bar{\sigma}$ can be viewed as an average over the coefficient functions b and σ with respect to η_0. Now consider a process X that is controlled by η_0 and would thereby have the continuous generator

$$\bar{A}_\alpha f(x) = \bar{b}(x)f'(x) + \frac{\bar{\sigma}(x)}{2}f''(x) - \alpha f(x) \tag{2.3}$$

with $\alpha > 0$ for the infinite horizon discounted criterion and $\alpha = 0$ for the long term average criterion. We simply denote $\bar{A}_0 = \bar{A}$. Note that for any α, $\bar{A}_\alpha f(x) = \int_U Af(x,u)\eta_1(du)$. As there is no control on the boundary behavior, we will keep the operator B in its given form. The particular form of B will have a certain influence on the following analysis, which will be indicated in the following examples.

We also keep the right-hand side functional of the linear constraints, R, which is defined by

$$Rf = \begin{cases} -\alpha f(x_0) & \alpha > 0 \\ 0 & \alpha = 0 \end{cases}. \tag{2.4}$$

Definition 2.5. *With $\bar{A}_\alpha$ as in (2.3), B as in one of the cases in Table II.1 and R as in (2.4), the linear constraints under fixed η_0 are given by*

$$\int_E \bar{A}_\alpha f(x)\,\bar{\mu}_0(dx) + \int_E Bf(x)\,\bar{\mu}_1(dx) = Rf \quad \forall f \in C_c^2(E) \tag{2.6}$$
$$\bar{\mu}_0 \in \mathcal{P}(E), \quad \bar{\mu}_1 \in M(E).$$

Remark 2.7. *The expected occupation measures $\bar{\mu}_0$ and $\bar{\mu}_1$ are now measures on (only) E since the control is fixed. As we will see, they can be considered as state-space marginals of solutions to the linear constraints given by (1.40).*

In order to show existence and uniqueness, we extract information from the linear constraints by considering solutions, in some sense, of the equation $\bar{A}_\alpha f = I_{[a,b]}$, where $[a,b]$ is an interval in $\mathscr{B}(E)$. As μ_1 only puts mass on the boundary points of the state space $E = [e_l, e_r]$, boundary conditions can be imposed on such solutions to make $\int_E Bf(x)\,\bar{\mu}_1(dx)$ vanish, at least partially. Hence, to find $\bar{\mu}_1$ we would set $[a,b] = [e_l, e_r]$, use that $\bar{\mu}_0(E) = 1$ and impose, for the example in the case of reflections at both e_l and e_r, that $f'(e_l) = 0$. This would yield $1 - f'(e_r)\bar{\mu}_1(\{e_r\}) = Rf$, which allows us to find $\bar{\mu}_1(\{e_r\})$. In similar manner, $\bar{\mu}_1(\{e_l\})$ could be computed. If $\bar{\mu}_1$ is determined, we again use solutions to $\bar{A}_\alpha f = I_{[a,b]}$, giving equations of the form $\bar{\mu}_0([a,b]) = Rf - \int_E Bf\, d\bar{\mu}_1$. Using classic results from measure theory, the knowledge of $\bar{\mu}$ on intervals that generate $\mathscr{B}(E)$ is enough to provide uniqueness and existence of $\bar{\mu}_0$.

A caveat has been omitted in this outline. The equation $\bar{A}_\alpha f = I_{[a,b]}$ will not have 'classical' solutions since $\bar{b}$ and $\bar{\sigma}$ are not continuous, and neither is $I_{[a,b]}$. This will be mitigated by a mollifying argument when presenting the two examples that are in order. First, we consider a long-term average problem with a reflection on the left endpoint and a jump on the right endpoint of the state space. Second, we investigate an infinite horizon discounted criterion with reflections at both ends of the state space.

II.2.1 Example: Long-Term Average Problem with Singular Behavior given by a Jump and a Reflection

We consider the long-term average criterion. The process of interest shall be reflected at the left endpoint e_l of the state space E, and it shall jump back to a point $s \in [e_l, e_r)$ upon entering the right endpoint of the state space e_r. Hence, the integral term with the generator of the singular behavior takes the form

$$\int_E Bf\, d\bar{\mu}_1 = f'(e_l)\bar{\mu}_1(\{e_l\}) + \big(f(s) - f(e_r)\big)\bar{\mu}_1(\{e_r\})$$

For sake of simplicity, we denote $\bar{A} \equiv \bar{A}_\alpha$. Recall that $R = 0$. As pointed out, we intend to find a function f solving the differential equation $Af(x) = g(x)$, where g is some indicator function. However, an indicator function is discontinuous (except for the case $g \equiv I_E$), as are the coefficient functions $\bar{b}$ and $\bar{\sigma}$. This situation does not allow for a classical solution of the differential equation. In other words, the desired function f does not exist. This situation is overcome as follows. In a first step, we will regard continuous functions g, b and σ and solve the differential equation

$$g(x) = b(x)f'(x) + \frac{1}{2}\sigma(x)f''(x), \tag{2.8}$$

as illustrated in the next proposition. The solution will dictate the desired structure of three functions h, h_1 and h_2 (which can formally be thought of as $h_1 = h'$ and $h_2 = h''$) such that $\bar{b}(x)h_1(x) + \frac{1}{2}\bar{\sigma}^2(x)h_2 = g(x)$. Then, a mollifying argument will be used to construct a sequence $\{f_k\}_{k\in\mathbb{N}} \in C_c^2(E)$ with $\bar{A}f_k \to \bar{b}(x)h_1(x) + \frac{\bar{\sigma}^2(x)}{2}h_2 = g(x)$ in the 'right' sense, such that arguments as outlined in the introduction to this section still work. This 'right' sense means that we require the convergence to be pointwise with a uniform bound on any of the expressions appearing in $\bar{A}f_k$. This allows to use the bounded convergence theorem, as $\bar{\mu}_0$ is a probability measure.

Proposition 2.9. *Assume g, b and σ are continuous. Then, a general solution to (2.8) is given by*

$$f_c(x) = \int_{c_2}^{x} \left[\int_{c_1}^{y} \frac{2g(z)}{\sigma^2(z)} e^{\int_y^z \frac{2b(t)}{\sigma^2(t)}dt} dz + K_1 e^{-\int_{c_1}^{y} \frac{2b(t)}{\sigma^2(t)}dt} \right] dy + K_2, \tag{2.10}$$

where $e_l \leq c_1, c_2 \leq e_r$ and $K_1, K_2 \in \mathbb{R}$.

Proof. Using integrating factors, we have since

$$f_c''(x) + \frac{2b(x)}{\sigma^2(x)} f'(x) = \frac{2g(x)}{\sigma^2(x)}$$

that

$$f_c'(y) = \int_{c_1}^{y} \frac{2g(z)}{\sigma^2(z)} e^{\int_y^z \frac{2b(t)}{\sigma^2(t)}dt} dz + K_1 e^{-\int_{c_1}^{y} \frac{2b(t)}{\sigma^2(t)}dt}.$$

The result follows from another integration from c_2 to x with respect to y. □

Remark 2.11. *Note that*

$$f_c''(x) = \frac{2g(x)}{\sigma^2(x)} - \frac{2b(x)}{\sigma^2(x)} \cdot \left(\int_{c_1}^{x} \frac{2g(z)}{\sigma^2(z)} e^{\int_x^z \frac{2b(t)}{\sigma^2(t)} dt} dz + K_1 e^{-\int_{c_1}^{x} \frac{2b(t)}{\sigma^2(t)} dt} \right).$$

Remark 2.11 shows that when we we replace g, b and σ by the respective discontinuous versions g, $\bar{b}$ and $\bar{\sigma}$, f_c'' is continuous everywhere except at a finite set of points, but is at least left or right continuous at any point. If we formally define f, f' and f'' by replacing g, b and σ by g, $\bar{b}$ and $\bar{\sigma}$ in the expressions for f_c, f_c' and f_c'', rather than actually taking derivatives, simple algebra reveals that

$$\bar{b}(x) f'(x) + \frac{\bar{\sigma}^2(x)}{2} f''(x) = g$$

still holds, as we intended to show.

A short digression will connect these considerations to the theory of diffusion processes. The following concepts can be used to express the generator in terms of derivatives taken with respect to measures.

Definition 2.12. *For $c_1 \in \mathbb{R}$, the scale density is defined by*

$$s(x) = e^{-\int_{c_1}^{x} \frac{2\bar{b}(z)}{\bar{\sigma}^2(z)} dz}.$$

The scale measure S is the measure which has density s.

Definition 2.13. *The speed density is defined by*

$$m(x) = \frac{1}{\bar{\sigma}^2(x) s(x)}.$$

The speed measure M is the measure which has density m.

Remark 2.14. *A little bit of algebra reveals that we can express the generator $\bar{A}$ by $\bar{A}f(x) \equiv \frac{1}{2} \frac{d}{dM} \left(\frac{df}{dS} \right)$ in an integral sense. If $\bar{b}$ and $\bar{\sigma}$ are continuous, both expressions are identical. If* *not,* $\frac{1}{2}\frac{d}{dM}\left(\frac{df}{dS}\right)$ *is still well defined and poses an equation*

$$\frac{1}{2}\frac{d}{dM}\left(\frac{df(x)}{dS}\right) = g(x)$$

whose solutions satisfy $Af(x) = g(x)$ *almost everywhere.*

Before we describe the mollifying approach to make the desired argument work, we investigate a choice for the free parameters c_1, c_2, K_1, K_2 such that the function constructed in the proof of Proposition 2.9 fulfills a couple of desired properties at the boundary. Again let $g(x) = I_D(x)$, where D is an interval in $\mathscr{B}(E)$. Denote

$$f_D(x) = \int_{c_2}^{x}\left[\int_{c_1}^{y}\frac{2I_D(z)}{\bar{\sigma}^2(z)}e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}dz + K_1 e^{-\int_{c_1}^{y}\frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\right]dy + K_2,$$

and again think of f_D'' as taking derivatives in the continuous case and replacing the respective quantities by their discontinuous ones.

Lemma 2.15. *Let* D *be an interval in* $\mathscr{B}(E)$*, and* $s \in E$*,* $e_l \leq s < e_r$*. There is a choice of* c_1, c_2, K_1, K_2 *such that* $f_D'(e_l) < 0$*,* $f_D(s) - f_D(e_r) = 0$*.*

Proof. Consider the intervals $I_1 = [e_l, s]$ and $I_2 = [s, e_r]$. We construct two versions of f_D, f_{I_1} and f_{I_2} on I_1 and I_2, respectively. For both of these functions, we set $c_1 = c_2 = s$ and choose the same constants K_1 and K_2, which will be determined later on. Then

$$\begin{aligned}
{f_{I_1}}''(s) &= \frac{2I_D(s)}{\bar{\sigma}^2(s)} - \frac{2\bar{b}(s)}{\bar{\sigma}^2(s)}\left(\int_s^s \frac{2I_D(z)}{\bar{\sigma}^2(z)}e^{-\int_s^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}dz + K_1 e^{-\int_s^s \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\right) = {f_{I_2}}''(s)\\
{f_{I_1}}'(s) &= K_1 = {f_{I_2}}'(s)\\
f_{I_1}(s) &= K_2 = f_{I_2}(s)
\end{aligned}$$

holds and thus the concatenation of f_{I_1} and f_{I_2}, now called f_D, is of such a form that if we replaced $\bar{b}$ and $\bar{\sigma}$ by continuous approximations, which we will do in the mollifying argument to be presented, the function and its two derivatives are continuous. Now we have to ensure that $f_{I_2}(s) - f_{I_2}(e_r) = 0$. This means that we need

$$f_{I_2}(e_r) = \int_s^{e_r} \int_s^y \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\, dy + K_1 \int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy + K_2 = K_2,$$

which we can solve for K_1. This gives

$$K_1 = -\left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy \right)^{-1} \cdot \left(\int_s^{e_r} \int_s^y \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\, dy \right) < 0.$$

The formula for f' reveals that

$$f'_D(e_l) = -\int_{e_l}^s \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} < 0, \tag{2.16}$$

since the integrand of the first term is positive. Note that the negative sign on the right hand side was introduced by changing the lower and upper limits of integration. □

Remark 2.17. *Given $D = (c, d]$ or $D = [c, d]$ for $e_l \le c < d \le e_r$, note that the value of $f'_D(e_l)$ is decreasing in d. Indeed, since in (2.16), both the first term and K_1 are decreasing in d. Further, the integrand in (2.16) will always be dominated by $\frac{2}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}$, which is bounded on a compact set, so by the dominated convergence theorem, the function $g : d \mapsto f'_D(e_l)$ is continuous and, if we set $c = e_l$, $g(e_l) = 0$ holds. Finally, note that*

$$\begin{aligned} &f'_{(e_l,d]}(e_l) - f'_{(e_l,c]}(e_l) \\ &\quad = \int_s^{e_l} \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz \\ &\qquad - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy \right)^{-1} \cdot \left(\int_s^{e_r} \int_s^y \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\, dy \right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}. \end{aligned}$$

Lemma 2.18. *Let D be an interval in $\mathscr{B}(E)$ and $s \in E$, $e_l \le s < e_r$. There is a choice of c_1, c_2, K_1, K_2 such that $f'_D(e_l) = 0$, $f_D(s) - f_D(e_r) < 0$.*

Proof. Follow the construction of f_{I_1} and f_{I_2} as seen in Lemma 2.15, and concatenate the two functions to form the function f_D. To ensure that $f_D'(e_l) = 0$, we need

$$\begin{aligned} f_D'(e_l) &= \int_s^{e_l} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} = 0 \\ \Leftrightarrow K_1 &= -\left(e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}\right)^{-1} \cdot \left(\int_s^{e_l} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\right) \\ &= \left(e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}\right)^{-1} \cdot \left(\int_{e_l}^{s} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\right) \end{aligned}$$

Note that by the construction of f_D, $f_D(s) = K_2$ and

$$f_D(e_r) = \int_s^{e_r} \left[\int_s^{y} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{y}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^{y} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}\right] dy + K_2 > K_2,$$

since both first and second summands inside of the first integral are positive. □

As previously stated, the functions f_D considered in these two lemmas are not twice differentiable, and thus it is not a solution to $Af(x) = g(x) \equiv I_D$. This is remedied by the following mollifying argument.

Proposition 2.19. *For any $D \in \mathscr{B}(E)$, there is a sequence $\{f_k\}_{k\in\mathbb{N}}$ of twice differentiable functions such that $f_k \to f_D$, $f_k' \to f_D'$ uniformly, and $f_k'' \to f_D''$ pointwise on the set where f_D'' is continuous, with $\|f_k''\|_\infty < M$ for some $M > 0$.*

Proof. This can be shown by applying a standard argument from Øksendal (1998) (see Appendix D thereof) onto the function f_D as constructed above. □

Corollary 2.20. *For any $D \in \mathscr{B}(E)$, there is a sequence $\{f_k\}_{k\in\mathbb{N}}$ of twice differentiable functions such that $Af_k \to I_D$ pointwise with $\|Af_k\|_\infty < M$ for some $M > 0$.*

Remark 2.21. *Note that in particular, Corollary 2.20 allows, by the bounded convergence theorem, to interchange limits in the following expression:*

$$\int_E \lim_{k\to\infty} Af_k \, d\bar{\mu}_0 = \lim_{k\to\infty} \int_E Af_k \, d\bar{\mu}_0.$$

The following theorem summarizes the considerations taken up to this point.

Theorem 2.22. *Let $(\bar{\mu}_0, \bar{\mu}_1)$ be a solution of the linear constraints (2.6). For any interval $D \in \mathscr{B}(E)$ we have that*

$$\bar{\mu}_0(D) = \theta_1 \bar{\mu}_1(\{e_l\}) + \theta_2 \bar{\mu}_1(\{e_r\})$$

for some $\theta_1, \theta_2 \geq 0$ that only depends on D. In particular, θ_1 and θ_2 can be chosen in such a way that either θ_1 or θ_2 is 0.

Proof. For $D \in \mathscr{B}(E)$, choose a sequence $\{f_k\}_{k \in N} \in C^2$ converging to f_D according to Corollary 2.20. Then, as $(\bar{\mu}_0, \bar{\mu}_1)$ solves the linear constraints (2.6) for $Rf = 0$,

$$\begin{aligned}
\bar{\mu}_0(D) &= \int_E I_D d\bar{\mu}_0 = \int_E \lim_{k\to\infty} Af_k d\bar{\mu}_0 = \lim_{k\to\infty} \int_E Af_k d\bar{\mu}_0 = -\lim_{k\to\infty} \int_E Bf_k d\bar{\mu}_1 \\
&= -\int_E \lim_{k\to\infty} Bf_k d\bar{\mu}_1 = -\int_E Bf d\bar{\mu}_1 = \theta_1 \bar{\mu}_1(\{e_l\}) + \theta_2 \bar{\mu}_1(\{e_l\})
\end{aligned}$$

where θ_1 and θ_2 depend on the values of $\lim_{k\to\infty} f_k = f_D$ at e_l and e_r. They can be chosen in such a way that either θ_1 or θ_2 vanishes, according to Lemmas 2.15 and 2.18. In the first case, we would have

$$\theta_1 = -f_D'(e_l) = \int_{e_l}^{s} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} > 0$$

and $\theta_2 = 0$. In the second case, we would have $\theta_1 = 0$ and

$$\theta_2 = -\left(f_D(s) - f_D(e_r)\right) = \int_s^{e_r} \left[\int_s^y \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}\right] dy > 0.$$

$\square$

Theorem 2.23. *Let $(\bar{\mu}_0, \bar{\mu}_1)$ and $(\hat{\mu}_0, \hat{\mu}_1)$ be two solutions to*

$$\begin{aligned}
&\int_E \bar{A}f \, d\bar{\mu}_0 + \int_E Bf \, d\bar{\mu}_1 = 0 \quad \forall f \in C_c^2(E) \\
&\bar{\mu}_0 \in \mathcal{P}(E), \bar{\mu}_1 \in M(E).
\end{aligned} \tag{2.24}$$

Then, $(\bar{\mu}_0, \bar{\mu}_1) = (\hat{\mu}_0, \hat{\mu}_1)$.

Proof. Consider the two solutions $(\bar{\mu}_0, \bar{\mu}_1)$ and $(\hat{\mu}_0, \hat{\mu}_1)$. Setting $D = E$, by Theorem 2.22, we have as $\bar{\mu}_0$ is a probability measure,

$$1 = \bar{\mu}_0(E) = \theta_1 \bar{\mu}_1(\{e_l\})$$

and likewise

$$1 = \hat{\mu}_0(E) = \theta_1 \hat{\mu}_1(\{e_l\}),$$

for some $\theta_1 \in \mathbb{R}$, from which we can follow that $\bar{\mu}_1(\{e_l\}) = \hat{\mu}_1(\{e_l\})$. Likewise, we show that $\bar{\mu}_1(\{e_r\}) = \hat{\mu}_1(\{e_r\})$. So, $\bar{\mu}_1 = \hat{\mu}_1$. Now, with $D = [c, d] \subset [e_l, e_r]$, again by Theorem 2.22,

$$\begin{aligned} \bar{\mu}_0(D) &= \theta_1 \bar{\mu}_1(\{e_l\}) + \theta_2 \bar{\mu}_1(\{e_r\}) = \theta_1 \hat{\mu}_1(\{e_l\}) + \theta_2 \hat{\mu}_1(\{e_r\}) \\ &= \hat{\mu}_0(D) \end{aligned}$$

for some $\theta_1, \theta_2 \in \mathbb{R}$. This shows that $\bar{\mu}_0$ and $\hat{\mu}_0$ agree on a π-system that generates $\mathscr{B}([e_l, e_r]) = \mathscr{B}(E)$, and hence $\bar{\mu}_0$ and $\hat{\mu}_0$ are identical. $\square$

We proceed to show the existence of solutions to (2.6). The proof of the following theorem will use the standard construction of a measure from an increasing, (right) continuous function of bounded variation.

Theorem 2.25. *There exists a solution* $(\bar{\mu}_0, \bar{\mu}_1)$ *to (2.6). Further,* $\bar{\mu}_0$ *is absolutely continuous with respect to Lebesgue measure.*

Proof. According to Lemma 2.15, choose $f \equiv f_E$ with $g \equiv I_E$, $f'_E(e_l) < 0$ and $f_E(s) - f_E(e_r) = 0$. Then, set $\bar{\mu}_1(\{e_l\}) = \frac{1}{-f'_E(e_l)} > 0$ and

$$\bar{\mu}_1(\{e_r\}) = \bar{\mu}_1(\{e_l\}) \frac{e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}\, dt}}{\int_s^{e_r} e^{-\int_s^{y} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy}.$$

Again by Lemma 2.15, choose $f \equiv f_{(e_l,d]}$ with $g \equiv I_{(e_l,d]}$, $f_d'(e_l) < 0$ and $f(s) - f(e_r) = 0$. Set $F(d) = -f_d'(e_l) \cdot \bar{\mu}_1(\{e_l\})$. Note that $f'_{[e_l,e_r]}(e_l) = f_E'(e_l)$, so $F(e_r) = 1$. By Remark 2.17, $F(d)$ is increasing in d (as $f_d'(e_l)$ decreases in d), continuous and

$$\begin{aligned} F(d) - F(c) \quad = \quad & -\bar{\mu}_1(\{e_l\}) \Bigg[\int_s^{e_l} \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz \\ & - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy \right)^{-1} \cdot \left(\int_s^{e_r} \int_s^y \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy\, dz \right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} \Bigg] \end{aligned}$$

By standard arguments from measure theory, there is a measure $\bar{\mu}_0$ on $E = [e_l, e_r]$ (equipped with $\mathscr{B}(E)$), such that $\bar{\mu}_0((c,d]) = F(d) - F(c)$. Due to the fact that $1 = F(e_r) = \bar{\mu}_0([e_l, e_r])$, $\bar{\mu}_0$ is a probability measure. Further note that the continuity of F implies that $\bar{\mu}_0((c,d]) = \bar{\mu}_0([c,d]) = \bar{\mu}_0([c,d)) = \bar{\mu}_0((c,d))$ for $c < d$.

We proceed to show that this measure $\bar{\mu}_0$, and the measure $\bar{\mu}_1$ as constructed in the first part of this proof, solve (2.6). To this end, observe that for any $f \in C_c^2(E)$, $\bar{A}f$ can be approximated by step functions in a bounded way, indeed, as continuous and bounded functions can be approximated by step functions by a standard argument from measure theory. Using the fact that $\bar{A}f$ is bounded and piecewise continuous, we can consider each of its continuous 'pieces' separately to prove this claim. In particular,

$$\bar{A}f(x) = \lim_{n\to\infty} \sum_{k=1}^n a_{k,n} \varphi_{k,n}(x),$$

where $\varphi_{k,n}$ are indicator functions of given intervals (which might be open to the left or right to match the discontinuities in $\bar{b}$ and $\bar{\sigma}$), and a_k, n are aptly chosen real numbers. By the dominated convergence theorem,

$$\int_E \bar{A}f d\bar{\mu}_0 = \lim_{n\to\infty} \sum_{k=1}^n a_{k,n} \int_E \varphi_{k,n} d\bar{\mu}_0.$$

Also, due to the fact that F is continuous and hence $\bar{\mu}_0((c,d]) = \bar{\mu}_0([c,d]) = \bar{\mu}_0([c,d)) = \bar{\mu}_0((c,d))$, we have for some $c<d$ that

$$\begin{aligned}
\int_E \varphi_{k,n} d\bar{\mu}_0 &= \bar{\mu}_0((c,d]) \\
&= F(d) - F(c) \\
&= -\bar{\mu}_1(\{e_l\})\Bigg[\int_s^{e_l} \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&\quad - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right)^{-1} \cdot \left(\int_s^{e_r}\int_s^y \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\, dz\right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\Bigg]
\end{aligned}$$

and again by dominated convergence

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^n a_{k,n}\int_E \varphi_{k,n} d\bar{\mu}_0 \\
&= -\bar{\mu}_1(\{e_l\}) \cdot \lim_{n\to\infty}\sum_{k=1}^n a_{k,n}\Bigg[\int_s^{e_l} \frac{2\varphi_{k,n}(z)}{\bar{\sigma}^2(z)} e^{\int_s^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&\quad - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right)^{-1} \cdot \left(\int_s^{e_r}\int_s^y \frac{2\varphi_{k,n}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\, dz\right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\Bigg] \\
&= -\bar{\mu}_1(\{e_l\}) \cdot \Bigg[\int_s^{e_l} \frac{2\bar{A}f(z)}{\bar{\sigma}^2(z)} e^{\int_s^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&\quad - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right)^{-1} \cdot \left(\int_s^{e_r}\int_s^y \frac{2\bar{A}f(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\, dz\right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\Bigg]
\end{aligned}$$

By the definition of $\bar{A}$ and using integration by parts,

$$\begin{aligned}
\int_s^y \frac{2\bar{A}f(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz &= \int_s^y f''(z) e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz + \int_s^y f'(z)\frac{2\bar{b}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&= \left[f'(z) e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\right]_s^y - \int_s^y f'(z)\frac{2\bar{b}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&\qquad + \int_s^y f'(z)\frac{2\bar{b}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&= f'(y) - f'(s) e^{\int_y^s \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt},
\end{aligned}$$

which directly implies that

$$
\begin{aligned}
\int_s^{e_r}\int_s^y \frac{2\bar{A}f(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz\,dy &= \int_s^{e_r} f'(y) - f'(s) e^{\int_y^s \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\,dy \\
&= f(e_r) - f(s) - f'(s)\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\,dy
\end{aligned}
$$

and hence

$$
\begin{aligned}
&-\left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\right)^{-1} \cdot \int_s^{e_r}\int_s^y \frac{2\bar{A}f(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\,dz \\
&\qquad = [f(s) - f(e_r)] \cdot \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\,dy\right)^{-1} + f'(s).
\end{aligned}
$$

Putting these results together, we deduce that

$$
\lim_{n\to\infty}\sum_{k=1}^n a_{k,n}\int_E \varphi_{k,n} d\bar{\mu}_0 = -\bar{\mu}_1(\{e_l\})\left(f'(e_l) + [f(s) - f(e_r)] \cdot \frac{e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}}{\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy}\right)
$$

and further

$$
\begin{aligned}
\int_E \bar{A}f\,d\bar{\mu}_0 &= -\bar{\mu}_1(\{e_l\}) \cdot \left(f'(e_l) + [f(s) - f(e_r)] \cdot \frac{e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}}{\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy}\right) \\
&= -\bar{\mu}_1(\{e_l\}) f'(e_l) - [f(s) - f(e_r)]\bar{\mu}_1(\{e_r\}) = -\int_E Bf d\bar{\mu}_1
\end{aligned}
$$

which shows that

$$
\int_E \bar{A}f\,d\bar{\mu}_0 + \int_E Bf\,d\bar{\mu}_1 = 0.
$$

Finally note that

$$
\begin{aligned}
F(d) \quad &= \quad -\bar{\mu}_1(\{e_l\})\Bigg[\int_s^{e_l} \frac{2I_{(e_l,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz \\
&\qquad - \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right)^{-1} \cdot \left(\int_s^{e_r}\int_s^y \frac{2I_{(e_l,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\, dz\right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt}\Bigg] \\
&= \quad \int_{e_l}^d \bar{\mu}_1(\{e_l\})\Bigg[\frac{2I_{(e_l,s]}(z)}{\bar{\sigma}^2(z)} e^{\int_s^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} \\
&\qquad + \left(\int_s^{e_r} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right)^{-1} \cdot \left(I_{[s,e_r]}(z)\int_s^y \frac{2I_{(e_l,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dy\right) e^{-\int_s^{e_l} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz\Bigg]
\end{aligned}
$$

and the integrand is integrable, hence $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure. □

This result finalizes the proof of existence and uniqueness of solutions to the linear constraints (2.6). We will make some useful deductions from this result, concerning the case when the control is not fixed. This means that the continuous generator takes the form

$$Af(x,u) = b(x,u)f'(x) + \frac{1}{2}\sigma(x,u)f''(x),$$

while the singular generator B stays the same due to the assumption that there is no control of the singular behavior. The linear constraints read as follows. Recall that these are the linear constraints appearing in the linear programming formulation for control problems as introduced in Section II.1.

$$
\begin{aligned}
&\int_{E\times U} Af\, d\mu_0 + \int_E Bf\, d\mu_1 = 0 \quad \forall f \in C_c^2(E) \\
&\mu_0 \in \mathcal{P}(E\times U), \mu_1 \in M(E).
\end{aligned}
\tag{2.26}
$$

Before we show the existence of a solution to these constraints (uniqueness cannot be guaranteed any longer), recall that we initially assumed that η_0 is such a relaxed control that $\bar{b}$ and $\bar{\sigma}$ are continuous everywhere except at finitely many points. This can, for example be achieved if η_0 is of the form

$$\eta_0(V,x) = \sum_{j=0}^{n_1} \sum_{i=0}^{n_2} \beta_{j,i} I_{E_j}(x) \delta_{u_i}(V)$$

where $E_1, E_2, \ldots, E_{n_1}$ is a partition of E, $u_1, u_2, \ldots, u_{n_2}$ are points in the control space U and the $\beta_{i,j}$ are aptly chosen coefficients. In the proposed numerical method, the relaxed controls of consideration will actually be of this form, compare (III.1.11) and (III.1.13).

Theorem 2.27. *There exists a solution to the linear constraints given by (2.26).*

Proof. For a regular conditional probability η_0 which makes $\bar{b}$ and $\bar{\sigma}$ continuous except for finitely many points, consider the unique solution $(\bar{\mu}_0, \bar{\mu}_1)$ to (2.6). Define two measures μ_0 and μ_1 on $\mathscr{B}(E \times U)$ by

$$\begin{aligned} \mu_0(dx \times du) &= \eta_0(du, x)\bar{\mu}_0(dx) \\ \mu_1(dx \times du) &= \bar{\mu}_1(dx). \end{aligned}$$

Then for any $f \in C^2(E)$,

$$\begin{aligned} \int_{E\times U} Af\, d\mu_0 + \int_E Bf\, d\mu_1 &= \int_E \int_U Af(x,u)\, \eta_0(du,x)\, \bar{\mu}_0(dx) + \int_E Bf(x)\, \bar{\mu}_1(dx) \\ &= \int_E \bar{A}f(x)\, \bar{\mu}_0(dx) + \int_E Bf(x)\, \bar{\mu}_1(dx) = 0 \end{aligned}$$

holds, so (μ_0, μ_1) solves (2.26). $\square$

Remark 2.28. *According to the theory of the linear programming approach, see Theorem 1.8 of Kurtz and Stockbridge (2017), the existence of a solution to (2.26) implies the existence of a solution to the relaxed singular controlled martingale problem as defined in Definition 1.24. Since we assumed that there is no control on the singular part, we indeed have that Bf only depends on x and Γ is a measure on only $([0,\infty)\times E)$. Theorem 1.8 of Kurtz and Stockbridge (2017) also states that Λ is of the form $\Lambda_t = \eta_0(\cdot, X_t)$.*

The theory for the given example concerned itself with a process that is reflected at the left endpoint e_l of the state space, and jumps away from the right endpoint e_r of the state space. Two additional cases can be dealt with in a similar manner. First, one could consider reflections at both e_l and e_r. Second, one could consider a reflection at e_r and a jump away from e_l. The respective results for these cases that are analogous to Lemmas 2.15 and 2.18 and Remark 2.17, can be found in Appendix B. The remaining derivations can be conducted similarly to the example presented in this section, and are omitted. For further reference, we state the following comprehensive result.

Theorem 2.29. *Let A be the generator of the continuous behavior of a process under consideration of the long-term average criterion, and let B take any of the forms described in Table II.1. Then, there exists a solution to the linear constraints given by (2.26). In the light of Remark 2.28, there also exists a solution to the relaxed singular martingale problem given in Definition 1.24.*

II.2.2 Example: Discounted Infinite Horizon Problem with Singular Behavior Given by Two Reflections

We turn our attention to the infinite horizon discounted criterion, and consider a process that is reflected at both boundary points e_l and e_r. The existence of solutions comes as an easy consequence of the existence of solutions to the long-term average problem as shown in the next theorem. To distinguish between the long term average case and the discounted case, we use the notation

$$\bar{A}_\alpha f(x) = \bar{b}(x)f'(x) + \frac{1}{2}\bar{\sigma}^2(x)f''(x) - \alpha f(x)$$

for some $\alpha > 0$. Recalling that in the discounted case, $Rf = -\alpha f(x_0)$, we regard the linear constraints given by

$$\begin{aligned}
&\int_E \bar{A}_\alpha f \, d\bar{\mu}_0 + \int_E Bf \, d\bar{\mu}_1 = -\alpha f(x_0) \quad \forall f \in C_c^2(E) \\
&\bar{\mu}_0 \in \mathcal{P}(E), \bar{\mu}_1 \in M(E).
\end{aligned} \tag{2.30}$$

The generator of the singular behavior B does not depend on α and is hence retained in this formulation. Note that for the given example, the integral term with the singular generator takes the form

$$\int_E Bf(x)\bar{\mu}_1(dx) = f'(e_l)\bar{\mu}_1(\{e_l\}) - f'(e_r)\bar{\mu}_1(\{e_r\})$$

Theorem 2.31. *There exists a solution to (2.30).*

Proof. By Theorem 2.29, there is a stationary solution (X, Γ, Λ) to the relaxed singular martingale problem defined by Definition 1.24 with $\Lambda_t = \eta_0(\cdot, X_t)$. In other words,

$$\begin{aligned}
f(X_t) - f(x_0) - &\int_0^t \bar{A}f(X_s)\, ds \\
&- \int_{[0,t]\times E\times U} Bf(X_s)\Gamma(ds \times dx \times du)
\end{aligned} \tag{2.32}$$

is a martingale for all $f \in C_c^2(E)$. If we consider $\bar{A}(x,u) \equiv \bar{A}(x)$, $\bar{A}_\alpha(x,u) \equiv \bar{A}_\alpha(x)$ and $B(x,u) \equiv B(x)$, we can regard a new martingale problem by demanding that

$$\begin{aligned}
f(X_t) - f(x_0) - &\int_0^t \int_U \bar{A}f(X_s,u)\Lambda_s(du)\, ds \\
&- \int_{[0,t]\times E\times U} Bf(X_s,u)\Gamma(ds \times dx \times du)
\end{aligned} \tag{2.33}$$

is a martingale for all $f \in C_c^2(E)$, which obviously has the solution (X, Λ, Γ), with $\Lambda_t \equiv \eta_0(\cdot, X_t)$. Note that since $\bar{A}$ and B do not depend on u, any relaxed control η is a solution. By the theory of the linear programming approach, compare Kurtz and Stockbridge (2017), the existence of a solution to the martingale problem implies a solution to

$$\begin{aligned} &\int_{E\times U} \bar{A}_\alpha f \, d\mu_0 + \int_{E\times U} Bf \, d\mu_1 = Rf \quad \forall f \in C_c^2(E) \\ &\mu_0 \in \mathcal{P}(E\times U), \mu_1 \in M(E\times U), \end{aligned} \tag{2.34}$$

but as $\bar{A}_\alpha$ and B do not depend on u, the state-space marginals of μ_0 and μ_1 solve (2.30). □

For uniqueness, we have to investigate solutions to the equation

$$\bar{A}_\alpha f(x) = \bar{b}(x)f'(x) + \frac{1}{2}\bar{\sigma}^2(x)f''(x) - \alpha f(x) = g(x),$$

where again g will be an indicator function of a Borel set of E. We will employ a stochastic solution approach to this equation and analyze the resulting solution in order to show that it fulfills a similar set of properties as presented in the previous section. In order to do so, we need to introduce several ideas from the theory of diffusion processes, as found in Rogers and Williams (2000). In the following, Y is a solution to the stochastic differential equation

$$dY_t = \bar{b}(Y_t)dt + \bar{\sigma}(Y_t)dW_t, \qquad Y_0 = x_0$$

It is important to point out that such a solution only represents the diffusion part of the problems we are considering, in other words, it is lacking any singular behavior. To be able to distinguish, we refer to the state space of Y as I. Note that if X is a solution to

$$dX_t = \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t + \xi_t, \qquad X_0 = x_0$$

with ξ modeling the singular behavior keeping X inside of its compact state space E, then $E \subset I$. In the following, we distinguish behavior based on different starting points of the diffusion. Hence, we refer to $\mathbb{P}_x$ as the law given by the stochastic process Y with $Y_0 = x$ and in the same way, denote $\mathbb{E}_x$ the expectation operator of the process Y starting at $Y_0 = x$.

Definition 2.35. *For $y \in I$, the first hitting time of y by the process Y is defined by*

$$\tau_y = \inf\{t \geq 0 : Y_t = y\}$$

Definition 2.36. *The process Y is called regular if for all x_0 in the interior of I and $y \in I$,*

$$\mathbb{P}_{x_0}(\tau_y < \infty) > 0.$$

A regular process is a process which, starting from a point in the interior of its state space I, can reach any point in the interior of the state space in finite time with positive probability. From here on, we will only consider regular processes. For a regular process, the following is well defined. Let $E = [e_l, e_r]$ and $I = [i_l, i_r]$.

Definition 2.37. *For $\alpha > 0$ and $y \in I$ define the functions $\phi_\alpha(x)$ and $\psi_\alpha(x)$ on I by*

$$\phi_\alpha(x) = \begin{cases} 1/\mathbb{E}_y\left[\exp(-\alpha\tau_x)\right] & i_l < x < y \\ \mathbb{E}_x\left[\exp(-\alpha\tau_y)\right] & y \leq x < i_r \end{cases}$$

and

$$\psi_\alpha(x) = \begin{cases} \mathbb{E}_x\left[\exp(-\alpha\tau_y)\right] & i_l < x < y \\ 1/\mathbb{E}_y\left[\exp(-\alpha\tau_x)\right] & y \leq x < i_r \end{cases}.$$

Proposition 2.38. *The functions ϕ_α and ψ_α are strictly convex, positive, and strictly decreasing and increasing, respectively.*

Proof. Rogers and Williams (2000), section V.50. Furthermore, compare Remark 2.14. □

More striking is the fact that ϕ_α and ψ_α solve the differential equation $\bar{A}_\alpha f(x) = 0$, in the sense discussed in Remark 2.14, which means it solves it almost everywhere.

Proposition 2.39. *ϕ_α and ψ_α solve*

$$\frac{1}{2}\frac{d}{dM}\left(\frac{df(x)}{dS}\right) = 0. \tag{2.40}$$

In particular, $\bar{A}_\alpha f(x) = 0$ holds almost everywhere.

Proof. Rogers and Williams (2000), section V.50. □

We proceed to investigate more of the regularity of ϕ_α and ψ_α. In particular, the literature only states that these functions are continuous. With the help of the differential equation (2.40) we are able to derive more regularity of these functions, in particular if the drift and diffusion functions are continuous.

Proposition 2.41. *The functions ϕ_α and ψ_α have continuous derivatives.*

Proof. Note that

$$\bar{A}_\alpha\phi_\alpha(x) = \bar{b}(x)\phi'_\alpha(x) + \frac{\bar{\sigma}^2(x)}{2}\phi''_\alpha(x) - \alpha\phi_\alpha(x) = 0 \quad \text{almost everywhere}$$

implies that

$$\phi''_\alpha(x) = \frac{2\alpha}{\bar{\sigma}^2(x)}\phi_\alpha(x) - \frac{2\bar{b}(x)}{\bar{\sigma}^2(x)}\phi'_\alpha(x) \quad \text{almost everywhere} \tag{2.42}$$

and further for some $c_1 \in R$,

$$\phi'_\alpha(x) = \phi'_\alpha(c_1) + \int_{c_1}^{x} \frac{2\alpha}{\bar{\sigma}^2(y)}\phi_\alpha(y) - \frac{2\bar{b}(y)}{\bar{\sigma}^2(y)}\phi'_\alpha(y)\,dy \quad \text{almost everywhere} \tag{2.43}$$

As ϕ_α is convex, the left derivative ϕ_- and the right derivative ϕ_+ exist, and both are continuous. Take $z \in \mathbb{R}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \leq z$ for all n, $\lim_{n\to\infty} x_n = z$, and (2.43) holds for all x_n. Hence, $\lim_{n\to\infty} \phi'_-(x_n) = \phi'_-(z)$, but

$$\phi'_-(x_n) = \phi'_\alpha(c_1) + \int_{c_1}^{x_n} \frac{2\alpha}{\bar{\sigma}^2(y)}\phi_\alpha(y) - \frac{2\bar{b}(y)}{\bar{\sigma}^2(y)}\phi'_\alpha(y)\,dy$$

and by dominated convergence,

$$\phi'_-(z) = \phi'_\alpha(c_1) + \int_{c_1}^{z} \frac{2\alpha}{\bar{\sigma}^2(y)}\phi_\alpha(y) - \frac{2\bar{b}(y)}{\bar{\sigma}^2(y)}\phi'_\alpha(y)\,dy$$

holds. Likewise, we can show that

$$\phi'_+(z) = \phi'_\alpha(c_1) + \int_{c_1}^{z} \frac{2\alpha}{\bar{\sigma}^2(y)}\phi_\alpha(y) - \frac{2\bar{b}(y)}{\bar{\sigma}^2(y)}\phi'_\alpha(y)\,dy.$$

Hence $\phi'_- = \phi'_+$ everywhere and ϕ'_α is continuous. The argument for ψ'_α is identical. □

Remark 2.44. *If $\bar{b}$ and $\bar{\sigma}$ are continuous (2.42) holds everywhere and indicates that ϕ''_α, and likewise ψ''_α, are continuous. If $\bar{b}$ and $\bar{\sigma}$ are not continuous, the discontinuities of ϕ''_α and ψ''_α are precisely the points where $\bar{b}$ and $\bar{\sigma}$ are discontinuous.*

In the case where drift $\bar{b}$ and diffusion $\bar{\sigma}$ are only piecewise continuous, with finitely many discontinuities, we again employ a mollifying approach. This is presented for ϕ_α, with the argument for ψ_α being identical. We will frequently consider the following type of convergence.

Definition 2.45. *Let (S, ν) be a finite measure space. A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ on (S, ν) is said to converge to a function f boundedly and pointwise (or pointwise almost everywhere), if $\sup_{k\in\mathbb{N}}\{\|f_k\|_\infty\} < \infty$ and $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in S$ (or almost every $x \in S$).*

Remark 2.46. *Let (S, ν) be a finite measure space, and assume that $\{f_k\}_{k\in\mathbb{N}}$ converges to f boundedly and pointwise almost everywhere. Then,*

$$\lim_{k\to\infty} \int_S f_k\,d\nu = \int_S f\,d\nu$$

by the bounded convergence theorem.

Proposition 2.47. *As $k \to \infty$, we have that*

$$\bar{A}_\alpha \phi_k(x) \to 0$$

boundedly and pointwise almost everywhere.

Proof. Construct a sequence ϕ_k by mollifying ϕ_α using Proposition 2.19. As $\phi_k \to \phi_\alpha$, $\phi_k' \to \phi_\alpha'$ uniformly, and in particular bounded and pointwise, and $\phi_k'' \to \phi_\alpha''$ pointwise and bounded almost everywhere, we have that

$$\begin{aligned}
\lim_{k\to\infty} \bar{A}_\alpha \phi_k &= \lim_{k\to\infty} \left(\bar{b}(x)\phi_k'(x) + \frac{\bar{\sigma}^2(x)}{2}\phi_k''(x) - \alpha\phi_k(x) \right) \\
&= \bar{b}(x)\phi_\alpha'(x) + \frac{\bar{\sigma}^2(x)}{2}\phi_\alpha''(x) - \alpha\phi_\alpha(x) \\
&= 0
\end{aligned}$$

□

Using ϕ_k and ψ_k, we construct a sequence of functions $\{f_k\}_{k\in N}$ such that $\lim_{k\to\infty} Af_k \to g$ boundedly and pointwise for some function g. Typically, g will be an indicator function and needs to be mollified as well.

Proposition 2.48. *Let s be the scale density, compare Definition 2.12, of a diffusion with drift $\bar{b}$ and diffusion $\bar{\sigma}$. Then, the Wronskian*

$$w_\alpha = \frac{\phi_\alpha'(x)\psi_\alpha(x) - \phi_\alpha(x)\psi_\alpha'(x)}{s(x)}$$

is constant and negative.

Proof. Note that for any x where $\bar{b}$ and $\bar{\sigma}$ are continuous, which is any x such that $\bar{A}_\alpha(x)\phi_\alpha(x) = \bar{A}_\alpha(x)\psi_\alpha(x) = 0$, we have that

$$\begin{aligned}
w'_\alpha(x) &= \frac{s(x)\left[\phi''_\alpha(x)\psi_\alpha(x) + \phi'_\alpha(x)\psi'_\alpha(x) - \phi_\alpha(x)\psi''_\alpha(x) - \phi'_\alpha(x)\psi'_\alpha(x)\right]}{s^2(x)} \\
&\quad - \frac{s'(x)\left[\phi'_\alpha(x)\psi_\alpha(x) - \phi_\alpha(x)\psi'_\alpha(x)\right]}{s^2(x)} \\
&= \frac{\left[\phi''_\alpha(x)\psi_\alpha(x) - \phi_\alpha(x)\psi''_\alpha(x)\right] + \frac{2\bar{b}(x)}{\bar{\sigma}^2(x)}\left[\phi'_\alpha(x)\psi_\alpha(x) - \phi_\alpha(x)\psi'_\alpha(x)\right]}{s(x)} \\
&= \frac{\psi_\alpha(x)\left[\phi''_\alpha(x) + \frac{2\bar{b}(x)}{\bar{\sigma}^2(x)}\phi'_\alpha(x)\right] - \phi_\alpha(x)\left[\psi''_\alpha(x) + \frac{2\bar{b}(x)}{\bar{\sigma}^2(x)}\psi'_\alpha(x)\right]}{s(x)} \\
&= \frac{\psi_\alpha(x)\left[\frac{2\alpha}{\bar{\sigma}^2(x)}\phi_\alpha(x)\right] - \phi_\alpha(x)\left[\frac{2\alpha}{\bar{\sigma}^2(x)}\phi_\alpha(x)\right]}{s(x)} = 0,
\end{aligned}$$

where we used that $\frac{s'_k(x)}{s_k(x)} = -\frac{2\bar{b}_k(x)}{\bar{\sigma}^2(x)}$. So, w_α is piecewise constant. But as it is is continuous, it is constant. To see that its value is negative, observe that s is positive, and by Proposition 2.38 we can easily deduce that the numerator of w_α is negative. $\square$

Theorem 2.49. *Let $c_1, c_2 \in \mathbb{R}$. For a piecewise continuous function g, define a sequence of functions $\{f_k\}_{k\in N}$ by*

$$f_k(x) = \frac{2}{w_\alpha}\left[\phi_k(x)\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy + \psi_k(x)\int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy\right] \tag{2.50}$$

where m_k is the mollified speed density, stemming from Definition 2.13 and g_k is a mollification of g, both using Proposition 2.19. Then, $\bar{A}_\alpha f_k \to g$ as $k \to \infty$, boundedly and pointwise almost everywhere. Moreover, $\lim_{k\to\infty} f'_k(x) > 0$ *for all $x \in I$.*

Proof. We begin by examining the derivatives of f_k for a fixed $k \in \mathbb{N}$.

$$
\begin{aligned}
f_k'(x) &= \frac{2}{w_\alpha}\left[\phi_k'(x)\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy + \phi_k(x)\psi_k(x)g_k(x)m_k(x)\right. \\
&\qquad \left. + \psi_k'(x)\int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy - \phi_k(x)\psi_k(x)g_k(x)m_k(x)\right] \\
&= \frac{2}{w_\alpha}\left[\phi_k'(x)\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy + \psi_k'(x)\int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
f_k''(x) &= \frac{2}{w_\alpha}\left[\phi_k''(x)\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy + \phi_k'(x)\psi_k(x)g_k(x)m_k(x)\right. \\
&\qquad \left. +\psi_k''(x)\int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy - \psi_k'(x)\phi_k(x)g_k(x)m_k(x)\right]
\end{aligned}
$$

Using these formulas, we have that

$$
\begin{aligned}
\bar{A}_\alpha f_k(x) &= \bar{b}(x)f_k'(x) + \frac{\bar{\sigma}^2(x)}{2}f_k''(x) - \alpha f_k(x) \\
&= \frac{2}{w_\alpha}\left[\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy\left(\bar{b}(x)\phi_k'(x) + \frac{\bar{\sigma}^2(x)}{2}\phi_k''(x) - \alpha\phi_k(x)\right)\right. \\
&\qquad \left. + \int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy\left(\bar{b}(x)\psi_k'(x) + \frac{\bar{\sigma}^2(x)}{2}\psi_k''(x) - \alpha\psi_k(x)\right)\right] \\
&\qquad +\frac{2}{w_\alpha}\left[\frac{\bar{\sigma}^2(x)}{2}\phi_k'(x)\psi_k(x)g_k(x)m_k(x) - \frac{\bar{\sigma}^2(x)}{2}\phi_k(x)\psi_k'(x)g_k(x)m_k(x)\right] \\
&= \frac{2}{w_\alpha}\left[\int_{c_1}^{x}\psi_k(y)g_k(y)m_k(y)\,dy\cdot\bar{A}_\alpha\phi_k(x) + \int_{x}^{c_2}\phi_k(y)g_k(y)m_k(y)\,dy\cdot\bar{A}_\alpha\psi_k(x)\right] \\
&\qquad +\frac{\bar{\sigma}^2(x)m_k(x)}{w_\alpha}g_k(x)\left[\phi_k'(x)\psi_k(x) - \phi_k(x)\psi_k'(x)\right]
\end{aligned}
$$

By Proposition 2.47, the term in the brackets converges to 0, bounded and pointwise almost everywhere as $k \to \infty$. Regarding the second term, note that $\bar{\sigma}^2(x)m_k(x) \to \frac{1}{s(x)}$ in a bounded way as $k \to \infty$ and hence

$$
\frac{\bar{\sigma}^2(x)m_k(x)}{w_\alpha} \to \frac{1}{w_\alpha s(x)} = \frac{1}{[\phi'(x)\psi(x) - \phi(x)\psi'(x)]}
$$

boundedly and pointwise almost everywhere. Also, $\phi_k'(x)\psi_k(x) - \phi_k(x)\psi_k'(x) \to \phi'(x)\psi(x) - \phi(x)\psi'(x)$ as $k \to \infty$. Making these ends meet, we see that, in a bounded way,

$$\frac{\bar{\sigma}^2(x)m_k(x)}{w_\alpha} g_k(x) \left[\phi_k'(x)\psi_k(x) - \phi_k(x)\psi_k'(x)\right] \to g(x)$$

as $k \to \infty$. This shows the first part of the assertion. The second part is easily derived from the formula for f_k' and the facts that $\phi_k \to \phi > 0$, $\psi_k \to \psi > 0$, $\phi_k' \to \phi' > 0$, $\psi_k' \to \psi' < 0$, $g_k \to g \geq 0$ and $g_k \to g > 0$ as $k \to \infty$, all boundedly and pointwise almost everywhere if necessary. $\square$

This result shows that we can set up a sequence of functions which lies in the domain of the generator, but its values under the map $\bar{A}_\alpha$ converge to any desired function g. Of course, we choose g to be an indicator function in the following to once again apply a result from measure theory to show the uniqueness and existence.

A short digression will allow us to prove that for a solution $(\bar{\mu}_0, \bar{\mu}_1)$ to (2.30), $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure. We refer the reader to Folland (1999), Section 3.5, for a discussion of absolutely continuous functions, and the nomenclature used in the next proposition. Consider the sequence of functions constructed in the proof of Theorem 2.49, with $g = I_D$ for an interval $D \in \mathscr{B}(E)$. It is easy to see that if f_D is the pointwise limit of this sequence of functions $\{f_{k,D}\}_{k\in\mathbb{N}}$, it has the form

$$f_D(x) = \frac{2}{w_\alpha}\left[\phi_\alpha(x)\int_{e_l}^{x} \psi_\alpha(y)I_D(y)m(y)\,dy + \psi_\alpha(x)\int_{x}^{e_l} \phi_\alpha(y)I_D(y)m(y)\,dy\right]. \tag{2.51}$$

Note that we set $c_1 = c_2 = e_l$. Further,

$$f_D'(x) = \frac{2}{w_\alpha}\left[\phi_\alpha'(x)\int_{e_l}^{x} \psi_\alpha(y)I_D(y)m(y)\,dy + \psi_\alpha'(x)\int_{x}^{e_l} \phi_\alpha(y)I_D(y)m(y)\,dy\right] \tag{2.52}$$

and indeed, $f_D' = \lim_{k\to\infty} f_{k,D}'$ holds pointwise.

Proposition 2.53. *Let $x_0, s \in E$. The functions defined using (2.51) and (2.52), given by*

$$
\begin{aligned}
F_1 &: \mathbb{R} \ni d \mapsto f_{(-\infty,d]}(x_0) \\
F_2 &: \mathbb{R} \ni d \mapsto f'_{(-\infty,d]}(e_l) \\
F_3 &: \mathbb{R} \ni d \mapsto f'_{(-\infty,d]}(e_r)
\end{aligned}
$$

are absolutely continuous.

Proof. Consider an interval $(a_k, b_k] \subset \mathbb{R}$. Then,

$$
\begin{aligned}
|f_{(a_k,b_k]}(x_0)| &= \left| \frac{2}{w_\alpha} \left[\phi_\alpha(x) \int_{e_l}^{x_0} \psi_\alpha(y) I_{(a_k,b_k]}(y) m(y)\, dy + \psi_\alpha(x) \int_{x_0}^{e_l} \phi_\alpha(y) I_{(a_k,b_k]}(y) m(y)\, dy \right] \right| \\
&\leq \left| \frac{2}{w_\alpha} \right| (\|\phi_\alpha\|_\infty + \|\psi_\alpha\|_\infty) \left[\int_{e_l}^{x_0} I_{(a_k,b_k]}(y) m(y)\, dy \right].
\end{aligned}
$$

The expression $(\|\phi_\alpha\|_\infty\| + \psi_\alpha\|_\infty)$ is indeed finite, since both functions are continuous, or piecewise continuous, and hence can be bounded on the compact set $[e_l, x_0]$. Let $M(x) = \int_{e_l}^{x} m(y)\, dy$. Note that M is increasing. Then,

$$
\int_{e_l}^{x_0} I_{(a_k,b_k]}(y) m(y)\, dy \leq \int_{e_l}^{e_r} I_{(a_k,b_k]}(y) m(y)\, dy = M(b_k) - M(a_k),
$$

but M is absolutely continuous with respect to Lebesgue measure, so for $\epsilon > 0$, there exists a $\delta > 0$ such that there is a finite sequence of disjoint intervals $\{(a_k, b_k]\}_{k=1}^{n}$ for which the implication

$$
\sum_{k=1}^{n} (b_k - a_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} (M(b_k) - M(a_k)) < \frac{\epsilon}{\left|\frac{2}{w_\alpha}\right| (\|\phi_\alpha\|_\infty + \|\psi_\alpha\|_\infty)}
$$

holds. But then,

$$\begin{aligned}\sum_{k=1}^{n}|F_1(b_k)-F_1(a_k)| &\leq \left|\frac{2}{w_\alpha}\right|(\|\phi_\alpha\|_\infty+\|\psi_\alpha\|_\infty)\cdot\sum_{k=1}^{n}\left[\int_{e_l}^{x_0} I_{(a_k,b_k]}(y)\,dy\right] \\ &\leq \left|\frac{2}{w_\alpha}\right|(\|\phi_\alpha\|_\infty+\|\psi_\alpha\|_\infty)\cdot\sum_{k=1}^{n}(M(b_k)-M(a_k))<\epsilon,\end{aligned}$$

which shows that F_1 is absolutely continuous. The arguments for F_2 and F_3 are similar, recalling that ϕ'_α and ψ'_α are continuous and bounded. □

Theorem 2.54. *The continuous expected occupation measure $\bar{\mu}_0$ of a solution $(\bar{\mu}_0, \bar{\mu}_1)$ to* (2.30) *is absolutely continuous with respect to Lebesgue measure.*

Proof. Choose a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $\bar{A}f_k \to I_{(\infty,d]}$ boundedly and pointwise as $k\to\infty$. If $(\bar{\mu}_0,\bar{\mu}_1)$ solves (2.30), we have that

$$\begin{aligned}\bar{\mu}_0((-\infty,d]) &= \int_E I_{(-\infty,d]}d\bar{\mu}_0 = \lim_{k\to\infty}\int_E Af_k d\bar{\mu}_0 \\ &= \lim_{k\to\infty}\left(-\alpha f_k(x_0)-\int_E Bf_k d\bar{\mu}_1\right) \\ &= -\alpha f_{(-\infty,d]}(x_0)-F_2(d)\bar{\mu}_1(\{e_l\})+F_3(d)\bar{\mu}_1(\{e_r\})\end{aligned}$$

Using this notation, we actually have

$$\bar{\mu}_0((-\infty,d]) = -\alpha F_1(d)+F_2(d)\bar{\mu}_1(\{e_l\}) = F_3(d)\bar{\mu}_1(\{e_r\})$$

and the right hand side is an absolutely continuous function. By standard results from measure theory, $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure. □

We show two results that will give us the uniqueness theorem.

Lemma 2.55. *Let D be an interval in $\mathscr{B}(E)$. Then, there is a sequence of functions $\{f_{D,k}\}_{k\in\mathbb{N}}$, such that $\bar{A}_\alpha f_k \to I_D$ as $k\to\infty$ boundedly and pointwise with $\lim_{k\to\infty} f'_{D,k}(e_l)=0$ and $\lim_{k\to\infty} f'_{D,k}(e_r)>0$.*

Proof. In (2.50), set $c_1 = c_2 = e_l$. Then, $f'_{D,k}(e_l) = 0$ for all $k \in \mathbb{N}$ and

$$\begin{aligned}\lim_{k\to\infty} f'_{D,k}(e_r) &= \lim_{k\to\infty} \frac{2}{w_\alpha}\left[\phi'_k(e_r)\int_{e_l}^{e_r} \psi_k(y)g_k(y)m_k(y)\,dy + \psi'_k(e_r)\int_{e_r}^{e_l} \phi_k(y)g_k(y)m_k(y)\,dy\right] \\ &= \frac{2}{w_\alpha}\left[\phi'_\alpha(e_r)\int_{e_l}^{e_r} \psi_\alpha(y)g(y)m(y)\,dy - \psi'_\alpha(e_r)\int_{e_l}^{e_r} \phi_\alpha(y)g(y)m(y)\,dy\right] > 0,\end{aligned}$$

since w_α is negative. As seen in Theorem 2.49, we have that $\bar{A}_\alpha f_k \to I_D$ as $k \to \infty$. □

Lemma 2.56. *Let D be an interval in $\mathscr{B}(E)$. Then, there is a sequence of functions $\{f_{D,k}\}_{k\in\mathbb{N}}$, such that $\bar{A}_\alpha f_k \to I_D$ as $k \to \infty$ boundedly and pointwise with* $\lim_{k\to\infty} f'_{D,k}(e_r) = 0$ *and* $\lim_{k\to\infty} f'_{D,k}(e_l) < 0$.

Proof. In (2.50), set $c_1 = c_2 = e_r$. Then, $f'_{D,k}(e_r) = 0$ for all $k \in \mathbb{N}$ and

$$\begin{aligned}\lim_{k\to\infty} f'_{D,k}(e_l) &= \lim_{k\to\infty} \frac{2}{w_\alpha}\left[\phi'_k(e_l)\int_{e_r}^{e_l} \psi_k(y)g_k(y)m_k(y)\,dy + \psi'_k(e_l)\int_{e_l}^{e_r} \phi_k(y)g_k(y)m_k(y)\,dy\right] \\ &= \frac{2}{w_\alpha}\left[-\phi'_\alpha(e_l)\int_{e_l}^{e_r} \psi_\alpha(y)g(y)m(y)\,dy + \psi'_\alpha(e_l)\int_{e_l}^{e_r} \phi_\alpha(y)g(y)m(y)\,dy\right] < 0.\end{aligned}$$

As seen in Theorem 2.49, we have that $\bar{A}_\alpha f_k \to I_D$ as $k \to \infty$. □

Theorem 2.57. *Let $(\bar{\mu}_0, \bar{\mu}_1)$ and $(\hat{\mu}_0, \hat{\mu}_1)$ be two solutions to* (2.30). *Then, $(\bar{\mu}_0, \bar{\mu}_1)$ and $(\hat{\mu}_0, \hat{\mu}_1)$ are identical.*

Proof. Consider two solutions $(\bar{\mu}_0, \bar{\mu}_1)$ and $(\hat{\mu}_0, \hat{\mu}_1)$. Setting $D = E$, by Lemma 2.55, choose a sequence $\{f_{k,E}\}_{k\in\mathbb{N}}$ such that

$$\begin{aligned}1 &= \bar{\mu}_0(E) = \int_E I_E d\bar{\mu}_0 = \int_E \lim_{k\to\infty} \bar{A}_\alpha f_{k,E} d\bar{\mu}_0 = \lim_{k\to\infty}\int_E \bar{A}_\alpha f_{k,E} d\bar{\mu}_0 \\ &= \lim_{k\to\infty}\left(-\alpha f_{k,E}(x_0) - \int_E B f_{k,E} d\bar{\mu}_1\right) \\ &= \lim_{k\to\infty}\left(-\alpha f_{k,E}(x_0) - B f_{k,E}(e_r)\bar{\mu}_1(\{e_r\})\right)\end{aligned}$$

and with the same reasoning,

$$1 = \lim_{k\to\infty} \left(-\alpha f_{k,E}(x_0) - Bf_{k,E}(e_r)\hat{\mu}_1(\{e_r\})\right)$$

which right away shows that $\bar{\mu}_1(\{e_r\}) = \hat{\mu}_1(\{e_r\})$. Then, again setting $D = E$, by Lemma 2.56, choose a sequence $\{f_{k,d}\}_{k\in\mathbb{N}}$ such that

$$1 = \lim_{k\to\infty} \left(-\alpha f_{k,E}(x_0) - Bf_{k,E}(e_l)\bar{\mu}_1(\{e_l\}) - Bf_{k,E}(e_r)\bar{\mu}_1(\{e_r\})\right)$$

and

$$1 = \lim_{k\to\infty} \left(-\alpha f_{k,E}(x_0) - \hat{B}f_{k,E}(e_l)\hat{\mu}_1(\{e_l\}) - Bf_{k,E}(e_r)\hat{\mu}_1(\{e_r\})\right)$$

which then implies that $\bar{\mu}_1(\{e_l\}) = \hat{\mu}_1(\{e_l\})$. By Theorem 2.49, setting $D = [c, d]$ for $e_l \leq c < d \leq e_r$, find a sequence of functions $\{f_{k,[c,d]}\}_{k\in\mathbb{N}}$ such that $\bar{A}_\alpha f_k \to I_{[c,d]}$ pointwise and bounded. Then

$$\begin{aligned}
\bar{\mu}_0([c,d]) &= \int_E I_{[c,d]} d\bar{\mu}_0 = \int_E \lim_{k\to\infty} Af_{k,[c,d]} d\bar{\mu}_0 = \lim_{k\to\infty} \int_E Af_{k,[c,d]} d\bar{\mu}_0 \\
&= \lim_{k\to\infty} \left(-\alpha f_{k,[c,d]}(x_0) - \int_E Bf_{k,[c,d]} d\bar{\mu}_1\right) \\
&= \lim_{k\to\infty} \left(-\alpha f_{k,[c,d]}(x_0) - \int_E Bf_{k,[c,d]} d\hat{\mu}_1\right) \\
&= \lim_{k\to\infty} \int_E Af_{k,[c,d]} d\bar{\mu}_0 = \ldots = \\
&= \hat{\mu}_0([c,d])
\end{aligned}$$

holds, which shows that $\bar{\mu}_0$ and $\hat{\mu}_0$ agree on a π-system that generates $\mathscr{B}(E)$, and hence they are identical. $\square$

II.3 Weak Convergence of Measures

The convergence proofs for the method proposed in this thesis will, in some parts, be based on the notion of weak convergence of measures. In the literature, weak convergence is frequently considered for probability measures, the classical reference is given by Billingsley (1999). We regard this concept in the slightly more general setting of finite Borel measures. An extensive discussion of this concept can be found in Bogachev (2007). In the following, let (S, d) be a metric space, and let $\mathscr{A}$ be the σ-algebra of Borel sets on S such that we can once consider the set of bounded, uniformly continuous functions from S to $\mathbb{R}$, denoted $C_b^u(S)$, and the measurable space $(S, \mathscr{A})$.

Definition 3.1. *Consider a sequence of finite measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $(S, \mathscr{A})$ and another finite measure μ on $(S, \mathscr{A})$. We say that μ_n converges weakly to μ, in symbol $\mu_n \Rightarrow \mu$, if for all $f \in C_b^u(S)$*

$$\int_S f(x)\mu_n(dx) \to \int_S f(x)\mu(dx) \quad \text{as } n \to \infty$$

holds.

Remark 3.2. *If we regard integration against μ_n as an operator on $C_b^u(S)$, weak convergence of measures can be viewed as pointwise convergence.*

Several statements that are equivalent to Definition 3.1 can be shown. This is referred to by Portmanteau's Theorem in the literature. The following equivalent condition to the weak convergence of measures will be used later on.

Proposition 3.3. *Consider a sequence of finite measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $(S, \mathscr{A})$ and another finite measure μ on $(S, \mathscr{A})$. μ_n converges to μ weakly if and only if for all bounded functions $\hat{f}$ that are continuous μ-almost everywhere,*

$$\int_S \hat{f}(x)\mu_n(dx) \to \int_S \hat{f}(x)\mu(dx) \quad \text{as } n \to \infty$$

holds.

Proof. The ‘if’ implication is trivial. The ‘only if’ implication is a simple generalization of the proof for Corollary 8.4.2. from Bogachev (2007), where the statement is shown for probability measures. □

The notion of sequential compactness can be generalized (from topological spaces) to the framework of weak convergence of measures. An important concept in this regard is the idea of tightness, which ensures that a sequence of measures does not push mass ‘out to infinity’. Since we are dealing with Borel measures that can have arbitrarily large (but finite) mass, we also need an idea of uniform boundedness of a sequence.

Definition 3.4. *A sequence of finite measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $(S, \mathscr{A})$ is called tight if for all $\epsilon > 0$, there is a compact set K_ϵ in S such that*

$$\mu_n(K_\epsilon^C) < \epsilon$$

holds for all $n \in \mathbb{N}$.

Definition 3.5. *A sequence of finite measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $(S, \mathscr{A})$ is called uniformly bounded if for some $l \geq 0$, $\mu_n(S) \leq l$ holds for all $n \in \mathbb{N}$.*

The following result is known as Prokhorov’s theorem in the literature. It gives a necessary and sufficient condition for the existence of convergent subsequences.

Theorem 3.6. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of finite measures on $(S, \mathscr{A})$. Then, the following are equivalent.*

i) $\{\mu_n\}_{n\in\mathbb{N}}$ contains a weakly convergent subsequence,

ii) $\{\mu_n\}_{n\in\mathbb{N}}$ is tight and uniformly bounded.

Proof. See Bogachev (2007), Theorem 8.6.2. □

Remark 3.7. *Trivially, a tight sequence of probability measures contains a weakly convergent subsequence.*

II.4 Cubic Spline Interpolation and B-Splines

In this section we review the literature on cubic spline interpolation and B-Spline basis functions in particular. These basis functions serve as a discrete basis in $C^2(E)$, forming the finite-dimensional space of constraint functions which is needed in order to attain a computationally tractable version of the infinite-dimensional linear program. The following results present a summary of the theory presented in de Boor (2001) and Atkinson (1989).

The idea of cubic spline interpolation is the following. Let $E = [e_l, e_r] \in \mathbb{R}$ be a closed interval. Consider a finite set of pointwise distinct points $\{e_j\}_{j=0}^n$ in E, usually called a grid or a mesh, and a set of values $\{f_j\}_{j=0}^n$ stemming from a function $f : E \mapsto \mathbb{R}$, such that $f(e_j) = f_j$ for $j = 0, 1, \ldots n$. Such a situation is encountered when no closed form of f is available, or we can only observe the values of f on a finite set of points. We want to reconstruct, or approximate, f by finding a function g that fulfills $g(e_j) = f_j$ for $j = 0, 1, \ldots n$ and can be written in a closed form. This closed form of g can than be used to estimate f at points which are not contained in $\{e_j\}_{j=0}^n$, or to derive an approximation of the integral of f.

What we have described to this point is generally called an interpolation problem, and we can convince ourselves that a linear interpolation on the intervals $[e_j, e_{j+1}]$ solves this problem. However, this only provides a continuous, and not differentiable function. Also, the quality of the approximation between the grid points might be poor, as it is not guaranteed that f is linear between two grid points. A variety of techniques to attain higher-order approximation has been developed, and we refer the interested reader to de Boor (2001) for a extensive discussion of the techniques. In this thesis, we focus on Cubic spline interpolation. Assume in the following that $e_j < e_{j+1}$ for $j = 0, 1, \ldots, n-1$.

Cubic spline interpolation seeks to interpolate $\{f_j\}_{j=0}^n$ on $\{e_j\}_{j=0}^n$ by a function that is twice continuously differentiable, and a piecewise polynomial of degree three on the intervals $[e_j, e_{j+1}]$ for $j = 0, 1, \dots n$. Usually, one poses boundary conditions on $s(e_0)$ and $s(e_n)$ to ensure that this problem is well posed. We will not dwell on the specifics of this problem and refer to the literature, see de Boor (2001) or Atkinson (1989) once more. However, these cubic splines can be expressed as a finite linear combination of a certain set of basis functions, and these basis functions will be of fundamental importance to the convergence analysis of the proposed method in Section III.2.

To construct such basis functions, enhance the grid with the introduction of additional points defined by

$$e_{-3} < e_{-2} < e_{-1} < e_0 \qquad \text{and} \qquad e_n < e_{n+1} < e_{n+2} < e_{n+3}.$$

Further, let $[h]^+ : E \ni x \mapsto [h]^+(x) := \max(h(x), 0)$ be the positive part of a function h.

Definition 4.1. *The set of B-spline basis functions for a grid* $\{e_j\}_{j=-3}^{n+3}$ *is defined by*

$$B_j(x) = (e_{j+4} - e_j) \sum_{i=j}^{j+4} \frac{[(e_i - x)^3]^+}{\Psi_j'(e_i)}, \quad j = -3, -2, \dots n-1$$

where

$$\Psi_j(x) = \prod_{i=j}^{j+4} (x - e_i).$$

The properties of B-spline basis functions are well understood. The most pertinent facts for our purposes are the following.

Proposition 4.2. *Let* $\{e_j\}_{j=0}^n$ *be a set of grid points and let* $\{B_j\}_{j=-3}^{n-1}$ *be the B-spline basis functions on the grid* $\{e_j\}_{j=-3}^{n+3}$.

i) The support of B_j *is given by* (e_j, e_{j+4}).

ii) The set of linear combinations of $\{B_j\}_{j=-3}^{n-1}$ *is the set of all cubic splines on the grid* $\{e_j\}_{j=0}^n$.

Proof. See de Boor (2001). $\square$

Remark 4.3. *The second property states that if we are given grid points $\{e_j\}_{j=0}^{n}$ with function values $\{f_j\}_{j=0}^{n}$, the cubic spline s through these data points is a linear combination of B-spline basis functions on the enhanced grid $\{e_j\}_{j=-3}^{n+3}$.*

The importance of B-spline basis functions for this thesis is revealed in the next theorem. First, we have to define the mesh gauge and the mesh ratio of a given grid $\{e_j\}_{j=-3}^{n+3}$.

Definition 4.4. *Let $\pi = \{e_j\}_{j=-3}^{n+3}$ be a set of given grid points. For $j = -3, \ldots, n+3$, set $\Delta e_j = e_{j+1} - e_j$. The mesh gauge γ_π of π is given by*

$$\gamma_\pi = \max_{j=-3,\ldots,n+3} \Delta e_j.$$

Definition 4.5. *Let $\pi = \{e_j\}_{j=-3}^{n+3}$ be a set of given grid points. For $j = -3, \ldots, n+3$, set $\Delta e_j = e_{j+1} - e_j$. The mesh ratio ρ_π of π is given by*

$$\rho_\pi = \frac{\max\limits_{j=-3,\ldots,n+3} \Delta e_j}{\min\limits_{j=-3,\ldots,n+3} \Delta e_j}.$$

Remark 4.6. *If the grid points $\{e_j\}_{j=-3}^{n+3}$ are chosen equidistant, the mesh ratio is equal to 1.*

Theorem 4.7. *Let $f \in C^2(E)$ and let $\{\pi_k\}_{k\in\mathbb{N}}$ be a sequence of meshes such that $\gamma_\pi \to 0$ and $\rho_{\pi_k} \to 1$ as $k \to \infty$. Let s_k be the cubic spline interpolating f_k on π_k. Then, the function and its derivatives $s^{(r)}$ converges to $f^{(r)}$ uniformly, for $r = 0, 1, 2$.*

Proof. See Hall and Meyer (1976), Theorem 1. $\square$

Corollary 4.8. *Consider $C_c^2(E)$ of twice differentiable functions with compact support and define a norm on this function space by $\|f\|_{\mathscr{D}} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$, where $\|\cdot\|_\infty$ denotes the uniform norm. Let $\{\pi_k\}_{k\in\mathbb{N}}$ be a sequence of meshes such that $\rho_{\pi_k} \to 1$ as $k \to \infty$, and let $B^{(k)}$ be the set of B-spline basis functions on π_k. Then, $\cup_{k\in\mathbb{N}} B^{(k)}$ is dense in $(C_c^2(E), \|\cdot\|_{\mathscr{D}})$.*

Approximation Techniques

This chapter proposes approximation techniques that are used to solve the stochastic control problems introduced in Chapter II. We distinguish between models with a bounded state space, and models with an unbounded state space. In respect to the convergence analysis of Chapter IV, models with a bounded state space are easier, and are presented first. The approach for models with an unbounded state space relies on an additional 'layer' of approximations, as presented in the second part of this chapter. This additional layer brings forth a reduced problem which effectively has a bounded state space, and minor adjustments to the techniques used to approximate models with bounded state spaces can be employed to obtain a computable version of this reduced problem.

III.1 Infinite Time Horizon Problems with Bounded State Space

This section introduces the approximation scheme for stochastic control problems that feature an infinite time horizon and a bounded state space. The singular behavior can be present or not, and if it is given by a jump, jump sizes are non-random. Thus, the singular behavior is deterministic. As in Section I.2, we mainly regard the case where singular behavior is present, as it is more general, and occasionally point out how the considerations have to be altered to tackle problems without singular behavior.

Several aspects of the approximation will be discussed. First, the discretization of the infinite-dimensional linear program is presented. Second, it is illustrated how the discretization can be used to set up a finite-dimensional linear program given in vector-matrix form. This is followed by a section describing how the approximation approach can be adapted to simply evaluate given controls, rather than finding an optimal control. The section closes with a couple of remarks on specific meshing and the choice of basis functions, which are omitted in the initial description of the approximation scheme for the sake of clarity.

Let the state space $E = [e_l, e_r] \subset \mathbb{R}$ be a closed and bounded interval, and let the control space $U = [u_l, u_r] \subset \mathbb{R}$ be a closed and bounded interval as well. Note that in this setting, we do not have to distinguish between $C_c^2(E)$ and $C^2(E)$. The cost criterion is expressed using two continuous functions $c_0, c_1 : E \mapsto \mathbb{R}_{\geq 0}$, with c_0 representing the cost induced by the continuous behavior of the process, and c_1 representing the costs induced by the singular behavior. We denote the set of probability measures on $E \times U$ by $\mathcal{P}(E \times U)$, and by $\mathcal{M}(E \times U)$ the set of finite Borel measures on $E \times U$.

III.1.1 Discretization of the Linear Program

The general linear program we wish to solve (compare Definitions II.1.36 and II.1.37) is

$$\begin{array}{ll} \text{Minimize} & \displaystyle\int_{E\times U} c_0 d\mu_0 + \int_{E\times U} c_1 d\mu_1 \\ \text{Subject to} & \begin{cases} \int Af d\mu_0 + \int Bf\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty \\ \mu_0 \in \mathcal{P}(E \times U) \\ \mu_1 \in \mathcal{M}(E \times U) \end{cases} \end{array}.$$

For a non-singular problem, simply assume $B \equiv 0$ and $\mu_1 \equiv 0$ in the following discussion. In the singular case, the expected occupation measure μ_1 puts mass on only a finite number of points in the state space. To be precise, it is concentrated on either $\{e_l\}$, $\{e_r\}$, or on both of them, being the only points where singular behavior occurs. For the ease of notation, we will use

$$\mathscr{M}_\infty = \left\{(\mu_0,\mu_1) \in \mathcal{P}(E\times U)\times\mathcal{M}(E\times U) : \int Af d\mu_0 + \int Bf\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty\right\},$$

recalling that $\mathscr{D}_\infty = (C^2(E), \|\cdot\|_{\mathscr{D}})$ and

$$J : \mathcal{P}(E\times U)\times\mathcal{M}(E\times U) \ni (\mu_0,\mu_1) \mapsto J(\mu_0,\mu_1) = \int_{E\times U} c_0\, d\mu_0 + \int_{E\times U} c_1\, d\mu_1 \in \mathbb{R}_{\geq 0}.$$

This allows for the following definition.

Definition 1.1. *The infinite-dimensional linear program is to find*

$$\min\left\{J(\mu_0,\mu_1)|(\mu_0,\mu_1) \in \mathscr{M}_\infty\right\}.$$

Note that this problem features infinitely many constraints, as the function space $\mathscr{D}_\infty$ is infinite-dimensional, and an infinite set of degrees of freedom, since the spaces $\mathcal{P}(E\times U)$ and $\mathcal{M}(E\times U)$ are not finitely representable. The approximation that follows proposes to break this infinite dimensional problem into a finite dimensional problem in two steps. First, we limit the number of constraints to a finite number by taking a countable basis of $\mathscr{D}_\infty$, and truncating it after finitely many basis elements. Second, we introduce measures $\hat{\mu}_0$ and $\hat{\mu}_1$ that are finite linear combinations of simple measures, yielding a finite dimensional linear program. For theoretic purposes, however, we restrict the set of measures (μ_0,μ_1) in $\mathscr{M}_\infty$ to such measures that $\mu_1(E\times U) \leq l$ for some $l > 0$. Set

$$\mathcal{M}^l(E\times U) = \left\{\mu_1 \in \mathcal{M}(E\times U) : \mu_1(E\times U) \leq l\right\}.$$

Then we can define

$$\mathscr{M}^l_\infty = \left\{(\mu_0,\mu_1) \in \mathcal{P}(E\times U)\times\mathcal{M}^l(E\times U) : \int Af d\mu_0 + \int Bf d\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty\right\}.$$

Definition 1.2. *The l-bounded infinite-dimensional linear program is to find*

$$\min \left\{ J(\mu_0, \mu_1) | (\mu_0, \mu_1) \in \mathscr{M}_\infty^l \right\}.$$

Remark 1.3. *In the case of a non-singular problem, the introduction of this bound on the mass of μ_1 is not necessary.*

For the discretization of the constraints, we require the following result.

Proposition 1.4. *$\mathscr{D}_\infty$ is separable.*

Proof. This follows directly from Corollary II.4.8, which states that a countable dense subset is given by B-spline basis functions. □

This allows us to discretize the constraints by truncating a countable dense subset of the constraint function space as stated in the following definition.

Definition 1.5. *Let $\{f_k\}_{k \in \mathbb{N}}$ be a countable dense subset in $\mathscr{D}_\infty$. For $n \in \mathbb{N}$, define the n-dimensional test function space by*

$$\mathscr{D}_n = (\operatorname{span}(\cup_{k=1}^n \{f_k\}), \|\cdot\|_{\mathscr{D}}).$$

How a specific choice of B-spline basis functions can be attained by dividing the state space into a mesh will be discussed in Section III.1.4. With the n-dimensional test function space, we can introduce

$$\mathscr{M}_n^l = \left\{ (\mu_0, \mu_1) \in \mathcal{P}(E \times U) \times \mathcal{M}^l(E \times U) : \int Af d\mu_0 + \int Bf d\mu_1 = Rf \quad \forall f \in \mathscr{D}_n \right\}$$

which by linearity of operators A and B, the functional R, as well as integration is the same as setting

$$\mathscr{M}_n^l = \left\{ (\mu_0, \mu_1) \in \mathcal{P}(E \times U) \times \mathcal{M}^l(E \times U) : \int Af_k d\mu_0 + \int Bf_k d\mu_1 = Rf_k, k = 1, \dots, n \right\}.$$

Definition 1.6. *The l-bounded (n,∞)-dimensional linear program is to find*

$$\min\left\{J(\mu_0,\mu_1)|(\mu_0,\mu_1)\in\mathscr{M}_n^l\right\}.$$

The notation '(n,∞)' refers to the fact that the constraints of the problem have been discretized, but the measures, which are the variables, have not yet been discretized. The convergence analysis that is to come will reveal that, using the notion of ϵ-optimality, an almost optimal solution of the l-bounded (n,∞)-dimensional problem is an almost optimal solution to the original problem if n and l are large enough. However, we still have to construct a computationally attainable subset of $\mathscr{M}_n^l$, in other words, a finite-dimensional approximation of $\mathscr{M}_n^l$. Theorems II.1.42 and II.1.43 state that an optimal relaxed control is given in feedback form, and the notion of a regular conditional probability (Definition II.1.26) is used to model feedback controls. In other words, we assert that $\mu_0(dx\times du)=\eta_0(du,x)\mu_{0,E}(dx)$, where $\mu_{0,E}$ is the state space marginal of μ_0 and η_0 represents the feedback control for the continuous behavior. In similar fashion, we can decompose the singular expected occupation measure μ_1 such that $\mu_1(dx\times du)=\eta_1(du,x)\mu_{1,E}(dx)$ with $\mu_{1,E}$ referring to the state space marginal of μ_1. The regular conditional probability η_1 models the relaxed control of the singular part, for example the jump sizes.

To make the approximation work, a couple of assumptions are introduced at this point. First, we need to require that for every $V\in\mathscr{B}(U)$, we have that $\eta_0(V,\cdot):E\mapsto[0,1]$ is continuous almost everywhere. Second, we assume that $\mu_{0,E}$, the state-space component of the measure μ_0, is absolutely continuous with respect to Lebesgue measure. If we denote the density by p, we have that $\mu_0(dx\times du)=\eta_0(du,x)p(x)dx$, and $p\in L^1(E)$. We construct the approximation in this chapter based on these hypotheses. However, we can note that by the derivations presented in Section II.2, these two hypotheses are fulfilled if there is no control on the singular behavior, and η_0 is such a control that the functions $\bar{b}$ and $\bar{\sigma}$ defined by

$$\bar{b} : E \ni x \mapsto \bar{b}(x) = \int_U b(x,u)\,\eta_0(du,x), \qquad \bar{\sigma} : E \ni x \mapsto \bar{\sigma}(x) = \sqrt{\int_U \sigma^2(x,u)\,\eta_0(du,x)}$$

are bounded and continuous in E except at finitely many points, and either left or right continuous at these finitely many points. Remarks II.2.1 and II.2.2 indicate possible classes of controls that make this assumption hold. As a third hypothesis, the aforementioned density p of $\mu_{0,E}$ has to fulfill the constraint that $\lambda\left(\{x : p(x) = 0\}\right) = 0$. In other words, p is equal to zero only on a set of Lebesgue measure 0.

Now we discuss how to approximate a measure $(\mu_0, \mu_1) \in \mathscr{M}_n^l$. To begin with, define the sequence $k_m^{(1)} = m$ for $m \in \mathbb{N}$. We define a second sequence $\{k_m^{(2)}\}_{m\in\mathbb{N}}$ as follows. Since c_0, b and σ are assumed to be continuous over a compact set, for all $m \in \mathbb{N}$, there is a $\delta_m > 0$ such that for all $u, v \in U$ with $|u - v| \leq \delta_m$, it is true that

$$\max\left\{|c_0(x,u) - c_0(x,v)|, |b(x,u) - b(x,v)|, \left|\frac{1}{2}\sigma^2(x,u) - \frac{1}{2}\sigma^2(x,v)\right|\right\} \leq \frac{1}{2^{m+1}},$$

uniformly in x. Set $k_m^{(2)}$ to be the integer such that $\frac{u_r - u_l}{2^{k_m^{(2)}}} \leq \delta_m$. This is done to have a sufficiently accurate approximate integration of the cost function c_0 and the function Af against the relaxed control η_0 in the convergence proof presented in Section IV.1.

Example 1.7. *Assume that* $U = [-1, 1]$*, let* $c_0(x,u) = x^2 + u^2$*,* $b(x,u) = u$ *and* $\sigma(x,u) \equiv \sigma$*. We have that, as* $|u + v| \leq 2$*,*

$$|c_0(x,u) - c_0(x,v)| = |u^2 - v^2| = |u + v| \cdot |u - v| \leq 2|u - v|$$

and hence we need $2|u - v| \leq \frac{1}{2^{m+1}}$*, so* $\delta_m = \frac{1}{2^{m+2}}$*. Set* $k_m^{(2)} = m + 3$*. Then,*

$$\frac{u_r - u_l}{2^{k_m^{(2)}}} = \frac{2}{2^{m+3}} = \frac{1}{2^{m+2}} = \delta_m.$$

Also,

$$|b(x,u) - b(x,v)| = |u - v| \leq \frac{1}{2^{m+2}} \leq \frac{1}{2^{m+1}}$$

holds and $|\sigma(x,u) - \sigma(x,v)|$ is trivially zero.

The sequences $k_m^{(1)}$ and $k_m^{(2)}$ determine the fineness of the following subsets of the control space and state space as defined by

$$\begin{aligned} E^{(k_m^{(1)})} &= \{e_j = e_l + \tfrac{e_r - e_l}{2^{k_m^{(1)}}} \cdot j, \quad j = 0, \ldots, 2^{k_m^{(1)}}\} \\ U^{(k_m^{(2)})} &= \{u_j = u_l + \tfrac{u_r - u_l}{2^{k_m^{(2)}}} \cdot j, \quad j = 0, \ldots, 2^{k_m^{(2)}}\}. \end{aligned} \tag{1.8}$$

Note that the union of these set over all $m \in \mathbb{N}$ is dense in the state space and the control space, respectively. A diagram of these two meshes can be found in Figure III.1

Remark 1.9. *The proposed way of partitioning U ensures that for given m and $u \in U$, there is a $v \in U^{(k_m^{(2)})}$ such that the expressions involving the cost function $|c_0(x,u) - c_0(x,v)|$ or the generator for the continuous behavior*

$$|Af(x,u) - Af(x,v)| = \left| f'(x)b(x,u) + \frac{1}{2}f''(x)\sigma^2(x,u) - f'(x)b(x,v) - \frac{1}{2}f''(x)\sigma^2(x,v) \right|,$$

which holds both for the long-term average criterion and the discounted infinite horizon criterion, can be uniformly bounded by $K\left(\frac{1}{2}\right)^{m+1}$, with $K = 1$ if we consider c_0 and $K = \max\{\|f_1\|_{\mathscr{D}}, \ldots, \|f_k\|_{\mathscr{D}}\}$ if we consider A and fixed $k \in \mathbb{N}$.

Finally, set $k_m^{(3)} = m$. This third sequence will be used to control the discretization of the density of $\mu_{0,E}$. From now on, m is called the discretization level, and we will regard the three sequences as components of a vector $k_m \equiv (k_m^{(1)}, k_m^{(2)}, k_m^{(3)})$. Since different choices for $k_m^{(1)}$ or $k_m^{(3)}$ could be made, for example to implement adaptive meshing, we retain the slightly involved notation from here on rather than just substituting $k_m^{(1)}$ and $k_m^{(3)}$ by m.

The approximation scheme is now constructed as follows. Choose a countable basis of $L^1(E)$, say $\{p_n\}_{n\in\mathbb{N}}$. Specific choices for this basis are discussed in Section III.1.4. We truncate this basis to $p_1, \ldots, p_{2^{k_m^{(3)}}}$ in order to have a finite-dimensional space of functions. The density p is approximated by

$$\hat{p}_{k_m}(x) = \sum_{i=0}^{2^{k_m^{(3)}}-1} \gamma_i p_i(x) \tag{1.10}$$

where $\gamma_i, i = 0, \ldots, 2^{k_m^{(3)}} - 1$ are weights to be chosen under the constraint that $\int_E \hat{p}_{k_m}(x)dx = 1$ and that $\gamma_i \geq 0$ for $i = 0, 2, \ldots, 2^{k_m^{(3)}}$. Set $E_j = [x_j, x_{j+1})$ for $j = 0, 1, \ldots 2^{k^{(1)}} - 2$, and $E_{2^{k^{(1)}}-1} = [x_{2^{k^{(1)}}-1}, x_{2^{k^{(1)}}}]$ and

$$\hat{\eta}_{0,k}\left(V, x\right) = \sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \beta_{j,i} I_{E_j}(x) \delta_{u_i}(V). \tag{1.11}$$

where δ_{u_i} denotes the Dirac measure on u_i, and $\beta_{j,i} \in \mathbb{R}_{\geq 0}, j = 0, \ldots, 2^{k_m^{(1)}} - 1, i = 0, \ldots, 2^{k_m^{(2)}}$ are weights yet to be chosen under the constraint that $\sum_{i=0}^{2^{k_m^{(2)}}} \beta_{j,i} = 1$ for $j = 0, \ldots, k_m^{(1)} - 1$. We approximate η_0 in the form of (1.11), which means that the relaxed control is approximated by point masses in the U-'direction' and piecewise constant in the E-'direction'. We set $\hat{\mu}_{0,k_m}(dx \times du) = \hat{\eta}_{0,k_m}(du, x)\hat{p}_{k_m}(x)dx$.

To approximate the singular expected occupation measure μ_1, we use the fact that we know a-priori where the process is going to show singular behavior. Thus, if we introduce the regular conditional probability η_1 and write $\mu_1(dx \times du) = \eta_1(du, x)\mu_{1,E}(dx)$, and for $F \in \mathscr{B}(E)$, we have

$$\mu_{1,E}(F) = \sum_{i=1}^{N} \alpha_i \delta_{s_i}(F) \tag{1.12}$$

where $s_1, \ldots, s_N$ are the fixed points in E at which the singular behavior of the process happens (usually these are just $\{e_l\}$ and $\{e_r\}$) and $\alpha_1, \ldots, \alpha_N$ are weights taking values in the non-negative real numbers, with $\sum_{i=1}^{N} \alpha_i \leq l$. For the convergence proof, it will be crucial that the points $s_1, \ldots, s_N$ lie in the set $E^{(k_m^{(1)})}$. If this is not the case using the dyadic points as stated above, we can simply add $s_1, \ldots, s_N$ into the set, without changing the analysis that is to follow. So, we assume in the following that $s_1, \ldots, s_N \subset E^{(k_m^{(1)})}$ for all m. We approximate the relaxed control η_1 for $j = 1, \ldots, N$ by

$$\hat{\eta}_{1,k_m}(V, s_j) = \sum_{i=0}^{2^{k_m^{(2)}}} \zeta_{j,i} \delta_{u_i}(V), \tag{1.13}$$

with $\sum_{i=0}^{2^{k_m^{(2)}}} \zeta_{j,i} = 1$ for all $j = 1, \ldots, N$. So, we have $\hat{\mu}_{1,k_m}(dx \times du) = \hat{\eta}_{1,k_m}(du, x)\mu_{1,E}(dx)$. Summing up, we consider measures of the form

$$\left(\hat{\mu}_{0,k_m}, \hat{\mu}_{1,k_m}\right)(dx \times du) = \left(\hat{\eta}_{0,k_m}(du, x)\hat{p}_{k_m}(x)dx, \hat{\eta}_{1,k_m}(du, x)\mu_{1,E}(dx)\right)$$

and we introduce the notation

$$\mathscr{M}^l_{n,k_m} = \left\{ \begin{array}{l} \left(\mu_{0,k_m}, \mu_{1,k_m}\right) \in \mathscr{M}^l_n \text{ such that} \\ \left(\mu_{0,k_m}, \mu_{1,k_m}\right)(dx \times du) = \left(\hat{\eta}_{0,k_m}(du, x)\hat{p}_{k_m}(x)dx, \hat{\eta}_{1,k_m}(du, x)\mu_{1,E}(dx)\right) \end{array} \right\}.$$

This finalizes the discretization of the measures and leaves us with the following linear program, identifying $\mathscr{M}^l_{n,k_m} \equiv \mathscr{M}^l_{n,m}$.

Definition 1.14. *The l-bounded (n, m)-dimensional linear program is to find*

$$\min\left\{J(\mu_0, \mu_1) | (\mu_0, \mu_1) \in \mathscr{M}^l_{n,m}\right\}.$$

Given that the number of variables, which is the number of degrees of freedom, controlled by m, is larger than the number of constraints, this gives a finite linear program, for which extensive solution theory exists. For a discussion, see Vanderbei (2014). In particular, it is known that a finite linear program has a solution, and sophisticated numerical algorithms to find such a solution exist. So, we can assume that we are able to find an optimal solution to the l-bounded (n, m)-dimensional linear program. The convergence analysis will reveal that an optimal solution for the l-bounded (n, m)-dimensional linear program is an ϵ-optimal solution to the l-bounded (n, ∞)-dimensional linear program, which, as already mentioned, is, in a specific sense, close to the optimal solution to the original problem.

For the sake of completeness, we state the final formulation of the discretized linear program for a non-singular problem. In this case, we have $b \equiv 0$. The set of admissible measures is

$$\mathscr{M}_{n,m} = \left\{\mu_{0,k_m} \in \mathscr{M}_n : \mu_{0,k_m}(dx \times du) = \hat{\eta}_{0,k_m}(du, x)\hat{p}_{k_m}(x)dx\right\},$$

where

$$\mathscr{M}_n = \{\mu_0 \in \mathcal{P}(E \times U) : \int Af_k d\mu_0 = Rf_k, k = 1, \ldots, n\}.$$

Definition 1.15. *The (n, m)-dimensional linear program in the non-singular case is to find*

$$\min \left\{J(\mu_0) | \mu_0 \in \mathscr{M}_{n,m}\right\}.$$

Deviations from previous work

We want to briefly outline how the proposed approximation relates to work that has been established in Kaczmarek et al. (2007), Rus (2009) and Lutz (2007). The respective papers have approached the discretization of measures and constraints in such a way that both base functions for the constraint space $C_c^2(E)$ and for the approximate density $\hat{p}$ are of the same type. In Kaczmarek et al. (2007) and Lutz (2007), this was done using continuous piecewise linear functions. In Rus (2009), this was done using Hermite polynomials. Both of these approaches have deficiencies in terms of the theoretical analysis.

First, discretizing the density p with continuous piecewise linear functions assumes that the density is indeed continuous. Given a specific control problem, this cannot be assured without a detailed analytic treatment of the problem. We decided to use $L^1(E)$ basis functions that pose the least possible set of assumptions, which is mere integrability of the density. Nevertheless, it is worth noting that analytic solutions, see Appendix C, suggest that the density is, at least piecewise, highly regular, and the use of higher-order elements such as linear elements or even quadratic elements could be justified.

Second, the choice of B-splines basis functions for the constraint function space is extremely crucial in establishing the convergence of the method, as elaborated in Remark IV.1.9. In particular, it is necessary to use the dominated convergence theorem when integrating sequences of functions given by Af_n and Bf_n, a theorem which can only be assumed if the underlying sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the $\|\cdot\|_{\mathscr{D}}$-norm. This boundedness condition is neither given with continuous piecewise linear basis functions, nor Hermite polynomial basis functions.

On a more general level, the proposed approximation succeeds in preserving the linear program structure of the infinite-dimensional problem, as is further elaborated in Section III.1.2. Analogous to Kaczmarek et al. (2007) and Lutz (2007), a finite-dimensional linear program is derived which can be solved with algorithms that rely heavily on this structure, and guarantee swift computation. The structure utilized in Rus (2009) uses a linear least square projection setup that only allows for the evaluation of a given control, and hence has to rely on fairly generic optimization algorithms to solve an unstructured global optimization problem.

III.1.2 Computational Set-up of Discretized Linear Program

A finite dimensional linear program features a set of constraints, given by equalities and inequalities that are linear in the unknowns, and a linear cost function. As we will see, the l-bounded, (n, m)-dimensional linear program coming from the proposed approximation is not linear in the coefficients α_i, $\beta_{j,i}$, γ_i and $\zeta_{j,i}$ which were introduced in Section III.1.1 (with i and j ranging respectively), but is linear in certain products of these coefficients. The first part of this section works out the precise structure of the linear program for these coefficients, the second part shows how the desired results, in particular the relaxed control, can be attained from the solution of the linear program.

As before, we regard a discretization level $k \equiv \left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ with three components (the dependence on m as presented in the previous section is omitted for sake of readability), with the first one controlling the number of points in the state space $E = [e_l, e_r]$, the second one controlling the number of points in the control space $U = [u_l, u_r]$ (both used to discretize the relaxed controls) and the third one controlling the number of basis functions $p_0, \dots, p_{2^{k^{(3)}}-1}$ for the density of the approximate state space marginal of the continuous expected occupation measure μ_0. We regard the constraint functions $f_1, \dots, f_n$, representing a finite basis for a subspace $\mathscr{D}_n$ of $\mathscr{D}_\infty$. Each of these basis functions represents an equality constraint in the linear program. An additional constraint is introduced due to the fact that the approximation of μ_0 has to be a probability measure.

In this section, each equality constraint is regarded as a row in a coefficient matrix which has $n+1$ rows. To specify the number of columns in this coefficient matrix we need to look closely into the specific expressions. Recall the discrete formulas of p, η_0, $\mu_{1,E}$ and η_1 as defined by (1.10) to (1.13) respectively. We refer to the discretized quantities by $\hat{p}_k$, $\hat{\eta}_{0,k}$, $\mu_{1,E}$ and $\hat{\eta}_{1,k}$ in the following. Note that $\mu_{1,E}$ is already discrete and thus is not approximated. A priori, the number of unknowns is equal to

$$M := \underbrace{\left(2^{k^{(1)}} - 1\right) \cdot 2^{k^{(2)}}}_{\text{coefficients for relaxed control } \eta_0} \cdot \underbrace{\left(2^{k^{(3)}}\right)}_{\text{coefficients for density } p} + N \cdot \underbrace{\left(2^{k^{(2)}}\right)}_{\text{coefficients for relaxed control } \eta_1},$$

where N refers to the number of points with singular behavior, such as jumps and reflections. Typically, $N = 2$ since singular behavior only occurs at $\{e_l\}$ and $\{e_r\}$. Still, we assume a more general setting and denote these points by $s_1, s_2, \dots, s_N$. They are assumed to lie within $E^{(k^{(1)})}$. For the sake of simplicity, we assume that the first N points of the set $E^{(k^{(1)})}$ are precisely $s_1, s_2, \dots, s_N$. According to these considerations, we consider a coefficient matrix $C \in \mathbb{R}^{n+1,M}$, and a right hand side vector $d \in \mathbb{R}^{n+1}$. Further constraints ensuring that $\hat{\mu}_{0,k}$ and $\hat{\mu}_{1,k}$ are non-negative, and $\hat{\mu}_{1,k}(E \times U) \leq l$ will be introduced in the form of upper and lower bounds on the coefficients.

We proceed to construct the coefficient matrix C. For fixed $1 \leq m \leq n$, the constraints dictate that

$$\int_E \int_U Af_m(x,u)\hat{\eta}_{0,k}(du,x)\hat{p}_k(x)dx + \int_E \int_U Bf_m(x,u)\hat{\eta}_{1,k}(du,x)\mu_{1,E}(dx) = Rf_m$$

which, given the discrete representation of the involved quantities (see again (1.10) to (1.13)), is equivalent to

$$\begin{aligned} Rf_m &= \int_E \left(\sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \beta_{j,i} Af_m(x,u_i) I_{E_j}(x) \delta_{u_i}(V) \right) \sum_{l=0}^{2^{k^{(3)}}-1} \gamma_l p_l(x) dx \\ &\qquad\qquad + \sum_{j=1}^{N} \left(\sum_{i=0}^{2^{k^{(2)}}} \zeta_{j,i} Bf_m(s_j,u_i) \right) \alpha_j \\ &= \sum_{j=0}^{2^{k^{(1)}}-1} \int_{E_j} \left(\sum_{i=0}^{2^{k^{(2)}}} \beta_{j,i} Af_m(x,u_i) \right) \sum_{l=0}^{2^{k^{(3)}}-1} \gamma_l p_l(x) dx + \sum_{j=1}^{N} \left(\sum_{i=0}^{2^{k^{(2)}}} \zeta_{j,i} Bf_m(s_j,u_i) \right) \alpha_j. \end{aligned}$$

To obtain a linear equation, consider the following reformulation of the right hand side of the previous equation.

$$\sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \beta_{j,i}\gamma_l \int_{E_j} Af_m(x,u_i)p_l(x)dx + \sum_{j=0}^{N} \sum_{i=0}^{2^{k^{(2)}}} \zeta_{j,i}\alpha_j Bf_m(s_j,u_i). \tag{1.16}$$

From this reformulation we can see that the equation is linear in $\beta_{j,i}\gamma_l$ and $\zeta_{j,i}\alpha_j$, but some re-indexing is required to set it up as a straight forward inner product for implementation. Identify the parameters in (1.16) using the following, purely notational, renumbering. The symbol % refers to modulo division, for example, $9\,\%\,5 = 4$.

$$\begin{aligned} \pi_i &= \left(\left\lfloor \lfloor i/2^{k^{(3)}} \rfloor / 2^{k^{(2)}} \right\rfloor \% \left(2^{k^{(1)}} - 1\right), \left\lfloor i/2^{k^{(3)}} \right\rfloor \% 2^{k^{(2)}}, i \% \, 2^{k^{(3)}} \right), \\ &\qquad\qquad \text{if } 0 \le i \le \left(2^{k^{(1)}} - 1\right) 2^{k^{(2)}} k^{(3)} - 1 \end{aligned}$$

$$\begin{aligned} \pi_i &= \left(\left\lfloor \left(i - (2^{k^{(1)}} - 1)2^{k^{(2)}} \cdot 2^{k^{(3)}}\right) / 2^{k^{(2)}} \right\rfloor \% \, N, \left(i - (2^{k^{(1)}} - 1)2^{k^{(2)}} \cdot 2^{k^{(3)}}\right) \% \, 2^{k^{(2)}} \right), \\ &\qquad\qquad \text{if } \left(2^{k^{(1)}} - 1\right) 2^{k^{(2)}} \cdot 2^{k^{(3)}} \le i \le \left(2^{k^{(1)}} - 1\right) \cdot 2^{k^{(2)}} \cdot 2^{k^{(3)}} + N \cdot 2^{k^{(2)}} - 1. \end{aligned}$$

Note that for $0 \le i \le \left(2^{k^{(1)}}-1\right) 2^{k^{(2)}} k^{(3)} - 1$, π_i is a triple, and for $\left(2^{k^{(1)}}-1\right) 2^{k^{(2)}} \cdot 2^{k^{(3)}} \le i \le \left(2^{k^{(1)}}-1\right) \cdot 2^{k^{(2)}} \cdot 2^{k^{(3)}} + N \cdot 2^{k^{(2)}} - 1$, π_i is a pair. This is necessary to reduce the triple and double sums of (1.16) to a single sum. Set $\phi^c_{j,i,l} = \beta_{j,i}\gamma_l$ and $\phi^s_{j,i} = \zeta_{j,i}\alpha_j$, $M_1 = \left(2^{k^{(1)}}-1\right) \cdot 2^{k^{(2)}} \cdot 2^{k^{(3)}}$ and $M_2 = N \cdot 2^{k^{(2)}}$. Note that $M = M_1 + M_2$. Define $\phi_i = \phi^c_{\pi(i)}$ if $0 \le i \le M_1 - 1$ and $\tilde{\phi}_i = \phi^s_{\pi(i)}$ if $M_1 + 1 \le i \le M_1 + M_2 - 1$. Then, we can rewrite (1.16) and use it to set up an equation as follows.

$$\sum_{i=0}^{M_1} \phi_i \int_{E_{\pi_{i,1}}} A f_m(x, u_{\pi_{i,2}}) p_{\pi_{i,3}}(x) dx + \sum_{i=M_1+1}^{M_1+M_2} \phi_i B f_m(s_{\pi_{i,1}}, u_{\pi_{i,2}}) = R f_m \tag{1.17}$$

Here $\pi_{i,j}$ denotes the jth component of the tuple π_i. (1.17) clearly is linear in the variables $\{\phi_i, 0 \le i \le M-1\}$. For $1 \le m \le n$, we define the coefficients of the matrix $C \in \mathbb{R}^{n+1,M}$ by

$$\begin{aligned} C_{m,i} &= \int_{E_{\pi_{i,1}}} A f_m(x, u_{\pi_{i,2}}) p_{\pi_{i,3}}(x) dx, \qquad 0 \le i \le M_1 - 1 \\ C_{m,i} &= B f_m(s_{\pi_{i,1}}, u_{\pi_{i,2}}), \qquad M_1 \le i \le M_1 + M_2 - 1. \end{aligned}$$

The integrals in this expression can be computed either analytically or using a quadrature rule. If the functions $f_1, \dots f_n$ and the functions $p_0, \dots, p_{2^{k^{(3)}}-1}$ are given by piecewise polynomials (compare Section III.1.4), quadrature rules of high enough order yield exact results. Note that we use a numbering starting at 0 only for the ease of notation. Finally, we need to incorporate an equation ensuring that $\hat{\mu}_{0,k}$ has a full mass of 1. Note that for a given solution ϕ, we will have the following continuous and singular expected occupation measures, given

$V \in \mathscr{B}(U)$ and $F \in \mathscr{B}(E)$.

$$
\begin{aligned}
\hat{\mu}_{0,k}(V \times F) &= \sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \phi^c_{j,i,l} \int_{E_j \cap F} p_l(x)\, dx \cdot \delta_{u_i}(V) \\
&= \sum_{i=0}^{M_1-1} \phi_i \int_{E_{\pi_{i,1}} \cap F} p_{\pi_{i,3}}(x)\, dx \cdot \delta_{u_{\pi_{i,2}}}(V) \qquad (1.18) \\
\hat{\mu}_{1,k}(V \times F) &= \sum_{j=0}^{N} \sum_{i=0}^{2^{k^{(2)}}} \phi^s_{j,i} \cdot \delta_{s_j}(F) \cdot \delta_{u_i}(V) \\
&= \sum_{i=M_1}^{M_1+M_2} \phi_i \cdot \delta_{s_{\pi_{i,1}}}(F) \cdot \delta_{u_{\pi_{i,2}}}(V) \qquad (1.19)
\end{aligned}
$$

Since $\hat{\mu}_{0,k}$ ought to have a full mass of 1, we have to assert that

$$
\begin{aligned}
\hat{\mu}_{0,k}(U \times E) &= \sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \phi^c_{j,i,l} \int_{E_j} p_l(x) dx \\
&= \sum_{i=0}^{M_1-1} \phi_i \int_{E_{\pi_{i,1}}} p_{\pi_{i,3}}(x) dx = 1,
\end{aligned}
$$

which again is a linear constraint and we set

$$
\begin{aligned}
C_{n+1,i} &= \int_{E_{\pi_{i,1}}} p_{\pi_{i,3}}(x), \qquad 0 \le i \le M_1 - 1 \\
C_{n+1,i} &= 0, \qquad M_1 \le i \le M_1 + M_2 - 1
\end{aligned}
$$

The right hand side $d \in \mathbb{R}^{n+1}$ is simply computed by $d_m = Rf_m$ for $1 \le m \le n$, and $d_{n+1} = 1$.

Depending on the choice of basis functions $f_1, f_2, \dots, f_n$ and $p_0, p_1, \dots, p_{2^{k^{(3)}}-1}$, in particular for the choice described in Section III.1.4, a lot of the coefficients in C are zero, as f_m and $p_{\pi_{i,3}}$ do not necessarily have joint support. As a matter of fact, this might cause complete rows of the matrix to be zero, which then can be dropped from the coefficient matrix, resulting in a drastic reduction of unknowns. It also allows for the use of sparse matrices in the implementation, lowering the computational efforts needed to solve the resulting linear problem.

Next, we need to assert that $\hat{\mu}_{0,k}$ and $\hat{\mu}_{1,k}$ are non-negative. For $\hat{\mu}_{1,k}$ this is straight forward by asserting that ϕ_i are non-negative for all $M_1 + 1 \leq i \leq M_1 + M_2$. For $\hat{\mu}_{0,k}$ some short observations have to be made regarding the choice of basis functions. Not every choice of basis functions ensures that if the coefficients are non-negative, the linear combination of the basis functions with those coefficients is non-negative. However, this holds, for example, for piecewise constant basis functions or Lagrange $p1$-elements, which linearly interpolate between two given (non-negative) points. Thus, we have to restrict ourselves to basis functions that ensure non-negativity of the resulting density p_k, and again demand that $\phi_i \geq 0$ are non-negative for all $0 \leq i \leq M_1 - 1$. The bound on the full mass of $\hat{\mu}_{1,k}$ can be ensured as follows, compare (1.19),

$$\hat{\mu}_{1,k}(E \times U) = \sum_{j=0}^{N} \sum_{i=0}^{2^{k^{(2)}}} \phi_{j,i}^{s} \cdot \delta_{s_j}(E) \cdot \delta_{u_i}(U) = \sum_{i=M_1}^{M_1+M_2-1} \phi_i \leq l.$$

The objective function can be expressed by

$$\begin{aligned} J(\phi) &= \sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \phi_{j,i,l}^{c} \int_{E_j} c_0(x,u_i) p_l(x) dx + \sum_{j=0}^{N} \sum_{i=0}^{2^{k^{(2)}}} \phi_{j,i}^{s} c_1(s_j, u_i) \\ &= \sum_{i=0}^{M_1-1} \phi_i \int_{E_{\pi_{i,1}}} c_0(x,u_i) p_{\pi_{i,3}}(x) dx + \sum_{i=M_1}^{M_1+M_2} \phi_i c_1(s_{\pi_{i,1}}, u_{\pi_{i,2}}) \end{aligned}$$

which is also linear in ϕ. Define a vector $h \in \mathbb{R}^{M_1+M_2}$ with

$$\begin{aligned} h_i &= \int_{E_{\pi_{i,1}}} c_0(x,u_i) p_{\pi_{i,3}}(x) dx, \qquad 0 \leq i \leq M_1 - 1 \\ h_i &= c_1(s_{\pi_{i,1}}, u_{\pi_{i,2}}), \qquad M_1 \leq i \leq M_1 + M_2 - 1. \end{aligned}$$

The finite dimensional program we have to solve is thus given by

$$\begin{array}{ll} \text{minimize} & h^T\phi \\ \text{such that} & \displaystyle\sum_{i=M_1}^{M_1+M_2-1} \phi_i \leq l, \qquad C\phi = d, \qquad \phi \geq 0. \end{array}$$

Now we illustrate how to process the results obtained by solving this linear program. Assume we have solved for $\phi = \{\phi_i\}_{i=0,1,\dots M_1+M_2-1}$. We cannot directly derive the parameters $\beta_{j,i}$, γ_l and α_j from ϕ. Because the relaxed controls, represented by the regular conditional probabilities $\hat{\eta}_{0,k_m}$ and $\hat{\eta}_{1,k_m}$ are of our central interest, we have to exploit the properties of regular conditional probabilities to obtain the controls. We are interested in the probability measures $\eta_0(\cdot, X = z)$ and $\eta_1(\cdot, X = z)$ which represent feedback controls, given that the process X is in the state z. Assume that $z \in E_j$ for some j, and recall that we set $E_j = [x_j, x_{j+1})$ for $j = 0, 1, \dots 2^{k^{(1)}} - 2$ as well as $E_{2^{k^{(1)}}-1} = [x_{2^{k^{(1)}}-1}, x_{2^{k^{(1)}}}]$. Due to the kind of approximation we chose for the regular conditional probability, it is clear that $\eta_0(\cdot, z_1) = \eta_0(\cdot, z_2)$ if $z_1, z_2 \in E_j$, and likewise, $\eta_1(\cdot, z_1) = \eta_1(\cdot, z_2)$ holds, due to the way we constructed $\hat{\eta}_{0,k}$ and $\hat{\eta}_{1,k}$. But by basic properties of the regular conditional probabilities, we have for a measure $\hat{\mu}_{0,k}$ with $\hat{\mu}_{0,k}(dx \times du) = \hat{\eta}_k(du, x)\hat{\mu}_{0,k,E}(dx)$ and $V \in \mathscr{B}(U)$,

$$\begin{aligned} \hat{\mu}_{0,k}\left(E_j \times V\right) &= \int_{E_j} \hat{\eta}_{0,k}(V, z)\hat{\mu}_{0,k,E}(dz) \\ &= \hat{\eta}_{0,k}(V, x_j) \int_{E_j} \hat{\mu}_{0,k,E}(dz) \\ &= \hat{\eta}_{0,k}(V, x_j)\hat{\mu}_{0,k,E}(E_j) \end{aligned}$$

which directly implies that

$$\hat{\eta}_{0,k}(V, x_j) = \frac{\hat{\mu}_{0,k}\left(E_j \times V\right)}{\hat{\mu}_{0,k,E}\left(E_j\right)} \tag{1.20}$$

thereby giving us a formula to compute the relaxed control $\hat{\eta}_{0,k}$. Note that we only need find $\hat{\eta}_{0,k}\left(\{u_l\}, z\right) = \beta_{j,l}$, $z \in E_l$ for $j = 0, 1, \dots, 2^{k^{(2)}}$ and $l = 0, 1, \dots, 2^{k^{(1)}} - 1$, as we assumed that $\hat{\eta}_{0,k}$ is a discrete measure on $u_0, u_1, \dots, u_{2^{k^{(1)}}}$. The value for $\hat{\mu}_{0,k}\left(\{u_l\} \times E_j\right)$ can be obtained using (1.18). It is given by

$$\hat{\mu}_{0,k}\left(\{u_l\} \times E_j\right) = \sum_{i=0}^{M_1-1} \phi_i \int_{E_{\pi_{i,1}} \cap E_j} p_{\pi_{i,3}}(x)\, dx \cdot \delta_{u_{\pi_{i,2}}}(\{u_l\}).$$

Similarly, we can deduce that

$$\hat{\eta}_{1,k}\left(V, \{s_j\}\right) = \frac{\hat{\mu}_{1,k}\left(V \times \{s_j\}\right)}{\mu_{1,E}\left(\{s_j\}\right)}, \tag{1.21}$$

if $\hat{\mu}_{1,k}$ is a measure with $\hat{\mu}_{1,k}(dx \times du) = \hat{\eta}_k(du, x)\mu_{1,E}(dx)$. The value for $\hat{\mu}_{1,k}(\{u_l\} \times s_j)$ can be obtained using (1.19). It is given by

$$\hat{\mu}_{1,k}(\{u_l\} \times \{s_j\}) = \sum_{i=M_1}^{M_1+M_2} \phi_i \cdot \delta_{s_{\pi_{i,1}}}(\{s_j\}) \cdot \delta_{u_{\pi_{i,2}}}(\{u_l\}).$$

In order to find the density of the state space marginal of $\hat{\mu}_{0,k}$, one has to analyze the expression

$$\begin{aligned}
\hat{\mu}_{0,k}(U \times F) &= \sum_{j=0}^{2^{k^{(1)}}-1} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \phi^c_{j,i,l} \int_{E_j \cap F} p_l(x) dx \cdot \delta_{u_i}(U) \\
&= \int_F \underbrace{\sum_{j=0}^{2^{k^{(1)}}} \sum_{i=0}^{2^{k^{(2)}}} \sum_{l=0}^{2^{k^{(3)}}-1} \phi^c_{j,i,l} I_{E_j}(x) p_l(x)}_{\text{density}} \, dx \\
&= \int_F \underbrace{\sum_{i=0}^{M} \phi_i I_{[x_{\pi_{i,1}}, x_{\pi_{i,1}+1})} p_{\pi_{i,3}}(x)}_{\text{density}} \, dx,
\end{aligned}$$

obviously using that $\delta_{u_i}(U) \equiv 1$. In particular, one can use this expression to find

$$\hat{\mu}_{0,k,E}\left(E_j\right) = \int_{E_j} \sum_{i=0}^{M} \phi_i I_{[x_{\pi_{i,1}}, x_{\pi_{i,1}+1})} p_{\pi_{i,3}}(x) dx$$

which is needed in (1.20).

To obtain the weights of the singular part of the expected occupation measure on the points of singular behavior, one simply uses

$$\begin{aligned}
\mu_{1,E}(\{s_l\}) = \hat{\mu}_{1,k}\left(U \times \{s_l\}\right) &= \sum_{j=0}^{N} \sum_{i=0}^{2^{k^{(2)}}} \phi_{j,i} \cdot \delta_{s_j}(\{s_l\}) \cdot \delta_{u_i}(U) \\
&= \sum_{i=0}^{2^{k^{(2)}}} \phi^{s}_{l,i} = \sum_{i=0}^{2^{k^{(2)}}} \phi_{M_1+(l-1)+i\cdot N}.
\end{aligned}$$

In particular, this formula is needed in (1.21).

Remark 1.22. *This setup can be reduced to attain the linear program when no singular behavior is present. In particular, $M_2 = 0$ and all quantities related to $\hat{\mu}_{1,k}$ and c_1 vanish from the constraint matrix and the cost vector. However, note that this comes at the cost of losing degrees of freedom.*

III.1.3 Evaluation of Cost Criterion

While the proposed approximation is used to derive a solvable optimization problem, the same approximation techniques can be used to approximate the cost criterion for a fixed control. The set-up described in this section can be used to analyze the accuracy of the evaluation of the cost criterion, for a control stemming from the proposed approximation, which is presented later in Section IV.1.3.

Consider two fixed relaxed controls η_0 and η_1 of the form (1.11) and (1.13). In this section, we do not consider η_0 and η_1 as approximations, and will not use the $\hat{}$ notation when referring to them. Set

$$\bar{A}f(x) = \int_U Af(x,u)\eta_0(du,x), \qquad \bar{B}f(x) = \int_U B(x,u)\eta_1(du,x). \tag{1.23}$$

We have seen in Section II.2 that under certain conditions on η_0 and η_1, there is a unique pair of expected occupation measures that solves the linear constraints given by

$$\begin{aligned} &\int_E \bar{A}f\, d\bar{\mu}_0 + \int_E \bar{B}f\, d\bar{\mu}_1 = Rf \quad \forall f \in \mathscr{D}_\infty \\ &\bar{\mu}_0 \in \mathcal{P}(E), \quad \bar{\mu}_1 \in \mathcal{M}(E). \end{aligned} \tag{1.24}$$

In particular, we worked out that $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure. The approximate forms of η_0 and η_1 (compare (1.11) and (1.13)) fulfill the conditions outlined in Section II.2. The approach of finding an approximate solution to (1.24) mirrors the approximation introduced to solve the actual control problem in Section III.1.1. To this end, we introduce set of feasible measures by identifying $\eta \equiv (\eta_0, \eta_1)$ and setting

$$\mathscr{M}_\infty^\eta = \left\{ (\bar{\mu}_0, \bar{\mu}_1) \in \mathcal{P}(E) \times \mathcal{M}(E) : \int_E \bar{A}f(x)\bar{\mu}_0(dx) + \int_E \bar{B}f(x)\bar{\mu}_1(dx) = Rf \quad \forall f \in \mathscr{D}_\infty \right\},$$

of which we know that it only contains one element, and for $1 \leq n < \infty$ the sets

$$\mathscr{M}_n^\eta = \left\{ (\bar{\mu}_0, \bar{\mu}_1) \in \mathcal{P}(E) \times \mathcal{M}(E) : \int_E \bar{A}f(x)\bar{\mu}_0(dx) + \int_E \bar{B}f(x)\bar{\mu}_1(dx) = Rf \quad \forall f \in \mathscr{D}_n \right\}.$$

As before, $\mathscr{D}_n$ is a finite-dimensional subspace of $\mathscr{D}_\infty$ spanned by basis functions $f_1, f_2, \ldots, f_n$. For theoretic purposes again, we introduce a bound on the full mass of $\bar{\mu}_1$. Choose $l > 0$ and define for $1 \leq n \leq \infty$

$$\mathscr{M}_n^{\eta,l} = \{ (\bar{\mu}_0, \bar{\mu}_1) \in \mathscr{M}_n^{\eta_0,\eta_1} \ : \ \bar{\mu}_1(E) \leq l \}$$

where we choose l large enough such that $\mu_1(E \times U) = \int_E \eta_1(U, x)\mu_{1,E}(dx) \leq l$ for the actual expected occupation measure $\mu_{1,E}$ that is associated with the relaxed control η_1. Later, we introduce an approximation for measures in $\mathscr{M}_n^{\eta,l}$ that is computationally tractable. First, we illustrate how the discretization of the constraints leads to an approximate value for the cost criterion that is different from the actual cost criterion. To this end, note that a solution

$(\bar{\mu}_0^\eta, \bar{\mu}_1^\eta)$ to (1.24) implies a solution

$(\mu_0^\eta(dx \times du), \mu_1^\eta(dx \times du)) \equiv (\eta_0(du,x)\mu_0^\eta(dx), \eta_1(du,x)\mu_1^\eta(dx))$ to the linear constraints

$$\begin{aligned} &\int_{E\times U} Af\, d\mu_0 + \int_{E\times U} Bf\, d\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty \\ &\mu_0 \in \mathcal{P}(E\times U), \quad \mu_1 \in \mathcal{M}(E\times U), \end{aligned} \tag{1.25}$$

which implies an equivalent solution to the relaxed martingale problem. In other words, there is indeed a process X that is controlled by η_0 and η_1. It gives a value of the cost criterion by the following formula. Let $\mathcal{A}$ be set of all admissible controls.

$$\bar{J} : \mathcal{A} \ni \eta \mapsto \bar{J}(\eta) = \int_{E\times U} c_0(x,u)\mu_0^\eta(dx\times du) + \int_{E\times U} c_1(x,u)\mu_1^\eta(dx\times du).$$

Note that $\bar{J}(\eta) = J(\mu_0^\eta, \mu_1^\eta)$. To compare this with the discretized case, fix $n \in \mathbb{N}$, take a pair of discrete measures on E, denoted $(\bar{\mu}_{0,n}^\eta, \bar{\mu}_{1,n}^\eta) \in \mathscr{M}_n^{\eta,l}$ and define two new measures on $E \times U$ by

$$\mu_{0,n}^\eta(dx\times du) = \eta_0(du,x)\bar{\mu}_{0,n}^\eta(dx), \quad \mu_{1,n}^\eta(dx\times du) = \eta_1(du,x)\bar{\mu}_{1,n}^\eta(dx)$$

and set

$$J_n : \mathcal{A} \ni \eta \mapsto J_n(\eta) = \int_{E\times U} c_0(x,u)\mu_{0,n}^\eta(dx\times du) + \int_{E\times U} c_1(x,u)\mu_{1,n}^\eta(dx\times du).$$

We must assume that for any $1 \leq n < \infty$, $J_n(\eta) \neq \bar{J}(\eta)$, as $(\mu_{0,n}^\eta, \mu_{1,n}^\eta)$ do not actually represent expected occupation measures of solutions to the relaxed martingale problem. The relation between these quantities for increasing n is investigated in Section IV.1.3.

The computationally tractable version of $\mathscr{M}_n^\eta$ is attained in the same fashion as demonstrated in Section III.1.1. We represent a discretized measure $\hat{\mu}_{0,k,E}(dx) \equiv \hat{p}_k(x)dx$ as in (1.10) and $\mu_{1,E}$ as in (1.12). In a similar manner as seen in Section III.1.2, we set up a coefficient matrix with $n+1$ constraints and $M_3 = 2^{k^{(3)}} + N$ unknowns, with coefficients $\{\gamma_i\}_{i=0,\ldots,2^{k^{(3)}}-1}$

stemming from the density of the continuous expected occupation measure, and $\{\alpha_i\}_{i=1,\ldots,N}$ stemming from the singular expected occupation measure. Form the vector of unknowns by defining $\phi \equiv (\gamma_0, \gamma_2, \ldots, \gamma_{2^{k(3)}-1}, \alpha_1, \alpha_2, \ldots, \alpha_N)^T$. The entries of the coefficient matrix $C \in \mathbb{R}^{n,M_3}$ are

$$
\begin{aligned}
C_{k,i} &= \int_E \int_U Af_k(x,u)\eta_0(du,x)p_i(x)dx, && 0 \le i \le 2^{k(3)} - 1, \quad 1 \le k \le n,\\
C_{k,i} &= \int_U Bf_k(x,u)\eta_1(du, s_{i-2^{k(3)}}), && 2^{k(3)} \le i \le M_3 - 1, \quad 1 \le k \le n,\\
C_{n+1,i} &= \int_E \int_U \eta_0(du,x)p_i(x)dx, && 0 \le i \le 2^{k(3)} - 1,\\
C_{n+1,i} &= 0, && 2^{k(3)} \le i \le M_3 - 1,
\end{aligned}
$$

while the right hand side $d \in R^{n+1}$ is given by $d_i = Rf_i$, $i = 1, \ldots, n$ and $d_{n+1} = 1$. Again we adopt a numbering scheme starting at 0 for convenience. As seen in the previous subsection, we also have to require that

$$
\sum_{i=2^{k(3)}}^{M_3} \phi_i \le l \tag{1.26}
$$

$$
\phi_i \ge 0, \quad i = 0, 1, \ldots, M_3 - 1. \tag{1.27}
$$

Assume $M_3 \gg n+1$ such that the under-determined equation system $C\phi = d$ has at least one solution that fulfills (1.27) and (1.26). Any solution serves as an approximate solution to $\mathscr{M}_\infty^\eta$. To solve for ϕ, linear least squares solvers can be used. They can be set up to minimize $\|\phi\|_2^2$ under the constraint that $C\phi = d$, and the constraints given by (1.27) and (1.26).

This set-up will serve as a tool to evaluate the cost criterion for any pair of controls η_0, η_1. Note that C in this section has significantly fewer columns than C in Section III.1.2, since the control is fixed. Hence a simple evaluation of the cost associated with a fixed control can be conducted more quickly, or alternatively, with higher accuracy in the same amount of time.

Remark 1.28. *As before, this setup can be reduced to accommodate problems with no singular behavior in an obvious way.*

III.1.4 Basis Functions and Meshing

The preceding considerations were formulated without specifics on the choice of basis functions for the density of the state-space marginal of the continuous expected occupation measure $\mu_{0,E}$ or the basis functions used to represent a discrete version of the constraint space $\mathscr{D}_\infty$. These basis functions are determined by a finite-element type approach, where each basis function is associated with a subinterval of the state space E. Hence, the choice of basis functions depends on the mesh, in other words, a finite number of points, that is put into the state space E and determines the subintervals.

So far, we have considered a discretization level $k \equiv \left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ which was used to discretize the expected occupation measures, see Section III.1.1. The parameter $k^{(3)}$ was used to discretize the density p of the state space marginal of μ_0, compare (1.10). However, a specific choice of basis functions was not given. We also used a parameter n to introduce a finite dimensional constraint space $\mathscr{D}_n$, where Definition 1.5 did not elaborate on the choice of basis functions beyond the fact that they are given by B-splines basis functions. This section will present the choices for these two types of basis functions.

To start with, we slightly alter the notion of a discretization level to have four components given by $k \equiv \left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}\right)$. The fourth component $k^{(4)}$ controls the discretization of the constraint space $\mathscr{D}_\infty$.

Consider the state space $E = [e_l, e_r]$. Analogous to the discretization of the state space for the controls, compare Section III.1.1, we consider the mesh

$$E_1^{(k^{(3)})} = \left\{ e_j = e_l + \frac{e_r - e_l}{2^{k^{(3)}}} \cdot j, j = 0, \dots, 2^{k^{(3)}} \right\} \tag{1.29}$$

consisting of $2^{k^{(3)}} + 1$ points.

Definition 1.30. *The set of functions*

$$F^{(p)}_{k^{(3)}} = \left\{ I_{[e_i, e_{i+1})}, i = 0, 1, \ldots, 2^{k^{(3)}} \right\}$$

is called the set of piecewise constant basis functions for the discretization level k.

Note that $F^{(p)}_{k^{(3)}}$ fulfills $F^{(p)}_{k^{(3)}} \subset F^{(p)}_{k^{(3)}+1}$ and $\cup_{k^{(3)} \in \mathbb{N}} F^{(p)}_{k^{(3)}}$ is dense subset of $L^1(E)$. This choice of basis functions poses minimal assumptions on the density of $\mu_{0,E}$, which is an integrable function. The finite elements in the state space are given by the intervals $[e_j, e_{j+1}], j = 0, \ldots, 2^{k^{(3)}}$. When considering linear combinations of functions in $F^{(p)}_{k^{(3)}}$, non-negativity of these linear combinations can be achieved by asserting that all coefficients of the linear combination are non-negative.

Remark 1.31. *The use of higher order basis functions, such as piecewise linear functions or even quadratic functions (see Solin (2006)) is only reasonable if more regularity of the state space density p can be assumed. Furthermore, ensuring non-negativity might become an issue when basis functions attain negative values, which, for example is the case for quadratic Lagrange elements. Still, if these caveats can be overcome, the use of higher order basis functions could lead to more accurate and efficient numerical schemes.*

Concerning the mesh for the constraint functions $f_1, f_2, \ldots, f_n$, whose linear span $\mathscr{D}_n$ forms a subspace of $\mathscr{D}_\infty$, we introduce another mesh

$$E_2^{(k^{(4)})} = \left\{ e_j = e_l + \frac{e_r - e_l}{2^{k^{(4)}}} \cdot j, j = -3 \ldots, 2^{k^{(4)}} + 3 \right\}$$

which we will use to define B-spline basis functions as introduced in Section II.4. Note that the index j indeed runs from -3 to $2^n + 3$, since this is required in the definition of B-spline basis functions, again compare Section II.4. Set $n = 2^{k^{(4)}} + 2$, which is the actual number of

B-spline basis functions obtained from this mesh, and define

$$\mathscr{D}_n = \text{span}\{B_j, j = -3, -2, \ldots 2^n - 1\}.$$

A schematic diagram of the different meshes obtained by setting $k^{(1)} = 2$, $k^{(2)} = 2$, $k^{(3)} = 3$ and $k^{(4)} = 3$ is shown in Figure III.1. It displays the mesh $E_2^{(k^{(4)})}$ on the constraint space in gray, the mesh $E_1^{(k^{(3)})}$ used for the discretization of the state space density in blue and the mesh used to discretize the relaxed control as defined in (1.8), in red.

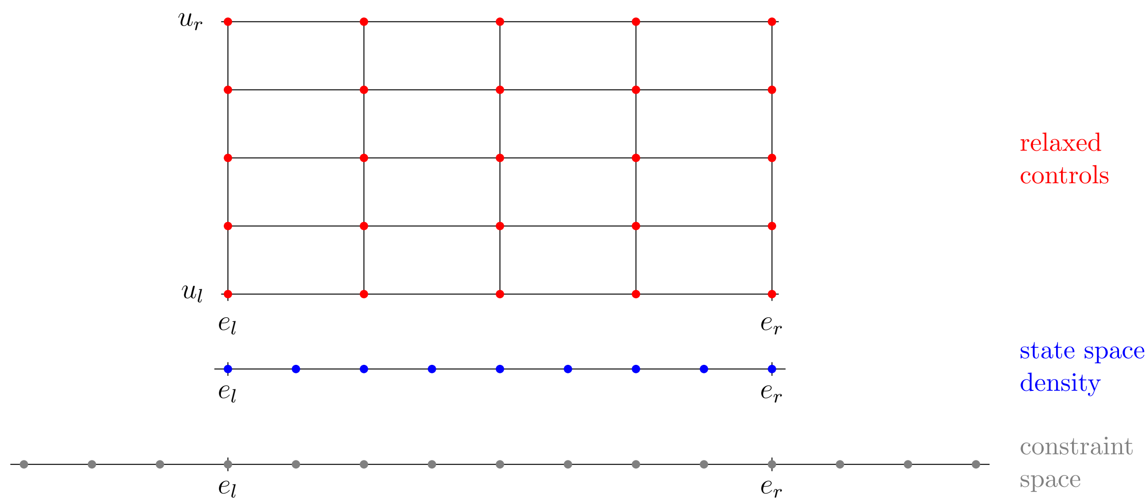

Figure III.1: Mesh example for $k^{(1)} = 2$, $k^{(2)} = 2$, $k^{(3)} = 3$ and $k^{(4)} = 3$

Remark 1.32. *In the scope of this work, all meshes are equidistant meshes given by the dyadic partitions of the state and control space. This is done to simplify programming efforts. However, it could be beneficial to choose the meshes adaptively in an iterative method which will provide higher accuracy without a steep increase in computation time. An investigation of such an approach is beyond the scope of this thesis.*

Remark 1.33. *As it will be revealed in the convergence proof, it may be necessary to introduce additional mesh points to (1.29) which will then give us a higher number of degrees of freedom. This is simply done by cutting selected intervals in half. In other words, we are* *considering the set of points*

$$E_1^{(k^{(3)},+)} = \left\{ e_j = e_l + \frac{e_r - e_l}{2^{k^{(3)}}} \cdot j, \quad j = 0, \dots, 2^{k^{(3)}} \right\} \cup \left\{ \frac{e_r + e_l}{2} - \frac{e_r - e_l}{2^{k^{(3)}} + 1} \right\} \tag{1.34}$$

and the set of indicator functions on the intervals given by this partition. If more mesh points are needed, we proceed by defining

$$E_1^{(k^{(3)},++)} = E_1^{(k^{(3)},+)} \cup \left\{ \frac{e_r + e_l}{2} + \frac{e_r - e_l}{2^{k^{(3)}} + 1} \right\} \tag{1.35}$$

and so forth. This choice of additional points is arbitrary, and more sophisticated approaches could be employed.

III.2 Infinite Time Horizon Problems with Unbounded State Space

This section introduces the approximation approach needed to solve problems with an unbounded state space, an infinite time horizon and no singular behavior. We restrict ourselves to the case in which $E = (-\infty, \infty) = \mathbb{R}$. The cases $E = [e_l, \infty)$ or $E = (-\infty, e_r]$ can be treated in a similar fashion, and are not dealt with explicitly in order to avoid repetition. However, we point out which modification would have to be made in order to accommodate for these cases. In terms of the linear programming approach, we seek to solve problems described as follows. We use the notation $\mathscr{D}_\infty(\mathbb{R}) = (C_c^2(\mathbb{R}), \|\cdot\|_{\mathscr{D}})$ to point out that we are considering constraint functions over all real numbers now. In contrast to the considerations in Section III.1, we must assert that the functions in $\mathscr{D}_\infty(\mathbb{R})$ have compact support, since this does not anymore follow from the fact that they are continuous functions on a compact space. The infinite-dimensional linear program is given by

$$\begin{aligned}
&\text{Minimize} \int_{\mathbb{R}\times U} c_0 \, d\mu \\
&\text{Subject to} \begin{cases} \displaystyle\int_{\mathbb{R}\times U} Af d\mu = Rf \quad \forall f \in \mathscr{D}_\infty(\mathbb{R}) \\ \mu \in \mathcal{P}(\mathbb{R}\times U) \end{cases}.
\end{aligned} \tag{2.1}$$

Once again, A is the generator of the continuous behavior of the process, and c_0 is a continuous cost function. R is the right hand functional. The idea of the presented approximation is to derive a linear programming formulation whose constraint functions have their support in a bounded interval $[-K, K]$, and the considered expected occupation measure μ has full mass on $[-K, K]$. The convergence analysis will reveal that for K large enough, this linear programming formulation admits solutions that are ϵ-optimal for the original formulation. Several assumptions are made to make the convergence of the approximation work. First, we must assume that there exists a solution μ to (2.1) with finite cost. Second, we need to impose two assumptions on the cost function c_0.

Definition 2.2. *A function $c_0 : \mathbb{R} \times U \mapsto \mathbb{R}_{\geq 0}$ is called increasing in $|x|$, if for any $L > 0$, there is a $K > 0$ such that*

$$c_0(x, u) > L \quad \forall\, x \notin [-K, K]$$

uniformly in $u \in U$.

Definition 2.3. *A function $c_0 : \mathbb{R} \times U \mapsto \mathbb{R}_{\geq 0}$ allows for compactification if there is a $K_0 > 0$ such that there is a function $u^- : [-K_0, K_0] \mapsto U$, which is continuous except on finitely many points, fulfilling the property that*

$$\inf \Big\{ c_0(x, u) | x \in [-K, K]^C,\, u \in U \Big\} \geq \sup \left\{ c_0(x, u^-(x)) | x \in [-K, K] \right\}.$$

If we assume that c_0 is increasing in $|x|$, we are able to show certain tightness properties of solutions to (2.1). In particular, μ can be shown to be tight. Albeit a rather opaque definition, the assumptions on a function that allows for compactification are easy to check. For example, given that U is a bounded interval, the functions $c_0(x,u) = x^2 + u^2$, $c_0(x,u) = |x| + |u|$ or $c_0(x,u) = u$, allow for compactification. Restricting the considerations on such functions, we are able to show that it is sufficient to consider measures which have support on a compact interval when looking for ϵ-optimal solutions in the following, see Lemma IV.2.11.

Similar to the notation from the previous section, denote the set of feasible measures by

$$\mathscr{M}_{\infty,\mathbb{R}} = \left\{ \mu \in \mathcal{P}(\mathbb{R} \times U) : \int Af d\mu = Rf \quad \forall f \in \mathscr{D}_\infty(\mathbb{R}) \right\}. \tag{2.4}$$

Further, we denote $\mathscr{D}([-K,K]) = (C_c^2((-K,K)), \|\cdot\|_{\mathscr{D}})$. A subtle distinction has to be remarked on here. By $C_c^2\,((-K,K))$ we understand the space of twice differentiable functions whose support is contained in a compact subset of $(-K,K)$. In other words, the support of a function $f \in C_c^2((-K,K))$ and the support of its first and second derivatives is a proper subset of $[-K,K]$. In the following, we will call $[-K,K]$ the *computed state space*.

Define two more sets of feasible measures, with reduced constraints, as follows. First, set

$$\dot{\mathscr{M}}_{\infty,[-K,K]} = \left\{ \mu \in \mathcal{P}(\mathbb{R} \times U) : \int Af d\mu = Rf \quad \forall f \in \mathscr{D}([-K,K]) \right\} \tag{2.5}$$

and then, set

$$\mathscr{M}_{\infty,[-K,K]} = \left\{ \mu \in \mathcal{P}([-K,K] \times U) : \int Af d\mu = Rf \quad \forall f \in \mathscr{D}([-K,K]) \right\}. \tag{2.6}$$

The difference between these two set is that a measure $\dot{\mu} \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ can have mass on all of $\mathbb{R} \times U$, whereas a measure $\mu \in \mathscr{M}_{\infty,[-K,K]}$ is concentrated on the set $[-K,K] \times U$.

With the help of these sets of feasible measures, we can define three different linear programs as follows.

Definition 2.7. *The infinite-dimensional linear program for a model with unbounded state space is to find*

$$\min \{J(\mu) | \mu \in \mathscr{M}_{\infty,\mathbb{R}}\}.$$

The reduced version of the linear program is defined next.

Definition 2.8. *Let $K > 0$ be a real number. The K-reduced infinite-dimensional linear program is to find*

$$\min \left\{ J(\mu) | \mu \in \dot{\mathscr{M}}_{\infty,[-K,K]} \right\}.$$

Finally, the reduced version with a measure concentrated on $[-K, K]$ reads as follows.

Definition 2.9. *Let $K > 0$ be a real number. The K-reduced-concentrated infinite-dimensional linear program is to find*

$$\min \left\{ J(\mu) | \mu \in \mathscr{M}_{\infty,[-K,K]} \right\}.$$

Under the described assumptions on μ, the convergence analysis reveals that there is a K large enough such that an ϵ-optimal solution to the K-reduced problem is almost optimal in regard to the original problem. Then, we show that for any measure $\dot{\mu} \in \dot{\mathscr{M}}_{\infty,[-K,K]}$, there exists a measure $\mu \in \mathscr{M}_{\infty,[-K,K]}$ such that $J(\mu) \leq J(\dot{\mu})$. This shows that it suffices to solve the K-reduced-concentrated infinite-dimensional linear program. However, an ϵ-optimal solution to the K-reduced-concentrated linear program can be obtained using the same approximation techniques as used for the problems with a bounded state space, and can be analyzed in identical manner. In particular, a dense subset of $\mathscr{D}([-K, K])$ is given by B-spline basis functions whose support is fully contained in $[-K, K]$. In the light of Section III.1.4, we can define a mesh for the basis functions as follows for some $n \in \mathbb{N}$.

$$E_2^{(n)} = \left\{ e_j = -K + \frac{2K}{2^n} \cdot j,\ j = 0 \ldots, 2^n \right\}$$

The union over all B-spline basis functions on the meshes given by all $n \in \mathbb{N}$ is dense in $\mathscr{D}([-K, K])$. Note that we do not introduce additional mesh points to the left of $-K$ and the right of K, respectively, in order to assure that the support of these basis functions and their derivatives indeed lies properly within $[-K, K]$.

Remark 2.10. *If $E = [e_l, \infty)$, we consider the mesh*

$$E_2^{(n)} = \left\{ e_j = e_l + \frac{K - e_l}{2^n} \cdot j,\ j = -3 \ldots, 2^n \right\}.$$

If $E = (-\infty, e_r]$, we consider the mesh

$$E_2^{(n)} = \left\{ e_j = -K + \frac{e_r - K}{2^n} \cdot j,\ j = 0 \ldots, 2^n + 3 \right\}.$$

For fixed n, we can set $\mathscr{D}_n$ to be the finite linear space of B-spline basis functions over the mesh $E_2^{(n)}$. This allows for the introduction of

$$\mathscr{M}_{n,[-K,K]} = \left\{ \mu \in \mathcal{P}([-K, K] \times U) : \int A f_k d\mu = Rf \quad \forall f_k \in \mathscr{D}_n([-K, K]) \right\}$$

and the following definition.

Definition 2.11. *Let $K > 0$ be a real number. The K-reduced-concentrated, (n, ∞)-dimensional linear program is to find*

$$\min \left\{ J(\mu) | \mu \in \mathscr{M}_{n,[-K,K]} \right\}.$$

Analogous to the considerations for bounded state spaces, we split μ by considering $\mu(dx \times du) = \eta(du, x)p(x)dx$ for a regular conditional probability η, and a L^1-function p. Then, η and p are approximated by $\hat{\eta}_{k_m}$ and $\hat{p}_{k_m}$ as presented in Section III.1.1, according to (1.10) and (1.11). Using the notation

$$\mathscr{M}_{n,m,[-K,K]} = \left\{ \mu_{k_m} \in \mathscr{M}_{n,[-K,K]} : \mu_{0,k_m}(dx \times du) = \hat{\eta}_{k_m}(du, x)\hat{p}_{k_m}(x)dx \right\}.$$

we can define the following, computationally attainable linear program.

Definition 2.12. *Let $K > 0$ be a real number. The K-reduced-concentrated, (n, m)-dimensional linear program is to find*

$$\min \left\{ J(\mu) | \mu \in \mathscr{M}_{n,m,[-K,K]} \right\}.$$

It is crucial to notice that the K-reduced-concentrated (n, ∞)-dimensional linear program has the same structure as the l-bounded, (n, ∞)-dimensional linear program and that the K-reduced-concentrated (n, m)-dimensional linear program has the same structure as the l-bounded, (n, m)-dimensional linear program, both of which were derived in the approximation of bounded state space models in Section III.1.1. In particular, the constraints are given by a finite number n of basis functions in $\mathscr{D}_n([-K, K])$, whose union over all n is dense in $\mathscr{D}([-K, K])$. In case of the (n, m)-dimensional program, the degrees of freedom are represented by the parameter choices in the approximation of η and p. Hence, the computational set up of the linear program and the meshing approaches remain analogous to the case of a bounded state space, compare Section III.1.2 and Section III.1.4. The evaluation of the cost criteria for a fixed control can be conducted in a similar fashion as presented in Section III.1.3. Most importantly, the convergence analysis for finding ϵ-optimal solutions for the K-reduced-concentrated, infinite-dimensional linear program can be carried out in a manner similar to the case of the infinite-dimensional linear program with a bounded state space. It thus remains to investigate how the K-reduced-concentrated, infinite-dimensional linear program relates to the original problem. This is discussed in Section IV.2.

Finally, note that the described technique can also be used in the presence of singular behavior, when the state space is either of the form $E = [e_l, \infty)$ or $E = (-\infty, e_r]$ and the processes is either reflected or jumps back into the interior of E upon entering $\{e_l\}$ or $\{e_r\}$, respectively. In order to do so, one has to introduce the generator of the singular behavior B and the singular expected occupation measure μ_1 into the formulation (2.1), and conduct the approximation scheme as illustrated in Section III.1.1. Again, for the sake of brevity, an explicit discussion of this is omitted.

Convergence Analysis

This chapter presents the convergence analysis of the approximation techniques laid out in Section II.4. First, the convergence analysis for the approximation of models with a bounded state space and the presence of singular behavior at the boundary of the state space is conducted. Most of these considerations can also be applied to analyze the discretization of the K-reduced, infinite-dimensional linear program for unbounded state spaces. Hence, the second part of this chapter will mainly focus on showing that an ϵ-optimal solution of the K-reduced, infinite-dimensional linear program is, 'almost' an optimal solution to the original unbounded problem.

IV.1 Infinite Time Horizon Problems with Bounded State Space

The error analysis for problems with infinite horizon and bounded state space will be conducted in three steps, following the line of thought in which the approximation was presented in Section III.1.1. We treat the general case which involves singular behavior on the boundaries, but the analysis still holds when setting the generator of the singular behavior B, the singular expected occupation measure μ_0 and its approximation $\hat{\mu}_0$ equal to zero. If we do so, the K-reduced infinite-dimensional linear program for unbounded state spaces can be analyzed in the very way presented in the following.

Recall that we seek to solve the infinite dimensional linear program given by

$$\min \{J(\mu_0, \mu_1) | (\mu_0, \mu_1) \in \mathscr{M}_\infty\}. \tag{1.1}$$

As described in Section III.1, the proposed approximation techniques includes the following steps. To start out, we introduced an upper bound on the full mass of μ_1. Then, the first discretization step dealt with finding a finite number of constraints, leading to the (n, ∞)-dimensional problem. The second discretization step introduced an approximation of the measures we are considering, defining the (n, m)-dimensional problem.

Theorems II.1.42 and II.1.43 ensured the existence of a minimizer in (1.1). Throughout this section, we refer to this minimizer by $J^* = \min \{J(\mu_0, \mu_1) | (\mu_0, \mu_1) \in \mathscr{M}_\infty\}$. However, this minimizer might not be attainable using the proposed discretization and we retreat to the slightly relaxed optimization problem of finding a ϵ-optimal solution to this problem in the sense of finding a pair of measures $(\mu_0^\epsilon, \mu_1^\epsilon) \in \mathscr{M}_\infty$ such that

$$J(\mu_0^\epsilon, \mu_1^\epsilon) - J^* \leq \epsilon.$$

Note that $\mathscr{M}_{n,m}^l \subset \mathscr{M}_n^l$ is the only space that is computationally attainable. But since $\mathscr{M}_n^l \supset \mathscr{M}_\infty^l$, and $\mathscr{M}_\infty^l \subset \mathscr{M}_\infty$, it is not clear how $\mathscr{M}_{n,m}^l$ relates to $\mathscr{M}_\infty$. The following results are shown to ensure that measures in $\mathscr{M}_{n,m}^l$ are close to the optimal solution in $\mathscr{M}_\infty$, and that we can find ϵ-optimal solutions to the infinite dimensional linear program by finding the optimal solutions to the l-bounded (n, m)-dimensional linear program. We show that

1. An ϵ-optimal solution to the l-bounded infinite-dimensional problem is a 2ϵ-optimal solution for the infinite-dimensional problem for l large enough (Theorem 1.5),

2. An ϵ-optimal solution to the (n, ∞)-dimensional problem is $2\epsilon + \delta$-optimal solution to the infinite dimensional problem, for some $\delta > 0$ and n large enough (Section IV.1.1),

3. An optimal solution to the (n, m)-dimensional problem will be an ϵ-optimal solution to the (n, ∞)-dimensional problem for m large enough (Section IV.1.2),

4. Considering a fixed m and two fixed relaxed controls η_0 and η_1, influencing the continuous and singular behavior, respectively, as well as n large enough, the approximate cost criterion value computed using n constraint functions is within ϵ of the actual value of the cost criterion value when the obtained control is implemented, (Section IV.1.3).

Note that the first three items on this list show that we can approximate the optimal cost criterion value with the proposed approximation. In order to check if the control suggested by the approximation scheme indeed controls a stochastic process in an almost optimal way, we need to evaluate the cost criterion with a large value of n for fixed controls and compare it to the value obtained when solving the linear program.

The first item on this list is easily shown and does not require its own subsection. But to begin, we need a precise definition of ϵ-optimality in different contexts.

Definition 1.2. *An ϵ-optimal solution to the infinite-dimensional linear program is a pair of measures $(\mu_0^\epsilon, \mu_1^\epsilon) \in \mathscr{M}_\infty$ such that*

$$J(\mu_0^\epsilon, \mu_1^\epsilon) - J^* \leq \epsilon.$$

Remark 1.3. *Observe that $(\mu_0^\epsilon, \mu_1^\epsilon)$ being ϵ-optimal is equivalent to*

$$J(\mu_0^\epsilon, \mu_1^\epsilon) - J(\mu_0, \mu_1) \leq \epsilon \qquad \text{for all } (\mu_0, \mu_1) \in \mathscr{M}_\infty.$$

Remark 1.4. *In the same way as seen in Definition 1.2, we can define ϵ-optimality for the l-bounded infinite dimensional linear program, the l-bounded (n, ∞)-dimensional linear program and the l-bounded (n, m)-dimensional linear program, by replacing $\mathscr{M}_\infty$ by $\mathscr{M}^l$, $\mathscr{M}_n^l$ or $\mathscr{M}_{n,m}^l$, respectively.*

Theorem 1.5. *There is an $l_0 > 0$ such that for all $l \geq l_0$, an ϵ-optimal solution to the l-bounded infinite-dimensional problem is a 2ϵ-optimal solution for the infinite-dimensional problem.*

Proof. By the theory of the linear programming approach (compare Kurtz and Stockbridge (2017)), we know that any solution to the infinite-dimensional linear program satisfies $\mu_1(E \times U) < \infty$. So, in particular, an ϵ-optimal solution $(\mu_0^\epsilon, \mu_1^\epsilon)$ to the infinite-dimensional linear program has a singular expected occupation measure μ_1^ϵ with full mass $\mu_1^\epsilon(E \times U) \leq l_0$ for some $l_0 > 0$. So, if $l \geq l_0$, $\mathscr{M}_\infty^l$ contains an ϵ-optimal solution to the infinite-dimensional problem. Obviously, such a solution, denoted $(\mu_0^\epsilon, \mu_1^\epsilon)$, is also an ϵ-optimal solution for the l-bounded infinite dimensional problem. Now consider an arbitrary (and possibly different) ϵ-optimal solution to the l-bounded infinite dimensional problem, denoted $(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon})$, and assume it would not be a 2ϵ-optimal solution to the infinite dimensional problem. This means there would be a pair of measures $(\mu_0, \mu_1) \in \mathscr{M}_\infty$ such that

$$J(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon}) - J(\mu_0, \mu_1) > 2\epsilon \Leftrightarrow J(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon}) > J(\mu_0, \mu_1) + 2\epsilon$$

Note, we have that $|J(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon}) - J(\mu_0^\epsilon, \mu_1^\epsilon)| \leq \epsilon$, since both pairs of measures are ϵ-optimal. Hence we can deduce

$$\begin{aligned} J(\mu_0, \mu_1) + 2\epsilon &< J(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon}) \leq |J(\mu_0^{l,\epsilon}, \mu_1^{l,\epsilon}) - J(\mu_0^\epsilon, \mu_1^\epsilon)| + |J(\mu_0^\epsilon, \mu_1^\epsilon)| \\ &< \epsilon + J(\mu_0^\epsilon, \mu_1^\epsilon) \\ \Leftrightarrow J(\mu_0, \mu_1) + \epsilon &< J(\mu_0^\epsilon, \mu_1^\epsilon) \end{aligned}$$

which would contradict $(\mu_0^\epsilon, \mu_1^\epsilon)$ being ϵ-optimal. □

Remark 1.6. *In the absence of singular behavior, that is if B is identical to zero, this part of the convergence proof is not needed.*

Remark 1.7. *This analysis addresses an important detail omitted in the proof of Theorem 3.1.2 of Rus (2009), and explains the introduction of the bound l in the full mass of the singular expected occupation measure μ_1. To be precise, we have to rely on results stating that a tight sequence of measures always contains a weakly convergent subsequence. Classical results regarding this situation can be found in Billingsley (1999), but they assume that the considered sequence of measures only contains probability measures, which have full mass of 1. As any singular expected occupation measure μ_1 is not a probability measure, a sequence approximating μ_1 does not necessarily consist of probability measures. The more general theory of Bogachev (2007) has to be applied, as reviewed in Section II.3. However, respective results assume the existence of a bound on the full mass of μ_1.*

IV.1.1 Optimality of the l-bounded (n, ∞)-dimensional problem

Recall that the l-bounded infinite dimensional linear program poses

$$\begin{array}{ll} \text{Minimize} & \displaystyle\int_{E\times U} c_0 d\mu_0 + \int_{E\times U} c_1 d\mu_1 \\ \text{Subject to} & \begin{cases} \displaystyle\int Af d\mu_0 + \int Bf d\mu_1 = Rf \quad \forall f \in C_c^2(E) \\ \mu_0 \in \mathcal{P}(E\times U) \\ \mu_1 \in \mathcal{M}^l(E\times U) \end{cases} \end{array}$$

We will use the notation

$$\mathscr{M}_\infty^l = \left\{ (\mu_0, \mu_1) \in \mathcal{P}(E\times U) \times \mathcal{M}^l(E\times U) : \int Af d\mu_0 + \int Bf d\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty \right\}$$

and

$$J : \mathcal{P}(E\times U) \times \mathcal{M}^l(E\times U) \ni (\mu_0,\mu_1) \mapsto J(\mu_0,\mu_1) = \int_{E\times U} c_0 d\mu_0 + \int_{E\times U} c_1 d\mu_1 \in \mathbb{R}_{\geq 0}$$

to refer to the feasible set of the l-bounded infinite dimensional linear program and the cost criterion. As seen in Section II.1, Propositions II.1.18 and II.1.41 the operators and the right hand side functional

$$\begin{aligned} A &: \mathscr{D}_\infty \mapsto (C_b^u(E\times U), \|\cdot\|_\infty) \\ B &: \mathscr{D}_\infty \mapsto (C_b^u(E\times U), \|\cdot\|_\infty) \\ R &: \mathscr{D}_\infty \mapsto \mathbb{R} \end{aligned}$$

are linear and continuous. Thus $A|_{\mathscr{D}_n}$, $B|_{\mathscr{D}_n}$ are also continuous linear operators for each n, and $R|_{\mathscr{D}_n}$ is a continuous linear functional for each n. Note that as E and U are compact, we have that $C_b(E\times U) = C_b^u(E\times U)$. We also use, for a countable basis $\{f_k\}_{k\in\mathbb{N}}$ of $\mathscr{D}_\infty$, the notation

$$\mathscr{M}_n^l = \left\{(\mu_0,\mu_1)\in\mathcal{P}(E\times U)\times\mathcal{M}^l(E\times U) : \int Af_k d\mu_0 + \int Bf_k d\mu_1 = Rf_k, k=1,\ldots,n\right\}.$$

Note that $\mathscr{M}_\infty^l \subset \mathscr{M}_n^l$ for any $n\in\mathbb{N}$.

The symbol "$\Rightarrow$" will be used to denote weak convergence of measures in the sequel. We start the convergence analysis by considering arbitrary sequences of measures in $\mathscr{M}_n^l$, and later move to sequences of ϵ-optimal measures.

Proposition 1.8. *Let $\{(\mu_{0,n},\mu_{1,n})\}_{n\in\mathbb{N}}$ be a sequence of measures such that $(\mu_{0,n},\mu_{1,n})\in\mathscr{M}_n^l$ for all $n\in\mathbb{N}$ and assume that $\mu_{0,n}\Rightarrow\mu_0$ and $\mu_{1,n}\Rightarrow\mu_1$ for some $(\mu_0,\mu_1)\in\mathcal{P}(E\times U)\times\mathcal{M}^l(E\times U)$. Then, $(\mu_0,\mu_1)\in\mathscr{M}_\infty^l$.*

Proof. For arbitrary $m\in\mathbb{N}$, take $g_m\in\mathscr{D}_m$. For $n>m$, it is true that

$$\int Ag_m d\mu_{0,n} + \int Bg_m d\mu_{1,n} = Rg_m$$

By assumption on A and B, the functions Ag_m and Bg_m are bounded and uniformly continuous (compare Remark II.1.19), so by weak convergence of the involved measures, we have

$$
\begin{aligned}
Rg_m = \lim_{n\to\infty} Rg_m &= \lim_{n\to\infty}\left(\int Ag_m d\mu_{0,n} + \int Bg_m d\mu_{1,n}\right) \\
&= \int Ag_m d\mu_0 + \int Bg_m d\mu_1.
\end{aligned}
$$

For $g \in \mathscr{D}_\infty$ pick a sequence $g_m \to g$ in $\mathscr{D}_\infty$ with $g_m \in \mathscr{D}_m$ for all $m \in \mathbb{N}$. By the continuity of the involved operators A and B and functional R, we have that

$$
\lim_{m\to\infty} Ag_m = Ag, \qquad \lim_{m\to\infty} Bg_m = Bg \qquad \text{and} \qquad \lim_{m\to\infty} Rg_m = Rg
$$

Further, $\{Ag_m\}_{m\in\mathbb{N}}$ and $\{Bg_m\}_{m\in\mathbb{N}}$ are convergent sequences in $C_b^u(E\times U, \|\cdot\|_\infty)$ by Proposition II.1.18 and are hence uniformly bounded. This allows for the application of the bounded convergence theorem in the following equation.

$$
\int Ag d\mu_0 + \int Bg d\mu_1 = \lim_{m\to\infty}\left(\int Ag_m d\mu_0 + \int Bg_m d\mu_1\right).
$$

Finally, we have that

$$
\lim_{m\to\infty}\left(\int Ag_m d\mu_0 + \int Bg_m d\mu_1\right) = \lim_{m\to\infty} Rg_m = Rg
$$

which implies that $(\mu_0, \mu_1) \in \mathscr{M}_\infty$. Note that $I_{E\times U}$ is bounded and uniformly continuous on $E \times U$, so by weak convergence, we have

$$
\mu_1(E\times U) = \int_{E\times U} I_{E\times U}\, d\mu_1 = \lim_{n\to\infty}\int_{E\times U} I_{E\times U}\, d\mu_{1,n} = \lim_{n\to\infty}\mu_{1,n}(E\times U) \le l,
$$

as $\mu_{1,n}(E\times U) \le l$ for all $n \in \mathbb{N}$, which shows that $(\mu_0, d\mu_1) \in \mathscr{M}_\infty^l$. □

Remark 1.9. *The idea of this proof, and similar results in the sequel is taken from Rus (2009), in particular Theorem 3.1.2. However, the application of the dominated convergence* *theorem in Rus' proof cannot clearly be justified with the Hermite polynomial basis functions which are used to approximate the constraints therein. In contrast, our choice of B-spline basis functions guarantees the applicability of the dominated convergence argument.*

Lemma 1.10. *For each $n \in \mathbb{N}$, assume that $(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n}) \in \mathscr{M}^l_n$ and that $(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n})$ is an ϵ-optimal solution for the l-bounded (n,∞)-dimensional problem. Assume that $\mu^\epsilon_{0,n} \Rightarrow \hat{\mu}_0$ and $\mu^\epsilon_{1,n} \Rightarrow \hat{\mu}_1$ for some $\hat{\mu}_0 \in \mathcal{P}(E \times U)$, $\hat{\mu}_1 \in \mathcal{M}^l(E \times U)$. Then, $(\hat{\mu}_0, \hat{\mu}_1)$ is an ϵ-optimal solution to the l-bounded infinite-dimensional problem.*

Proof. By Proposition 1.8, we know that $(\hat{\mu}_0, \hat{\mu}_1) \in \mathscr{M}^l_\infty$. Assume $(\hat{\mu}_0, \hat{\mu}_1)$ would not be ϵ-optimal. Then, there exists an $(\mu_0, \mu_1) \in \mathscr{M}^l_\infty$ such that

$$\begin{aligned} J(\hat{\mu}_0, \hat{\mu}_1) &> J(\mu_0, \mu_1) + \epsilon. \qquad (1.11) \\ \Leftrightarrow \int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 &> \int c_0 d\mu_0 + \int c_1 \mu_1 + \epsilon. \end{aligned}$$

As c_0 and c_1 are continuous functions over a compact set, and hence uniformly continuous and bounded, by weak convergence, we know that $\int c_0 d\mu^\epsilon_{0,n} \to \int c_0 d\hat{\mu}_0$ and $\int c_1 d\mu^\epsilon_{1,n} \to \int c_1 d\hat{\mu}_1$ in $\mathbb{R}$, as $n \to \infty$. So, there is an N large enough such that for all $n \geq N$

$$\int c_0 d\mu^\epsilon_{0,n} + \int c_1 d\mu^\epsilon_{1,n} > \int c_0 d\mu_0 + \int c_1 d\mu_1 + \epsilon.$$

But because $\mathscr{M}^l_\infty \subset \mathscr{M}^l_n$, this contradicts the fact that $(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n})$ is ϵ-optimal, and (1.11) is false. This implies that

$$J(\hat{\mu}_0, \hat{\mu}_1) \leq J(\mu_0, \mu_1) + \epsilon$$

for all $(\mu_0, \mu_1) \in \mathscr{M}^l_\infty$. $\square$

The following result will show that a sequence of ϵ-optimal sequences of measures generates a sequence of values for the cost criterion which lies inside of a certain interval, independent of the sequence of measures converging or not.

Lemma 1.12. *For each $n \in \mathbb{N}$, assume that $(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n}) \in \mathscr{M}^l_n$ and that for $n \in \mathbb{N}$, $(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n})$ is an ϵ-optimal solution for the l-bounded (n, ∞)-dimensional problem. Then, for $\delta > 0$, there exists a $z \in \mathbb{R}$ and an $N(\delta) \in \mathbb{N}$ such that*

$$J(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n}) \in \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta\right) \qquad n \geq N(\delta)$$

Proof. Observe that as $E \times U$ is compact, both $\{\mu^\epsilon_{0,n}\}_{n\in\mathbb{N}}$ and $\{\mu^\epsilon_{1,n}\}_{n\in\mathbb{N}}$ are tight sequences of measures, and both can be uniformly bounded. So, if $\{\mu^\epsilon_{0,n_j}\}_{j\in\mathbb{N}}$ is a convergent subsequence, which is guaranteed to exist due to the tightness (see Remark II.3.7), $\{\mu^\epsilon_{1,n_j}\}_{j\in\mathbb{N}}$ is still a tight sequence and furthermore its mass is uniformly bounded by l. Hence it has a convergent sub-subsequence $\{\mu^\epsilon_{1,n_{j_k}}\}_{k\in\mathbb{N}}$ by Theorem II.3.6 and $\{\mu^\epsilon_{0,n_{j_k}}\}_{k\in\mathbb{N}}$ is still convergent. Thus, we can consider two convergent subsequences of $\{(\mu^\epsilon_{0,n}, \mu^\epsilon_{1,n})\}_{n\in\mathbb{N}}$, denoted $\{(\mu^\epsilon_{0,n_j}, \mu^\epsilon_{1,n_j})\}_{j\in\mathbb{N}}$ and $\{(\mu^\epsilon_{0,n'_j}, \mu^\epsilon_{1,n'_j})\}_{j\in\mathbb{N}}$. Assume that

$$\mu^\epsilon_{0,n_j} \Rightarrow \hat{\mu}_0, \quad \mu^\epsilon_{1,n_j} \Rightarrow \hat{\mu}_1, \quad \mu^\epsilon_{0,n'_j} \Rightarrow \tilde{\mu}_0, \quad \mu^\epsilon_{1,n'_j} \Rightarrow \tilde{\mu}_1$$

and that $\hat{\mu}_0 \neq \tilde{\mu}_0$ or $\hat{\mu}_1 \neq \tilde{\mu}_1$. Assume that

$$\int c_0 d\tilde{\mu}_0 + \int c_1 d\tilde{\mu}_1 > \int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 + \epsilon. \tag{1.13}$$

Choose $N \in \mathbb{N}$ large enough such that $\forall j \geq N$,

$$\int c_0 d\mu^\epsilon_{0,n'_j} + \int c_1 d\mu^\epsilon_{1,n'_j} > \int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 + \epsilon$$

which is possible because

$$\int c_0 d\mu^\epsilon_{0,n'_j} + \int c_1 d\mu^\epsilon_{1,n'_j} \to \int c_0 d\tilde{\mu}_0 + \int c_1 d\tilde{\mu}_1 \text{ as } j \to \infty,$$

which again is due to to the fact that c_0 and c_1 are uniformly continuous as they are continuous maps over a compact set. However, this clearly contradicts the assumption that $(\mu^{\epsilon}_{0,n'_j}, \mu^{\epsilon}_{1,n'_j})$ is ϵ-optimal, as $(\hat{\mu}_0, \hat{\mu}_1) \in \mathscr{M}^l_{n_j}$ for all $j \in \mathbb{N}$. Hence, (1.13) is false, and in particular, we have that

$$\int c_0 d\tilde{\mu}_0 + \int c_1 d\tilde{\mu}_1 \leq \int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 + \epsilon$$

Similarly, we can show that

$$\int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 \leq \int c_0 d\tilde{\mu}_0 + \int c_1 d\tilde{\mu}_1 + \epsilon$$

from which we can conclude that

$$\left| \int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 - \left(\int c_0 d\tilde{\mu}_0 + \int c_1 d\tilde{\mu}_1 \right) \right| \leq \epsilon.$$

In other words, the cost criterion values of any two limits of convergent subsequences of $(\{\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n})\}_{n\in\mathbb{N}}$ are only ϵ apart. Hence, there exists a $z \in \mathbb{R}$ such that

$$J(\hat{\mu}_0, \hat{\mu}_1) \in \left[z - \frac{\epsilon}{2}, z + \frac{\epsilon}{2} \right]$$

for any limit $(\hat{\mu}_0, \hat{\mu}_1)$ of a convergent subsequence of $\{(\mu_{0,n}, \mu_{1,n})\}_{n\in\mathbb{N}}$. For $\delta > 0$ it is therefore obviously true that

$$\int c_0 d\hat{\mu}_0 + \int c_1 d\hat{\mu}_1 \in \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta \right).$$

Now assume that there is a non-convergent subsequence $\{(\mu^{\epsilon}_{0,n_j}, \mu^{\epsilon}_{1,n_j})\}_{j\in\mathbb{N}}$ such that for any given $N \in \mathbb{N}$, there is a $j \geq N$ with

$$\int c_0 d\mu^{\epsilon}_{0,n_j} + \int c_1 d\mu^{\epsilon}_{1,n_j} \notin \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta \right) \tag{1.14}$$

Thus, there exists a sub-subsequence $\{(\mu^{\epsilon}_{0,n_{j_k}}, \mu^{\epsilon}_{1,n_{j_k}})\}_{k\in\mathbb{N}}$ with

$$\int c_0 d\mu^{\epsilon}_{0,n_{j_k}} + \int c_1 d\mu^{\epsilon}_{1,n_{j_k}} \notin \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta\right) \qquad \forall k \in \mathbb{N}.$$

This sequence is tight and uniformly bounded, and hence contains a convergent 'sub-sub'-subsequence $\{(\mu^{\epsilon}_{0,n_{j_{k_l}}}, \mu^{\epsilon}_{1,n_{j_{k_l}}})\}_{l\in\mathbb{N}}$ with $\mu^{\epsilon}_{0,n_{j_{k_l}}} \Rightarrow \hat{\mu}_0$ and $\mu^{\epsilon}_{1,n_{j_{k_l}}} \Rightarrow \hat{\mu}_1$, But then,

$$J(\hat{\mu}_0, \hat{\mu}_1) \in \left[z - \frac{\epsilon}{2}, z + \frac{\epsilon}{2}\right]$$

and hence there is an N large enough such that for $l \geq N$

$$\int c_0 d\mu^{\epsilon}_{0,n_{j_{k_l}}} + \int c_1 d\mu^{\epsilon}_{1,n_{j_{k_l}}} \in \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta\right)$$

which contradicts the construction of $\{(\mu^{\epsilon}_{0,n_{j_k}}, \mu^{\epsilon}_{1,n_{j_k}})\}_{k\in\mathbb{N}}$. So, the assumption of (1.14) is false, and the claim is proven. $\square$

Remark 1.15. *This treatment of diverging subsequences presents a novelty in the convergence analysis in comparison to what was established in Rus (2009). In particular, the proof of Theorem 3.1.2 therein omits the fact that a sequence of tight measures can have diverging subsequences, and only the existence of a converging subsequence is guaranteed by results as presented in II.3.*

While Lemma 1.12 gives us some information about the location of the value of the cost criterion of ϵ-optimal sequences, the next result relates the infimum of the cost criterion, taken over all measures in $\mathscr{M}^l_\infty$, to this interval.

Proposition 1.16. *Let* $\left[z - \frac{\epsilon}{2}, z + \frac{\epsilon}{2}\right]$ *be the interval from Lemma 1.12, and set*

$$J^{l,*} = \inf\left(J(\mu_0, \mu_1) : (\mu_0, \mu_1) \in \mathscr{M}^l_\infty\right)$$

Then,

$$z - \frac{3\epsilon}{2} \leq J^{l,*} \leq z + \frac{\epsilon}{2}$$

Proof. Take a sequence $\{(\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n})\}_{n\in\mathbb{N}}$ of ϵ-optimal solutions to $\mathscr{M}^l_n$, respectively, and consider a convergent subsequence $\{(\mu^{\epsilon}_{0,n_j}, \mu^{\epsilon}_{1,n_j})\}_{j\in\mathbb{N}}$ with limit $(\hat{\mu}^{\epsilon}_0, \hat{\mu}^{\epsilon}_1)$. Then, by Lemma 1.10 $(\hat{\mu}^{\epsilon}_0, \hat{\mu}^{\epsilon}_1)$ is ϵ-optimal and

$$J(\hat{\mu}^{\epsilon}_0, \hat{\mu}^{\epsilon}_1) \in \left[z - \frac{\epsilon}{2}, z + \frac{\epsilon}{2}\right]$$

holds by the proof of Lemma 1.12. Hence it is true that

$$J^{l,*} \leq J(\hat{\mu}_0, \hat{\mu}_1) \leq z + \frac{\epsilon}{2}.$$

On the other hand, as $(\hat{\mu}_0, \hat{\mu}_1)$ is optimal,

$$\begin{aligned} J^{l,*} + \epsilon &\geq J(\hat{\mu}^{\epsilon}_0, \hat{\mu}^{\epsilon}_1) \geq z - \frac{\epsilon}{2} \\ \Leftrightarrow J^{l,*} &\geq z - \frac{3\epsilon}{2}, \end{aligned}$$

which proves the claim. □

The following theorem summarizes the obtained results.

Theorem 1.17. *For each $n \in \mathbb{N}$, assume that $(\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n}) \in \mathscr{M}^l_n$ and that for $n \in \mathbb{N}$, $(\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n})$ is an ϵ-optimal solution for the (n,∞)-dimensional problem. Then, for $\delta > 0$, there exists an $N(\delta)$ such that*

$$|J(\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n}) - J^{l,*}| \leq 2\epsilon + \delta.$$

for all $n \geq N(\delta)$.

Proof. Select N large enough such that

$$J(\mu^{\epsilon}_{0,n}, \mu^{\epsilon}_{1,n}) \in \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta\right) \qquad n \geq N$$

which is possible due to Lemma 1.12. By Proposition 1.16, we have that

$$J(\mu_{0,n}, \mu_{1,n}) - J^{l,*} \leq z + \frac{\epsilon}{2} + \delta - \left(z - \frac{3\epsilon}{2}\right) = 2\epsilon + \delta$$

and

$$J^{l,*} - J(\mu_{0,n}, \mu_{1,n}) \leq z + \frac{\epsilon}{2} - \left(z - \frac{\epsilon}{2} - \delta\right) = \epsilon + \delta,$$

from which the claim follows. □

Remark 1.18. *In conjunction with Theorem 1.5, an $\bar{\epsilon}$-optimal value of the infinite dimensional linear program can be found as follows. Select $l > 0$ large enough such that an $\frac{\bar{\epsilon}}{2}$-optimal solution in $\mathscr{M}_\infty^l$ is an $\bar{\epsilon}$-optimal solution in $\mathscr{M}_\infty$. Take $\epsilon > 0$ and $\delta > 0$ such that $2\epsilon + \delta \leq \frac{\bar{\epsilon}}{2}$. Use Proposition 1.16 to pick $n \in \mathbb{N}$ large enough such that an $\bar{\epsilon}$-optimal solution in $\mathscr{M}_n^l$ is a $2\epsilon + \delta$-optimal solution in $\mathscr{M}_\infty^l$. Then, find an ϵ-optimal solution in $\mathscr{M}_n^l$.*

While this remark illustrates how the main result in this section serves in the overall convergence analysis, the next two remarks illustrate how the given derivation can be applied to models without singular behavior, and to the K-reduced (n, ∞)-dimensional linear program that is used to approximate models with a unbounded state space.

Remark 1.19. *This part of the convergence proof can be generalized to the case of no singular behavior by setting $B \equiv 0$, as well as c_1 and $\mu_{1,n}$ (or any other considered singular expected occupation measure) to zero. The statement of Remark 1.18 remains identical.*

Remark 1.20. *Having made the observation in Remark 1.19, we can see that the analysis carried out here applies to the optimality of the K-reduced, (n, ∞)-dimensional linear program for unbounded state spaces. Indeed, as the arguments presented rely only on the fact that the set of constraint functions $\{f_k\}_{k \in \mathbb{N}}$ forms a basis of $\mathscr{D}_\infty$, or $\mathscr{D}_\infty([-K, K])$ in the case of an unbounded state space, and on the fact that μ_1, or μ in the case of an unbounded state space, has full mass on a closed interval.*

IV.1.2 Optimality of the l-bounded (n, m)-dimensional problem

The previous subsection ended by stating that an ϵ-optimal solution to the l-bounded (n,∞)-dimensional problem is almost optimal, for n and l large enough. We now show that an optimal solution to the l-bounded (n, m)-dimensional problem is an ϵ-optimal solution to the l-bounded (n,∞)-dimensional problem, for m large enough. Throughout this section, the number of constraints n will be fixed.

Key in proving this claim is the fact that we can approximate the cost functional of arbitrary measures in $\mathscr{M}_n^l$ by discrete measures in $\mathscr{M}_{n,m}^l$. To be precise, we seek to show that for any $(\mu_0, \mu_1) \in \mathscr{M}_n^l$ and $\epsilon > 0$, there is a pair of discretized measures $(\hat{\mu}_{0,k_m}, \hat{\mu}_{1,k_m}) \in \mathscr{M}_{n,m}^l$ such that, recalling that we assume $\mu_{0,E}(dx) = p(x)dx$ for some $p \in L^1(E)$,

$$\begin{aligned} \left| \int_E \int_U c_0(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| &< \frac{\epsilon}{2} \\ \left| \int_{U\times E} c_1(x,u)\mu_1(dx \times du) - \int_E \int_U c_1(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \right| &< \frac{\epsilon}{2}, \end{aligned}$$

where $\hat{\mu}_{0,k_m}(dx \times du) = \hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx$ and $\hat{\mu}_{1,k_m}(dx \times du) = \hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx)$, using the approximation defined in (III.1.10), (III.1.11) and (III.1.13). This implies that

$$|J(\mu_0, \mu_1) - J(\hat{\mu}_{0,k_m}, \hat{\mu}_{1,k_m})| < \epsilon.$$

To achieve this, some technical considerations are in order.

Lemma 1.21. *Assume $a < b$ and $c < d$ are real numbers. Let $p : [a, b] \mapsto [0,\infty)$ be a probability density function. Let $f : [a, b] \mapsto [c, d]$ be a function that is continuous almost everywhere with respect to the measure $p(x)\,dx$. Let*

$$\{x_k = a + \frac{b-a}{2^n} \cdot k, k = 0, \ldots, 2^n\}.$$

Set

$$\hat{f}_n(x) = \sum_{k=0}^{2^n-1} f(x_k) I_{[x_k, x_{k+1})}(x) + f(x_{2^n-1}) \cdot I_{\{x_{2^n}\}}(x).$$

Then, for any $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that for all* $n \geq N$*,*

$$\int |f(x) - \hat{f}_n(x)| p(x) dx < \epsilon.$$

Proof. We will use the fact that $F = \{x \in [a, b] : f \text{ is discontinuous at } x\}$ is a $p(x)dx$-null set, and thus

$$\int_{[a,b]} |f(x) - \hat{f}_n(x)| p(x) dx = \int_{[a,b) \cap F^C} |f(x) - \hat{f}_n(x)| p(x) dx.$$

So, take $\epsilon > 0$ and pick $x \in [a, b) \cap F^C$ arbitrarily. Then, there is a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Choose n large enough such that $\frac{1}{2^n} < \delta$. Then, since $x \in [x_k, x_{k+1}]$ for some k, we have

$$|\hat{f}_n(x) - f(x)| = |f(x_k) - f(x)| < \epsilon,$$

since $|x_k - x| < \delta$. But this means that $\hat{f}_n \to f$ pointwise, p-almost everywhere. As $|f(x) - \hat{f}_n(x)| < [d - c]$ for all $x \in [a, b] \cap F^C$, the bounded convergence theorem implies that there is an $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\int |f(x) - \hat{f}_n(x)| p(x) dx < \epsilon,$$

which proves the claim. $\square$

In the current situation, the number n of test functions in $\mathscr{D}_n$ representing the constraints is fixed. Hence, we can prove the following two results, that will be needed to analyze the approximation of the cost criterion on the one hand, and the extent to which the constraints are fulfilled on the other hand.

Lemma 1.22. *For $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\max\{|c_0(x,u) - c_0(x,v)|, |Af_1(x,u) - Af_1(x,v)|, \ldots, |Af_n(x,u) - Af_n(x,v)|\} < \epsilon$$

holds whenever $|u - v| < \delta$, uniformly in x.

Proof. As c_0 is continuous on a compact set, there is a $\delta_1 > 0$ such that if $|u - v| < \delta_1$, then $|c_0(x,u) - c_0(x,v)| < \epsilon$. For $k = 1, 2, \ldots, n$ we have in the case of the long-term average criterion that

$$Af_k(x,u) = f_k'(x) \cdot b(x,u) + \frac{1}{2} f_k''(x)\sigma^2(x,u).$$

In the case of the infinite-horizon discounted criterion with discounting rate α, we have

$$Af_k(x,u) = f_k'(x) \cdot b(x,u) + \frac{1}{2} f_k''(x)\sigma^2(x,u) - \alpha f(x)$$

In both cases

$$|Af_k(x,u) - Af_k(x,v)| \leq |f_k'(x)| \cdot |b(x,u) - b(x,v)| + \frac{1}{2} f_k''(x)|\sigma^2(x,u) - \sigma^2(x,v)|$$

holds. For each of the finitely many k, f_k' and f_k'' are continuous on a compact set, and hence the expressions $f_k'(x)$ and $\frac{1}{2} f_k''(x)$ can be bounded uniformly in x and k, say by a constant $C > 0$. Choose δ_2 with $0 < \delta_2 < \delta_1$ such that $|b(x,u) - b(x,v)| < \frac{\epsilon}{2C}$ whenever $|u - v| < \delta_2$ and δ_3 with $0 < \delta_3 < \delta_2$ such that $|\sigma^2(x,u) - \sigma^2(x,v)| < \frac{\epsilon}{2C}$ whenever $|u - v| < \delta_3$. This again is possible since b and σ^2 are continuous functions on a compact set. But then,

$$|Af_k(x,u) - Af_k(x,v)| \leq C \cdot \big(|b(x,u) - b(x,v)| + |\sigma^2(x,u) - \sigma^2(x,v)|\big) < C\left(\frac{\epsilon}{2C} + \frac{\epsilon}{2C}\right) = \epsilon.$$

Set $\delta_3 = \delta$ to complete the proof. $\square$

Lemma 1.23. *Let $s_1, \ldots, s_N$ be the points of singular behavior. For $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\max\{|c_1(x,u) - c_1(x,v)|, |Bf_1(s_j,u) - Bf_1(s_j,v)|, \ldots, |Bf_n(s_j,u) - Bf_n(s_j,v)|\} < \epsilon$$

holds whenever $|u - v| < \delta$, independent of $j = 1, \ldots N$.

Proof. As c_1 is continuous on a compact set, there is a $\delta_1 > 0$ such that if $|u - v| < \delta_1$, then $|c_1(x,u) - c_1(x,v)| < \epsilon$. If the singular behavior is given by reflections, we are done at this point, setting $\delta = \delta_1$, since we do not consider controlled reflections. If the singular behavior is given by a jump from s_j to $s_j + h(s_j,u)$, the following analysis holds. For $k = 1, 2, \ldots, n$

$$Bf_k(s_j,u) = f_k(s_j + h(s_j,u)) - f_k(s_j)$$

and thereby

$$\begin{aligned} |Bf_k(x,u) - Bf_k(x,v)| &= |f_k(s_j + h(s_j,u)) - f(s_j) - f_k(s_j + h(s_j,v)) + f(s_j)| \\ &\leq |f_k(s_j + h(s_j,u)) - f_k(s_j + h(s_j,v))| \end{aligned}$$

holds. Each of the f_k are continuous, so there is a $\theta > 0$ such that $|f_k(x) - f_k(y)| < \frac{\epsilon}{N}$ if $|x - y| < \theta$ uniformly in k. Pick $\delta_2 > 0$ small enough such that for all $|u - v| < \delta_2$, $|h(s_j,u) - h(s_j,v)| < \theta$ uniformly in j. This is possible due to the continuity of h and the fact that we only consider finitely many points s_j. Set $\delta_2 = \delta$ to complete the proof. $\square$

The next lemmas examine the properties of the proposed approximation of a control η_0 of the form (III.1.11) and η_1 of the form (III.1.13). We will consider specific approximations according to the following definitions.

Definition 1.24. *For a discretization level k_m, the usual approximation of a control on the continuous part η_0 given by the general form*

$$\hat{\eta}_{0,k_m}(V,x) = \sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \beta_{j,i} I_{E_j}(x) \delta_{u_i}(V)$$

is defined by setting $U_i = [u_i, u_{i+1})$, $U_{2^{k_m^{(2)}}} = \{u_{2^{k_m^{(2)}}}\}$ and $\beta_{j,i} := \eta_0(U_i, x_j) = \int_{U_i} \eta_0(du, x_j)$ for $i = 0, 1, \ldots, 2^{k_m^{(2)}} - 1$ and $j = 0, 1, \ldots, 2^{k_m^{(1)}}$.

Definition 1.25. *For a discretization level k_m, the usual approximation of a control on the singular part η_1 given by the general form,*

$$\hat{\eta}_{1,k_m}(V, s_j) = \sum_{i=0}^{2^{k_m^{(2)}}} \zeta_{j,i} \delta_{u_i}(V), \tag{1.26}$$

is defined by setting $U_i = [u_i, u_{i+1})$, $U_{2^{k_m^{(2)}}} = \{u_{2^{k_m^{(2)}}}\}$ and $\zeta_{j,i} := \eta_1(U_i, s_j) = \int_{U_i} \eta_1(du, s_j)$ for $i = 0, 1, \ldots, 2^{k_m^{(2)}} - 1$ and $j = 1, \ldots, N$.

Lemma 1.27. *Consider a relaxed control given by a regular conditional probability η_0, and a probability density function p. Let $g(x,u) = c_0(x,u)$ or $g(x,u) = Af_k(x,u)$ for any $k = 1, 2, \ldots, n$. For $\epsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,*

$$\left| \int_E \int_U g(x,u) \eta_0(du, x) p(x) dx - \int_E \int_U g(x,u) \hat{\eta}_{0,k_m}(du, x) p(x) dx \right| < \epsilon,$$

where $\hat{\eta}_{0,k_m}$ is given by the usual approximation of η_0, as defined by Definition 1.24.

Proof. For the usual approximation of η_0, observe that

$$\begin{aligned} |I| &\equiv \left| \int_E \int_U g(x,u) \eta_0(du,x) p(x) dx - \int_E \int_U g(x,u) \hat{\eta}_{0,k_m}(du,x) p(x) dx \right| \\ &= \left| \int_E \int_U g(x,u) \eta_0(du,x) p(x) dx - \int_E \left(\sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \beta_{j,i} I_{E_j}(x) g(x, u_i) \right) p(x) dx \right|. \end{aligned}$$

and by the definition of $\beta_{j,i}$ it follows that

$$\begin{aligned}
|I| &= \left| \int_E \int_U g(x,u)\eta_0(du,x)p(x)dx - \int_E \left(\sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} \eta_0(du,x_j) I_{E_j}(x) g(x,u_i) \right) p(x)dx \right| \\
&\leq \left| \int_E \int_U g(x,u)\eta_0(du,x)p(x)dx - \int_E \left(\sum_{i=0}^{2^{k_m^{(2)}}} g(x,u_i) \int_{U_i} \eta_0(du,x) \right) p(x)dx \right| \\
&\quad + \left| \int_E \left(\sum_{i=0}^{2^{k_m^{(2)}}} g(x,u_i) \int_{U_i} \eta_0(du,x) \right) p(x)dx \right. \\
&\qquad \left. - \int_E \left(\sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} \eta_0(du,x_j) I_{E_j}(x) g(x,u_i) \right) p(x)dx \right| \\
&\equiv |I_1| + |I_2| .
\end{aligned}$$

Observe that

$$|I_1| = \left| \int_E \left(\sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} \left(g(x,u) - g(x,u_i) \right) \eta_0(du,x) \right) p(x)dx \right| .$$

By Lemma 1.22, there is a $\delta > 0$ such that for all $|u-v| < \delta$, we have that $|g(x,u)-g(x,v)| < \frac{\epsilon}{2}$, independent of x. Choose m_1 large enough such that for all $m \geq m_1$ it is true that $\frac{1}{2^{k_m^{(2)}}} < \delta$. Then

$$|I_1| < \left| \int_E \left(\sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} \frac{\epsilon}{2} \eta_0(du,x) \right) p(x)dx \right| = \frac{\epsilon}{2},$$

as for any $x \in E$, $\eta_0(\cdot, x)$ is a probability measure, and so is $p(x)dx$. We now examine the term $|I_2|$ which can be simplified to

$$|I_2| = \left| \int_E \sum_{i=0}^{2^{k_m^{(2)}}} g(x,u_i) \left(\int_{U_i} \eta_0(du,x) - \sum_{j=0}^{2^{k_m^{(1)}}-1} I_{E_j}(x) \int_{U_i} \eta_0(du,x_j) \right) p(x)dx \right| . \tag{1.28}$$

To begin with, observe that, independent of x_j and thus independent of our choice of $E^{(k_m^{(1)})}$, we have

$$\begin{aligned}\sum_{i=0}^{2^{k_m^{(2)}}} |\eta_0(U_i,x)-\eta_0(U_i,x_j)| &\leq \sum_{i=0}^{2^{k_m^{(2)}}} (|\eta_0(U_i,x)|+|\eta_0(U_i,x_j)|) \\ &\leq \sum_{i=0}^{2^{k_m^{(2)}}} (\eta_0(U_i,x)+\eta_0(U_i,x_j)) = 2, \qquad (1.29)\end{aligned}$$

due to the fact regular conditional probabilities are indeed probability measures for fixed x or x_j, and thus are additive in their first argument. In the following analysis, we regard I_2 as a sequence with two indices, say $I_2(a,b) \equiv I_2(k_a^{(1)}, k_b^{(2)}) \equiv I_2$ (with a slight abuse of notation), where a and b are two discretization levels.

Our first claim is that $I_2(a,b)$ is a Cauchy sequence in b when a is fixed. To see this, we analyze two successive elements of the sequence. Regard

$$\begin{aligned}|I_2(a,b+1)-I_2(a,b)| &= \left|I_2(k_a^{(1)}, k_{b+1}^{(2)}) - I_2(k_a^{(1)}, k_b^{(2)})\right| \\ &= \left|\int_E \left[\sum_{i=0}^{2^{k_{b+1}^{(2)}}} g(x,\tilde{u}_i)\left(\eta_0(\tilde{U}_i,x) - \sum_{j=0}^{2^{k_a^{(1)}}-1} I_{E_j}(x)\eta_0(\tilde{U}_i,x_j)\right)\right.\right. \\ &\qquad \left.\left. - \sum_{i=0}^{2^{k_b^{(2)}}} g(x,u_i)\left(\eta_0(U_i,x) - \sum_{j=0}^{2^{k_a^{(1)}}-1} I_{E_j}(x)\eta_0(U_i,x_j)\right)\right] p(x)dx\right|\end{aligned}$$

where $\tilde{U}_i$ and $\tilde{u}_i$ are used to indicate the partition of U and the points of the discrete set in U of the next discretization level $b+1$. Due to the additivity of measures, the two sums, if regarded as a Riemann-type approximation of an integral, only differ by a more accurate choice of 'rectangle-height' in the Riemann sum. To formalize this, for $i \in \{0,\dots,2^{k_{b+1}^{(2)}}\}$ let

$\pi(i) \in \{0, \ldots, 2^{k_b^{(2)}}\}$ be the index such that $\tilde{U}_i \subset U_{\pi(i)}$. Then,

$$
\begin{aligned}
&|I_2(a, b+1) - I_2(a, b)| \\
&= \left| \int_E \left[\sum_{i=0}^{2^{k_{b+1}^{(2)}}} \left(g(x, \tilde{u}_i) - g(x, u_{\pi(i)}) \right) \cdot \left(\eta_0(\tilde{U}_i, x) - \sum_{j=0}^{2^{k_a^{(1)}}-1} I_{E_j}(x) \eta_0(\tilde{U}_i, x_j) \right) \right] p(x) dx \right| \\
&\leq \int_E \left[\sum_{i=0}^{2^{k_{b+1}^{(2)}}} \left| \, g(x, \tilde{u}_i) - g(x, u_{\pi(i)}) \right| \cdot \left| \eta_0(\tilde{U}_i, x) - \sum_{j=0}^{2^{k_a^{(1)}}-1} I_{E_j}(x) \eta_0(\tilde{U}_i, x_j) \right| \right] p(x) dx \\
&\leq K \left(\frac{1}{2} \right)^{b+1} \int_E \left[\sum_{i=0}^{2^{k_{b+1}^{(2)}}} \left| \eta_0(\tilde{U}_i, x) - \sum_{j=0}^{2^{k_a^{(1)}}-1} I_{E_j}(x) \eta_0(\tilde{U}_i, x_j) \right| \right] p(x) dx \\
&\leq K \left(\frac{1}{2} \right)^{b+1} \sum_{j=0}^{2^{k_a^{(1)}}-1} \int_{E_j} \underbrace{\left[\sum_{i=0}^{2^{k_{b+1}^{(2)}}} \left| \eta_0(\tilde{U}_i, x) - \eta_0(\tilde{U}_i, x_j) \right| \right]}_{\leq 2, \text{ by } (1.29)} p(x) dx \\
&\leq K \left(\frac{1}{2} \right)^{b}
\end{aligned}
$$

due to the fact that $\left| \, g(x, \tilde{u}_i) - g(x, u_{\pi(i)}) \right|$ is uniformly bounded by $K \left(\frac{1}{2}\right)^{b+1}$, with $K = 1$ if $g(x, u) = c_0(x, u)$, and $K = \max\{\|f_1\|_{\mathscr{D}}, \ldots, \|f_k\|_{\mathscr{D}}\}$, by our choice of $U^{\left(k_b^{(2)}\right)}$, compare Remark III.1.9.

Now, for some $\vartheta > 0$, choose b large enough such that $\sum_{j=b}^{\infty} \left(\frac{1}{2}\right)^j < \frac{\vartheta}{K}$. Then, for all $b_1, b_2 \geq b$, we have

$$
\begin{aligned}
|I_2(a, b_1) - I_2(a, b_2)| &= \left| \sum_{j=b_2+1}^{b_1} I_2(a, j) - I_2(a, j-1) \right| \\
&\leq \sum_{j=b_2+1}^{b_1} |I_2(a, j) - I_2(a, j-1)| \\
&\leq K \sum_{j=b_2+1}^{b_1} \left(\frac{1}{2} \right)^{j-1} < \vartheta,
\end{aligned}
$$

which shows that I_2 is Cauchy in b, which, is the same as saying it is Cauchy in $k_b^{(2)}$. However, note that this bound on the increment of I_2 in $k_b^{(2)}$ is independent of a, so it does not depend on the choice of $E^{(k_a^{(1)})}$. Given this result, choose $m_2 \geq m_1$ such that for all $m \geq m_2$, we have that $k_m^{(2)}$ is large enough such that for all $b_1, b_2 \geq k_m^{(2)}$, $|I_2(k_m^{(1)}, b_1) - I_2(k_m^{(1)}, b_2)| < \frac{\epsilon}{4}$. Now observe that for each $i \in \{0, 1, \ldots, 2^{k_m^{(2)}}\}$ the function $x \mapsto \eta_0(U_i, x)$ can be approximated as described in Lemma 1.21. To see this, note that

$$\sum_{j=0}^{2^{k_m^{(1)}}-1} I_{E_j}(x)\eta_0(U_i, x_j) = \sum_{j=0}^{2^{k_m^{(1)}}-1} \eta_0(U_i, x_j) I_{[x_j, x_{j+1})}(x) + \eta_0(U_i, x_{2^{k_m^{(1)}}-1}) I_{\left\{x_{2^{k_m^{(1)}}}\right\}}(x)$$

has the form of the approximate function used in Lemma 1.21. Hence, for $k_m^{(2)}$ fixed and for each $i \in \{0, 1, \ldots, 2^{k_m^{(2)}}\}$, there is a $k^{(1,i)}$ large enough such that for all $k^{(i)} \geq k^{(1,i)}$, we have

$$\int_E \left| \eta_0(U_i, x) - \sum_{j=0}^{2^{k^{(i)}}-1} I_{E_j}(x)\eta_0(U_i, x_j) \right| p(x)dx < \frac{\epsilon}{4 \max\left\{\|c_0\|_\infty, \|Af_1\|_\infty, \ldots \|Af_n\|_\infty\right\} (2^{k_m^{(2)}+1})}.$$

Set $\tilde{k} = \max\left\{\max_{i=1,\ldots,2^{(k_m^{(2)})}} k^{(1,i)}, k_{m_2}^{(1)}\right\}$. Then,

$$\begin{aligned}
|I_2(\tilde{k}, k_m^{(2)})| &\leq \int_E \|g\|_\infty \sum_{i=0}^{2^{k_m^{(2)}}} \left| \eta_0(U_i, x) - \sum_{j=0}^{2^{\tilde{k}}-1} I_{E_j}(x)\eta_0(U_i, x_j) \right| p(x)dx \\
&\leq \|g\|_\infty \sum_{i=0}^{2^{k_m^{(2)}}} \int_E \left| \eta_0(U_i, x) - \sum_{j=0}^{2^{\tilde{k}}-1} I_{E_j}(x)\eta_0(U_i, x_j) \right| p(x)dx \\
&\leq \frac{\epsilon}{4} < \frac{\epsilon}{2}.
\end{aligned}$$

Also, note that $I_2(\tilde{k}, k_m^{(2)})$ is decreasing in $\tilde{k}$, by the approximation properties analyzed in the proof of Lemma 1.21. Also, for $l \geq k_m^{(2)}$, we have

$$\begin{aligned}
\left| I_2(\tilde{k}, l) \right| &\leq \left| I_2(\tilde{k}, l) - I_2(\tilde{k}, k_m^{(2)}) \right| + \left| I_2(\tilde{k}, k_m^{(2)}) \right| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}
\end{aligned}$$

which means that $I_2 \equiv I_2(\tilde{k}, k_m^{(2)})$ does not exceed $\frac{\epsilon}{2}$ when $\tilde{k}$ or $k_m^{(2)}$ increase. Choose $m_3 \geq m_2$ such that for all $m \geq m_3$, $k_{m_3}^{(1)} \geq \tilde{k}$. Then, for all $m \geq m_3$, we have that $I_2 < \frac{\epsilon}{2}$. □

Lemma 1.30. *Consider a relaxed control on the singular part given by a regular conditional probability η_1. Let $g(x,u) = c_1(x,u)$ or $g(x,u) = Bf_k(x,u)$ for any $k = 1, 2, \ldots, n$. For $\epsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,*

$$\left| \int_E \int_U g(x,u)\eta_1(du,x)\mu_{1,E}(dx) - \int_E \int_U g(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \right| < \epsilon, \tag{1.31}$$

where $\hat{\eta}_{1,k_m}$ is given by the usual approximation of η_1, as defined in Definition 1.25.

Proof. We only have to show that

$$\left| \int_U g(s_j,u)\eta_1(du,s_j) - \int_U g(s_j,u)\hat{\eta}_{1,k_m}(du,s_j) \right| < \frac{\epsilon}{\mu_{1,E}(E)}$$

uniformly in $j = 1, \ldots, N$, since $\mu_{1,E}$ only puts mass on the points $s_1, \ldots, s_N$. By the definition of the usual approximation of η_1,

$$\begin{aligned}
&\left| \int_U g(s_j,u)\eta_1(du,s_j) - \int_U g(s_j,u)\hat{\eta}_{1,k_m}(du,s_j) \right| \\
&\qquad = \left| \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} g(s_j,u)\eta_1(du,s_j) - \int_{U_i} g(s_j,u)\hat{\eta}_{1,k_m}(du,s_j) \right| \\
&\qquad = \left| \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} g(s_j,u)\eta_1(du,s_j) - g(s_j,u_i)\zeta_{j,i} \right| \\
&\qquad = \left| \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} g(s_j,u)\eta_1(du,s_j) - g(s_j,u_i) \int_{U_i} \eta_1(du,s_j) \right| \\
&\qquad \leq \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} |g(s_j,u) - g(s_j,u_i)|\, \eta_1(du,s_j).
\end{aligned}$$

According to Lemma 1.23, there is a $\delta > 0$ such that $|g(s_j, u) - g(s_j, v)| < \frac{\epsilon}{\mu_{1,E}(E)}$ whenever $|u - v| < \delta$. Choose m large enough such that for all $m \geq m_0$ it is true that $\frac{1}{2^{k_m^{(2)}}} < \delta$ to ensure that $|u - u_i| < \delta$. Then,

$$\sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} |g(s_j, u) - g(s_j, u_i)| \, \eta_1(du, s_j) < \frac{\epsilon}{\mu_{1,E}(E)} \cdot \sum_{i=0}^{2^{k_m^{(2)}}} \int_{U_i} \eta_1(du, s_j) = \frac{\epsilon}{\mu_{1,E}(E)}$$

holds, which proves the claim. $\square$

Remark 1.32. *If no singular behavior is present, the expression seen in (1.31) is trivially zero.*

The results given in Lemma 1.27 and Lemma 1.30 serve us in two ways. First, they ensure that the given approximation can approximate the cost criterion well enough, second, they are instrumental in showing that the constraints given by the (n, ∞)-dimensional program are 'almost' fulfilled. This allows us to make small adjustments to the approximation, fulfilling the constraints but still approximating the cost criterion well enough. In the following, we will restrict our considerations to a basis $\{p_j\}_{j \in \mathbb{N}}$ in $L^1(E)$ that is given by the indicator functions on the intervals $[x_j, x_{j+1}), j = 0, \ldots 2^{k_m^{(3)}} - 2$ and $[x_{2^{k_m^{(3)}}-1}, x_{2^{k_m^{(3)}}}]$ given by the discretization of E, for increasing m as described in Section III.1.4. We refer to the basis functions given for fixed m by $\{p_0, p_1, \ldots p_{2^{k_m^{(3)}}-1}\}$.

Definition 1.33. *Let η_0 be any relaxed control. For $n, m \in \mathbb{N}$, define the constraint matrix $C^{(m)} \in \mathbb{R}^{n+1, 2^{k_m^{(3)}}}$ by*

$$\begin{aligned} C_{k,j}^{(m)} &= \int_E \int_U Af_k(x,u)\eta_0(du,x)p_j(x)dx \qquad \text{if } k = 1,2,\ldots,n, j = 0,1,\ldots,2^{k_m^{(3)}} - 1 \\ C_{n+1,j}^{(m)} &= \int_E p_j(x)dx \qquad \text{if } j = 1,2,\ldots,2^{k_m^{(3)}} - 1. \end{aligned}$$

Remark 1.34. *Note that one can increase m so far that $C^{(m)}$ has full rank $n+1$.*

Definition 1.35. *For $m \in \mathbb{N}$ and $\hat{\eta}_{0,k_m}$ of the form* (III.1.11)*, $\hat{\eta}_{1,k_m}$ of the form* (III.1.13) *and some $\tilde{p}_{k_m}$ in the span of $\{p_0, p_2, \dots, p_{2^{k_m^{(3)}}-1}\}$, the constraint error $d^{(m)} \in \mathbb{R}^{n+1}$ is defined for $k = 1, \dots, n$ by*

$$d_k^{(m)}(\tilde{p}_{k_m}) = Rf_k - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\tilde{p}_{k_m}(x)dx - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx).$$

and for $k = n+1$ by $d_{n+1}^{(m)} = 1 - \int_E \tilde{p}_{k_m}(x)(dx)$.

Remark 1.36. *For some $\tilde{p}_{k_m}$, the constraint error specifies how 'far' $\tilde{p}_{k_m}$ is from fulfilling the constraints given by the test functions $f_1, f_2, \dots, f_n$, and how 'far' it is from being a probability density integrating to 1. In particular, if $d^{(m)}(\tilde{p}_{k_m}) = 0$, $\tilde{p}_{k_m}$ fulfills the constraints and is a probability density.*

Remark 1.37. *If $\gamma \in \mathbb{R}^{k_m^{(3)}}$ is a vector with the coefficients of $\tilde{p}_{k_m}$ in terms of the basis $p_0, p_1, \dots p_{2^{k_m^{(3)}}-1}$, the constraint error for $k = 1, \dots, n$ is given by the components*

$$d_k^{(m)}(\tilde{p}_{k_m}) = Rf_k - \left(C^{(m)}\gamma\right)_k - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx). \tag{1.38}$$

and for $n+1$ by $d_{n+1}^{(m)}(\tilde{p}_{k_m}) = 1 - \left(C^{(m)}\gamma\right)_{n+1} = 1 - \|\tilde{p}_{k_m}\|_{L^1(E)}$.

Remark 1.39. *In the absence of singular behavior, the expression seen in (1.38) is equal to*

$$d_k^{(m)}(\tilde{p}_{k_m}) = Rf_k - \left(C^{(m)}\gamma\right)_k.$$

Now we investigate the size of the discretization error. First, we prove an $L^1(E)$-approximation result for the density p, that relies on the assumption that $\lambda(\{x : p(x) = 0\}) = 0$.

Lemma 1.40. *Let p be a probability density function with $\lambda(\{x : p(x) = 0\}) = 0$. Then, for any $\epsilon > 0$ and $D_1 > 0$, there exists an $\hat{\epsilon}_1 < \epsilon$ and an m_0 such that for all $m \geq m_0$, there is a $\tilde{p}_{k_m}$ in the span of $\{p_0, p_, \dots, p_{2^{k_m^{(3)}}-1}\}$ with $\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_1}{D_1}$ and $\tilde{p}_{k_m} \geq \hat{\epsilon}_1$ on E.*

Proof. Find $\hat{\epsilon}_1 < \epsilon$ such that $\lambda\left(\{x : f(x) \leq \hat{\epsilon}_1\}\right) < \frac{1}{2D_1}$, which is possible due to continuity from above of measures. Define

$$\bar{p}(x) = \begin{cases} p(x) & p(x) > \hat{\epsilon}_1 \\ \hat{\epsilon}_1 & p(x) \leq \hat{\epsilon}_1 \end{cases}.$$

Then, $\|p - \bar{p}\|_{L^1(E)} \leq \hat{\epsilon}_1 \cdot \lambda\left(\{x : f(x) \leq \hat{\epsilon}_1\}\right) \leq \frac{\hat{\epsilon}_1}{2D_1}$. Now, choose m_0 large enough such that for all $m \geq m_0$, there is a $\tilde{p}_{k_m} \in \operatorname{span}\left(p_0, p_1, \dots, p_{2^{k_m^{(3)}}-1}\right)$ with $\|\bar{p} - \tilde{p}_{k_m}\|_{L^1(E)} \leq \frac{\hat{\epsilon}_1}{2D_1}$. Then,

$$\|p - \tilde{p}_{k_m}\|_{L^1(E)} \leq \|p - \bar{p}\|_{L^1(E)} + \|\bar{p} - \tilde{p}_{k_m}\|_{L^1(E)} \leq \frac{\hat{\epsilon}_1}{2D_1} + \frac{\hat{\epsilon}_1}{2D_1} = \frac{\hat{\epsilon}_1}{D_1},$$

but also, $\tilde{p}_{k_m} \geq \hat{\epsilon}_1$ holds. □

Proposition 1.41. *Consider a pair of measures $(\mu_0, \mu_1) \in \mathscr{M}_{n,\infty}$, and let $\mu_0(dx \times du) = \eta_0(du, x)p(x)dx$ as well as $\mu_1(dx \times du) = \eta_1(du, x)\mu_{1,E}(dx)$. Let $\bar{A} = \max_{k=1,\dots,n} \|Af_k\|_\infty$. For $\epsilon > 0$ and $D_2 > 0$, there exists an $\hat{\epsilon}_2 < \epsilon$ and an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, there is a function $\tilde{p}_{k_m}$ in the span of $\{p_0, p_1, \dots, p_{2^{k_m^{(3)}}-1}\}$ with $\|d^{(m)}(\tilde{p}_{k_m})\|_\infty < \hat{\epsilon}_2$, where $d^{(m)}(\tilde{p}_{k_m})$ is the constraint error using the usual approximations $\hat{\eta}_{0,k_m}(du, x)$ and $\hat{\eta}_{1,k_m}(du, x)$ of the given controls η_0 and η_1, defined by Definition 1.24 and Definition 1.25, respectively. In particular, $\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3\max\{\bar{A},1\}}$ as well as $\tilde{p}_{k_m} \geq D_2 \cdot \hat{\epsilon}_2$ holds.*

Proof. Since $(\mu_0, \mu_1) \in \mathscr{M}_{n,\infty}$, we have that for each $k = 1, 2, \dots, n$

$$Rf_k = \int_E \int_U Af_k(x, u)\eta_0(du, x)p(x)dx + \int_E \int_U Bf_k(x, u)\mu_1(dx \times du)$$

and thereby

$$
\begin{aligned}
& d_k^{(m)}(\tilde{p}_{k_m}) \\
&= Rf_k - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\tilde{p}_{k_m}(x)dx - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \\
&= \int_E \int_U Af_k(x,u)\eta_0(du,x)p(x)dx + \int_E \int_U Bf_k(x,u)\eta_1(du,x)\mu_{1,E}(dx) \\
&\quad - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\tilde{p}_{k_m}(x)dx - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \\
&= \int_E \int_U Af_k(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\tilde{p}_{k_m}(x)dx \\
&\quad + \int_E \int_U Bf_k(x,u)\eta_1(du,x)\mu_{1,E}(dx) - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx)
\end{aligned}
$$

holds. The triangle inequality reveals that

$$
\begin{aligned}
& |d_k^{(m)}(\tilde{p}_{k_m})| \\
&\quad = \left| \int_E \int_U Af_k(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)p(x)dx \right| \\
&\qquad + \left| \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)p(x)dx - \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\tilde{p}_{k_m}(x)dx \right| \\
&\qquad + \left| \int_E \int_U Bf_k(x,u)\eta_1(du,x)\mu_{1,E}(dx) - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \right| \\
&\quad \equiv \left| d_{k,1}^{(m)} \right| + \left| d_{k,2}^{(m)} \right| + \left| d_{k,3}^{(m)} \right|
\end{aligned}
$$

Apply Lemma 1.40 with ϵ and $D_1 = D_2 \cdot 3 \cdot \max\{\bar{A}, 1\}$. Take $\hat{\epsilon}_1$ and m_1 from this statement. Set $\hat{\epsilon}_2 = \hat{\epsilon}_1 / D_2$. Then, $\hat{\epsilon}_2 < \epsilon$ and for all $m \geq m_1$, there is a $\tilde{p}_{k_m} \in \operatorname{span}\{p_0, p_1, \ldots, p_{2^{k_m^{(3)}}-1}\}$ such that $\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3 \cdot \max\{\bar{A},1\}}$ as well as $\tilde{p}_{k_m} \geq D_2 \cdot \hat{\epsilon}_2$ holds. Then,

$$
\left| d_{k,2}^{(m)} \right| \equiv \left| \int_E \int_U Af_k(x,u)\hat{\eta}_{0,k_m}(du,x)\left(p(x) - \tilde{p}_{k_m}(x)\right)dx \right| \leq \bar{A}\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3}
$$

By Lemma 1.27, we can choose $m_2 \geq m_1$ such that for all $m \geq m_2$, $|d_{k,1}|$ is bounded by $\frac{\hat{\epsilon}_2}{3}$. By Lemma 1.30, we can choose $m_3 \geq m_2$ such that $|d_{k,3}|$ is bounded by $\frac{\hat{\epsilon}_2}{3}$ for all $m \geq m_3$, which shows that $\left| d_k^{(m)}(\tilde{p}_{k_m}) \right| < \hat{\epsilon}_2$ for $k = 1, 2, \ldots, n$. For $k = n+1$, since p is a probability density

$$\|\tilde{p}_{k_m}\|_{L^1(E)} \leq \|\tilde{p}_{k_m} - p\|_{L^1(E)} + \|p\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3\max\{\bar{A}, 1\}} + 1 < \hat{\epsilon}_2 + 1.$$

Now assume that $\|\tilde{p}_{k_m}\|_{L^1(E)} < 1 - \hat{\epsilon}_2$. Then,

$$\|p\|_{L^1(E)} \leq \|p - \tilde{p}_{k_m}\|_{L^1(E)} + \|\tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3\max\{\bar{A}, 1\}} + 1 - \hat{\epsilon}_2 < \hat{\epsilon}_2 + 1 - \hat{\epsilon}_2 = 1,$$

a contradiction, and we have that

$$1 - \hat{\epsilon}_2 \leq \|\tilde{p}_{k_m}\|_{L^1(E)} \leq 1 + \hat{\epsilon}_2$$

Hence, $|d_{n+1}^{(m)}(\tilde{p}_{k_m})| < \hat{\epsilon}_2$, which completes the proof, setting $m_0 = m_3$. □

Lemma 1.42. *For any $\epsilon > 0$, there is an $\hat{\epsilon} < \epsilon$ and an $m_0 \in \mathbb{N}$ large enough such that for all $m \geq m_0$, there is a $\tilde{p}_{k_m} \in \operatorname{span}\{p_0, p_1, \ldots, p_{2^{k_m^{(3)}}-1}\}$, and the equation $C^{(m)}y = -d^{(m)}(\tilde{p}_{k_m})$ has a solution with $\|y\|_\infty < \hat{\epsilon}$. In particular, $\tilde{p}_{k_m} \geq \hat{\epsilon}$ and $\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \hat{\epsilon}$ hold.*

Proof. Select $m_1 \in \mathbb{N}$ large enough such that for all $m \geq m_1$, $C^{(m)}$ has full rank and thus $n+1$ independent columns. Let $\bar{C}^{(m)} \in \mathbb{R}^{n+1,n+1}$ be the matrix consisting of these $n+1$ independent columns. Set $\delta = \frac{\epsilon}{\max\left\{1, \left\|\left(\bar{C}^{(m_1)}\right)^{-1}\right\|_\infty\right\}}$ and by Proposition 1.41, with $\epsilon = \delta$ and $D_2 = \|\left(\bar{C}^{(m_1)}\right)^{-1}\|_\infty$, find $m_2 \geq m_1$ such that for all $m \geq m_2$, there is a $\tilde{p}_{k_m}$, with $\|d^{(m)}(\tilde{p}_{k_m})\|_\infty < \hat{\epsilon}_2 < \delta$ for some $\hat{\epsilon}_2 > 0$, fulfilling $\tilde{p}_{k_m} > \|\left(\bar{C}^{(m_1)}\right)^{-1}\|_\infty \cdot \hat{\epsilon}_2$ as well as $\|p - \tilde{p}_{k_m}\|_{L^1(E)} < \frac{\hat{\epsilon}_2}{3\cdot\max\{\bar{A},1\}} < \hat{\epsilon}$. Let $\hat{\epsilon} = \|\left(\bar{C}^{(m_1)}\right)^{-1}\|_\infty \cdot \hat{\epsilon}_2$. Consider the solution $y \in \mathbb{R}^{2^{k_{m_1}^{(2)}}}$ for $C^{(m_1)}y = d^{(m_2)}(\tilde{p}_{k_m})$ that is given by injecting $\bar{y} = \left(\bar{C}^{(m_1)}\right)^{-1}\left(-d^{(m_2)}(\tilde{p}_{k_m})\right)$ into $\mathbb{R}^{2^{k_{m_1}^{(2)}}}$. Then,

$$\|y\|_\infty = \|\bar{y}\|_\infty = \|\left(\bar{C}^{(m_1)}\right)^{-1} d^{(m_2)}(\tilde{p}_{k_m})\|_\infty \leq \|\left(\bar{C}^{(m_1)}\right)^{-1}\|_\infty \|d^{(m_2)}(\tilde{p}_{k_m})\|_\infty < \hat{\epsilon}.$$

We now show that there is a solution $\hat{y}$ to $C^{(m_2)}y = d^{(m_2)}(\tilde{p}_{k_m})$ that fulfills $\|\hat{y}\|_\infty < \hat{\epsilon}$, where $C^{(m_2)}$ is a full constraint matrix rather that just a selections of columns from a constraint matrix. By the definition of the continuous constraint matrix, we have that for $k = 1, 2, \ldots, n+1$ and $i = 0, 1, \ldots, 2^{k_m^{(3)}} - 1$,

$$C_{k,2i-1}^{(m+1)} + C_{k,2i}^{(m+1)} = C_{k,i}^{(m)}$$

holds. Indeed, since if $1 \leq k \leq n$ by the choice of basis functions $\{p_0, p_1, \ldots, p_{2^{k_m^{(3)}}-1}\}$, the coefficients are given by integration of the functions Af_k over intervals that are cut in half, and if $k = n+1$, the coefficients are simply given by the interval lengths $(x_{j+1} - x_j)$ since $p_j = 1$ on $[x_{j+1}, x_j)$. Hence, if y is a solution to $C^{(m)}x = -d$, the vector $\bar{y}$ with components

$$\bar{y}_{2i-1} = \bar{y}_{2i} = y_i$$

where $i = 0, 1, \ldots, 2^{k_m^{(3)}} - 1$, fulfills $C^{(m+1)}y = -d$, and $\|y\|_\infty = \|\bar{y}\|_\infty$ holds. Inductively, this reveals that for any $m \geq m_1$, there is a solution $\hat{y}$ to $C^{(m)}y = -d^{(m_1)}(\tilde{p}_{k_m})$ which fulfills $\|\hat{y}\|_\infty = \|y\|_\infty < \hat{\epsilon}$. In particular, this means that there is a solution $\hat{y}$ to $C^{(m_2)}y = -d^{(m_2)}(\tilde{p}_{k_m})$, with $\|\hat{y}\|_\infty = \|y\|_\infty < \hat{\epsilon}$. For any $m \geq m_2$, this analysis can be conducted similarly, showing the result for $m_0 = m_2$. $\square$

Proposition 1.43. *For $(\mu_0, \mu_1) \in \mathscr{M}_n^l$ and every $\epsilon > 0$, there is an m_0 such that for all $m \geq m_0$ there is a $(\hat{\mu}_{0,k_m}, \hat{\mu}_{1,k_m}) \in \mathscr{M}_{n,k_m}^l$, with*

$$|J(\mu_0, \mu_1) - J(\hat{\mu}_{0,k_m}, \hat{\mu}_{1,k_m})| < \epsilon.$$

Proof. Pick $\epsilon > 0$ arbitrarily. For $(\mu_0, \mu_1) \in \mathscr{M}_n^l$, let $\mu_{0,E}$ be the state space marginal of μ_0 and let η_0 be the regular conditional probability such that $\mu_0(dx \times du) = \eta_0(du, x)p(x)dx$. Likewise, let $\mu_1(dx \times du) = \eta_1(du, x)\mu_{1,E}(dx)$. Define $\hat{\eta}_{0,m_k}$ and $\hat{\eta}_{1,m_k}$ by the usual approximations, see Definition 1.24 and Definition 1.25, respectively. First, by Lemma 1.30, we have that there is an m_1 large enough such that for all $m \geq m_1$,

$$\left| \int_E \int_U c_1(x,u)\eta_1(du,x)d\mu_{1,E} - \int_E \int_U c_1(x,u)\hat{\eta}_{1,k_m}(du,x)d\hat{\mu}_{1,E} \right| < \frac{\epsilon}{2}$$

Now, we consider the approximation of the cost accrued by c_0. We show that

$$\left| \int_E \int_U c_0(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| < \frac{\epsilon}{2},$$

for the given choice of $\hat{\eta}_{0,k_m}$ and a choice of $\hat{p}_{k_m}$ to be identified. This will be done by a successive application of the triangle inequality. First, observe that

$$\begin{aligned}
&\left| \int_E \int_U c_0(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| \\
&\quad \leq \quad \left| \int_E \int_U c_0(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)p(x)dx \right| \\
&\qquad + \left| \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| \\
&\quad \equiv |I_1| + |I_2|.
\end{aligned}$$

Now set

$$\vartheta = \min\left\{ \frac{\epsilon}{8\|c_0\|_\infty(e_r - e_l)}, \frac{3\epsilon \max\{\bar{A}, 1\}}{8\|c_0\|_\infty} \right\}.$$

By Lemma 1.42 we can choose an $m_3 \geq m_2$ such that for all $m \geq m_3$, there is a function $\bar{p}_{k_m} = \sum_{i=0}^{2^{k_m^{(3)}}-1} \bar{\gamma}_i p_i$ in $L^1(E)$ that allows for a solution $\bar{y}$ to $C^{(m)}y = -d^{(m)}(\bar{p}_{k_m})$ with $\|\bar{y}\|_\infty < \hat{\vartheta} < \vartheta$ for some $\hat{\vartheta} < \vartheta$, but $\bar{p}_{k_m} > \hat{\vartheta}$. This m_3 is also large enough to approximate p by $\bar{p}_{k_m}$ with an accuracy of $\frac{\epsilon}{8\|c_0\|_\infty}$ for all $m \geq m_3$. Define new coefficients $\gamma_i = \bar{\gamma}_i - \bar{y}_i$ and set $\hat{p}_{k_m} = \sum_{i=0}^{2^{k_m^{(3)}}-1} \gamma_i p_i$. Then for all $k = 1, 2, \ldots, n$,

$$\begin{aligned}
d_k^{(m)}(\hat{p}_{k_m}) &= Rf_k - \left(C^{(m)}\gamma\right)_k - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \\
&= Rf_k - \left(C^{(m)}(\bar{\gamma} - \bar{y})\right)_k - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) \\
&= Rf_k - \left(C^{(m)}\bar{\gamma}\right)_k - \int_E \int_U Bf_k(x,u)\hat{\eta}_{1,k_m}(du,x)\mu_{1,E}(dx) + \left(C^{(m)}\bar{y}\right)_k \\
&= d_k^{(m)}(\bar{p}_{k_m}) - d_k^{(m)}(\bar{p}_{k_m}) = 0
\end{aligned}$$

and

$$d_{n+1}^{(m)}(\hat{p}_{k_m}) = 1 - (C^{(m)}\gamma)_{n+1} - (C^{(m)}\bar{\gamma})_{n+1} = d_{n+1}^{(m)}(\bar{p}_{k_m}) - d_{n+1}^{(m)}(\bar{p}_{k_m}) = 0$$

which shows that $\hat{p}_{k_m}$ fulfills the constraints. But also, $\hat{p}_{k_m} \geq 0$. So $(\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx, \hat{\eta}_{0,k_m}(du,x)\mu_{1,E}(dx)) \in \mathscr{M}_{n,m}$. Furthermore,

$$\begin{aligned}
\|p - \hat{p}_{k_m}\|_{L^1(E)} &\leq \|p - \bar{p}_{k_m}\|_{L^1(E)} + \|\bar{p}_{k_m} - \hat{p}_{k_m}\|_{L^1(E)} \\
&\leq \frac{\epsilon}{8\|c_0\|_\infty} + \frac{\epsilon}{8\|c_0\|_\infty} = \frac{\epsilon}{4\|c_0\|_\infty}
\end{aligned}$$

This shows that

$$\begin{aligned}
|I_2| \equiv \quad & \left| \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| \\
&\leq \int_E \|c_0\|_\infty \int_U \hat{\eta}_{0,k_m}(du,x)|p(x) - \hat{p}_{k_m}(x)|dx < \|c_0\|_\infty \|p - \hat{p}_{k_m}\|_{L^1} \\
&< \frac{\epsilon}{4},
\end{aligned}$$

since $\int_U \hat{\eta}_{0,k_m}(du,x) = 1$. Turning to $|I_1|$, we have seen in Lemma 1.27 that there is an $m_4 \geq m_3$ such that for all $m \geq m_4$, $|I_1| < \frac{\epsilon}{4}$. To summarize,

$$\begin{aligned}
&\left| \int_E \int_U c_0(x,u)\eta_0(du,x)p(x)dx - \int_E \int_U c_0(x,u)\hat{\eta}_{0,k_m}(du,x)\hat{p}_{k_m}(x)dx \right| \\
&\quad \leq I_1 + I_2 \\
&\quad \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

But this gives us the assertion, and finishes the proof, setting $m_0 = m_4$. □

So far, we have analyzed the properties of the proposed approximation only with regard to an arbitrary pair of measures $(\mu_0, \mu_1) \in \mathscr{M}^l_{n,m}$. Now we return our attention to ϵ-optimal measures.

Lemma 1.44. *Let $\{\mu_{0,n,k_m}, \mu_{1,n,k_m}\}$ be a sequence of measures with $(\mu_{0,n,k_m}, \mu_{1,n,k_m}) \in \mathscr{M}^l_{n,m}$, for all $m \in \mathbb{N}$. Assume that $\mu_{0,n,k_m} \Rightarrow \hat{\mu}_{0,n}$ and $\mu_{1,n,k_m} \Rightarrow \hat{\mu}_{1,n}$ as $m \to \infty$. Then, $(\hat{\mu}_{0,n}, \hat{\mu}_{1,n}) \in \mathscr{M}^l_n$.*

Proof. Take $f_n \in \mathscr{D}_n$. Since Af_n and Bf_n are bounded and uniformly continuous, we have

$$\begin{aligned}
\int Af_n d\hat{\mu}_{0,n} + \int Bf_n d\hat{\mu}_{1,n} &= \lim_{m\to\infty} \left(\int Af_n d\mu_{0,n,k_m} + \int Bf_n d\mu_{1,n,k_m} \right) \\
&= \lim_{m\to\infty} Rf_n = Rf_n.
\end{aligned}$$

Note that $I_{E\times U} \equiv 1$ is bounded and uniformly continuous on $E \times U$, so by weak convergence, we have

$$\hat{\mu}_{1,n}(E\times U) = \int_{E\times U} I_{E\times U}\, d\hat{\mu}_{1,n} = \lim_{m\to\infty} \int_{E\times U} I_{E\times U}\, d\hat{\mu}_{1,n,k_m} = \lim_{m\to\infty} \hat{\mu}_{1,n,k_m}(E\times U) \leq l,$$

which shows that $(\hat{\mu}_{0,n}, \hat{\mu}_{1,n}) \in \mathscr{M}^l_n$. □

With these technicalities sorted out, we can prove that optimal solutions in $\mathscr{M}^l_{n,k}$ are ϵ-optimal solutions in $\mathscr{M}^l_n$.

Proposition 1.45. *Let* $\{\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}\}$ *be a sequence of optimal measures with* $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) \in \mathscr{M}^l_{n,m}$*, for all* $m \in \mathbb{N}$*. Assume that* $\mu^*_{0,n,k_m} \Rightarrow \hat{\mu}^*_n$ *and* $\mu^*_{1,n,k_m} \Rightarrow \hat{\mu}^*_{1,n}$ *as* $m \to \infty$*. Then,*

$$J(\hat{\mu}^*_{0,n}, \hat{\mu}^*_{1,n}) = \inf_{(\mu_{0,n},\mu_{1,n})\in\mathscr{M}^l_n} J(\mu_{0,n}, \mu_{1,n})$$

Proof. Assume not. Obviously,

$$J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) < \inf_{(\mu_{0,n},\mu_{1,n})\in\mathscr{M}^l_n} J(\mu_{0,n}, \mu_{1,n})$$

cannot hold, since by Lemma 1.44, $(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) \in \mathscr{M}^l_n$. So, we can assume that

$$J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) > \inf_{(\mu_{0,n},\mu_{1,n})\in\mathscr{M}^l_n} J(\mu_{0,n}, \mu_{1,n}).$$

Then, there is a $(\mu_{0,n}, \mu_{1,n}) \in \mathscr{M}^l_n$ and an $\epsilon > 0$ such that

$$J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) \geq J(\mu_{0,n}, \mu_{1,n}) + \epsilon$$

By Proposition 1.43, for some m_0 large enough there is a $(\tilde{\mu}_{0,n,k_{m_0}}, \tilde{\mu}_{1,n,k_{m_0}}) \in \mathscr{M}^l_{n,k_{m_0}}$ such that

$$\left|J(\mu_{0,n}, \mu_{1,n}) - J(\tilde{\mu}_{0,n,k_{m_0}}, \tilde{\mu}_{1,n,k_{m_0}})\right| < \epsilon$$

But then,

$$\begin{aligned}
0 \leq J(\tilde{\mu}_{0,n,k_{m_0}}, \tilde{\mu}_{1,n,k_{m_0}}) &\leq \left|J(\tilde{\mu}_{0,n,k_{m_0}}, \tilde{\mu}_{1,n,k_{m_0}}) - J(\mu_{0,n}, \mu_{1,n})\right| + |J(\mu_{0,n}, \mu_{1,n})| \\
&< \epsilon + J(\mu_{0,n}, \mu_{1,n}) \\
&< J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}).
\end{aligned}$$

Observe that since $\mathscr{M}^l_{n,k_m} \subset \mathscr{M}^l_{n,k_{m+1}}$ we have that the sequence $\{J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})\}_{m\in\mathbb{N}}$ decreases to $J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n})$. So,

$$J(\tilde{\mu}_{0,n,k_{m_0}}, \tilde{\mu}_{1,n,k_{m_0}}) < J(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) \leq J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})$$

which contradicts that $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})$ is the optimal solution in $\mathscr{M}^l_{n,k_m}$, if $m \geq m_0$. □

We have only shown a convergence result of the values for the optimality criteria under the assumption that the sequence of measures converges. The following result shows how the sequence of values for the optimality criterion behaves independently from the convergence of the underlying sequence of measures.

Proposition 1.46. *Let $\{(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})\}_{m\in\mathbb{N}}$ be a sequence of optimal solutions with $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) \in \mathscr{M}^l_{n,k_m}$ for all $m \in \mathbb{N}$. respectively. Then, the sequence of numbers $\{J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})\}_{m\in\mathbb{N}}$ converges to $J^*_n := \inf_{(\mu_{0,n},\mu_{1,n})\in\mathscr{M}^l_n} J(\mu_{0,n}, \mu_{1,n})$.*

Proof. As mentioned before, $\{J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})\}_{m\in\mathbb{N}}$ is a decreasing sequence. Also, it is bounded from below, so it converges. As $\{\mu^*_{0,n,k_m}\}_{m\in\mathbb{N}}$ and $\{\mu^*_{1,n,k_m}\}_{m\in\mathbb{N}}$ are sequences of measures over a compact space, they are tight, and $\{\mu^*_{1,n,k_m}\}_{m\in\mathbb{N}}$ is uniformly bounded by l. Consequently there is a converging subsequence $\{(\mu^*_{n,k_{m_j}}, \mu^*_{1,n,k_{m_j}})\}_{j\in\mathbb{N}}$ with

$$\mu^*_{0,n,k_j} \Rightarrow \hat{\mu}^*_{0,n} \qquad \text{and} \qquad \mu^*_{1,n,k_j} \Rightarrow \hat{\mu}^*_{1,n}$$

for some $(\hat{\mu}^*_n, \hat{\mu}^*_{1,n}) \in \mathscr{M}_n$. Then, we have by Proposition 1.45, as c_0 and c_1 are bounded and uniformly continuous,

$$\begin{aligned} J^*_n = J(\hat{\mu}^*_{0,n}, \hat{\mu}^*_{1,n}) &= \int c_0 d\hat{\mu}^*_{0,n} + \int c_1 d\hat{\mu}^*_{1,n} \\ &= \lim_{j\to\infty} \left(\int c_0 d\hat{\mu}^*_{0,n,k_{m_j}} + \int c_1 d\hat{\mu}^*_{1,n,k_{m_j}} \right) \\ &= \lim_{j\to\infty} J(\mu^*_{0,n,k_{m_j}}, \mu^*_{1,n,k_{m_j}}), \end{aligned}$$

but $\{J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})\}_{m\in\mathbb{N}}$ converges, and any subsequence has to converge to its very limit. So,

$$\lim_{m\to\infty} J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) = J^*_n$$

□

This result shows that if we can find optimal solutions to $\mathscr{M}^l_{n,k_m}$, for large enough m, we are sufficiently close to the optimal value one can achieve in $\mathscr{M}^l_n$. In other words, for large enough m, we have found an ϵ-optimal solution for the l-bounded, (n,∞)-dimensional linear program. However, this is precisely what we needed to find according to the first part of the convergence proof. The following theorem summarizes the results obtained in this section.

Theorem 1.47. *For $n \in \mathbb{N}, \epsilon > 0$ there is an $M \equiv M(\epsilon, n)$ such that for all $m \geq M$, $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) \in \mathscr{M}^l_n$ is an optimal solution to the l-bounded (n,m)-dimensional linear problem, then $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})$ is an ϵ-optimal solution to the l-bounded (n,∞)-dimensional linear program.*

Proof. We have that $J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m})$ is a decreasing sequence and hence by Proposition 1.46, for a fixed $\epsilon > 0$ and M large enough, for all $m \geq M$,

$$\epsilon \geq J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) - J^*_n = J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) - \inf_{(\mu_0,\mu_1)\in\mathscr{M}^l_n} J(\mu_0,\mu_1) \geq 0$$

holds. In particular, we have that

$$|J(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) - \inf_{(\mu_0,\mu_1)\in\mathscr{M}^l_n} J(\mu_0,\mu_1)| \leq \epsilon.$$

□

Remark 1.48. *In combination with Remark 1.18, an $\bar{\epsilon}$-optimal solution to the infinite dimensional linear program can be found as follows. Select $l > 0$ large enough such that an $\frac{\bar{\epsilon}}{2}$-optimal solution in $\mathscr{M}^l_\infty$ is a $\bar{\epsilon}$-optimal solution in $\mathscr{M}_\infty$. Take $\epsilon > 0$ and $\delta > 0$ such that*

$2\epsilon + \delta \leq \frac{\bar{\epsilon}}{2}$. Use Proposition 1.16 to pick $n \in \mathbb{N}$ large enough such that an $\bar{\epsilon}$-optimal solution in $\mathscr{M}_n^l$ is a $2\epsilon + \delta$-optimal solution in $\mathscr{M}_\infty^l$. Then, find an ϵ-optimal solution in $\mathscr{M}_n^l$ by selecting m large enough such that an optimal solution in $\mathscr{M}_{n,m}^l$ is ϵ-optimal in $\mathscr{M}_n^l$, by Theorem 1.47.

Remark 1.49. *Note that an optimal solution $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) \in \mathscr{M}_{n,m}^l$ does not necessarily represent a solution to the infinite dimensional linear program, in other words, $(\mu^*_{0,n,k_m}, \mu^*_{1,n,k_m}) \notin \mathscr{M}_\infty$ has to be assumed. This situation is addressed in the following subsection.*

Remark 1.50. *In Remarks 1.32 and 1.39 we indicated modification to the analysis needed to be made to show the convergence for problems that do not feature singular behavior. Using these modifications, the optimality of the K-reduced, (n,∞)-dimensional linear program can be shown using the derivations presented. The conclusion of Remark 1.49 holds in this case, as well.*

IV.1.3 Accuracy of Evaluation

Up to this point, we have proven that for any $\epsilon > 0$ there are n and m large enough, such that for some pair of measures $(\hat{\mu}_{0,n,k_m}, \hat{\mu}_{1,n,k_m}) \in \mathscr{M}_{n,m}^l \subset \mathscr{M}_n^l$, $|J(\hat{\mu}_{0,n,k_m}, \hat{\mu}_{1,n,k_m}) - J^*| \leq \epsilon$, where $J^* = \inf_{(\mu_0,\mu_1)\in\mathscr{M}_\infty} J(\mu_0,\mu_1)$. The measures $(\hat{\mu}_{0,n,k_m}, \hat{\mu}_{1,n,k_m})$ have a certain structure due to the fact that they lie in $\mathscr{M}_{n,m}^l$. This structure, in particular, is given by the fact that the regular conditional probabilities representing the relaxed controls are of the form

$$\hat{\eta}_0(V,x) = \sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{i=0}^{2^{k_m^{(2)}}} \beta_{j,i} I_{E_J}(x)\delta_{u_i}(V) \tag{1.51}$$

$$\hat{\eta}_1(V,s_j) = \sum_{i=1}^{2^{(2)}_{k_m}} \zeta_{j,i}\delta_{u_i}(V), \qquad j = 1,\dots,N \tag{1.52}$$

where $E_j = [x_j, x_{j+1})$ for $j = 0, 1, \dots 2^{k_m^{(1)}} - 2$, and $E_{2^{k^{(1)}}-1} = [x_{2^{k_m^{(1)}}-1}, x_{2^{k_m^{(1)}}}]$, with the choices of $\beta_{i,j}$ and $\zeta_{i,j}$ determined by the usual approximation, see Definitions 1.24 and 1.25. The parameters $k_m^{(1)}$ and $k_m^{(2)}$ stem from the discretization level $k_m \equiv (k_m^{(1)}, k_m^{(2)}, k_m^{(3)})$, where $k_m^{(1)}$ and $k_m^{(2)}$ control the discretization of the relaxed control in E- and U-'directions', respectively, and $k_m^{(3)}$ controls the discretization of $\mu_{0,E}$, which is the state-space marginal of μ_0. While $\hat{\eta}_0$ and $\hat{\eta}_1$ can indeed be implemented into the process which is considered in the stochastic control problem, the analytic expected occupation measures stemming from this process are not necessarily identical to $(\hat{\mu}_{0,n,k_m}, \hat{\mu}_{1,n,k_m})$ for any m. It is consequently of interest to see how accurately we can approximate the cost criterion of a process that is controlled by $\hat{\eta}_0$ and $\hat{\eta}_1$. To this end, we analyze the approximation of the cost criterion value of a process that is controlled by a fixed pair of controls $(\hat{\eta}_0, \hat{\eta}_1) \equiv \eta$. In terms of notation and particular approximation schemes, we refer back to Section III.1.3.

In the following, we drop the ˆ-notation for (η_0, η_1) due to the fact that this analysis is independent of (η_0, η_1) being an approximate control or not. However, the marginal $\hat{\mu}_{0,E}$ which is of interest in this subsection will still be an approximation, and the ˆ-notation is sustained for $\hat{\mu}_{0,E}$. The dependency on m will not be of specific interest in this section as m will be fixed, so we also drop the subscript in the subsequent formulas.

It is vital to point out that in absence of a control on the singular behavior, η_0 fulfills the conditions formulated in Section II.2, in particular Remark II.2.1. Hence we assume that the set $\mathscr{M}_\infty^\eta$ contains precisely one pair of measures, referred to as $(\bar{\mu}_0, \bar{\mu}_1)$ in the following. Further, $\bar{\mu}_0$ is absolutely continuous with respect to Lebesgue measure.

To start, we have to investigate some regularity properties.

Proposition 1.53. *Let η_0 be of the form (1.51) and η_1 be of the form (1.51). Let A be the generator of the continuous behavior of X, and B be the generator of the singular behavior* *of X. Then, for $f \in C_c^2(E)$, $c_0, c_1 \in C(E \times U)$, the functions*

$$\begin{aligned} \bar{A}f(x) :\, x &\mapsto \int_U Af(x,u)\eta_0(du,x) \qquad & \bar{c}_0 :\, x &\mapsto \int_U c_0(x,u)\eta_0(du,x) \\ \bar{B}f(x) :\, x &\mapsto \int_U Bf(x,u)\eta_1(du,x) \qquad & \bar{c}_1 :\, x &\mapsto \int_U c_1(x,u)\eta_1(du,x) \end{aligned}$$

are bounded and continuous almost everywhere with respect to any measure that is absolutely continuous with respect to Lebesgue measure.

Proof. By the definition of η_0, compare (1.51), $x \mapsto \eta_0(\cdot, x) \in \mathcal{P}(U)$ is piecewise constant, and for each fixed x, $\eta_0(\cdot, x)$ is a discrete probability measure taking only finitely many values. Let $[x_j, x_{j+1})$ be an interval of the partition of E such that η_0 is constant. If $x \in [x_j, x_{j+1})$ we have that

$$\int_U b(x,u)\eta_0(du,x) = \sum_{i=1}^{2^{k_m^{(2)}}} b(x,u_i) \cdot \beta_{j,i}$$

is continuous, as b is continuous by assumption, and it is also bounded since b is bounded on $[x_j, x_{j+1}) \subset [x_j, x_{j+1}]$ as a continuous function on a compact set. So,

$$x \mapsto \int_U b(x,u)\eta_0(du,x)$$

can be split up over a finite number of intervals $[x_j, x_{j+1}]$, $j = 0, \ldots, 2^{k^{(1)}} - 1$, and on each of these intervals, it is bounded and continuous. That shows that this function is bounded, and continuous everywhere but at finitely many points of the state space E. This means that it is almost everywhere continuous with respect to Lebesgue measure, from which we can conclude that it is almost everywhere continuous with respect to any measure that is absolutely continuous with respect to Lebesgue measure. The same analysis can be carried out for the maps

$$x \mapsto \int_U \sigma^2(x,u)\eta_0(du,x), \;\; \bar{c}_0 : x \mapsto \int_U c_0(x,u)\eta_0(du,x) \text{ and } \bar{c}_1 : x \mapsto \int_U c_1(x,u)\eta_1(du,x).$$

This already proves two parts of the claim. For the other two parts, we readily conclude that the map

$$x \mapsto \int_U Af(x,u)\eta_0(du,x)$$

is bounded and almost everywhere continuous. Indeed, since in the case of the long-term average cost criterion, we have that

$$\begin{aligned}\int_U Af(x,u)\eta_0(du,x) &= \int_U f'(x)b(x,u) + \frac{1}{2}\sigma^2(x,u)f''(x)\eta_0(du,x)\\ &= f'(x)\int_U b(x,u)\eta_0(du,x) + \frac{1}{2}f''(x)\int_U \sigma^2(x,u)\eta_0(du,x)\end{aligned}$$

and f' and f'' are both bounded and continuous. Similarly, in case of the infinite horizon discounted cost criterion, we have

$$\int_U Af(x,u)\eta_0(du,x) = f'(x)\int_U b(x,u)\eta_0(du,x) + \frac{1}{2}f''(x)\int_U \sigma^2(x,u)\eta_0(du,x) - \alpha f(x)$$

and the conclusion is the same. The same analysis holds for the generator of the singular behavior, where $B(f,u) = f'(x)$ or $B(f,u) = f(a) - f(x)$. $\square$

Corollary 1.54. *The operators $\bar{A}$ and $\bar{B}$ defined by*

$$\begin{aligned}\bar{A} : \mathscr{D}_\infty \ni f \mapsto \bar{A}f(x) &:= \int_U Af(x,u)\eta_0(du,x)\\ \bar{B} : \mathscr{D}_\infty \ni f \mapsto \bar{B}f(x) &:= \int_U Bf(x,u)\eta_1(du,x)\end{aligned}$$

are continuous mappings into the space of bounded functions on E.

Proof. By Proposition 1.53, we have that $\bar{A}f$ and $\bar{B}f$ are indeed bounded functions for $f \in \mathscr{D}_\infty$. The analysis of Proposition II.1.18 can easily be applied to the given case, proving the claim. $\square$

With these results we can proceed to analyze measure that lie in the setting

$$\mathscr{M}_n^{\eta,l} = \{(\bar{\mu}_0, \bar{\mu}_1) \in \mathscr{M}_n^{\eta_0,\eta_1} | \bar{\mu}_1(E) \leq l\}$$

as introduced in Section III.1.3. In particular, we want to compare the cost criterion $J_n(\eta)$ to the exact cost criterion $\bar{J}(\eta)$.

Proposition 1.55. *Choose l large enough such that $\bar{\mu}_1(E) \leq l$. For $n \in \mathbb{N}$, let $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \in \mathscr{M}_n^{\eta,l}$, and assume that $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \Rightarrow (\tilde{\mu}_0, \tilde{\mu}_1)$ for some $(\tilde{\mu}_0, \tilde{\mu}_1)$. Then, $(\tilde{\mu}_0, \tilde{\mu}_1) = (\bar{\mu}_0, \bar{\mu}_1)$.*

Proof. By the uniqueness result, it suffices to show that $(\tilde{\mu}_0, \tilde{\mu}_1) \in \mathscr{M}_\infty^\eta$. For $f \in \mathscr{D}_\infty$, let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of functions with $f_k \in \mathscr{D}_k$ such that $f_k \to f$ in $\mathscr{D}_\infty$. As we have seen in Corollary 1.54, $\bar{A}$ and $\bar{B}$ are continuous operators mapping into the bounded functions, and we can use the dominated convergence theorem to deduce that

$$\int_E \bar{A}f(x)\bar{\mu}_0(dx) + \int_E \bar{B}f(x)\bar{\mu}_1(dx) = \lim_{k\to\infty} \left(\int_E \bar{A}f_k(x)\bar{\mu}_0(dx) + \int_E \bar{B}f_k(x)\bar{\mu}_1(dx) \right)$$

By Proposition 1.53, $\bar{A}f$ is continuous almost everywhere with respect to $\bar{\mu}_0$, and $\bar{B}f$ is continuous almost everywhere with respect to $\bar{\mu}_1$. Hence, by weak convergence (compare Proposition II.3.3) we have that

$$\begin{aligned}
&\lim_{k\to\infty} \left(\int_E \bar{A}f_k(x)\bar{\mu}_0(dx) + \int_E \bar{B}f_k(x)\bar{\mu}_1(dx) \right) \\
&\qquad = \lim_{k\to\infty} \lim_{n\to\infty} \left(\int_E \bar{A}f_k(x)\bar{\mu}_{0,n}(dx) + \int_E \bar{B}f_k(x)\bar{\mu}_{1,n}(dx) \right) \\
&\qquad = \lim_{k\to\infty} Rf_k = Rf
\end{aligned}$$

as $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \in \mathscr{M}_n^{\eta,l}$. This proves the claim. $\square$

Corollary 1.56. *Choose l large enough such that $\bar{\mu}_1(E) \leq l$. For $n \in \mathbb{N}$, let $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \in \mathscr{M}_n^{\eta,l}$, and assume that $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \Rightarrow (\tilde{\mu}_0, \tilde{\mu}_1)$ for some $(\tilde{\mu}_0, \tilde{\mu}_1)$. Then, $J_n(\eta) \to \bar{J}(\eta)$ as $n \to \infty$.*

Proof. By Proposition 1.55, we have that $(\tilde{\mu}_0, \tilde{\mu}_1) = (\bar{\mu}_0, \bar{\mu}_1)$. Again using Proposition 1.53, we can deduce the following statement, which proves the claim.

$$\begin{aligned}
\bar{J}(\eta) &= \int_E \bar{c}_0 \, d\bar{\mu}_0 + \int_E \bar{c}_1 \, d\bar{\mu}_1 \\
&= \lim_{n\to\infty} \left(\int_E \bar{c}_0 \, d\bar{\mu}_{n,0} + \int_E \bar{c}_1 \, d\bar{\mu}_{n,1} \right) = \lim_{n\to\infty} J_n(\eta)
\end{aligned}$$

□

The next results considers the situation when the convergence of the sequence $(\bar{\mu}_0, \bar{\mu}_1)$ is not given.

Theorem 1.57. *Choose l large enough such that $\bar{\mu}_1(E) \leq l$. For $n \in \mathbb{N}$, let $(\bar{\mu}_{0,n}, \bar{\mu}_{1,n}) \in \mathscr{M}_n^{\eta,l}$. Then, $J_n(\eta) \to \bar{J}(\eta)$ as $n \to \infty$.*

Proof. Assume the contrary. Then there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n \geq N$ such that $|\bar{J}(\eta) - J_n(\eta))| > \epsilon$. This allows for the construction of a subsequence $(\bar{\mu}_{0,n_k}, \bar{\mu}_{1,n_k})$ with

$$|\bar{J}(\eta) - J_{n_k}(\eta)| > \epsilon \quad \forall k \in \mathbb{N}. \tag{1.58}$$

Since $(\bar{\mu}_{0,n_k}, \bar{\mu}_{1,n_k})$ is a uniformly bounded sequence of measures over a compact space, and thus tight, by Theorem II.3.6, there is a weakly convergent sub-subsequence $(\bar{\mu}_{n_{k'}}, \bar{\mu}_{1,n_{k'}})$ with $(\bar{\mu}_{n_{k'}}, \bar{\mu}_{1,n_{k'}}) \Rightarrow (\bar{\mu}_0, \bar{\mu}_1)$. From Corollary 1.56 we know that $J_{n_{k'}}(\eta) \to \bar{J}(\eta)$, which is a contradiction to (1.58). From this contradiction we can conclude that $J_n(\eta) \to \bar{J}$ as $n \to \infty$. □

This result shows that a sequence of measures in $\mathscr{M}_n^{l,\eta}$ give cost criterion values that converge to the cost criterion value that is actually associated with η. Section III.1.3 show how such sequences can be attained by the approximate scheme introduced therein. Note that the results presented in the current section can be used in two ways. First, they can be used to set up an approximation scheme that simply evaluates the cost criterion for a given control. Second, they can be used to check the approximate cost criterion that was obtained by running a linear optimization with the approximation which was analyzed in Sections IV.1.1 and IV.1.2. This would be done in such a way that the obtained ϵ-optimal control would be fixed, and its cost criterion would be evaluated with more basis functions, which means a finer mesh for the discretization for the discretization of $\bar{\mu}_E$ is used. As no degrees of freedom for the controls have to be reserved, this comes at a smaller computational cost, and can provide a higher accuracy.

IV.2 Infinite Time Horizon Problems with Unbounded State Space

This section presents a convergence argument for the approximation for models with an unbounded state space as introduced in Section III.2. As pointed out in Section IV.1, particularly in Remark 1.50, the analysis carried out therein assures that for every $\epsilon > 0$, there is an m large enough such that an optimal solution to the K-reduced-concentrated, (n, m)-dimensional linear program is an ϵ-optimal solution to the K-reduced-concentrated, (n, ∞)-dimensional linear program. Further, for every $\delta > 0$ there is an n large enough such that an ϵ-optimal solution to the K-reduced-concentrated, (n, ∞)-dimensional linear program, is an $\hat{\epsilon} = 2\epsilon + \delta$-optimal solution to the K-reduced-concentrated, infinite dimensional linear program. Hence it remains to show that an $\hat{\epsilon}$-optimal solution to the K-reduced-concentrated, infinite dimensional linear program is an almost optimal solution to the original, infinite-dimensional linear program with an unbounded state space. This case is addressed in this section.

The following analysis shows similar features to the proofs seen in Section IV.1, but some subtleties have to be discussed due to the fact that we are considering an unbounded state space. Mainly, we will analyze how the K-reduced linear program relates to the infinite-dimensional linear program. From there, the desired result is readily concluded from an argument that connects optimal solutions of the K-reduced linear program to the solutions of the K-reduced-concentrated linear program.

Recall that we introduced the spaces $\mathscr{D}([-K,K]) = (C_c^2((-K,K)), \|\cdot\|_{\mathscr{D}})$ and $\mathscr{D}_\infty(\mathbb{R}) = (C_c^2(\mathbb{R}), \|\cdot\|_{\mathscr{D}})$. Clearly, $\cup_{K\in\mathbb{N}}\mathscr{D}([-K,K]) = \mathscr{D}_\infty(\mathbb{R})$. Hence, $\cap_{K\in\mathbb{N}}\dot{\mathscr{M}}_{\infty,[-K,K]} = \mathscr{M}_{\infty,\mathbb{R}}$ with the notation established in (III.2.4) and (III.2.5). Thus, we first analyze convergent sequences of measures in $\dot{\mathscr{M}}_{\infty,[-K,K]}$.

Lemma 2.1. *Let $\{\mu_K\}_{K\in\mathbb{N}}$ be a sequence of measures such that $\mu_K \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ for all $K \in \mathbb{N}$. Assume that $\mu_K \Rightarrow \mu$ for some $\mu \in \mathcal{P}(E\times U)$. Then, $\mu \in \mathscr{M}_{\infty,\mathbb{R}}$.*

Proof. Take $f \in \mathscr{D}_\infty(\mathbb{R})$. Then there is a $K_0 \in \mathbb{N}$ such that $\operatorname{supp}(f) \subset [-K,K]$ for all $K \geq K_0$. Also, Af is uniformly continuous and bounded since f, f' and f'' have compact support. Hence, by weak convergence of measures,

$$\int_{E\times U} Af\, d\mu = \lim_{K\to\infty}\int_{E\times U} Af\, d\mu_K = \lim_{K\to\infty} Rf = Rf.$$

□

The investigation of ϵ-optimal, converging sequences $\{\mu_K^\epsilon\}_K \in \mathbb{N}$ is in order. In particular, a measure μ_K^ϵ is ϵ-optimal in $\dot{\mathscr{M}}_{\infty,[-K,K]}$, or an ϵ-optimal solution to the K-reduced infinite-dimensional linear program if

$$\int_{E\times U} c_0 d\mu_K^\epsilon - \int_{E\times U} c_0 d\mu_K < \epsilon$$

holds. Similar definitions hold for the K-reduced-concentrated infinite-dimensional and the infinite-dimensional linear program. From now on, we have to rely on the assumption that there exists a $\mu \in \mathscr{M}_{\infty,\mathbb{R}}$ such that $\int_{E\times U} c_0\, d\mu < \infty$. This immediately ensures the existence of an ϵ-optimal solution in $\mathscr{M}_{\infty,\mathbb{R}}$ with finite costs. As $\mathscr{M}_{\infty,\mathbb{R}} \subset \dot{\mathscr{M}}_{\infty,[-K,K]}$, we also have that $\int_{E\times U} c_0 \, d\mu_K < \infty$ for some $\mu_K \in \dot{\mathscr{M}}_{\infty,[-K,K]}$. This guarantees the existence of an ϵ-optimal solution in $\mathscr{M}_{\infty,[-K,K]}$ with finite costs.

We need to ensure that a sequence of ϵ-optimal measures in $\dot{\mathscr{M}}_{\infty,[-K,K]}$ is tight. This will use the fact that c_0 is increasing in $|x|$, as introduced in Definition III.2.2.

Lemma 2.2. *For each $K \in \mathbb{N}$, assume that $\mu_K^\epsilon \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ and that μ_K^ϵ is an ϵ-optimal solution to the K-reduced infinite-dimensional linear program. Then, $\{\mu_K^\epsilon\}_{K\in\mathbb{N}}$ is tight.*

Proof. Assume the opposite. Then, there exists a $\delta > 0$ such that for all $L \in \mathbb{N}$, there exists a $K \in \mathbb{N}$ with $\mu_K^\epsilon \left([-L, L]^C \times U\right) \geq \delta$. Using that c_0 is increasing in $|x|$, choose L large enough such that $c_0(x, u) > \left(\int_{E\times U} c_0 \, d\mu + \epsilon\right) \cdot \frac{1}{\delta}$ for all $x \in [-L, L]^C$, uniformly in u, where μ is the measure in $\dot{\mathscr{M}}_{\infty,[-K,K]}$ with finite costs which is assumed to exist. By the hypotheses that $\{\mu_K^\epsilon\}_{K\in\mathbb{N}}$ is not tight, and $\mu_K^\epsilon \left([-L, L]^C \times U\right) \geq \delta$, for some $K \in \mathbb{N}$,

$$\int_{E\times U} c_0 \, d\mu_K^\epsilon > \int_{[-L,L]^C\times U} c_0 \, d\mu_K^\epsilon > \delta \cdot \left(\int_{E\times U} c_0 \, d\mu + \epsilon\right) \cdot \frac{1}{\delta} = \int_{E\times U} c_0 \, d\mu + \epsilon,$$

which is a contradiction since μ_K^ϵ is ϵ-optimal and $\mu \in \dot{\mathscr{M}}(\mathbb{R})$. $\square$

Remark 2.3. *The tightness can also be achieved by introducing the constraint that with some inf-compact function d, $\int_{E\times U} d(x, u) \, \mu(dx \times du) \leq M < \infty$ has to hold for any measure in $\mathscr{M}_{\infty,\mathbb{R}}$, using the same argument as above. This will become relevant in the numerical example presented in Section V.2.2.*

We proceed to investigate the optimality of limits of ϵ-optimal sequences in $\dot{\mathscr{M}}_{\infty,[-K,K]}$.

Lemma 2.4. *For each $K \in \mathbb{N}$, assume that $\mu_K^\epsilon \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ and that μ_K^ϵ is an ϵ-optimal solution to the K-reduced infinite-dimensional linear program. Assume that $\mu_K^\epsilon \Rightarrow \hat{\mu}$ for some $\hat{\mu} \in \mathcal{P}(E \times U)$. Then $\hat{\mu}$ is ϵ-optimal in $\mathscr{M}_{\infty,\mathbb{R}}$.*

Proof. By Lemma 2.1, $\hat{\mu} \in \mathscr{M}_{\infty,\mathbb{R}}$. Now assume that $\hat{\mu}$ would not be ϵ-optimal. Then, there exists a $\mu \in \mathscr{M}_{\infty,\mathbb{R}}$ such that

$$\infty > \int_{E\times U} c_0 \, d\hat{\mu} > \int_{E\times U} c_0 \, d\mu + \epsilon.$$

But as $\int_{E\times U} c_0 \, d\hat{\mu} < \infty$, there is an L_1 large enough such that for all $L \geq L_1$,

$$\int_{E\times U} c_0 \, d\hat{\mu} > \int_{[-L,L]\times U} c_0 \, d\hat{\mu} > \int_{E\times U} c_0 \, d\mu + \epsilon$$

holds. Find an $L \geq L_1$ such that $c_0(x,u) \geq L_1$ on $[-L,L]^C$, uniformly in u, which is possible due to the fact that c_0 is increasing in $|x|$. Define

$$\bar{c}_0(x) = \begin{cases} c_0(-L,u) & x < -L \\ c_0(x,u) & x \in [-L,L] \\ c_0(L,u) & x > L \end{cases}.$$

Observe that $\bar{c}_0$ is continuous and bounded, but also

$$\int_{E\times U} c_0 \, d\hat{\mu} \geq \int_{E\times U} \bar{c}_0 \, d\hat{\mu} \geq \int_{[-L,L]\times U} c_0 \, d\hat{\mu} > \int_{E\times U} c_0 \, d\mu + \epsilon \tag{2.5}$$

holds. Clearly, for any $K \in \mathbb{N}$ we have that

$$\int_{E\times U} c_0 \, d\mu_K^\epsilon \geq \int_{E\times U} \bar{c}_0 \, d\mu_K^\epsilon. \tag{2.6}$$

On the other hand, by weak convergence, which means that $\displaystyle\int_{E\times U} \bar{c}_0 \, d\mu_K^\epsilon \to \int_{E\times U} \bar{c}_0 \, d\hat{\mu}$, and by (2.5) there is a K large enough such that

$$\int_{E\times U} \bar{c}_0 \, d\mu_K^\epsilon > \int_{E\times U} c_0 \, d\mu + \epsilon.$$

Together with (2.6) this contradicts the fact that μ_K^ϵ is ϵ-optimal, since $\mu \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ for all K. $\square$

Remark 2.7. *If c_0 is bounded in x, and the tightness of the measures is attained by using another function d, see Remark 2.3, the construction of the function $\bar{c}_0$ is not needed.*

Lemma 2.4 is the counterpart of Lemma 1.10 in Section IV.1.1. It is used to prove the final statements regarding the K-reduced linear program, given by Lemma 2.8, Proposition 2.9 and Theorem 2.10. The arguments for these three results are identical to those given in the proofs of Lemma 1.12, Proposition 1.16 and Theorem 1.17. Hence, we only state the results.

Lemma 2.8. *For each $K \in \mathbb{N}$, assume that $\mu_K^\epsilon \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ and that μ_K^ϵ is an ϵ-optimal solution to the K-reduced infinite-dimensional linear program. Then, for $\delta > 0$, there is a $z \in \mathbb{R}$ and a $N(\delta) \in \mathbb{N}$ such that*

$$J(\mu_K^\epsilon) \in \left(z - \frac{\epsilon}{2} - \delta, z + \frac{\epsilon}{2} + \delta\right) \quad \forall K \geq N(\delta).$$

Proposition 2.9. *Let $\left[z - \frac{\epsilon}{2}, z + \frac{\epsilon}{2}\right]$ be the interval from Lemma 2.8, and set*

$$J^* = \inf\left(J(\mu) : (\mu) \in \mathscr{M}_{\infty,\mathbb{R}}\right).$$

Then,

$$z - \frac{3\epsilon}{2} \leq J^* \leq z + \frac{\epsilon}{2}.$$

Theorem 2.10. *For each $n \in \mathbb{N}$, assume that $\mu_n^\epsilon \in \mathscr{M}_{n,\mathbb{R}}$ and that for $n \in \mathbb{N}$, μ_n^ϵ is an ϵ-optimal solution for the K-reduced ∞-dimensional problem. Then, for $\delta > 0$, there exists an $N(\delta)$ such that*

$$|J(\mu_n^\epsilon) - J^*| \leq 2\epsilon + \delta$$

for all $n \geq N(\delta)$.

Theorem 2.10 proves the ϵ-optimality of the K-reduced infinite-dimensional problem. However, this includes measures that can have positive mass on any subset in $\mathbb{R}$, and are thus not computationally attainable. The next results reveal that it suffices to look only for solutions of the K-reduced-concentrated infinite-dimensional linear program, which only includes measures that have full mass on $[-K, K]$.

Lemma 2.11. *Let K be larger than the constant K_0 introduced in Definition III.2.3. Let μ_K be a measure in $\dot{\mathscr{M}}_{\infty,[-K,K]}$ with $\mu_K([-K, K]) < 1$. Then there exists a measure $\tilde{\mu}_K \in \mathscr{M}_{\infty,[-K,K]}$ with $\int_{E\times U} c_0\, d\tilde{\mu}_K \leq \int_{E\times U} c_0\, \mu_K$.*

Proof. Let $\tau = \mu_K([-K, K])$, and let u^- be the function described in Definition III.2.3. By Theorem II.2.29 in the case of the long-term average cost criterion, and by Theorem II.2.31 in the case of the discounted infinite horizon criterion, take a solution $(\hat{\mu}_0, \hat{\mu}_1)$ to the singular linear program with a bounded state space $[-K, K]$, with $x_0 = K$ and reflections at both end of the state space $\{-K\}$ and $\{K\}$, under a control satisfying $\eta_0(u^-(x), x) = 1$. Then, for $f \in \mathscr{D}([-K, K]) = (C^2((-K, K)), \|\cdot\|_{\mathscr{D}}) \subset \mathscr{D}_\infty = (C^2([-K, K]), \|\cdot\|_{\mathscr{D}})$, for either the long-term average cost criterion (upper row) or the infinite horizon discounted criterion (lower row),

$$\int_{E\times U} Af\, d\hat{\mu}_0 = \begin{cases} -\int_{E\times U} Bf\, \hat{\mu}_1 \\ -\alpha f(K) - \int_{E\times U} Bf\, \hat{\mu}_1 \end{cases} = 0$$

holds, as the support of f and its derivatives is fully contained in $(-K, K)$. Set $\tilde{\mu}_K^A = (1-\tau)\hat{\mu}_0$ and for $F \times V \in \mathscr{B}([-K, K] \times U)$, define $\tilde{\mu}_K(F \times V) = \tilde{\mu}_K^A(F \times V) + \mu_K(F \times V)$. Clearly, $\tilde{\mu}_K([-K, K]) = 1$ and for $f \in \mathscr{D}([-K, K])$

$$\int_{E\times U} Af\, d\tilde{\mu}_K = \int_{E\times U} Af\, \tilde{\mu}_K^A + \int_{E\times U} Af\, d\hat{\mu}_K = 0 + Rf = Rf,$$

so $\tilde{\mu}_K \in \mathscr{M}_{\infty,[-K,K]}$. But also,

$$\begin{aligned}
\int_{E\times U} c_0\, d\tilde{\mu}_K &= \int_{E\times U} c_0\, d\tilde{\mu}_K^A + \int_{E\times U} c_0\, d\mu_K \\
&= \int_{[-K,K]\times U} c_0\, d\tilde{\mu}_K^A + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq \int_{[-K,K]\times U} \sup\left\{c_0(x,u^-(x))|x\in[-K,K]\right\} d\tilde{\mu}_K^A + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq \tilde{\mu}_K^A\left([-K,K]\times U\right)\cdot \inf\left\{c_0(x,u)|x\in[-K,K]^C,\, u\in U\right\} + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq (1-\tau)\cdot \inf\left\{c_0(x,u)|x\in[-K,K]^C,\, u\in U\right\} + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq \mu_K\left([-K,K]^C\times U\right)\cdot \inf\left\{c_0(x,u)|x\in[-K,K]^C,\, u\in U\right\} + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq \int_{[-K,K]^C\times U} \inf\left\{c_0(x,u)|x\in[-K,K]^C,\, u\in U\right\} d\mu_K + \int_{[-K,K]\times U} c_0\, d\mu_K \\
&\leq \int_{[-K,K]^C\times U} c_0(x,u)\, d\mu_K + \int_{[-K,K]\times U} c_0\, d\mu_K = \int_{E\times U} c_0 d\mu_K,
\end{aligned}$$

which proves the claim. □

Proposition 2.12. *An ϵ-optimal solution $\mu_K^\epsilon \in \mathscr{M}_{\infty,[-K,K]}$ to the K-reduced-concentrated infinite dimensional linear program is an ϵ-optimal solution in $\dot{\mathscr{M}}_{\infty,[-K,K]}$, to the K-reduced infinite dimensional linear program.*

Proof. Assume the existence of $\mu_K \in \dot{\mathscr{M}}_{\infty,[-K,K]}$ with $J(\mu_K) < J(\mu_K^\epsilon) + \epsilon$. By Lemma 2.11, there is a measure $\tilde{\mu}_K \in \mathscr{M}_{\infty,[-K,K]}$ with $J(\tilde{\mu}_K) \leq J(\mu_K)$. But this contradicts the ϵ-optimality of μ_K^ϵ. □

By this result, it suffices to solve the K-reduced-concentrated problem ϵ-optimally. However, this can be attained using the same approximation technique used for the problems with a bounded state space. The following theorem summarizes the situation.

Theorem 2.13. *For every $K > 0$, let μ_K^ϵ be a ϵ-optimal solution to the K-reduced-concentrated infinite dimensional problem. Then for K large enough, μ_K^ϵ is a $2\epsilon + \delta$-optimal for the infinite-dimensional linear program with unbounded state space.*

Numerical Examples

This chapter discusses several stochastic control problems in infinite time and presents approximate solutions which were obtained with the numerical scheme introduced in Section II.4. A variety of different classes of problems is considered. In particular, we consider models with either bounded or unbounded state space, problems that do or do not feature costs of control, and problems featuring singular control, given either by an adjustable reflection, or a jump of adjustable size.

When analytic solutions are attainable, we analyze the numerical convergence of the approximation scheme. In other cases we limit the considerations to parameter sweeps, investigating how different parameter choices influence the performance of the numerical scheme, and the approximate solution to the various problems.

V.1 Infinite Time Horizon Problems with Bounded State Space

This section will present results using the presented approximation technique on stochastic control problems that feature a bounded state space. Analytic solutions are available for selected problems. This allows for a numerical analysis of the convergence. To this end, we frequently consider varying levels of discretization. We adopt the notion of the discretization level $k \equiv (k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)})$ in its enhanced version as introduced in Section III.1.4. Frequently, we will consider $\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ and $k^{(4)}$ separately, since the earlier parameters represent the degrees of freedom, and the later parameter represents the number of constraints.

The models regarded in this section embrace the control of a Brownian motion process, without (Section V.1.1) or with (Section V.1.2) costs of control, the 'classic' modified bounded follower problem (Section V.1.3) and a modification thereof featuring a variable jump size (Section V.1.4). Further, an example featuring Stochastic Logistic Growth and singular control is presented (Section V.1.5).

V.1.1 The Simple Particle Problem without Cost of Control

Starting with a first simple example, consider a stochastic control problem with state space $E = [-1, 1]$ and control space $U = [-1, 1]$, such that the process is governed by the SDE

$$X_t = x_0 + \int_0^t u(X_s)ds + \sigma W_t + \xi_t, \tag{1.1}$$

where ξ_t is a process capturing the singular behavior of X, given by a reflection at both boundaries of E. In other words, we have that $\xi_t = L^X_{\{-1\}} - L^X_{\{1\}}$ where $L^X_{\{a\}}$ is the local time of X at $a \in E$. We adopt the long-term average cost criterion

$$J \equiv \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t c_0(X_s, u_s)\, ds + \int_0^t c_1(X_s, u_s)\, d\xi_s\right]$$

with cost functions given by $c_0(x, u) = x^2$ and $c_1(x, u) \equiv c_1$ for some $c_1 \in \mathbb{R}$. Note that there is no control of the singular behavior, so the optimal control problem posed by this set-up only deals with finding the relaxed control of the continuous part. We can readily convince ourselves that with the given cost structure, the optimal control is given by a bang-bang control. To be precise, for every $x \in E$, the optimal control $\eta^*_0(\cdot, x)$ has to be a degenerate probability distribution putting full mass on the values $\{-1, 0, 1\}$ as specified in Table V.1. Hence it can be described by a deterministic function $u : E \mapsto U$. Note that the value for $x = 0$ can technically be chosen arbitrarily, since the continuous expected occupation

u	$-1 \leq x < 0$	$x = 0$	$0 < x \leq 1$
1	1	0	0
0	0	1	0
-1	0	0	1

Table V.1: Probabilities for optimal control, simple particle problem without costs of control

measure μ_0 does not put mass on this point.

An analytic treatment of this problem can be found in Appendix C.1. In particular, it is shown that for an arbitrary constant $-1 \leq c_2 \leq 1$,

$$p(x) = \frac{e^{\int_{c_2}^{x} \frac{2}{\sigma^2} u(v) dv}}{\int_{-1}^{1} e^{\int_{c_2}^{w} \frac{2}{\sigma^2} u(v) dv} dw}. \tag{1.2}$$

is the density function of the state space marginal of μ_0 under the control that is specified in Table V.1.

The results of an sample computation are shown in Figures V.1 and V.2. The first figure shows a visualization of the relaxed control, the second figure shows the approximate state space density, which is the density of the state space marginal $\hat{\mu}_{0,E}$. Figure V.1 has to be understood as follows. The x-axis, labeled 'state space' specifies the state the process is in. The y-axis, labeled 'control space' specifies the control value from the given discrete set approximating the control space, in this case, $\{-1, 0, 1\}$. The z-axis, labeled 'probability' gives the probability that the specified control is chosen when the process is in state x, in other words, the z-axis gives $\hat{\eta}_0(\{y\}, x)$. The red dots indicate the mesh points of the state space E. These are the points where the control can change its behavior, and the solid black lines in between the red dots indicate that the control is constant along the x-axis until it hits the next red dot. In Figure V.2, the actual state space density (1.2) was omitted from the plot since it would be barely distinguishable from the approximate density.

The parameter configuration for the SDE in this computation is displayed on the left in Table V.2. The choice of the starting point x_0 is irrelevant for the long-term average criterion, but is given for the sake of completeness. The discretization level of this computation was $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (2, 1, 10)$ and $k^{(4)} = 10$. Recall that n, the number of constraint functions, equals $n = 2^{k^{(4)}} + 2$, by the meshing approach presented in Section III.1.4. The right part of Table V.2 shows the results of a computation with this particular configuration. It displays the analytic values v, the approximate values $\hat{v}$, the absolute error e_a and the relative error e_r of the objective function J and the reflection weights w_1 and w_2, respectively. The last column gives the L^1-error for the state space density.

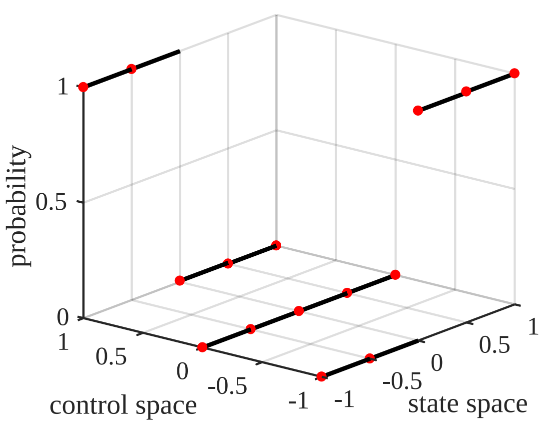

Figure V.1: Computed control, simple particle problem without costs of control

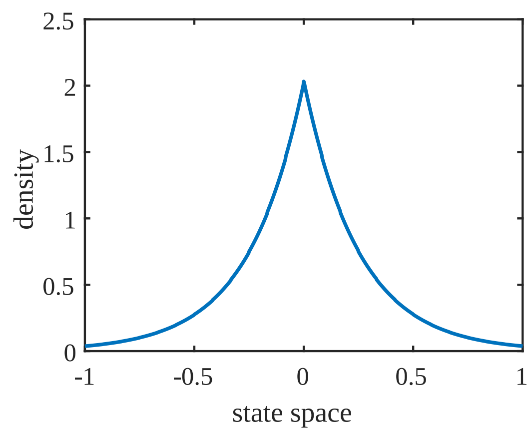

Figure V.2: State space density, simple particle problem without costs of control

parameter	value
x_0	0.1
σ	$\sqrt{2}/2$
c_1	0.01

	J	w_1	w_2	$\Vert p - \hat{p} \Vert_{L^1}$
v	$9.7201 \cdot 10^{-2}$	$9.3287 \cdot 10^{-3}$	$3.882878 \cdot 10^{-3}$	$9.3287 \cdot 10^{-3}$
$\hat{v}$	$9.7202 \cdot 10^{-2}$	$9.2926 \cdot 10^{-3}$	$9.3651 \cdot 10^{-3}$	–
e_a	$1.5036 \cdot 10^{-6}$	$3.612 \cdot 10^{-5}$	$3.6409 \cdot 10^{-5}$	–
e_r	$1.5468 \cdot 10^{-6}$	$3.872 \cdot 10^{-5}$	$3.9029 \cdot 10^{-5}$	–

Table V.2: Parameter configuration (left), results and errors (right), simple particle problem without costs of control

Since an analytic solution to the problem is known, we illustrate the numerical performance of the proposed approximate scheme in two ways. First, we investigate the various errors, as seen in the right part of Table V.2, for increasing discretization levels, starting at $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (3, 1, 3)$ and $k^{(4)} = 3$ and going up to $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (12, 1, 12)$ and

$k^{(4)} = 12$. Second, we fix $k^{(1)} = 2$, $k^{(3)} = k^{(4)} = 10$ and analyze the performance of the algorithm from $k^{(2)} = 1$ to $k^{(2)} = 8$.

In the first case, the fact that $k^{(2)}$ remains fixed means we assert that the optimal control has to have bang-bang form. Increasing $k^{(3)}$ introduces a higher accuracy for the computation, while increasing $k^{(1)}$ tests how well the solver finds the optimal control, which switches from 1 to -1 at $x = 0$, even when more choices of 'switching points' are available. The computations were conducted for two different meshing approaches. First, we use a mesh for the density $\hat{p}$ of the continuous expected occupation measure given by $2^{k^{(3)}} + 1$ mesh points. The results are displayed in Tables V.3 and V.4. Then, we use a mesh that adds in one additional mesh point, see Remark III.1.33. These results are displayed in Tables V.5 and V.6.

$k^{(4)}$	$\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$	T	J	e_a	e_r	e_{L^1}
v	-	-	0.09720	-	-	-
3	$(3, 1, 3)$	0.0135	0.12136	$2.4156 \cdot 10^{-2}$	$2.4852 \cdot 10^{-1}$	$4.0328 \cdot 10^{-1}$
4	$(4, 1, 4)$	0.0094	0.10333	$6.1245 \cdot 10^{-3}$	$6.3009 \cdot 10^{-2}$	$2.2326 \cdot 10^{-1}$
5	$(5, 1, 5)$	0.0101	0.09874	$1.5374 \cdot 10^{-3}$	$1.5817 \cdot 10^{-2}$	$1.1775 \cdot 10^{-1}$
6	$(6, 1, 6)$	0.0121	0.09759	$3.8477 \cdot 10^{-4}$	$3.9585 \cdot 10^{-3}$	$6.0509 \cdot 10^{-2}$
7	$(7, 1, 7)$	0.0184	0.09730	$9.6218 \cdot 10^{-5}$	$9.8989 \cdot 10^{-4}$	$3.0678 \cdot 10^{-2}$
8	$(8, 1, 8)$	0.0421	0.09722	$2.4056 \cdot 10^{-5}$	$2.4749 \cdot 10^{-4}$	$1.5445 \cdot 10^{-2}$
9	$(9, 1, 9)$	0.1197	0.09721	$6.0141 \cdot 10^{-6}$	$6.1873 \cdot 10^{-5}$	$7.7512 \cdot 10^{-3}$
10	$(10, 1, 10)$	0.4164	0.09720	$1.5035 \cdot 10^{-6}$	$1.5468 \cdot 10^{-5}$	$3.8811 \cdot 10^{-3}$
11	$(11, 1, 11)$	1.5987	0.09720	$3.7588 \cdot 10^{-7}$	$3.8671 \cdot 10^{-6}$	$1.9436 \cdot 10^{-3}$
12	$(12, 1, 12)$	7.7851	0.09720	$9.3977 \cdot 10^{-8}$	$9.6684 \cdot 10^{-7}$	$9.7091 \cdot 10^{-4}$

Table V.3: Results for optimality criterion and density, simple particle problem without costs of control, varying discretization levels, no extra mesh point

In Tables V.3 and V.5, respectively, we show the approximate value of the cost criterion J, its absolute and relative errors e_a and e_r as well as the error of the density in L^1-norm, denoted e_{L^1}. T is the execution time in seconds, which is an average time taken from 1000 repetitions of the same program run. In Tables V.4 and V.6, respectively, we display the

$k^{(4)}$	w_1	e_a	e_r	w_2	e_a	e_r
v	0.0093	-	-	0.0093	-	-
3	0.0170	$7.6660 \cdot 10^{-3}$	$8.2176 \cdot 10^{-1}$	0.0063	$3.0675 \cdot 10^{-3}$	$3.2882 \cdot 10^{-1}$
4	0.0123	$3.0036 \cdot 10^{-3}$	$3.2198 \cdot 10^{-1}$	0.0075	$1.8304 \cdot 10^{-3}$	$1.9621 \cdot 10^{-1}$
5	0.0106	$1.3201 \cdot 10^{-3}$	$1.4151 \cdot 10^{-1}$	0.0083	$1.0252 \cdot 10^{-3}$	$1.0990 \cdot 10^{-1}$
6	0.0088	$5.4484 \cdot 10^{-4}$	$5.8404 \cdot 10^{-2}$	0.0099	$6.1868 \cdot 10^{-4}$	$6.6320 \cdot 10^{-2}$
7	0.0090	$2.8107 \cdot 10^{-4}$	$3.0130 \cdot 10^{-2}$	0.0096	$2.9954 \cdot 10^{-4}$	$3.2110 \cdot 10^{-2}$
8	0.0095	$1.4739 \cdot 10^{-4}$	$1.5800 \cdot 10^{-2}$	0.0092	$1.4277 \cdot 10^{-4}$	$1.5305 \cdot 10^{-2}$
9	0.0093	$7.1954 \cdot 10^{-5}$	$7.7132 \cdot 10^{-3}$	0.0094	$7.3109 \cdot 10^{-5}$	$7.8370 \cdot 10^{-3}$
10	0.0094	$3.6409 \cdot 10^{-5}$	$3.9029 \cdot 10^{-3}$	0.0093	$3.6120 \cdot 10^{-5}$	$3.8720 \cdot 10^{-3}$
11	0.0093	$1.8096 \cdot 10^{-5}$	$1.9398 \cdot 10^{-3}$	0.0093	$1.8168 \cdot 10^{-5}$	$1.9476 \cdot 10^{-3}$
12	0.0093	$9.0751 \cdot 10^{-6}$	$9.7282 \cdot 10^{-4}$	0.0093	$9.0570 \cdot 10^{-6}$	$9.7088 \cdot 10^{-4}$

Table V.4: Results for weights of singular occupation measure, simple particle problem without costs of control, varying discretization levels, no extra mesh point

weights $w_1 = \hat{\mu}_{1,n,m}(\{e_l\})$ and $w_2 = \hat{\mu}_{1,n,m}(\{e_r\})$ of the singular expected occupation measure on the left and right boundary, respectively, with their absolute and relative errors e_a and e_r. In all tables, the first rows display the exact values v of the quantities under consideration. Tables V.3 and V.4 show a strong performance of the numerical method, with relative errors in the cost criterion being as low as $9.6684 \cdot 10^{-7}$ for the highest discretization level, for a rather short computation time. However, note that the computation time drastically increases between $k^{(4)} = 11$ and $k^{(4)} = 12$, which could indicate that the method becomes increasingly ill-conditioned. However, the L^1-error of the density and the computed values for w_1 and w_2 continue to show good convergence.

The introduction of an additional mesh point allows for a dramatic increase in accuracy, as Tables V.5 and V.6 reveal. The absolute error for the cost criterion is brought down to $3.0321 \cdot 10^{-12}$ for the discretization level with $k^{(4)} = 12$, with the machine accuracy of the utilized computer sitting at $2.2204 \cdot 10^{-16}$. This comparatively stronger performance can also be observed in the error measures for w_1 and w_2. However, the L^1-error of the density

$k^{(4)}$	$(k^{(1)}, k^{(2)}, k^{(3)})$	T	J	e_a	e_r	e_{L^1}
v	-	-	0.09720	-	-	-
3	$(3, 1, 3)$	0.0086	0.09762	$4.2414 \cdot 10^{-4}$	$4.3635 \cdot 10^{-3}$	$2.5784 \cdot 10^{-1}$
4	$(4, 1, 4)$	0.0087	0.09723	$3.4260 \cdot 10^{-5}$	$3.5247 \cdot 10^{-4}$	$1.2168 \cdot 10^{-1}$
5	$(5, 1, 5)$	0.0096	0.09720	$2.3294 \cdot 10^{-6}$	$2.3964 \cdot 10^{-5}$	$6.1183 \cdot 10^{-2}$
6	$(6, 1, 6)$	0.0115	0.09720	$1.5012 \cdot 10^{-7}$	$1.5444 \cdot 10^{-6}$	$3.0887 \cdot 10^{-2}$
7	$(7, 1, 7)$	0.0180	0.09720	$9.5006 \cdot 10^{-9}$	$9.7742 \cdot 10^{-8}$	$1.5538 \cdot 10^{-2}$
8	$(8, 1, 8)$	0.0412	0.09720	$5.9711 \cdot 10^{-10}$	$6.1431 \cdot 10^{-9}$	$7.7950 \cdot 10^{-3}$
9	$(9, 1, 9)$	0.1199	0.09720	$3.7450 \cdot 10^{-11}$	$3.8529 \cdot 10^{-10}$	$3.9042 \cdot 10^{-3}$
10	$(10, 1, 10)$	0.4134	0.09720	$2.6571 \cdot 10^{-12}$	$2.7336 \cdot 10^{-11}$	$1.9538 \cdot 10^{-3}$
11	$(11, 1, 11)$	1.5820	0.09720	$9.1287 \cdot 10^{-13}$	$9.3916 \cdot 10^{-12}$	$9.7734 \cdot 10^{-4}$
12	$(12, 1, 12)$	6.4320	0.09720	$3.0321 \cdot 10^{-12}$	$3.1194 \cdot 10^{-11}$	$4.8878 \cdot 10^{-4}$

Table V.5: Results for optimality criterion and density, simple particle problem without costs of control, varying discretization levels, one additional mesh point

remains on the same level. The cause of this strong convergence is not known to the author, and the given example of the simple particle problem remains the only example dealt with in this thesis which benefits from the introduction of an additional mesh point in such a drastic way. In both cases, with or without an additional mesh point, the constraint matrix was checked and indeed had full rank, compare Remark IV.1.34. Thus, this example fulfills the assumptions of the convergence theory of Section III.2, and we can only conjecture that this phenomenon hints at the effectiveness of adaptive meshing approaches, which lie beyond the scope of this thesis. The reason for this conjecture is that the additional mesh point was introduced into the first interval right of the middle of the state space, where the density decreases rapidly, and it is believed that the accuracy benefits the most when an additional mesh point is introduced in this area.

With the second example, where we fix $k^{(1)} = 2$, $k^{(3)} = k^{(4)} = 10$ and vary $k^{(2)} = 1$ to $k^{(2)} = 8$, we can investigate how well the numerical scheme performs when the assumption that the optimal control is of bang-bang type is dropped. In other words, we assume no

$k^{(4)}$	w_1	e_a	e_r	w_2	e_a	e_r
v	0.0093	-	-	0.0093	-	-
3	0.0098	$4.6337 \cdot 10^{-4}$	$4.9671 \cdot 10^{-2}$	0.0086	$7.4802 \cdot 10^{-4}$	$8.0185 \cdot 10^{-2}$
4	0.0095	$1.7265 \cdot 10^{-4}$	$1.8508 \cdot 10^{-2}$	0.0091	$1.9565 \cdot 10^{-4}$	$2.0973 \cdot 10^{-2}$
5	0.0094	$4.7030 \cdot 10^{-5}$	$5.0415 \cdot 10^{-3}$	0.0093	$4.8593 \cdot 10^{-5}$	$5.2090 \cdot 10^{-3}$
6	0.0093	$1.2005 \cdot 10^{-5}$	$1.2869 \cdot 10^{-3}$	0.0093	$1.2106 \cdot 10^{-5}$	$1.2977 \cdot 10^{-3}$
7	0.0093	$3.0168 \cdot 10^{-6}$	$3.2339 \cdot 10^{-4}$	0.0093	$3.0232 \cdot 10^{-6}$	$3.2408 \cdot 10^{-4}$
8	0.0093	$7.5518 \cdot 10^{-7}$	$8.0952 \cdot 10^{-5}$	0.0093	$7.5558 \cdot 10^{-7}$	$8.0995 \cdot 10^{-5}$
9	0.0093	$1.8886 \cdot 10^{-7}$	$2.0245 \cdot 10^{-5}$	0.0093	$1.8888 \cdot 10^{-7}$	$2.0247 \cdot 10^{-5}$
10	0.0093	$4.7218 \cdot 10^{-8}$	$5.0616 \cdot 10^{-6}$	0.0093	$4.7219 \cdot 10^{-8}$	$5.0617 \cdot 10^{-6}$
11	0.0093	$1.1805 \cdot 10^{-8}$	$1.2655 \cdot 10^{-6}$	0.0093	$1.1805 \cdot 10^{-8}$	$1.2654 \cdot 10^{-6}$
12	0.0093	$2.9517 \cdot 10^{-9}$	$3.1642 \cdot 10^{-7}$	0.0093	$2.9506 \cdot 10^{-9}$	$3.1630 \cdot 10^{-7}$

Table V.6: Results for weights of singular occupation measure, simple particle problem without costs of control, varying discretization levels, one additional mesh point

a priori knowledge on the control and see if the scheme still picks the optimal bang-bang control as a result. Table V.7 shows the execution time T in seconds, averaged over 1000 executions and the computed value for the cost criterion for varying $k^{(2)}$. For the displayed computation runs, we did not introduce an additional mesh point to the mesh. Naturally, the computing time increases as the number of unknowns increases with $k^{(2)}$, recalling that we consider $2^{k^{(2)}} + 1$ possible control values. The computed cost function value remains identical, as (thereby) does the absolute error e_a. For any discretization level, the switching point of the control was located at $x = 0.5$, hence the computed optimal controls were indeed the analytic optimal controls, showing that the numerical scheme works well even when operating on fewer *a priori* assumptions.

V.1.2 The Simple Particle Problem with Cost of Control

We investigate how introducing costs of control influences the solution of a stochastic control problem. We consider the process and its configuration from Section V.1.1, but use the cost

$k^{(2)}$	T	J	e_a
1	1.6379	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
2	2.7625	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
3	4.8887	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
4	8.8061	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
5	14.8037	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
6	29.267	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
7	58.7547	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$
8	115.2589	$9.7202036596 \cdot 10^{-2}$	$1.503538 \cdot 10^{-6}$

Table V.7: Results, simple particle problem without costs of control, increasing number of control values

function $c_0(x,u) = x^2 + u^2$. In other words, we will be 'charged' for using the control, and the optimal control is not of bang-bang type any longer. The cost of the reflections are still constant, with $c_1(x,u) \equiv c$. Following Example III.1.7, we have to choose $k^{(2)} = m+3$ in order to have a sufficiently good approximation of the cost criterion. The highest discretization to produce results in reasonable time was $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (7, 10, 7)$ and $k^{(4)} = 7$, which results in $n = 2^{k^{(4)}} + 2$ constraint functions. In order to increase accuracy of the evaluation, a two-step approach was utilized. First, the optimization problem was solved using $(k_1^{(1)}, k_1^{(2)}, k_1^{(3)}) = (7, 10, 7)$ and $k_1^{(4)} = 7$ (referred to as discretization level k_1 in the following). Then, the resulting control was fixed and the cost criterion was evaluated with the discretization level $(k_2^{(1)}, k_2^{(2)}, k_2^{(3)}) = (7, 10, 10)$ and $k_2^{(4)} = 10$ (referred to as discretization level k_2 in the following), according to the evaluation of cost criteria for fixed controls as discussed in Sections III.1.3 and IV.1.3.

No analytic solution of this problem is known to the author, and we thus restrict ourselves to investigating the behavior of the optimal control under varying costs for the reflection, c. The following plots show the control at discretization level k_1 and the state space density at discretization level k_2. For the sake of readability, we have chosen not to display the full relaxed control but the average control. That is, for each point $x \in E$ we display

$\int_U u\,\eta_0(du, x)$. Since the resulting controls all turned out to be deterministic controls, up to some numerical noise, this reduction can be justified. Figure V.3 shows the average of the

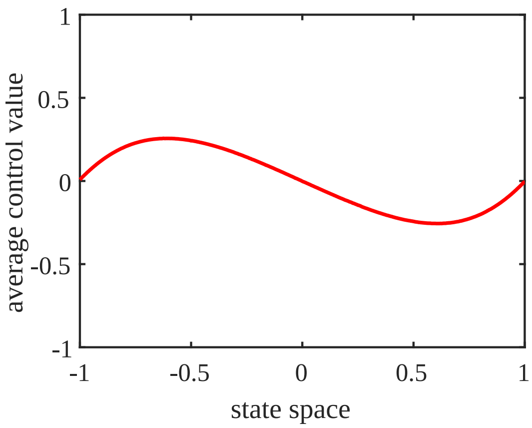

Figure V.3: Computed control, simple particle problem with costs of control, $c = 0.01$

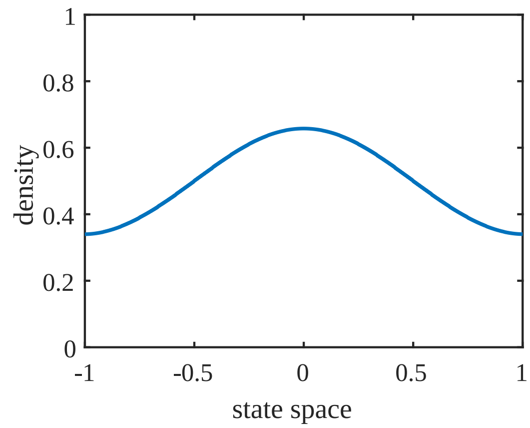

Figure V.4: State space density, simple particle problem with costs of control, $c = 0.01$

optimal control for $c = 0.01$. This is a rather mild penalty for the reflection, compared to the costs induced by the control, which exceed 0.01 as soon as $u > 0.1$. This explains the particular shape of the approximate optimal control. If the process is close to the boundary at -1 or 1, it is more efficient to let the process reflect back to the origin than to pay for a high control value pushing the process back to the origin. The state space density of this computation is shown in Figure V.4, showing a fairly 'wide' distribution of its mass over the interval $[-1, 1]$. Using the discretization level k_2, the value of the cost criterion is approximated by $J = 0.30259$. Figures V.5 and V.6 show the optimal control and state space density for $c = 1$. Note that with this higher penalty for reflection, the control is used more heavily, as its costs are considerably less than the costs inflicted by the reflection. This results in the state space density being more concentrated around the origin. For the discretization level k_2, the value of the cost criterion was $J = 0.42746$. To show an extreme case, consider the results for $c = 6$ as displayed in Figures V.7 and V.8. In the outer parts of the state space, the control is at the maximal allowed value, 1 or -1, respectively, since the costs caused by a reflection would be significantly higher than the costs caused by using

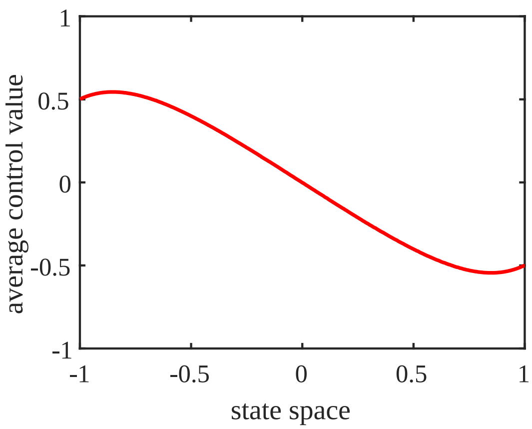

Figure V.5: Computed control, simple particle problem with costs of control, $c = 1$

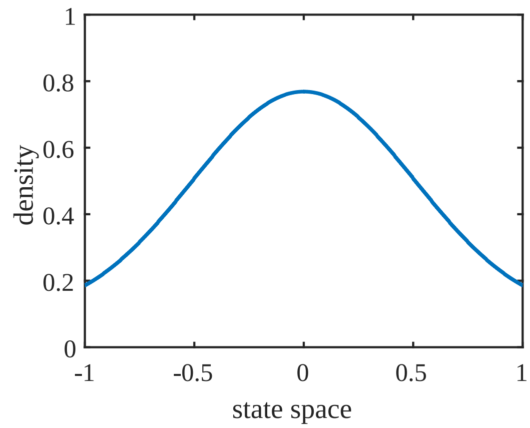

Figure V.6: State space density, simple particle problem with costs of control, $c = 1$

the control. Close to the origin, it is prudent to make less use of the control. Again, the state space density was concentrated even more around the origin. The approximate value for the cost criterion at discretization level k_2 is given by $J = 0.66403$.

V.1.3 The Modified Bounded Follower

The model considered in this section is a variation of the so-called bounded follower problem, as analyzed in Benes et al. (1980). The classic bounded follower problem comes from the idea of imitating a Brownian motion process with a finite variation process. This finite variation process is represented by an integral over the control process, and is of such a type that it can only make bounded increments, corresponding to the boundedness of the control space. This type of problem would be fairly similar to that presented in Section V.1.1. However, Helmes and Stockbridge (2008) introduced a modified version of this problem that features singular behavior and a bounded state space. This problem has a state space of $E = [0, 1]$. The process is reflected at the left endpoint of the state space, $\{0\}$, and performs a jump from $\{1\}$ to $\{0\}$. The control space is given $U = [-1, 1]$ and the process is governed by the

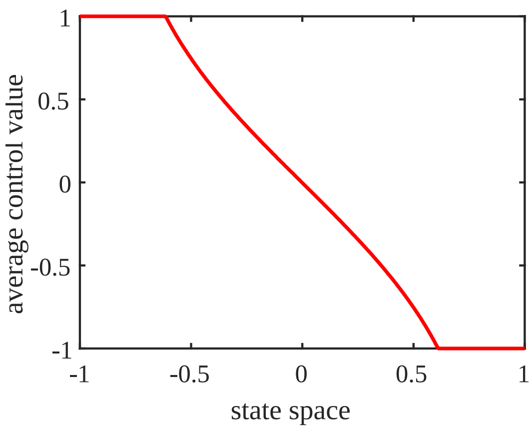

Figure V.7: Computed control, simple particle problem with costs of control, $c = 6$

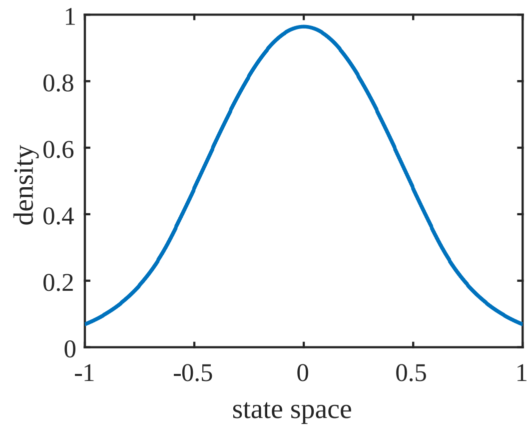

Figure V.8: State space density, simple particle problem with costs of control, $c = 6$

SDE

$$X_t = x_0 + \int_0^t u(X_s)ds + \sigma W_t + \xi_t.$$

Again we adopt the long-term average cost criterion

$$J \equiv \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t c_0(X_s, u_s)\, ds + \int_0^t c_1(X_s, u_s)\, d\xi_s\right].$$

Hence the starting value x_0 can be chosen arbitrarily to lie in E, with no influence on the results. ξ_t captures the singular behavior of the process given by the reflection and the jump part. In terms of imitating the Brownian motion process W_t, we must again find a control that keeps X_t close to the origin. The continuous cost function is set to $c_0(x, u) = x^2$. A jump cost is introduced by $c_1(x, u) = cI_{\{1\}}(x)$ for some $c \in \mathbb{R}_{\geq 0}$, if $x = e_r$. The given dynamics and cost structure suggest an optimal control as follows. In the vicinity of the origin, it is best to push the process to the left by setting $u = -1$. To the right of some 'switching point' a however, it is reasonable to push to the right to make use of the jump behavior, which takes the process back to the origin immediately, while accepting the penalty of c triggered by a jump. As no costs are charged for using the control, the optimal control is a bang-bang control of the form

$$u_a(x) = \begin{cases} -1 & x < a \\ +1 & x \geq a \end{cases}$$

where the 'switching point' a depends on the coefficients of the SDE. This allows us to consider only two mesh points in U, given by -1 and 1, in the subsequent calculations. For a derivation of the analytic solution, we refer the reader to Appendix C.2. In there, it is also illustrated how a closed form of the optimality criterion, depending on a, can be attained. Numerically optimizing the criterion with respect to a then gives a value for the optimal 'switching point', as used in the following. The state space density under the optimal control is given by

$$p_a(x) = \frac{\int_x^1 \exp\left(\int_x^y -\frac{2}{\sigma^2} u_a(z)\,dz\right)\,dy}{\int_0^1 \int_x^1 \exp\left(\int_x^y -\frac{2}{\sigma^2} u_a(z)\,dz\right)\,dy\,dx}. \tag{1.3}$$

We again proceed with analyzing the performance of the proposed numerical method for this example. Table V.8 shows the configuration of the problem which was solved. For this particular configuration, the switching point a is located at $a = 0.7512$. For this configuration,

x_0	σ	c
0.1	$\sqrt{2}$	0.01

Table V.8: Configuration, modified bounded follower problem

Figure V.9 shows the computed relaxed control for $k^{(4)} = 4$ and $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (4, 0, 4)$. The x-axis, labeled 'state space' specifies the state where the process is in. The y-axis, labeled 'control space' specifies the control value from the given discrete set approximating the control space, in this case, $\{-1, 0, 1\}$. The z-axis, labeled 'probability' gives the probability that the specified control is chosen when the process is in state x, in other words, the z-axis gives $\hat{\eta}_0(\{y\}, x)$. The red dots indicate the mesh points of the state space E. These are the points where the control can change its behavior, and the solid black lines in between indicate that the control is constant along the x-axis until it hits the next red dot. The switching point, where the control would switch from -1 to $+1$ is clearly visible.

Figure V.10 show the analytic solution of the state space density in red, and the approximate solution in blue. The approximate state space density clearly shows the features inherited from the piecewise constant basis functions. Its irregular pattern is due to the fact that we introduced an additional mesh point for the piecewise constant basis functions in the middle of the state space. Figure V.11 shows the computed relaxed control for $k^{(4)} = 10$

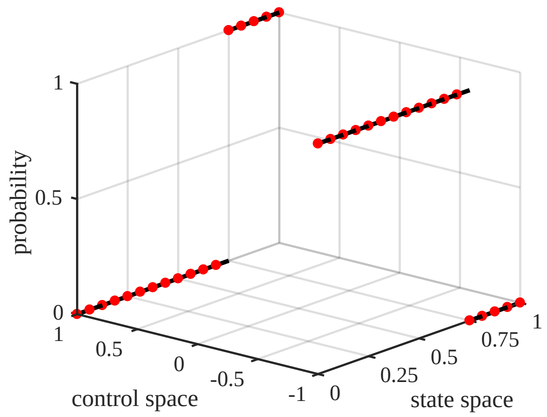

Figure V.9: Computed optimal control, modified bounded follower, coarse grid

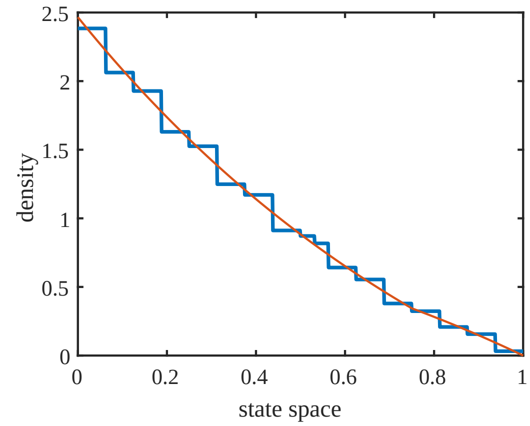

Figure V.10: State space density, modified bounded follower, coarse grid

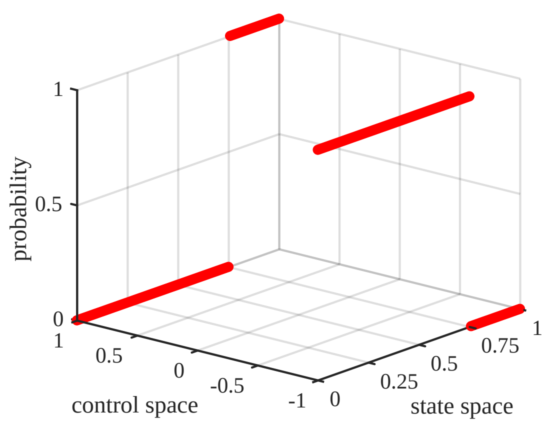

Figure V.11: Computed control, modified bounded follower, fine grid

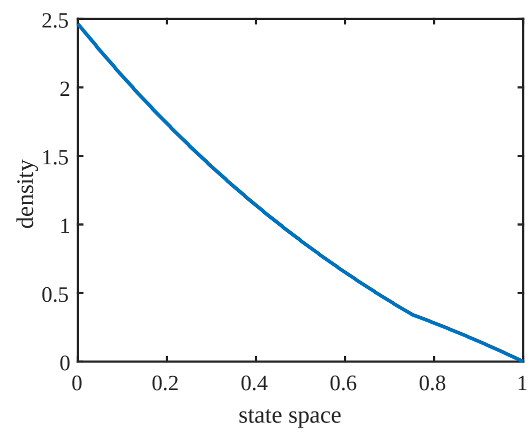

Figure V.12: State space density, modified bounded follower, fine grid

and $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (10, 0, 10)$. Due to the high number of mesh points in U, the solid black lines between the red dots are not visible. Figure V.12 shows the state space density for this discretization level. The exact solution could not be visually distinguished from the approximate solution and is thus omitted from the diagram. Note the change from a convex function to a concave function around the switching point $a = 0.7512$, and the fact that the density goes to zero approaching the right endpoint of the state space. The change in convexity can be explained by the fact that the drift of the process is changing, and together with the fact that the jump works like a 'sink' for the process, this explains why the density has to go to zero when approaching 1.

To analyze the convergence, the optimization was run with different discretization levels.

$k^{(4)}$	$\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$	T	J	e_a	e_r	e_{L^1}
v	-	-	0.1540	-	-	-
3	$(3,0,3)$	0.0089	0.15400	$9.7980 \cdot 10^{-6}$	$6.3629 \cdot 10^{-5}$	$9.0822 \cdot 10^{-2}$
4	$(4,0,4)$	0.0091	0.15399	$1.1993 \cdot 10^{-6}$	$7.7885 \cdot 10^{-6}$	$4.5628 \cdot 10^{-2}$
5	$(5,0,5)$	0.0101	0.15399	$5.4468 \cdot 10^{-7}$	$3.5372 \cdot 10^{-6}$	$2.2874 \cdot 10^{-2}$
6	$(6,0,6)$	0.0124	0.15399	$4.8085 \cdot 10^{-7}$	$3.1227 \cdot 10^{-6}$	$1.1452 \cdot 10^{-2}$
7	$(7,0,7)$	0.0173	0.15399	$4.7132 \cdot 10^{-7}$	$3.0608 \cdot 10^{-6}$	$5.7302 \cdot 10^{-3}$
8	$(8,0,8)$	0.0338	0.15399	$4.6935 \cdot 10^{-7}$	$3.0480 \cdot 10^{-6}$	$2.8661 \cdot 10^{-3}$
9	$(9,0,9)$	0.0844	0.15399	$3.3062 \cdot 10^{-7}$	$2.1471 \cdot 10^{-6}$	$1.4511 \cdot 10^{-3}$
10	$(10,0,10)$	0.2754	0.15399	$2.6546 \cdot 10^{-7}$	$1.7239 \cdot 10^{-6}$	$7.2582 \cdot 10^{-4}$
11	$(11,0,11)$	0.9586	0.15399	$3.5500 \cdot 10^{-7}$	$2.3054 \cdot 10^{-6}$	$4.5089 \cdot 10^{-4}$

Table V.9: Results (1), modified bounded follower, varying discretization levels

Tables V.9 and V.10 show the results and performance measures for various discretization levels $k^{(4)}$ and $\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$. Recall that n, the number of constraint functions, is $n = 2^{k^{(4)}}+2$. We also introduced an additional mesh point in the middle of the state space in an attempt to increase accuracy, compare Remark III.1.33. In contrast to the results from Section V.1.1, the introduction of an additional mesh point does not improve the error measure by several orders of magnitude. Therefore, a separate consideration of the results with and without an additional mesh point is omitted. However, numerical artifacts in the solution, that would appear when no additional mesh point was introduced, could be eliminated.

Table V.9 shows the approximate value of the cost criterion J, its absolute and relative errors e_a and e_r as well as the error of the density in L^1-norm, denoted e_{L^1}. T is the execution time, which is an average time taken from 1000 executions of the same program run. Table V.10 displays the weights $\hat{w}_1 = \hat{\mu}_{1,n,m}(\{e_l\})$ and $\hat{w}_2 = \hat{\mu}_{1,n,m}(\{e_r\})$ of the singular expected occupation measure on the left and right boundary, respectively, with their absolute and relative errors e_a and e_r. In both tables, the first rows display the exact values v of the quantities under consideration. Note that the method produces already fairly accurate

$k^{(4)}$	w_1	e_a	e_r	w_2	e_a	e_r
v	2.4659	-	-	1.5555	-	-
3	2.4667	$7.8873 \cdot 10^{-4}$	$3.1986 \cdot 10^{-4}$	1.5577	$2.1566 \cdot 10^{-3}$	$1.3865 \cdot 10^{-3}$
4	2.4661	$1.9843 \cdot 10^{-4}$	$8.0471 \cdot 10^{-5}$	1.5560	$5.4136 \cdot 10^{-4}$	$3.4803 \cdot 10^{-4}$
5	2.4659	$4.9692 \cdot 10^{-5}$	$2.0152 \cdot 10^{-5}$	1.5556	$1.3548 \cdot 10^{-4}$	$8.7099 \cdot 10^{-5}$
6	2.4659	$1.2428 \cdot 10^{-5}$	$5.0401 \cdot 10^{-6}$	1.5555	$3.3880 \cdot 10^{-5}$	$2.1781 \cdot 10^{-5}$
7	2.4659	$3.1074 \cdot 10^{-6}$	$1.2601 \cdot 10^{-6}$	1.5555	$8.4705 \cdot 10^{-6}$	$5.4455 \cdot 10^{-6}$
8	2.4659	$7.7684 \cdot 10^{-7}$	$3.1503 \cdot 10^{-7}$	1.5555	$2.1178 \cdot 10^{-6}$	$1.3615 \cdot 10^{-6}$
9	2.4650	$8.5772 \cdot 10^{-4}$	$3.4783 \cdot 10^{-4}$	1.5532	$2.3313 \cdot 10^{-3}$	$1.4987 \cdot 10^{-3}$
10	2.4655	$4.2860 \cdot 10^{-4}$	$1.7381 \cdot 10^{-4}$	1.5543	$1.1666 \cdot 10^{-3}$	$7.4998 \cdot 10^{-4}$
11	2.4631	$2.8329 \cdot 10^{-3}$	$1.1488 \cdot 10^{-3}$	1.5543	$1.1649 \cdot 10^{-3}$	$7.4887 \cdot 10^{-4}$

Table V.10: Results (2), modified bounded follower, varying discretization levels

approximations in almost negligible time for $k^{(4)} = 5$ or $k^{(4)} = 6$. The over-proportional increase in computing time for higher discretization levels $k^{(4)} = 10$ and $k^{(4)} = 11$ is due to longer execution time of the linear program solver (and not to the time needed setting up the coefficient matrix), and might indicate that the problem is becoming ill-conditioned. For $k^{(4)} = 12$ and $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (12, 0, 12)$, no reliable solution could be produced. In this case, the linear programming solver could find no point satisfying the constraints, which could be circumvented by increasing the degrees of freedom given by $\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ without increasing the number of constraints determined by $k^{(4)}$. However, this did not show better performance than the presented cases. The absolute error for $k^{(4)} = 11$ is on comparable level to results obtained in Rus (2009). Both the error of the cost criterion value and the L^1-error of the state space density are steadily decreasing. Even though the computing times obtained here cannot be compared directly to the ones presented in Rus (2009), it is worth mentioning that the drastic speed up is in part related to the fact that the approximation used here preserves the linear structure of the problem, while Rus' method loses this structure, requiring more complicated optimization approaches. The inferior approximation quality at $k^{(4)} = 11$ compared to $k^{(4)} = 10$ is believed to be due to the method becoming ill-conditioned. Considering the weights of the singular expected occupation measure, this seems to have a negative effect on the results starting from $k^{(4)} = 8$.

To further analyze the model of the modified bounded follower, we change the cost of the jump and compare cost criteria and switching points. All of the following calculations are conducted with $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (11, 0, 11)$ and $k^{(4)} = 11$. Table V.11 shows the results of these computations. It displays the cost of the jump c, the computed optimality criterion J, the optimal switching point a and w_1 and w_2, the weights on the reflection and jump, respectively. Naturally, the jump is used most extensively if it is free of costs, and hence the smallest switching point a is at 0.7070, when $c = 0$. This means that early on, the control pushes the process to the right to make use of the jump behavior. Increasing the jump costs leads to a larger switching point, which leads to a lower weight on the jump as well as a lower weight on the reflection, since the process spends more time in the interior of the state space. Naturally, the cost criterion increases with higher jump costs. For $c = 0.8$, it is not feasible to use the jump anymore, and the optimal control is constant at $u = -1$ throughout the state space.

Next, we introduce a cost for the reflection. The cost structure is hence given by $c_1(x, u) = c^{(r)} I_{\{0\}}(x) + c^{(j)} I_{\{1\}}(x)$. This will alter the optimal control in such a way that depending on

$c^{(j)}$	J	a	w_1	w_2
0	0.13817539704	0.7070	2.4841563111	1.6103053631
0.005	0.14615078462	0.7295	2.4743796941	1.5808085514
0.01	0.15398628449	0.7510	2.4649420981	1.5543338789
0.015	0.16169543646	0.7720	2.4555254491	1.5301491200
0.02	0.16169543646	0.7920	2.4471522640	1.5086852357
0.03	0.18418365489	0.8311	2.4311425436	1.4714530533
0.04	0.19874734491	0.8677	2.4174958174	1.4424059030
0.05	0.21305444916	0.9028	2.4065144750	1.4202163206
0.06	0.22717203551	0.9370	2.3982607443	1.4044204069
0.07	0.24116431693	0.9697	2.3932170306	1.3951440719
0.075	0.24813341509	0.9863	2.3918918256	1.3928225298
0.08	0.25509535018	1	2.3915311561	1.3922114438

Table V.11: Results, modified bounded follower, varying jump costs

the reflection costs $c^{(r)}$, the control will try to keep X away from the origin, but in its vicinity. For the subsequent computations, we use a discretization level of $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (11, 0, 11)$ and $k^{(4)} = 11$, with an additional mesh point, and fix the cost of the jump to $c^{(j)} = 0.01$.

Figure V.13 and Figure V.14 show the optimal control and the state space density for a reflection cost of $c^{(r)} = 0.06$. We can clearly see two points where the behavior of the control changes. First, from 0 to a point referred to as a_1 in the following (approximately at 0.2), it pushes the process away from the origin to avoid the costs associated with a reflection. After this point is crossed, the control pushes the process in the direction of the origin to avoid costs induced by the continuous cost function $c_0(x,u) = x^2$. As soon as the process crosses a second switching point, from here on referred to as a_2 (approximately at 0.8), the process is being pushed to the right endpoint of the state space, in order to benefit from the immediate jump back to the origin.

The state space density, on the one hand, shows the features we observed in the case with no reflection costs, with a change in convexity as soon as the second switching point a_2 is

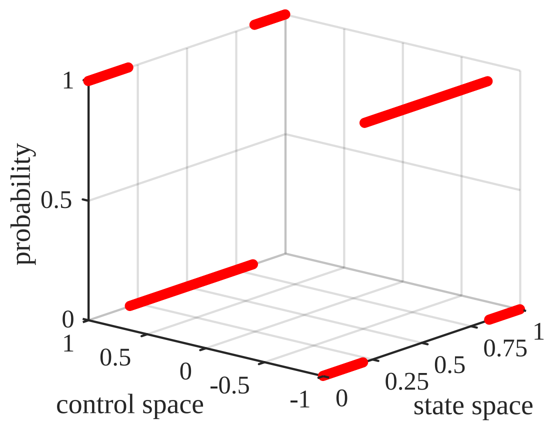

Figure V.13: Computed control, modified bounded follower, with costs of reflection

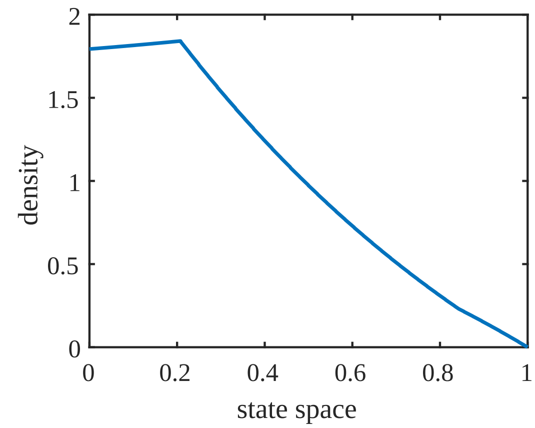

Figure V.14: State space density, modified bounded follower, with costs of reflection

crossed. Furthermore, we can identify the mode of the distribution close to the first switching point a_1, which underlines that the optimal controls try to keep the process far enough away from the origin, to avoid costly reflections, but in an area where the costs accrued by the continuous control $c_0(x, u) = x^2$ are also not too high.

Table V.12 shows the results for several computations where the reflection costs $c^{(r)}$ vary and the jump costs remain fixed at $c^{(j)}$. It displays the cost of the reflection $c^{(r)}$, the computed optimality criterion J, the optimal switching points a_1 (from $u = 1$ to $u = -1$) and a_2 (from $u = 1$ to $u = -1$) as well as w_1 and w_2, the weights on the reflection and jump, respectively. Evidently, the first switching point a_1 is zero when no costs are charged for a reflection. With higher costs for the reflection, the switching point a_1 then increases up to 0.335, at a reflection cost of $c^{(r)} = 0.16$. For this cost, the second switching point a_2 is at 1, which is due to the fact that the reflection is now so expensive, that a jump to the origin and the reflection that happens immediately afterwards are too costly compared to the lower costs one faces when the process is close to the origin. Again, and not surprisingly, the cost criterion increases with higher costs of the reflection.

$c^{(r)}$	J	a_1	a_2	w_1	w_2
0	0.15398626456	0	0.7510	2.4654719566	1.5543336778
0.02	0.19872591568	0.09766	0.7817	2.0723547958	1.5510431053
0.04	0.23824187369	0.1606	0.812	1.8968174008	1.5672960913
0.06	0.27505860018	0.2075	0.8433	1.7933796723	1.5843270386
0.08	0.31019724306	0.2441	0.8755	1.7252930125	1.5996596944
0.1	0.34419475360	0.2739	0.9087	1.6772735162	1.6144085540
0.12	0.37738203781	0.2983	0.9429	1.6429692772	1.6298117173
0.14	0.40999172579	0.3184	0.9775	1.6192383628	1.6478312950
0.16	0.44220948561	0.3354	1	1.6036249672	1.6709903450

Table V.12: Results, modified bounded follower, varying reflection costs

V.1.4 The Modified Bounded Follower with Variable Jump Size

In this section we investigate the performance of the proposed numerical scheme on a control problem which features a variable jump size, in other words, not only the continuous, but also the singular behavior of the process can be controlled. An adaption of the modified bounded Follower as presented in Section V.1.3 serves as an example. The size of the jump from the right endpoint of the state space is no longer fixed to be constant 1, but can range between 0 and 1 depending on the control input. In accordance with the description of controlled jump processes in Appendix A.1, we set

$$h : E \times U \ni (x,u) \mapsto h(x,u) = -1/2 \cdot (u+1) \cdot I_{\{1\}}(x).$$

The factor of $\frac{1}{2}$ is required to map the control space $U = [-1,1]$ onto the set of possible jump sizes given by $[0,1]$. The generator of the jump part of the singular behavior is now given by $Bf(x,u) = f(1-h(x,u)) - f(x)$. The continuous costs are again given by $c_0(x,u) = x^2$. We consider no costs for the reflection of the process at 0, but adopt a jump cost that is proportional to the length of the jump. In particular, we set $c_1(x,u) = -c^{(j)} \cdot h(x,u)$ if

$x = 1$ for some $c \in \mathbb{R}$. This will force the optimal control to make a trade-off between the benefits of bringing the process close to the origin and avoiding costs charged by the continuous part of the cost function c_1, but being charged for the length of the jump. For the subsequent computations we use a discretization level of $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (10, 0, 10)$ and $k^{(4)} = 10$, without an additional mesh point. For the jump size we use a different discretization of U with 512 mesh points. Figure V.15 and Figure V.16 show the results

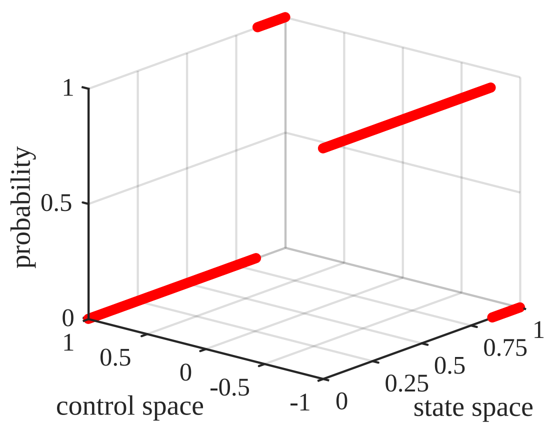

Figure V.15: Computed control, modified bounded follower with variable jump size

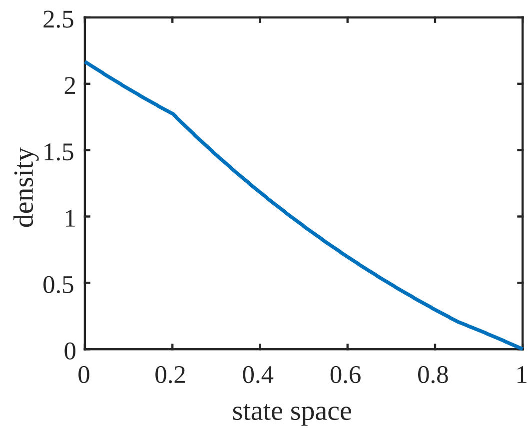

Figure V.16: State space density, modified bounded follower with variable jump size

of an example calculation with $c^{(j)} = 0.04$. The continuous control, Figure V.15, shows a behavior similar to that seen for a fixed jump size, with the switching point located at $a = 0.8535$. The optimal jump size in this example was calculated to -0.7988281, in other words, the process jumped to 0.2011719. The approximate state space density, Figure V.16, shows two features. Again, the change in convexity around the switching point is (slightly) observable at $x = 0.8535$. Another 'corner' of the graph is visible at $x = 0.2011719$, to which the process jumps from $x = 1$.

For various values of the cost coefficient $c^{(j)}$, Table V.13 shows the value of the cost criterion J, the switching point a, the optimal jump size h as well as w_1 and w_2, the weights on the reflection and jump, respectively. As one could expect, the switching point a increases with the jump costs, as it is less beneficial to use the jump. At the same time, the optimal jump size h decreases to compensate for higher costs of the jumps. Interestingly, the weight of

c	J	a	h	w_1	w_2
0	0.13817544839	0.7070	1	2.4836575948	1.6103060559
0.01	0.15348844793	0.75	0.9355	2.3691725147	1.5606528470
0.02	0.16753299261	0.7871	0.8848	2.2878151493	1.5297581715
0.03	0.18061852431	0.8223	0.8398	2.2217667982	1.5095122261
0.04	0.19294312467	0.8535	0.7988	2.1666821674	1.5001606321
0.05	0.20464590438	0.8828	0.7637	2.1224318980	1.49702472520
0.06	0.21583121777	0.9102	0.7285	2.0816826293	1.5023561336
0.07	0.22658069190	0.9355	0.6973	2.0477662898	1.5130052374
0.08	0.23696136554	0.96	0.6680	2.0179314613	1.5292589572
0.09	0.24702997878	0.9834	0.6406	1.9920476224	1.5509638088
0.1	0.25683597519	1	0.6133	1.9679798575	1.5801336706

Table V.13: Results, modified bounded follower with variable jump size

the jump w_2 does not show monotone behavior. First, it decreases, which is due to the fact the switching point a increases, thus the process is less likely to be pushed towards the right endpoint of the state space. However, starting from $c = 0.06$, the weight increases again, which is believed to be caused by the smaller jump size and the fact that from the point to which the process jumps, it is more likely to hit the right boundary of the state space again.

V.1.5 Optimal Harvesting in a Stochastic Logistic Growth Example

This section will present how the proposed numerical method can be used to solve an optimal harvesting problem featuring stochastic logistic growth. An exemplary model was investigated in Lungu and Øksendal (1997). As in the deterministic case, a logistic growth model features a carrying capacity K, which is regarded as the maximal value to which a quantity X can grow. The growth rate is proportional to the product of X with the 'free' carrying capacity $K - X$. Hence, the quantity grows slowly when it is either small or close to the carrying capacity. Examples of such behavior can be found in simple market models, assuming that there is a finite number of consumers able to buy a certain product, causing sales numbers to stagnate at K, or in modeling growth of animals in somewhat confined environments - a fish farm would be a specific example. In this case, K would be considered as the maximal number to which the fish population can grow, since space in a fish pond is limited.

In contrast to the deterministic logistic growth model, we introduce a diffusion part to the evolution of X, which, in the first example, would model fluctuation in the consumer's demand, and in the second example, fluctuations in the procreation of the fish. This diffusion part is also proportional to the product of X and the 'free' carrying capacity $K - X$. The stochastic differential equation of interest, in integral form, is

$$X_t = x_0 + \int_0^t rX_s(K - X_s)ds + \int_0^t \sigma X_s(K - X_s)dW_s \tag{1.4}$$

The constant $r \in \mathbb{R}$ models the deterministic growth rate, while $\sigma \in \mathbb{R}$ is used to model the scale of the diffusion part. Note that in this model, both deterministic growth and diffusion tend to 0 as X_t tends to either 0 or K. This ensures that the state space is actually given by $E = [0, K]$. Although the classic conditions for existence and uniqueness are not fulfilled, Lungu and Øksendal (1997) proves the existence of a solution. Further, they worked out that the optimal harvesting strategy under a infinite horizon discounted criterion is a reflection at some point $y \in (0, K)$. This can be viewed as 'instantaneous' harvesting, which takes out an infinitesimally small amount of X as soon as X reaches y. However, a closed form for y remains unclear, up to the point that for a discounting rate of α, the bounds

$$\frac{K}{2} - \frac{\alpha}{2r} \leq y \leq \frac{K}{2} - \frac{\alpha}{2r} + \frac{1}{2r}\left|rK - \alpha\right| \tag{1.5}$$

hold. The contribution of the proposed method to this problem is as follows. As described in Section III.1.3, we can utilize the linear programming approach to evaluate cost criteria of a given diffusion. We will be considering the reflected SDE in integral form

$$X_t = x_0 + \int_0^t r X_s(K - X_s)ds + \int_0^t \sigma X_s(K - X_s)dW_s - L^X_{\{y\}}.$$

Here, $L^X_{\{y\}}$ denotes the local time of X at y. Due to this reflection, the state space of the

$k^{(4)}$	$(k^{(1)}, k^{(2)}, k^{(3)})$	T	$J(y)$	w
4	$(5, 0, 5)$	0.022190	0.068229257083	0.0068229257083
5	$(6, 0, 6)$	0.016260	0.068228226679	0.0068228226679
6	$(7, 0, 7)$	0.020150	0.068228156312	0.0068228156312
7	$(8, 0, 8)$	0.022660	0.068228151991	0.0068228151991
8	$(9, 0, 9)$	0.033550	0.068228151719	0.0068228151719
9	$(10, 0, 10)$	0.060210	0.068228152113	0.0068228152113
10	$(11, 0, 11)$	0.125080	0.068228118784	0.0068228118784
11	$(12, 0, 12)$	0.296870	0.068228155828	0.0068228155828
12	$(13, 0, 13)$	0.925790	0.068228483923	0.0068228483923
13	$(14, 0, 14)$	3.249380	0.068229971432	0.0068229971432
14	$(15, 0, 15)$	15.301780	0.068240964541	0.0068240964541

Table V.14: Convergence, stochastic logistic growth, fixed reflection point

given problem is $[0, y]$. This problem can be viewed as an uncontrolled problem, as there is no influence by the control on the drift and diffusion rates, and the minimization has to be conducted with respect to the point of reflection y. The 'cost' criterion (representing negative yield) of interest is, for a discounting rate $\alpha > 0$,

$$J(y) = -\mathbb{E}\left[\int_0^\infty e^{-\alpha s}\, dL^X_{\{y\}}\right].$$

Using the linear programming approach we can evaluate this cost criterion for a fixed y and optimize y using a standard one-dimensional optimization solver. Intuitively, the goal of the optimization must be to keep the process at the point where it has the highest growth rate. Left and right of this point it will take longer for the population to recover, resulting in a diminished yield. This leads to the assumption that the cost criterion is unimodal over $(0, K)$, justifying the use of both golden section and parabolic optimization techniques as described in Brent (2002).

This model will be analyzed in the following ways. First, we will numerically investigate the convergence of our approach for a fixed value for y. An analytic error analysis cannot be conducted since the analytic solution remains unknown. Still, the result of an actual optimization of y can be checked against the analytic bounds given by (1.5). Secondly, we will perform the optimization of y for a line of different parameter choices and analyze the influence of the parameters r, σ and α on the optimal solution. Of special interest is how the computed optimal reflection points compare to the 'intuitive' choice of $y = 0.5$, which guarantees the largest deterministic growth rate.

To numerically analyze the convergence of the method, we fix the reflection at $y = 0.5$ and

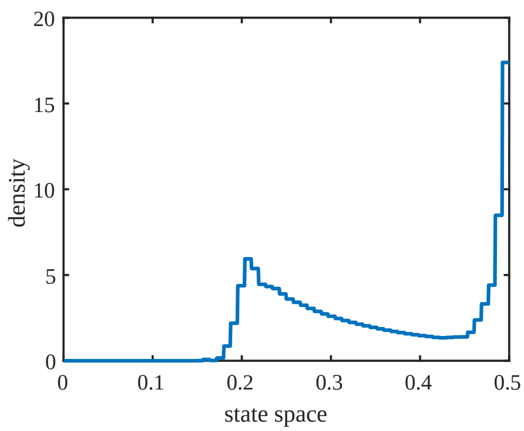

Figure V.17: Computed control, coarse grid, stochastic logistic growth

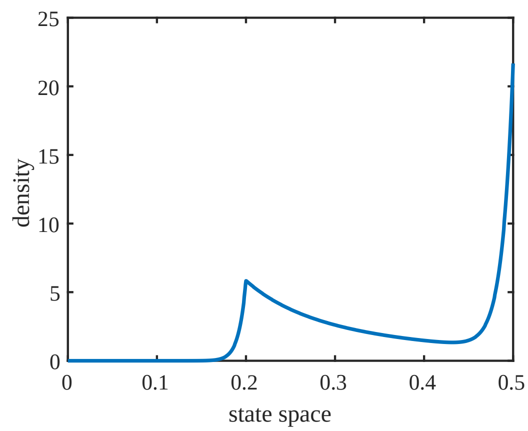

Figure V.18: State space density, fine grid, stochastic logistic growth

the drift coefficient, the diffusion coefficient as well as the discounting rate at $r = \sigma = \alpha = 0.1$. If we consider the yield obtained as some monetary value, a discount rate of $\alpha = 0.1$ is rather high, however, it helps to display some characteristic of the solution in a first example. Since we are considering the discounted infinite horizon criterion, the choice of the starting point actually plays a role, and is set to $x_0 = 0.2$. We will retain the notation for the discretization level from the previous sections, with $\left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ determining the degrees of freedom and $k^{(4)}$ determining the number of constraints, bearing in mind that for the given example, which does not have any continuous control, only $k^{(3)}$ and $k^{(4)}$ play a role in the analysis.

Table V.14 shows the computing time T averaged over 1000 executions, the computed cost criterion $J(y)$ as well as the reflection weight w for different levels of accuracy. The method shows a fairly convergent behavior, with fluctuation mainly happening in the 7th decimal digit, until it becomes increasingly ill-conditioned, starting at $k^{(4)} = 12$. The computed density for a discretization level of $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (6, 0, 6)$ and $k^{(4)} = 5$ is shown in Figure V.17. It obviously exhibits the 'step function' features given by the discretization of the density using indicator functions. Two things worth pointing out are visible in this plot. First, we have a concentration of mass around the starting point $x_0 = 0.2$. This is due to the fact that we are considering a relatively high discount factor of $\alpha = 0.1$, and the behavior of the process right after the start of the evolution is weighted heavier than later behavior. Second, we can see the density concentrating around the reflection point, which is the behavior that is expected from the given SDE model. After being harvested, the quantity X grows repeatedly against y, with only little probability of making downwards moves. Figure V.17 shows the density for a discretization level of $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (11, 0, 11)$ and $k^{(4)} = 10$, which is fine enough to make the features caused by the indicator basis functions disappear.

Next we investigate the influence of the model parameters b, σ and α on the optimal solution. With the context of a fish farm modeled by (1.4), a couple of interesting conclusions can be drawn. For the subsequent calculations we fix $\left(k^{(1)}, k^{(2)}, k^{(3)}\right) = (12, 0, 12)$ and $k^{(4)} = 11$. As a first example, we vary the drift coefficient r and investigate its influence on the optimal reflection point. The diffusion coefficient remains at $\sigma = 0.1$ and the discount rate is set to $\alpha = 0.01$. Table V.15 shows the result for these computations. The optimal position of the reflection increases with the drift coefficient. This is explained by the fact that we use a discounted reward criterion, and it is therefore important to harvest 'rather' early, which re-

b	y	$J(y)$
0.05	0.4235226	0.97906691
0.10	0.4625687	2.19317776
0.20	0.4812090	4.67383232
0.30	0.4875718	7.16719189
0.40	0.4905254	9.66378803
0.50	0.4946607	12.16154627
1.00	0.4962992	24.65763595
10.00	0.4996371	249.65202762

Table V.15: Approximate solutions for varying drift rate b

sults in a optimal position for the reflection where the drift $bX_t(K - X_t)$ is not maximal, but is reached early enough to benefit from a smaller discount. As the drift coefficient increases, it is possible to use larger reflection positions since the process grows to this point more quickly. The value of the optimality criterion increases, which is easily explained by the fact that the process recovers more quickly after harvesting. Note that the optimal position of the reflection y is slightly lower than the 'intuitive' choice of $y = 0.5$, but within the bounds established by (1.5).

Next, we vary the diffusion coefficient σ and investigate its influence on the optimal reflec-

σ	y	$J(y)$
0.02	0.4503935	2.1981714904
0.05	0.4533961	2.1973397002
0.10	0.4625687	2.1931777609
0.15	0.4779099	2.1820521946
0.20	0.5113975	2.2054183122
0.25	0.5259329	2.1601371377
0.30	0.5605976	2.0786552100

Table V.16: Approximate solutions for varying diffusion rate σ

α	y	$J(y)$
0.01	0.46256872	2.1931777609
0.15	0.43711173	1.3809111582
0.02	0.41173494	0.9834288529
0.03	0.36188956	0.6008668284
0.04	0.31037410	0.4224228605
0.05	0.25920076	0.3244632037
0.06	0.20776940	0.2665692517

Table V.17: Approximate solutions for varying discount rate α

tion point. The drift coefficient remains at $b = 0.1$ and the discount rate remains at $\alpha = 0.01$. Table V.16 shows the results from these calculations. Interestingly, the optimal position of the reflection increases with the diffusion coefficient, as the value of the reward criterion decreases. This could be interpreted in such a way that the harvesting is done at a higher point such that large increments coming from the diffusion term do not push the process too far down into a region of the state space with tiny drift rates. In other words, a relatively steady growth against the reflection point is 'bought' by accepting a higher discount with the process needing more time to reach the reflection point. For diffusion coefficients of 0.4 or higher, the numerical approximation represented a process being absorbed at $X = 0$. These results are omitted, as they show a fundamentally different dynamic behavior compared to the other cases. However, this supports our claim that with larger diffusion coefficients, the process tends to spend more time in lower regions of the state space.

Note how the optimal positions for the reflection deviate, in both directions, from the 'intuitive' choice of $y = 0.5$. Still, they remain in the theoretic boundaries of (1.5).

Finally, we vary the discount rate α and investigate its influence on the optimal reflection point. The drift coefficient remains at $b = 0.1$ and the diffusion coefficient remains at $\sigma = 0.1$. Table V.17 shows the results of this computation run. Naturally, the value of the optimality criterion decreases with the introduction of higher discount rates. This trend is countered by lowering the point of optimal reflection successively. Note that this will cause a slower recovery rate around the point of harvesting, but will make sure the process reaches this point earlier in time when starting at $x_0 = 0.2$. This case, especially for high discount rates, shows the largest discrepancy of the intuitive choice of $y = 0.5$ and the actual optimal value for y while maintaining the analytic bounds of (1.5).

V.2 Infinite Time Horizon Problems with Unbounded State Space

This section illustrates the performance of the proposed numerical method being applied to models that feature an unbounded state space. As in Section V.1, we use the notion of the discretization level $k \equiv \left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}\right)$ in its enhanced version as introduced in Section III.1.4. In addition, we refer to the 'computed state space' as the interval $[-K, K]$ which is used to approximate the unbounded state spaces appearing in the following examples.

The considered models include the optimal control of a Cox-Ingersol-Ross model in Section V.2.1 and an optimal asset allocation model in Section V.2.2, where the underlying asset price is either modeled by a geometric Brownian motion or an Ornstein-Uhlenbeck process.

V.2.1 Optimal Control of a Cox-Ingersol-Ross Model

This section presents a control problem for the Cox-Ingersol-Ross model, which is frequently used in financial modeling. Let $\rho, \mu, \sigma, x_0 > 0$. Consider the SDE in integral form

$$X_t = x_0 + \int_0^t \rho\left(\mu - X_s\right) \, ds + \int_0^t u_s \, ds + \int_0^t \sigma \sqrt{X_s} \, dW_s.$$

This model was first introduced in Cox et al. (1985). The given SDE features two important components. First, the drift $\int_0^t \rho\left(\mu - X_s\right) \, ds$ forces a 'mean reversion'. This means that X is forced towards the value μ, which is called the long-term mean. The coefficient ρ specifies the strength of this mean reversion. Derivations from μ are introduced by the stochastic component $\int_0^t \sigma \sqrt{X_s} \, dW_s$. While the coefficient σ specifies the intensity of the diffusion, it is also proportional to $\sqrt{X_s}$, which means that higher values of X_t result in higher diffusions, while smaller values of X_t show smaller diffusions. In particular, this structure ensures that $X_t > 0$ at any time t. Hence, we can consider the state space $E = [0, \infty)$. The control space in which u takes values is given by $U = [u_l, u_r]$ for some $u_l < u_r \in \mathbb{R}$.

We seek to optimize the discounted infinite horizon cost criterion

$$J \equiv \mathbb{E}\left[\int_0^\infty e^{-\alpha s} c_0(X_s, u_s)\, ds\right]$$

with $c_0(x, u) = (x - \mu)^2$. This means the process must be controlled in such a way that it stays as close as possible to the long-term mean. In an application, X could model the price of a commodity, for example oil, and it would be in the interest of the producer to keep the price at a certain level. The influence on the price can be enacted by increasing or lowering production. Herein, it is assumed that this can be done without introducing any costs.

We proceed by investigating the influence of the diffusion coefficient σ and the mean reversion coefficient ρ on the optimal control. However, we begin with a discussion of some numerical phenomena that have to be overcome in order to achieve reasonable results for this control problem. Table V.18 specifies the parameter choices for the SDE in this section. It was found to be necessary, at least for some parameter choices, to choose $k^{(3)} > k^{(4)}$ to

x_0	α	ρ	μ	σ	u_l	u_r
10	0.1	0.02	10	varies	-0.05	0.05

Table V.18: Configuration (1), Cox-Ingersol-Ross model

give the linear program solver enough degrees of freedom to find a feasible solution. For the discretization level $k = (10, 0, 12, 11)$, a computed state space $[-K, K]$ of $[0, 60]$ and $\sigma = 0.5$, Figure V.19 and Figure V.20 show the results for the computed optimal control and the density of the occupation measure, respectively. While the computed control looks very regular, the density looks highly irregular and needs to be examined more closely. A more detailed view on the situation is provided by Figure V.21, which shows a closer view of the state space density on the interval $[9.5, 10]$. To impart further insight, the mesh points of the constraint mesh are plotted with red dots. One can easily imagine the mesh of the expected occupation measure by dividing the intervals indicated by the red dots in half (recall that

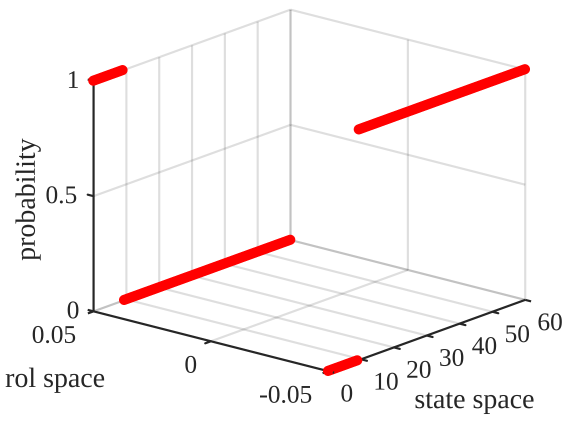

Figure V.19: Computed control, Cox-Ingersol-Ross model

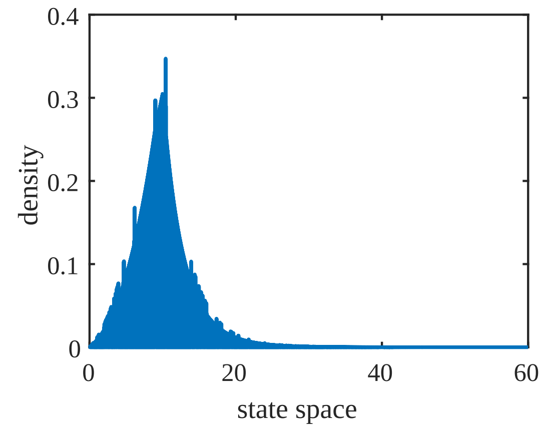

Figure V.20: State space density, Cox-Ingersol-Ross model

$k^{(3)} = 12$ and $k^{(4)} = 11$, and so the mesh for the expected occupation measure has twice as many points). These mesh points are omitted in the plot for the sake of visibility. Note that the situation is such that two intervals of the expected occupation measure mesh always lie within one interval of the constraint measure mesh. In other words, two degrees of freedom interact with the exact same constraints, which might indicate that these two degrees of freedom are, to some level, linearly dependent, and hence the solver can choose to assign full mass to one degree of freedom and assign zero mass to the other, in such a way that it favors the minimization of the cost criterion., without violating the linear constraints.

One can easily be convinced that such a expected occupation measure cannot be the expected occupation measure of an actual solution to the SDE, since the continuity of the paths would prohibit the process spending time inside the two separated intervals of the state space, but not inside the interval lying between them. The displayed expected occupation measure is only a solution to a discretized version of the problem given by the SDE, thus the observed behavior does not necessarily indicate a faulty approximation. Further, the error analysis presented in Section III.2 does not consider the L^1 error of the approximate densities, so again previous results are not contradicted.

In the light of the analysis provided in Kushner and Dupuis (2001) for discretization obtained by using a finite difference approximation of the HJB equation, it is conjectured that the presented approximation could as well be related to a discrete-state Markov chain, which would actually allow the observed behavior. A detailed investigation of this conjecture lies beyond the scope of this thesis. However, it is still desirable to obtain an approximation to

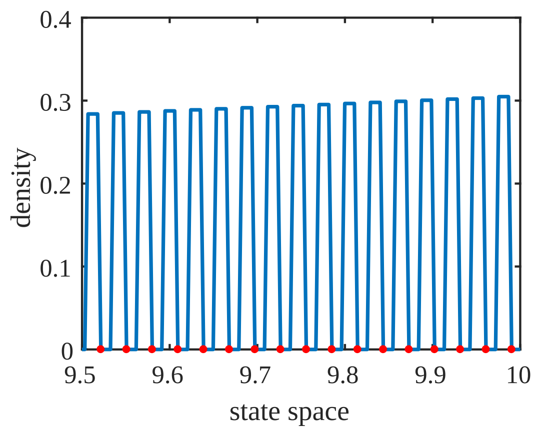

Figure V.21: Zoom of state space density, Cox-Ingersol-Ross model

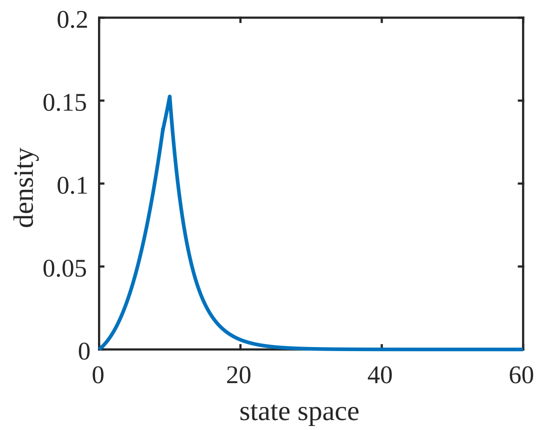

Figure V.22: Re-evaluated state space density, Cox-Ingersol-Ross model

the density that looks like the density of the expected occupation measure of a continuous process. This can be achieved by fixing the control we obtained in the computation, displayed in Figure V.19, and re-evaluating the cost criterion using the methods described in Section III.1.3. Figure V.22 shows the density of the expected occupation measure from this re-evaluation with $k^{(3)} = 12$ and $k^{(4)} = 11$, which are the same discretization levels as in the initial run of the linear program solver. This function is far more regular than its counterpart in Figure V.20, which is explained as follows. Recall from Section III.1.3, that we seek to minimize the Euclidean norm of the solution under the linear constraints obtained by fixing the control for the re-evaluation, as opposed to the linear program solver minimizing the cost criterion. As we pointed out above, it might be beneficial in terms of the cost criterion to assign all mass to one subinterval. However, this is disadvantageous when minimizing the Euclidean norm. This can easily be seen by comparing the Euclidean norms of the vectors $(1, 0)^T$ and $(0.5, 0.5)^T$ which have the same mass of a - it is obvious that it is beneficial to distribute the mass as evenly as possible over all subintervals to achieve a small Euclidean norm. Hence, the state space density given by the re-evaluation is more regular.

As the preceding discussion suggests, a two-step computation approach should be used to obtain regular results for the state space density. First, we run the linear programming solver with a given discretization level, then re-evaluate the resulting control with the same discretization levels. This produces reasonable results that allow us to move the focus towards the analysis of the model itself. In particular, we investigate the influence of the diffusion coefficient σ and the mean reversion coefficient ρ onto the optimal control and the cost criterion. We begin with the diffusion coefficient σ.

A close examination of the sample run with the configuration from Table V.18, see Figure V.19, reveals that the point where the optimal control switches its value from 0.05 to -0.05 lies slightly lower than the mean of $\mu = 10$. To make this clearer, Figures V.23 and V.24 show the average optimal control and the state space density for a diffusion coefficient of $\sigma = 0.5$. We notice that the control switches from 0.05 to -0.05 below the mean.

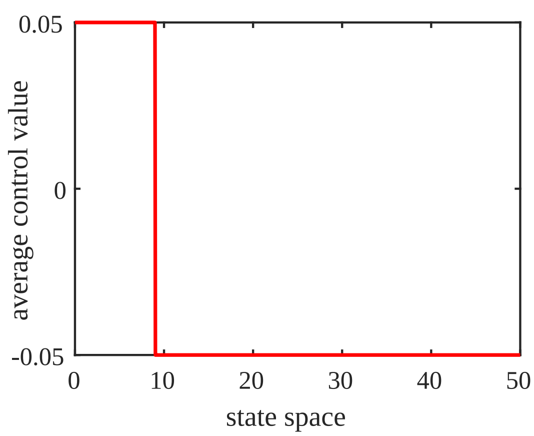

Figure V.23: Average optimal control, Cox-Ingersol-Ross model

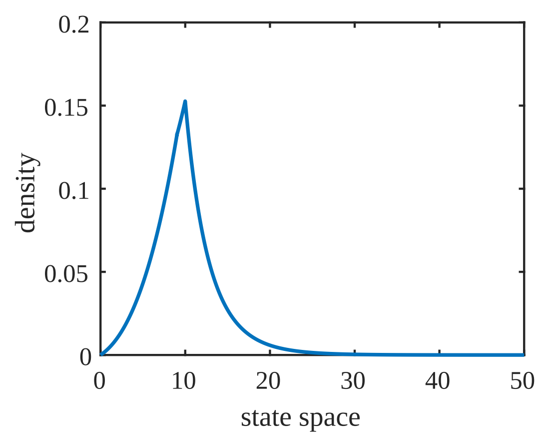

Figure V.24: State space density, Cox-Ingersol-Ross model

Second, we can see that this has a certain influence on the state space density in such a way that we can see a kink around the point where the control switches. The behavior of the control can be explained by the fact that while the mean reversion term of the SDE given by $\rho(x-\mu)$ and the cost function $c_0(x,u) = (x-\mu)^2$ are symmetric around the mean μ, the diffusion of the process is given by $\sigma\sqrt{x}$, and hence increases with x. To avoid larger-scale diffusions that could occur above the mean and would have a undesired impact on the cost criterion, the control seeks to keep the process slightly lower than the mean μ. To support this explanation, Table V.19 shows computation results for a parameter sweep of the diffusion coefficient σ. In particular, the point a, where the control switches from 0.05 to -0.05

σ	a	J	k	$[-K, K]$
0.1	9.946	3.9062266365	$(10, 0, 10, 10)$	$[0, 15]$
0.2	9.844	20.990368976	$(10, 0, 10, 10)$	$[0, 20]$
0.3	9.619	52.101463651	$(10, 0, 10, 10)$	$[0, 25]$
0.4	9.336	97.823195297	$(10, 0, 12, 11)$	$[0, 40]$
0.5	8.984	156.74606251	$(10, 0, 12, 11)$	$[0, 40]$
0.6	8.496	230.40191011	$(10, 0, 13, 12)$	$[0, 50]$
0.7	8.057	313.98748092	$(10, 0, 13, 12)$	$[0, 50]$

Table V.19: Results, Cox-Ingersol-Ross model, varying diffusion coefficient

is shown, along with the approximate value for the cost criterion J. The discretization level k and the computed state space $[-K, K]$ had to be adjusted when changing the diffusion coefficient σ, and are shown as well.

Two observations can be made. First, we clearly see that with an increasing diffusion coefficient, the behavior of the optimal control becomes increasingly 'cautious' in a way that the switch occurs at lower state in order to avoid the large diffusions that come along with higher states. Second, a significant influence of the diffusion coefficient on the cost criterion is visible, which increases by two orders of magnitude in the given range of σ.

For a diffusion coefficient of $\sigma = 0.8$, the state space density shows an absorption of the process at $x = 0$. While analytically possible, this special case is not considered here. We

x_0	α	ρ	μ	σ	u_l	u_r
10	0.1	varies	10	0.5	-0.05	0.05

Table V.20: Configuration (2), Cox-Ingersol-Ross model

proceed to investigate the influence of the mean reversion coefficient ρ on the optimal control and the optimal cost criterion. To do so, we use the configuration of the model specified in Table V.20. Table V.21 shows the results of several computations, specifying the mean reversion coefficient ρ, the switching point of the control a, the approximate value of the cost criterion J as well as the discretization level k and the computed state space $[-K, K]$. Naturally, the computed costs are shrinking with an increasing mean reversion, since the

ρ	a	J	k	$[-K, K]$
0.01	8.906	179.80636207	$(10, 0, 12, 11)$	$[0, 40]$
0.02	8.984	156.74606251	$(10, 0, 12, 11)$	$[0, 40]$
0.03	9.062	138.60839167	$(10, 0, 12, 11)$	$[0, 40]$
0.04	9.102	124.11426052	$(10, 0, 12, 11)$	$[0, 40]$
0.05	9.180	112.33381737	$(10, 0, 12, 11)$	$[0, 40]$
0.06	9.219	102.59701292	$(10, 0, 12, 11)$	$[0, 40]$
0.07	9.258	94.424699129	$(10, 0, 13, 11)$	$[0, 40]$
0.08	9.297	87.470247370	$(10, 0, 13, 11)$	$[0, 40]$
0.09	9.336	81.480455805	$(10, 0, 13, 11)$	$[0, 40]$
0.10	9.375	76.266998677	$(10, 0, 14, 11)$	$[0, 40]$

Table V.21: Results, Cox-Ingersol-Ross model, varying mean reversion coefficient

mean reversion works in favor of the cost criterion. Notably, the switching point a increases with the mean reversion. For the tested values of ρ, it remains under the mean $\mu = 10$, which is still assumed to be a precautionary behavior to avoid the large diffusion values for higher values of X. However, a stronger mean reversion helps mitigate large diffusion, as larger fluctuations are swiftly corrected, hence the precautionary action of switching the control to push the process away from the long-term mean μ is triggered at higher values of the process.

V.2.2 Optimal Asset Allocation Problems with Budget Constraints

In this section we consider an asset allocation problem, in which an entity seeks to maximize its position in an asset whose price is governed by a stochastic process, but only has a limited budget to spend. Such a problem, with an underlying geometric Brownian motion, is extensively discussed in Lu et al. (2017), where a numerical approach based on the dynamic programming principle is utilized. The budget constraints are therein treated using the method of Lagrange multipliers. Here we present how our technique, based on the linear programming approach, can be used to tackle this control problem. In particular, we show that the budget constraints are easily incorporated into the linear constraints on the expected occupation measures, which appears to be less cumbersome in terms of the numerical optimization.

First, we consider the example given by Lu et al. (2017), where the share price of the asset is modeled by a geometric Brownian motion process. However, the (on-average) exponential growth of such a process requires rather unrealistic discounting factors to allow a numerical treatment of this model, making it rather difficult to connect the results to applications. Second, we consider an asset allocation problem with the share price being modeled by an Ornstein-Uhlenbeck process. Such a process features mean reversion, which automatically makes the process spend most time in a confined area about its mean, resulting in a smoother numerical treatment.

We discuss the optimality criterion employed in this section as well as the budget constraint without being specific about the underlying process. Assume that X is some stochastic process with state space E, which is controlled by a process u taking values in $U = [0, u_r]$. X represents the price of the considered asset, and u represents the rate at which we purchase shares of the asset. We are not allowed to sell stock, hence u is inherently non-negative. To be precise, although it is considered as a control, u merely expresses the influence of purchasing stock on the price. Our objective is not to influence the asset price in a certain way. Instead, our goal is to maximize the discounted share position

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha_1 s} u_s \, ds\right] \tag{2.1}$$

for a discounting rate α_1. This discounting rate can be understood in two different ways. First, it can simply model inflation, for example if the asset is a currency. Second, it could model the discounting of the personal utility of maximizing the share position, meaning that we would prefer to have a large share position earlier rather than later.

The maximization of the share position has to be done in such a way that

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha_2 s} X_s u_s \, ds\right] \leq \theta \tag{2.2}$$

is satisfied for some $\theta > 0$. This is the previously mentioned budget constraint. We are not allowed to spend more than θ to purchase shares. However, the unexpended funds earn interest at rate α_2 so the budget constraint (2.2) considers the present value of funds. To make the linear programming approach applicable, we have to assume that $\alpha_1 = \alpha_2 \equiv \alpha$ and take this to be the interest rate. Then, it is easy to see that we can express the budget constraints with $d(x, u) = xu$ by

$$\mathbb{E}\left[\int_{E \times U} d(x, u) \, d\mu\right] \leq \theta$$

with the discounted expected occupation measure μ. Indeed, the structure of (2.2) is identical to that of the cost criterion (2.1). Using the approximation given by (III.1.10) and (III.1.11) we obtain the linear inequality

$$\sum_{i=1}^{2^{k_m^{(3)}}} \sum_{j=0}^{2^{k_m^{(1)}}-1} \sum_{l=0}^{2^{k_m^{(2)}}} \beta_{j,l} \gamma_i \int_{E_j} d(x, u_l) p_i(x) \, dx \leq \theta$$

which is easily added into the linear program of Section III.1.2. In particular, standard solvers include functionality for linear inequalities which allows for an straight forward incorporation of the budget constraint in the discretized setting. This is different from the approach proposed in Lu et al. (2017), where the treatment of the budget constraints requires a nested optimization scheme for the actual control problem and the Lagrange multipliers.

In the given setup, neither the cost function $c_0(x,u) = u$ nor the budget constraint function $(x,u) \mapsto x \cdot u$ are increasing in x as described in Definition III.2.2 (note that if $u = 0$, $x \cdot u = 0$). Hence, the tightness argument as presented in Lemma IV.2.2 is not applicable. According to Remark IV.2.3, it is sufficient to introduce a function d that is increasing in x and demand that for some $M > 0$,

$$\int_{E\times U} d(x,u)\, \hat{\mu}(dx \times du) \leq M. \tag{2.3}$$

Set $d(x,u) = |x|$, and we assume that the optimal solution of the analytic problem μ^* fulfills

$$\int_{E\times U} |x|\, \mu^*(dx \times du) < \infty.$$

We can simply pick a large value for M, and restrict the linear program to measures that fulfill (2.3). As this bound was never violated when using the numerical approximations, it is not further discussed in the sequel.

Asset Price Modeled by Geometric Brownian Motion

Consider a state space $E = [0,\infty)$ and a control space $U = [0, u_r]$. For this example, the price of the asset under consideration is governed by the following SDE in integral form, given $\mu, \sigma, b > 0$,

$$X_t = \underbrace{x_0 + \int_0^t \mu X_s\, ds + \int_0^t \sigma X_s\, dW_s}_{\text{geometric Brownian motion part}} + \underbrace{\int_0^t b X_s u_s\, ds}_{\text{control part}}.$$

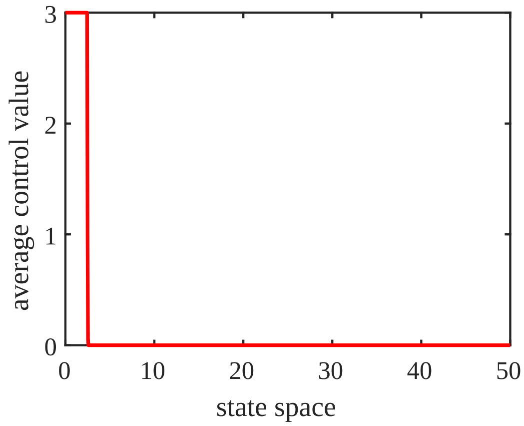

Figure V.25: Computed control, asset allocation problem under geometric Brownian motion

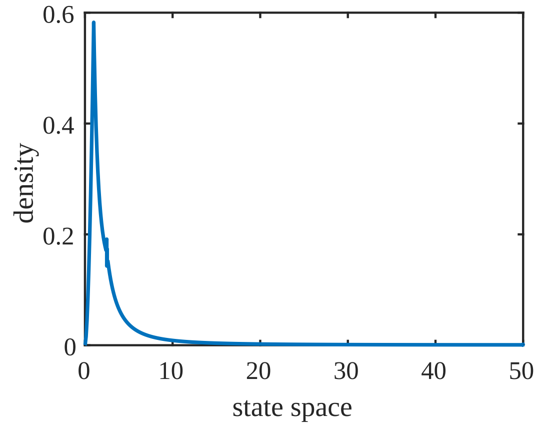

Figure V.26: State space density, asset allocation problem under geometric Brownian motion

Employing a control $u_t = 0$ means that we are not acquiring shares of the asset, while a control value $u_t > 0$ means that we are buying shares of the asset. Hence, the process X behaves as a geometric Brownian motion with a mean drift $\mu > 0$ and diffusion $\sigma > 0$ when we are not purchasing shares. If we do so, the share price is further influenced by an additional drift term $\int_0^t bX_s u_s \, ds$. This additional drift models the impact on the share price caused by the control purchasing shares, which is assumed to be proportional to the product of the share price and the number of shares purchased with proportionality coefficient $b > 0$. From a modeling point of view, X could be an illiquid asset, where the purchases made by the control represent a significant number of the currently available shares, and thus have a considerable impact on the price.

We use the optimality criterion (2.1) and the budget constraint (2.2) as presented in the

x_0	α	μ	σ	θ	u_r	b
1	0.25	0.2	0.5	10	3	0.06

Table V.22: Configuration, asset allocation problem under geometric Brownian motion

introduction of this section. As we will see, the optimal control for this problem is of bang-bang type, which means that up to a certain share price a, we purchase stocks at the highest possible rate u_r, and do not purchase stock if the share price is higher. It has to be noted that we have to assume rather high discounting factors α in order to make results computational attainable. This is due to the fact that the process X behaves similarly to a geometric Brownian motion, which -in expectation- grows unboundedly as time progresses. It is only due to the discounting that we can expect the expected occupation measure to be tight (and hence almost all mass can be contained in $[0, K]$ for some $K > 0$). Discounting rates of less than 0.2 were found to require vast computational resources, especially in terms of memory, which would go beyond the capacities of a standard computer. Especially in the light of considering α as inflation, or interest, the feasible values for α are fairly unrealistic. Thus we restrict ourselves to a few numerical examples for the given problem, and provide a more detailed parameter analysis for a model where the asset price is modeled by an Ornstein-Uhlenbeck process later on.

Table V.22 shows a sample configuration for the asset allocation problem. Note that $b - \frac{\sigma^2}{2} > 0$ holds, which ensures that the process X grows in expectation. The optimal control under this configuration and the state space density for a computation using the computed state space of $[-K, K] = [0, 50]$ are shown in Figures V.25 and V.26. To compute the approximate solution, a discretization level of $k = (10, 0, 15, 13)$ was used, and a re-evaluation step was conducted with the discretization level $k = (10, 0, 16, 13)$, in order to obtain a smooth state space density, similar to the discussion in Section V.2.1. The switching point of the control was at $a = 2.441$, which means that the approximate purchasing strategy dictates to buy shares when the price is at 2.441 or lower. We observed that the state space density shows some oscillation around this point which could not be eliminated with different choices for the discretization levels. Apart from this, note that the sharp mode of the state space density at the starting point $x_0 = 1$ is due to the fact that the discounting values the early positions of the state space more highly than later positions. The approximate value of the optimality criterion, which represents the expected discounted share position, was given by $J = 7.6375667435$.

x_0	a	J	k	$[-K, K]$
0.3	5.127	10.922451010	$(10, 0, 15, 13)$	$[0, \ 50]$
0.4	3.711	10.361620044	$(10, 0, 15, 13)$	$[0, \ 50]$
0.5	3.170	9.8157612398	$(10, 0, 15, 13)$	$[0, \ 50]$
0.6	2.832	9.3028927880	$(10, 0, 15, 13)$	$[0, \ 50]$
0.7	2.637	8.8292053506	$(10, 0, 15, 13)$	$[0, \ 50]$
0.8	2.539	8.3951362123	$(10, 0, 15, 13)$	$[0, \ 50]$
0.9	2.441	7.9988282420	$(10, 0, 15, 13)$	$[0, \ 50]$
1	2.441	7.6375667435	$(10, 0, 15, 13)$	$[0, \ 50]$
2	2.695	5.2990887981	$(10, 0, 15, 13)$	$[0, \ 60]$
3	3.213	4.1338099271	$(10, 0, 15, 13)$	$[0, \ 70]$
4	3.828	3.4453980821	$(10, 0, 15, 13)$	$[0, \ 80]$
5	4.395	2.9897877604	$(10, 0, 15, 13)$	$[0, \ 90]$
6	4.980	2.6620613757	$(10, 0, 15, 13)$	$[0, 100]$
7	5.508	2.4139654007	$(10, 0, 15, 13)$	$[0, 120]$
8	5.967	2.2174476342	$(10, 0, 15, 13)$	$[0, 130]$
9	6.426	2.0572882397	$(10, 0, 15, 13)$	$[0, 140]$

Table V.23: Results, asset allocation problem with geometric Brownian motion, varying starting value x_0

To analyze the model further, we investigate the influence of the starting value x_0 on optimal control and cost criterion value. We otherwise retain the parameter configuration given in Table V.22 and compute the optimal control and the respective expected discounted share position for starting values between 0.3 and 9. Table V.23 shows the results for these numerical solutions. In particular, the starting value x_0, the switching point of the control a (giving the price at which we stop purchasing) and the approximate discounted share position J are shown along with the discretization level k and the computed state space $[-K, K]$. To interpret this result, it is important to keep in mind that the process X grows (in expectation) exponentially from the starting value x_0. Hence, considering that the available budget remains constant, it is not surprising that the optimality criterion J, which represents the expected share position, shrinks when x_0 increases. In the same way, it is intuitively clear that in order to build up the stock position, one has to buy at higher prices as the starting value increases. However, we also observe an increase in the switching point when the starting value approaches 0. This is explained by recalling that we assume that our available budget pays interest when not being spent, and the fact that for small starting values, the initial growth of the share price is rather slow. The available budget grows more quickly than the share price, and hence we are able to pay fairly high prices to maximize our position. Albeit an interesting observation, this behavior is only notable due to the rather high (and unrealistic) interest rate. It is worth observing that for the starting values of 4 or higher, the optimal purchasing price is lower than the starting value, which means that we would have to 'hope' that the stock price drops below its initial value in order to purchase the stock.

Asset Price Modeled by Ornstein-Uhlenbeck Process

To present a model which allows for more realistic interest rates, consider the state space $E = (-\infty, \infty)$ and the control space $U = [0, u_r]$. For this example, the price of the asset under consideration is governed by the following SDE in integral form, given $\mu, \rho, \sigma, b > 0$,

$$X_t = \underbrace{x_0 + \int_0^t \rho(\mu - X_s)\, ds + \sigma W_t}_{\text{Ornstein-Uhlenbeck part}} + \underbrace{\int_0^t b X_s u_s\, ds}_{\text{control part}}.$$

In the absence of purchasing, this process is a Ornstein-Uhlenbeck process, where the drift $\int_0^t \rho(\mu - X_s)\, ds$ 'pulls' the process back to the mean μ as soon as deviations from μ caused by the stochastic part σW_t occur. Again the influence of the control represents the impact of purchasing a considerable number of the available shares, assuming the asset to be illiquid. Note that in contrast to the Cox-Ingersol-Ross model of Section V.2.1, negative values are possible. X could describe quantities like energy prices, which can be negative given low demand and high supply, or interest rates (which can be negative in the case of government bonds). The fact that X features mean reversion allows for smaller and more realistic interest rates α, as most of the mass of the expected occupation measure will be concentrated in a relatively small interval around μ.

An sample configuration for this problem is shown in Table V.24. The optimal control and

x_0	α	μ	ρ	σ	θ	u_r	b
100	0.02	100	0.1	5	8	1	0.1

Table V.24: Configuration, asset allocation problem under Ornstein-Uhlenbeck process

the state space density for a computation using the discretization level $k = (10, 0, 10, 10)$ with one additional mesh point, and a computed state space of $[-K, K] = [60, 140]$ are shown in Figures V.27 and V.28. The switching point a of the control is located at 72.660.

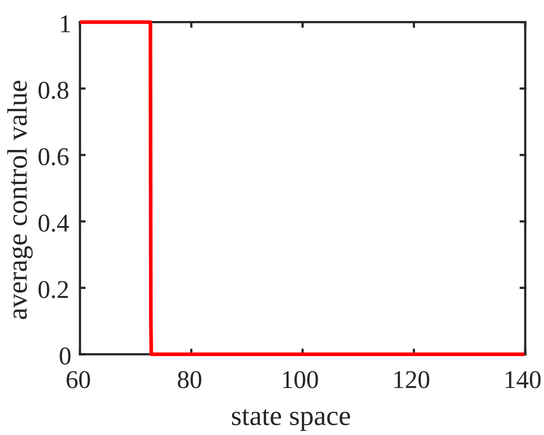

Figure V.27: Computed control, asset allocation problem under Ornstein-Uhlenbeck process

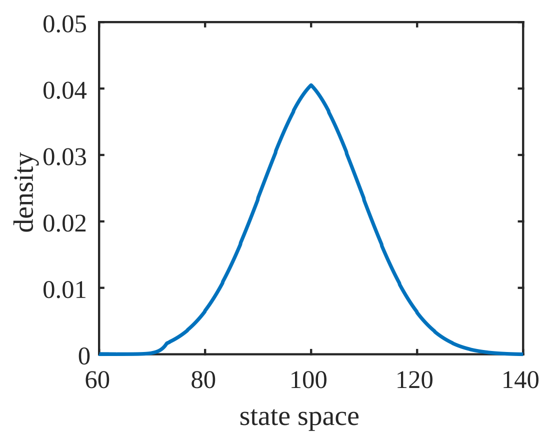

Figure V.28: State space density, asset allocation problem under Ornstein-Uhlenbeck process

A small kink in the state space density is visible at this point. The approximate discounted expected share position is $J = 0.11268988042$. This fairly low value is explained by the fact that we have a rather small fund availability of $\theta = 8$ compared to the mean of the process $\mu = 100$. Hence, purchases can only be made at fairly low share prices.

In the following, we provide a detailed investigation of the influence of the model parameters

σ, α, u_r and b on the optimal control and the optimality criterion given by the expected, discounted share position.

We begin by varying the diffusion coefficient σ. There are two different situations to consider. First, we consider a case with high fund availability θ, which allows us to buy shares at prices higher than the mean μ. Then, we look at a case with low fund availability, which forces us to purchase shares at a price lower than the mean μ. The complete parameter choices are shown in Table V.25. The results of the computations with high funds, $\theta = 4000$, are

x_0	α	μ	ρ	σ	θ	u_r	b
100	0.02	100	0.1	varies	1000 or 4000	1	0.001

Table V.25: Configuration (2), asset allocation problem under Ornstein-Uhlenbeck process

shown in Table V.26. The results for the low funded situation, $\theta = 1000$, are shown in Table V.27. Both tables show the diffusion coefficient σ, the switching point of the control a, the approximate expected share position J alongside the discretization level k and the computed state space $[-K, K]$.

σ	a	J	k	$[-K, K]$
3	106.0	40.577502624	$(10, 0, 13, 12)$	$[\ \ 70, 130]$
5	110.2	41.103046129	$(10, 0, 13, 12)$	$[\ \ 50, 150]$
7	114.5	41.591174213	$(10, 0, 13, 12)$	$[\ \ 30, 170]$
9	119.3	42.042247714	$(10, 0, 15, 12)$	$[\ \ \ 5, 195]$
11	124.2	42.474524830	$(10, 0, 16, 12)$	$[\ \ \ 0, 200]$
13	129.5	42.920942203	$(10, 0, 17, 12)$	$[\ -5, 205]$
15	129.5	43.345890065	$(10, 0, 17, 12)$	$[-15, 215]$

Table V.26: Results, asset allocation problem with Ornstein-Uhlenbeck process, varying diffusion coefficient, high fund availability ($\theta = 4000$)

Generally speaking, we can conclude that higher diffusion coefficients work in favor of the optimality criterion. This is due to the fact that with more volatility, lower prices are

σ	a	J	k	$[-K, K]$
3	95.31	10.899530407	$(10, 0, 13, 12)$	$[\ \ 70, 130]$
5	91.80	11.686953030	$(10, 0, 13, 12)$	$[\ \ 50, 150]$
7	90.02	12.303414645	$(10, 0, 14, 13)$	$[\ \ 30, 170]$
9	88.05	13.067920037	$(10, 0, 15, 13)$	$[\ \ 10, 190]$
11	86.33	13.878421197	$(10, 0, 15, 13)$	$[\ \ \ 0, 200]$
13	85.16	14.785432119	$(10, 0, 15, 13)$	$[\ \ \ 0, 200]$
15	84.53	15.717658230	$(10, 0, 16, 12)$	$[-10, 210]$

Table V.27: Results, asset allocation problem with Ornstein-Uhlenbeck process, varying diffusion coefficient, low fund availability ($\theta = 1000$)

more likely, and there are more chances to buy shares inexpensively. However, the way the optimal control adapts to different diffusion coefficients differs when larger or smaller funds are available. With high fund availability, see Table V.26, higher volatility causes the switching point a to increase. A possible explanation of this behavior could be that the money which is 'saved' by purchasing at the lower prices which are occurring more frequently due to the high diffusion, now is spent on purchases with higher share prices. Such behavior is advantageous due to the discounting. The discounted, expected share position benefits from earlier purchases more than it does from later purchases, and if we buy at higher prices, we can purchase more stock earlier.

The opposite behavior can be observed in the case of low fund availability. As stated before, the optimal control has to rely on the price dropping significantly below the mean μ to make purchases. A higher diffusion coefficient results in a purchasing strategy with a rather low switching point a, representing the maximal purchase price, since it is more likely that the share price actually drops that low in a time period which is short enough such that the discounting does not have a drastic impact on the optimality criterion. So, a decreases, and the expected stock position increases with a higher diffusion coefficient. From a numerical point of view, observe that we need to enlarge the computed state space $[-K, K]$ when the diffusion coefficient increases. This is to be expected, since a higher uncertainty distributes the mass of the expected occupation measure on a wider interval.

Next, we analyze the influence of the discounting factor α. We have already indicated that the discounting has some influence on the optimal purchasing strategy, as it punishes us for purchasing at price which is too low, with the process not dropping so far over long periods of time. The configuration for this parameter sweep can be found in Table V.28. Differently from the previous parameter sweep, we now use a budget of $\theta = 2000$ and $b = 0.002$ to model the impact on the share price when purchasing. The results for various discounting factors

x_0	α	μ	ρ	σ	θ	u_r	b
100	varies	100	0.1	5	2000	1	0.002

Table V.28: Configuration (3), asset allocation problem under Ornstein-Uhlenbeck process

are given in Table V.29. In particular, it shows the discounting factor α, the switching point of the control a, the approximate expected share position J alongside the discretization level k and the computed state space $[-K, K]$. Clearly, the expected discounted share position

α	a	J	k	$[-K, K]$
0.005	87.40	24.320020503	$(10, 0, 14, 13)$	$[50, 150]$
0.01	92.48	23.142813987	$(10, 0, 14, 13)$	$[50, 150]$
0.015	96.09	22.409298613	$(10, 0, 14, 13)$	$[50, 150]$
0.02	98.93	21.865960757	$(10, 0, 14, 13)$	$[50, 150]$
0.025	101.5	21.431711036	$(10, 0, 14, 13)$	$[50, 150]$
0.03	103.9	21.081483466	$(10, 0, 14, 13)$	$[60, 140]$
0.035	106.6	20.752395535	$(10, 0, 14, 13)$	$[60, 140]$
0.04	109.7	20.480356949	$(10, 0, 15, 13)$	$[65, 135]$

Table V.29: Results, asset allocation problem with Ornstein-Uhlenbeck process, varying discounting factor

shrinks with an increasing discounting factor. On the other hand, the switching point a increases with the discounting factor in order to counter the faster discounting. In order to purchase more shares early on, which contributes to the optimality criterion more significantly than later purchases, we have to accept higher prices. This happens in such a way that we start with a maximal purchasing price of 87.40, which is lower than the mean of $\mu = 100$, and end with a maximal purchasing price 109.7, which is higher than the mean.

The next parameter which is analyzed is the maximal purchasing rate u_r. The configuration of this parameter sweep is shown in Table V.30. The results for the computation with

x_0	α	μ	ρ	σ	θ	u_r	b
100	0.02	100	0.1	5	3000	varies	0.001

Table V.30: Configuration (4), asset allocation problem under Ornstein-Uhlenbeck process

varying maximal purchasing rate are displayed in Table V.31. In particular, it shows the maximal purchasing rate u_r, the switching point of the control a, the approximate expected share position J alongside the discretization level k and the computed state space $[-K, K]$. The effect of an increase in the maximal purchasing rate is such that the maximal purchasing

u_r	a	J	k	$[-K, K]$
1	103.8	31.797497033	$(10, 0, 14, 13)$	$[40, 160]$
1.5	98.71	32.878597048	$(10, 0, 14, 13)$	$[40, 160]$
2	96.02	33.569457233	$(10, 0, 14, 13)$	$[40, 160]$
2.5	94.25	34.0750508457	$(10, 0, 15, 13)$	$[40, 150]$
3	92.85	34.467140282	$(10, 0, 15, 13)$	$[40, 150]$
3.5	91.76	34.894183962	$(10, 0, 15, 13)$	$[40, 140]$
4	90.88	35.171673036	$(10, 0, 15, 13)$	$[40, 140]$

Table V.31: Results, asset allocation problem with Ornstein-Uhlenbeck process, varying maximal purchasing rate

price decreases, and the discounted maximal share position increases. Since we are able to buy more shares per unit of time, we can benefit more from low prices, and are able to buy more shares before the price increases again. Hence, we can wait for lower share prices and afford the additional discounting that comes into play when buying later.

Finally, we analyze the influence of the drift-per-buy coefficient b on the optimal control and the optimality criterion. Before we do so, we discuss some phenomena that appear in the numerical calculations for this model. Figure V.29 shows a computed optimal control and Figure V.30 show the respective state space density for a configuration where b is rather large compared to previous examples. The complete configuration is shown in Table V.32. To compute these results, we used a discretization level of $k = (10, 0, 15, 13)$ for both the

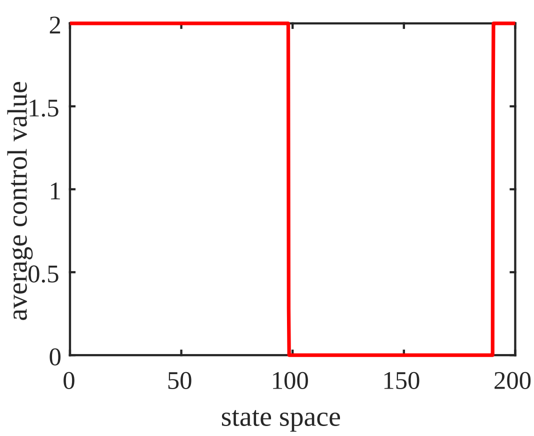

Figure V.29: Computed control, asset allocation problem under Ornstein-Uhlenbeck process, abnormal control behavior

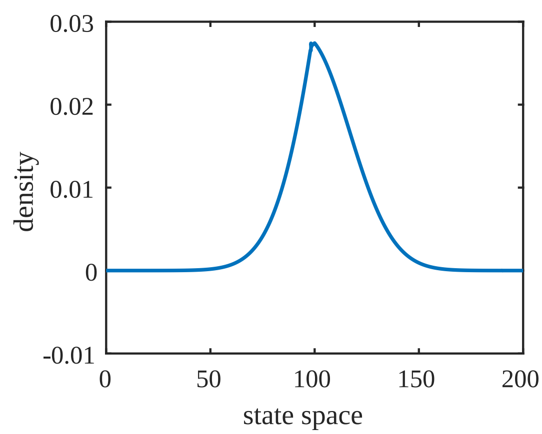

Figure V.30: State space density, asset allocation problem under Ornstein-Uhlenbeck process, abnormal control behavior

x_0	α	μ	ρ	σ	θ	u_r	b
100	0.02	100	0.1	9	3000	2	0.001

Table V.32: Configuration, asset allocation problem under Ornstein-Uhlenbeck process, abnormal control behavior

linear program solver and the re-evaluation step. The approximate value for the expected stock position J reached 34.2007. We can observe the ‘usual’ behavior of the optimal control for this model, which suggest to buy at the maximal possible purchasing rate as long as the share price is not higher than 98.05. However, the computed control also suggests that we should again purchase shares when the price rises above 190.2. One can readily see that such behavior cannot be optimal, as it is highly disadvantageous to purchase shares at very high prices. Hence we have to assume that this behavior is due to an artifact of the numerical method rather than the model itself. Two possible explanations are given in the following. To begin with, note that the state space density in the region of the state space which shows this anomaly is lower than $6.743 \cdot 10^{-7}$. One the one hand, this ensures that the impact of the abnormal control behavior on the optimality criterion is rather small, and might possibly be neglected. On the other hand, this might indicate that the computation of the control is numerically unstable. Recalling (III.1.20), the control for this example is computed by

$$\hat{\eta}_k(V, x_j) = \frac{\mu\left(V, E_j\right)}{\mu_E\left(E_j\right)},$$

where μ_E is the measure whose density is displayed in Figure V.30. A small denominator in this expression causes instability in this calculation, and hence could be a possible explanation for the unexpected behavior of the computed control.

The second explanation focuses on the way in which we approximate models with an unbounded state space using models with a bounded state space. We do this by discretizing the measures in the set

$$\mathscr{M}_{\infty,[-K,K]} = \left\{\mu \in \mathcal{P}([-K,K]\times U) \middle| \int Af d\mu = Rf \quad \forall f \in \mathscr{D}_\infty([-K,K])\right\},$$

compare (III.2.6). Crucially, for a function $f \in \mathscr{D}([-K,K])$ we stipulated that the support of f, f' and f'' is a proper subset of $[-K,K]$. In particular, we have that $f'(-K) = f'(K) = 0$. Hence, the set $\mathscr{M}_{\infty,[-K,K]}$ is indistinguishable from the set

$$\hat{\mathscr{M}}_{\infty,[-K,K]} = \left\{\mu \in \mathcal{P}([-K,K]\times U) \middle| \int Af d\mu + \int Bf d\mu_1 = Rf \quad \forall f \in \mathscr{D}_\infty([-K,K])\right\}$$

with $Bf(x) = f'(x) \cdot I_{\{-K\}}(x) - f'(x) \cdot I_{\{K\}}(x)$ and μ_1 being a finite Borel measure concentrated on $\{-K\} \cup \{K\}$. In other words, the approximate problem with a finite state space could actually feature reflections at $-K$ and K, which we are not able to pick up with the expected occupation measure μ due to the structure of $\mathscr{D}([-K, K])$. As seen in Section V.1.2, such reflections can be used in favor of bringing the process back towards the origin, by using the reflection to push the process back into the opposite direction. Hence, a second explanation to the unexpected control behavior is that the optimal control tries to use the rather high drift induced by the purchasing, given by $b \cdot X_t u_t$ and the reflection to lower prices. Obviously, such behavior is in consistent with the actual model featuring an unbounded state space. To

x_0	α	μ	ρ	σ	θ	u_r	b
10	0.015	10	0.1	0.75	5	0.5	varies

Table V.33: Configuration (5), asset allocation problem under Ornstein-Uhlenbeck process

remedy this situation, we tweak the model in such a way that we eliminate the drift-per buy term for larger values of the state space values, and further introduce a penalty for using the control at such high state space values. To be precise, for some $a_r \in [-K, K]$, we now seek to maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha_1 s}\left(u_s - I_{\{X_s > a_r\}} u_s\right)\, ds\right].$$

In other words, we do not benefit from using the control if X is larger than a_r, with a_r aptly chosen. The SDE is changed to

$$X_t = x_0 + \int_0^t \rho(\mu - X_s)\, ds + \sigma W_t + \int_0^t I_{\{X_s > a_r\}} b X_s u_s\, ds.$$

The parameter sweeps presented so far were configured in such a way that the described phenomena would not appear. However, when varying b it is likely that we see this undesired behavior, and the use of this penalty method is necessary.

This configuration which is used to analyze the influence of the drift-per-buy term b is specified in Table V.33, and the results of the computation are shown in Table V.34. We introduced an additional mesh point in the middle of the mesh for the state space density for all of the computations. Table V.34 shows the drift-per-buy rate b, the switching point of the control a, the approximate expected share position J alongside the discretization level k, the computed state space $[-K, K]$ and the point a_r at which we introduced the penalty for using the control. As to be expected, the value of the optimality criterion decreases when the

b	a	J	k	$[-K, K]$	a_r
0.01	6.797	0.80422691171	$(10, 0, 12, 12)$	$[0, 20]$	16
0.1	7.109	0.74245274521	$(10, 0, 12, 12)$	$[0, 20]$	16
0.2	7.422	0.69712180542	$(10, 0, 12, 12)$	$[0, 20]$	16
0.3	7.695	0.66559165530	$(10, 0, 12, 12)$	$[0, 20]$	16
0.4	7.91	0.64189415350	$(10, 0, 12, 12)$	$[0, 20]$	16
0.5	8.105	0.62300645684	$(10, 0, 12, 12)$	$[0, 20]$	16
0.6	8.301	0.60734080954	$(10, 0, 12, 12)$	$[0, 20]$	16
0.7	8.477	0.59392214278	$(10, 0, 12, 12)$	$[0, 20]$	16

Table V.34: Results, asset allocation problem with Ornstein-Uhlenbeck process, varying drift-per-buy rate

drift-per-buy rate b increases. As the impact of purchasing is more significant with higher values of b, the maximal purchasing price increases with it. In other words, if the considered asset is illiquid, or if we purchase considerable market shares of the asset, we have to accept higher prices and expect a lower position of shares to be attainable.

Conclusion

We conclude this thesis by first summarizing the achieved work. Then, we point out areas of future research which we find to be instrumental in further broadening the applicability and quality of finite element methods for linear programming formulations in stochastic control.

VI.1 Summary

This thesis contributed to the research on linear programming formulations for stochastic control both in terms of analytical and numerical considerations. While the analytical contributions in the form of the existence and uniqueness of linear constraints under certain conditions on the controls came as a necessary by-product of the convergence proof for the proposed discretization, the main focus of this thesis lay in establishing a numerical method for solving the infinite-dimensional linear programs that are associated with stochastic control problems in infinite time, with either a bounded or an unbounded state space. To some extent it could be shown that analogous to the analytic treatment of control problems using the linear programming approach, numerical techniques based on it can serve as an alternative to classic dynamic programming approaches. To support this claim, the proposed method was formulated in fairly general terms, analyzed in fine detail and applied to a wide range of control problems.

When stating the initial problem we used a formulation of the linear program that allows for bounded as well as unbounded state spaces, the existence or absence of singular behavior, singular control, and two different cost criteria. Then, we proposed an approximation which retained the underlying structure of a linear program in order to allow for efficient solution. In addition, it was demonstrated how the linear programming techniques can be used to evaluate the cost criteria for fixed controls, which opens the approach up to a broader class of problems, as seen in the stochastic growth example. The convergence considerations examined the defined approximation in detail and provided an analysis of the approximation errors in each respective step of the approximation. In particular, the discretization of the constraint space was analyzed with techniques from the theory of weak convergence of measure, and the approximation of the relaxed controls was dissected with a repeated application of the triangle inequality and several techniques from analysis. Furthermore, the convergence of the approximate evaluation of cost criteria for fixed controls was considered. With the theoretical treatment concluded, the performance of the proposed method was demonstrated by applying it to several problems in stochastic control. We began with solving some simple model problems of Brownian motion with bounded state spaces, for which analytic solutions are attainable and the actual convergence could be analyzed. Then, we introduced more involved features into the models, like costs of control and singular control. Finally, models with unbounded state spaces were investigated, and the underlying dynamics became more complicated by considering geometric Brownian Motion and mean-reverting models. The numerical experiments not only underlined the applicability of the models, but also contributed to the analysis of the models themselves.

VI.2 Outlook

The presented research can be carried on in several directions. A generalization of the method to state spaces of higher dimension would drastically increase the applicability to more challenging control problems. The consideration of other cost criteria, like finite time cost criteria or optimal stopping cost criteria would also promote the solution of more appealing problems. Finally, one could improve the numerics of the presented method by introducing higher order elements for the discretization of the function spaces, as well as by researching adaptive meshing approaches for the discretization procedure. On the theoretical side, a derivation of error bounds for the proposed approximations would be very interesting. The generalization of the proposed numerical method to higher dimensional state spaces will dramatically enhance the applicability of the method. The theory of the linear programming approach can as well be used in higher dimensions, but several challenges loom in regard to the numerical approximation. First, a countable dense set for the space of multivariate functions $(C_c^2(E), \|\cdot\|_{\mathscr{D}})$ has to be identified, with $\|f\|_{\mathscr{D}} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$ now picking up the uniform norm of the gradient and Jacobian of f. A convenient set of basis functions has to be defined in such a way that the sparsity of the coefficient matrix would be guaranteed in order to keep computation times low. Second, in the case of a bounded state space, the discretization of the singular expected occupation measure μ_1 would be more involved, as the boundaries in higher dimensions actually have non-trivial geometry, for example a line in a two-dimensional state space. Finally, the discretization approaches for both the densities and the controls would have to be adjusted to match this more general setting.

The introduction of finite time cost criteria, or an optimal stopping cost criterion comes with an increased complexity of the analytic problem. In both cases, the expected occupation measures would be time-dependent, which is expressed by introducing another constraint space for the time component, as showcased in Helmes and Stockbridge (2007). An approximation approach would now have to discretize the time dependence in both the constraints and the measures, increasing the complexity of the discretization scheme. Regarding the constraints, it is conjectured that the discretization can be conducted in similar fashion as presented here, as the additional constraint functions lie in C^1. For this space of functions, analogues of B-spline basis functions are available, see de Boor (2001). However, a time-discretization of the expected occupation measures would have to be devised from scratch. In particular, the time dependence of the control has to be addressed. When regarding optimal stopping problems, a discrete approximate version of the distribution of the stopping time has to established as well. An initial, and numerically successful, take on this can be found in Lutz (2007). The convergence theory of this approach remains an open research problem. A fairly straightforward enhancement of the presented approach is given by the introduction of higher order elements when discretizing the density of the expected occupation measure. Although this would, in some cases, require more assumption on the density, it is likely that such an approach would improve convergence rates. Key challenges would remain in guaranteeing the non-negativity of the density when using certain finite elements. This is guaranteed for piecewise constant or piecewise linear elements, but not for standard quadratic elements. Also, the convergence proof would have to be adjusted. Regarding the latter, recall that the proof of Proposition IV.1.43 assumes that the approximate density is zero only on a set with a Lebesgue measures of 0, which is not as easily concluded for non-constant elements. The possible benefits of higher-order elements were shown in Kaczmarek et al. (2007) and Rus (2009). Solving the modified bounded follower problem, they attained comparable or even better errors compared to those attained in this thesis, while still using an insufficient approximation of the constraints space with piecewise linear or Hermite polynomial basis functions for $C_c^2(E)$. In that matter, note that little flexibility of discretizing the constraint space $\mathscr{D}_\infty$ is given, as it is crucial to ensure the interchangeability of limit and integral in expressions as the following, given a sequence of functions $f_k \to f$ in some way:

$$\lim_{k\to\infty}\int_{E\times U} Af_k(x,u)\mu(dx\times du) = \int_{E\times U}\lim_{k\to\infty} Af_k(x,u)\mu(dx\times du) = \int_{E\times U} Af(x,u)\mu(dx\times du)$$

Given the displayed densities in Section IV.2, adaptive meshing approaches seem like a very natural way to reduce computational efforts by introducing fewer mesh points where the density remains constant, and more mesh point wherever it changes rapidly. Similar heuristics could be used on the meshes for the constraints or the relaxed control. However, it is worth noting that with initial attempts in this direction conducted during the research for this thesis, it was hard to ensure that the discrete linear program would have feasible solutions.

To elevate adaptive meshing over a heuristic state, an analytic treatment of local errors would be instructive. This would also factor into establishing error bounds for the proposed approximation, which remained completely unconsidered in this work. So far, only the convergence 'as $n \to \infty$' is proven, and it is unclear how the approximation behaves for finite discretization levels. If error bounds could be established, recommendations for the choice of discretization levels could be given, enhancing the applicability of the proposed method in real-world situations.

Appendix

A Theoretic Aspects of Singular Stochastic Processes

In this appendix, we derive Itô's formula for stochastic processes with singular behavior. This motivates the definition of the singular generator B in Section II.1. Consider a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[t_0,\infty)}, \mathbb{P})$ which fulfills the 'usual conditions', that is, the filtration $\{\mathscr{F}_t\}_{t\in[t_0,\infty)}$ is right-continuous and complete. Also consider a 1-dimensional $\{\mathscr{F}_t\}_{t\in[t_0,\infty)}$-Brownian motion process W. The set $E \subset \mathbb{R}$ is called the state space. Until specified differently, we will assume that $E = \mathbb{R}$. We are further given three functions, $\bar{b}$, called the drift, $\bar{\sigma}$, called the diffusion and $\bar{h}$, called the jump size, such that $\bar{b}, \bar{\sigma}, \bar{h} : [0,\infty) \times E \times \Omega \mapsto E$. Assume that $\bar{b}(\cdot,\cdot,\omega)$, $\bar{\sigma}(\cdot,\cdot,\omega)$ and $\bar{h}(\cdot,\cdot,\omega)$ are Borel measurable for all $\omega \in \Omega$, and $\bar{b}(\cdot, x, \cdot)$ and $\bar{\sigma}(\cdot, x, \cdot)$ and $\bar{h}(\cdot, x, \cdot)$ are progressively measurable for all $x \in E$. For details on the solvability of SDEs using the coefficients $\bar{b}$ and $\bar{\sigma}$, we refer to Karatzas and Shreve (1991) or Pham (2009).

Remark 1.1. *In the setting of Section II.1, consider a fixed (non-relaxed) control process u. Define $\bar{b}(t, x, \omega) = b(x, u_t(\omega))$, and $\bar{\sigma}$ as well as $\bar{h}$ similarly to match the setting of this section.*

A.1 Jump Processes

The goal of this subsection is to derive Itô's formula for a jump processes. Consider an almost surely increasing sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ and the right-continuous, increasing process of bounded variation given by

$$\xi_t = \sum_{k=1}^{\infty} I_{\{\tau_k \le t\}}.$$

This process can be used as an integrator of a Lebesgue-Stieltjes integral, giving the following stochastic differential equation for an Itô jump process a well-defined meaning. For the sake of exposition, the dependence on ω is dropped in the subsequent formulas.

$$\begin{aligned} dX_t &= \bar{b}(t, X_t)dt + \bar{\sigma}(t, X_t)dW_t + \bar{h}(t, X_{t-})d\xi_t, \quad t \ge t_0 \\ X_{t_0} &= x_0. \end{aligned} \tag{1.2}$$

The integral formulation for (1.2) is

$$X_t = x_0 + \int_{t_0}^t \bar{b}(s, X_s)\, ds + \int_{t_0}^t \bar{\sigma}(s, X_s)\, dW_s + \int_{t_0}^t \bar{h}(s, X_{s-})\, d\xi_s.$$

Proposition 1.3. *Let X be a solution to (1.2), and let $f \in C^2(E)$ be bounded, with bounded first and second derivatives. Then,*

$$\begin{aligned} f(X_t) &= f(x_0) + \int_{t_0}^t \bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s)\, ds + \int_{t_0}^t \bar{\sigma}(s, X_s) f'(X_s)\, dW_s \\ &\quad + \int_{t_0}^t f(X_{s-} + \bar{h}(s, X_{s-})) - f(X_{s-})\, d\xi_s. \end{aligned}$$

Proof. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of stopping times such that τ_k is the time of the k-th jump of X. We will use these stopping times to separate the continuous part of X from the non-continuous, so-called singular part. For $t \geq t_0$, assume that $n \in \mathbb{N}$ is the smallest natural number such that $\tau_n \geq t$. Then

$$\begin{aligned} f(X_t) - f(x_0) &= \underbrace{f(X_t) - f(X_{\tau_n}) + \sum_{k=2}^n \big(f(X_{\tau_k-}) - f(X_{\tau_{k-1}})\big) + f(X_{\tau_1-}) - f(x_0)}_{I_1,\ \text{continuous part}} \\ &\quad \underbrace{+ \sum_{k=1}^n f(X_{\tau_k}) - f(X_{\tau_k-})}_{I_2,\ \text{singular part}}. \end{aligned}$$

Using the classical Itô formula, we can deduce that

$$\begin{aligned} I_1 &= \int_{\tau_n}^t \bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s)\, ds + \int_{\tau_n}^t \bar{\sigma}(s, X_s) f'(X_s)\, dW_s \\ &\quad + \sum_{k=2}^n \left(\int_{\tau_{k-1}}^{\tau_k-} \bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s)\, ds + \int_{\tau_{k-1}}^{\tau_k-} \bar{\sigma}(s, X_s) f'(X_s) dW_s \right) \\ &\quad + \int_{t_0}^{\tau_1-} \bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s)\, ds + \int_{t_0}^{\tau_1-} \bar{\sigma}(s, X_s) f'(X_s)\, dW_s \end{aligned}$$

Observe that for any k

$$\int_{\tau_k-}^{\tau_k} \bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s)\, ds + \int_{\tau_k-}^{\tau_k} \bar{b}(s, X_s) f'(X_s)\, dW_s = 0$$

holds. Indeed, since both integrals are taking over one point in the time domain, which is a Lebesgue null set (hence, the first integral is zero) and by the continuity of the paths of W, this point has also no weight in the stochastic integral. Hence, we can remove the "$-$"

indicating the left limit in the expression for I_1, revealing that

$$\begin{aligned}
I_1 &= \int_{\tau_n}^{t} \bar{b}(s,X_s)f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s,X_s)f''(X_s)ds + \int_{\tau_n}^{t} \bar{\sigma}(s,X_s)f'(X_s)dW_s \\
&\quad + \sum_{k=2}^{n}\left(\int_{\tau_{k-1}}^{\tau_k} \bar{b}(s,X_s)f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s,X_s)f''(X_s)ds + \int_{\tau_{k-1}}^{\tau_k} \bar{\sigma}(s,X_s)f'(X_s)dW_s\right) \\
&\quad + \int_{t_0}^{\tau_1} \bar{b}(s,X_s)f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s,X_s)f''(X_s)ds + \int_{t_0}^{\tau_1} \bar{\sigma}(s,X_s)f'(X_s)dW_s \\
&= \int_{t_0}^{t} \bar{b}(s,X_s)f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s,X_s)f''(X_s)ds + \int_{t_0}^{t} \bar{\sigma}(s,X_s)f'(X_s)dW_s.
\end{aligned}$$

The term I_2 describes the increment in $f(X)$ when X jumps. This can be rewritten as follows.

$$I_2 = \sum_{k=1}^{n} f(X_{\tau_k}) - f(X_{\tau_k -}) = \int_0^t f(X_s) - f(X_{s-})d\xi_s = \int_0^t f\left(X_{s-} + \bar{h}(s,X_{s-})\right) - f(X_{s-})d\xi_s.$$

Adding I_1 and I_2 together, and adding $f(x_0)$, we obtain the desired result. □

Remark 1.4. *Proposition 1.3 reveals that the generator B of singular behavior given by a jump ought to be defined by $Bf(x) = f(x + h(s,x)) - f(x)$.*

A.2 Reflection Processes

To model a reflection of a stochastic process X, we use the concept of a local time process. An introduction to Brownian local time and the closely related Tanaka formula can be found in Øksendal (1998), exercise 4.10. Here, we will regard the same concept for Itô-processes, that is, solutions to the differential equation

$$\begin{aligned}
dX_t &= \bar{b}(t,X_t)dt + \bar{\sigma}(t,X_t)dW_t \\
X_{t_0} &= x_0,
\end{aligned}$$

and derive a generalization of Itô's lemma for processes featuring reflections.

Definition 1.5. *Let X be an Itô process. Let λ denote the Lebesgue measure on $[0,\infty)$. The local time of X at x is the process*

$$L_x^X(t) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\lambda(\{s\in[0,t] : |X_s - x| \le \epsilon\}).$$

Remark 1.6. *L_x^X can be shown to be an almost surely continuous, increasing process of bounded variation. Hence it can be used as an integrator in a Lebesgue-Stieltjes integral.*

A reflected process is modeled as follows. Assume that $r_1, r_2, \ldots, r_n \in E$ are the points at which the process is reflected to the right, and that $l_1, l_2, \ldots, l_n \in E$ are the points at which the process is reflected to the left. Define

$$\xi_t = \sum_{k=0}^{n} L^X_{r_k}(t) - \sum_{k=0}^{n} L^X_{l_k}(t)$$

As a linear combination of process of bounded variation, ξ can still be used as an integrator, giving the following stochastic differential equation for reflected process proper meaning.

$$\begin{aligned} dX_t &= \bar{b}(t, X_t)dt + \bar{\sigma}(t, X_t)dW_t + d\xi_t, \quad t \geq t_0 \\ X_{t_0} &= x_0. \end{aligned} \tag{1.7}$$

with integral meaning

$$X_t = x_0 + \int_{t_0}^{t} \bar{b}(s, X_s)ds + \int_{t_0}^{t} \bar{\sigma}(s, X_s)dW_s + \xi_t,$$

as $\xi_0 = 0$.

Proposition 1.8. *Let X be a solution to (1.7), and let $f \in C^2(E)$ be bounded, with bounded first and second derivatives. Then,*

$$\begin{aligned} f(X_t) &= f(x_0) + \int_{t_0}^{t} \left(\bar{b}(s, X_s) f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s) f''(X_s) \right) ds + \int_{t_0}^{t} \bar{\sigma}(s, X_s) f'(X_s) dW_s \\ &\quad + \int_{t_0}^{t} f'(X_s) d\xi_s. \end{aligned}$$

Proof. Consider the increment $f(X_t) - f(x_0)$, and a partition $t_0 < t_1 < t_2 < \ldots < t_n = t$ of $[0, t]$. Assume that $t_{j+1} - t_j \to 0$ as $n \to \infty$. By Taylor's formula, we have that

$$f(X_t) - f(x_0) = \underbrace{\sum_{j=0}^{n-1} f'(X_{t_j}) \left(X_{t_{j+1}} - X_{t_j} \right)}_{I_1} + \underbrace{\sum_{j=0}^{n-1} \frac{1}{2} f''(\eta_j) \left(X_{t_{j+1}} - X_{t_j} \right)^2}_{I_2}$$

where $\eta_j \in [X_{t_{j+1}} - X_{t_j}]$, for $j = 1, 2, \ldots n-1$, respectively. The term I_1 can be split up as follows.

$$I_1 = \sum_{j=1}^{n-1} f'(X_{t_j}) \left[\int_{t_j}^{t_{j+1}} \bar{b}(s, X_s)ds + \int_{t_j}^{t_{j+1}} \bar{\sigma}(s, X_s)dW_s \right] + \sum_{j=1}^{n-1} f'(X_{t_j}) \left(\xi_{t_{j+1}} - \xi_{t_j} \right)$$

As can be seen in the proof of the classical Itô formula, the first part of the sum converges to

$$\int_0^t \bar{b}(s, X_s) f'(X_s) ds + \int_{t_0}^{t} \bar{b}(s, X_s) f'(X_s) dW_s$$

as $n \to \infty$. The second part of the sum converges to the Lebesgue-Stieltjes integral $\int_0^t f'(X_s) d\xi_s$ as $n \to \infty$.

Let us hence move our attention to the term I_2. Observe that

$$\begin{aligned}\left(X_{t_{j+1}} - X_{t_j}\right)^2 &= \left(\int_{t_j}^{t_{j+1}} \bar{b}(s, X_s)ds + \int_{t_j}^{t_{j+1}} \bar{\sigma}(s, X_s)dW_s\right)^2 + \left[\xi_{t_{j+1}} - \xi_{t_j}\right]^2 \\ &\quad +2\left[\int_{t_j}^{t_{j+1}} \bar{b}(s, X_s)ds + \int_{t_j}^{t_{j+1}} \bar{\sigma}(s, X_s)dW_s\right]\left[\xi_{t_{j+1}} - \xi_{t_j}\right]\end{aligned}$$

Thereby, we have that

$$\begin{aligned}2 \cdot I_2 &= \underbrace{\sum_{j=0}^{n-1} f''(\eta_j)\left(\int_{t_j}^{t_{j+1}} \bar{b}(s, X_s)ds + \int_{t_j}^{t_{j+1}} \bar{\sigma}(s, X_s)dW_s\right)^2}_{I_{2,1}} + \underbrace{\sum_{j=0}^{n-1} f''(\eta_j)\left[\xi_{t_{j+1}} - \xi_{t_j}\right]^2}_{I_{2,2}} \\ &\quad + \underbrace{\sum_{j=0}^{n-1} 2f''(\eta_j)\left[\int_{t_j}^{t_{j+1}} \bar{b}(s, X_s)ds + \int_{t_j}^{t_{j+1}} \bar{\sigma}(s, X_s)dW_s\right]\left[\xi_{t_{j+1}} - \xi_{t_j}\right]}_{I_{2,3}}.\end{aligned}$$

As can be seen in the proof of the classical Itô formula, $I_{2,1}$ converges to $\int_{t_0}^{t} \bar{\sigma}^2(s, X_s) f''(X_s)ds$ as $n \to \infty$. Regarding $I_{2,2}$, we note that

$$I_{2,2} \leq \sup_{\alpha \in E} |f''(\alpha)| \left(\sup_{j=1,\ldots,n-1} |\xi_{t_{j+1}} - \xi_{t_j}|\right)(\xi_t - \xi_0), \tag{1.9}$$

which converges to 0 as $n \to \infty$ for each ω. Indeed, as ξ is a continuous function on the compact set $[0, t]$, and hence uniformly continuous and thereby $\sup_{j=1,\ldots,n-1} |\xi_{t_{j+1}} - \xi_{t_j}| \to 0$ as $n \to \infty$. Finally, $I_{2,3}$ can be bounded as follows.

$$I_{2,3} \leq 2 \sup_{\alpha \in E} |f''(\alpha)| \left(\sup_{j=1,\ldots,n-1} |\xi_{t_{j+1}} - \xi_{t_j}|\right)\left[\int_{t_0}^{t} \bar{b}(s, X_s)ds + \int_{t_0}^{t} \bar{\sigma}(s, X_s)dW_s\right]$$

Hence for any $\omega \in \Omega$, $I_{2,3}$ goes to 0 as $n \to \infty$, as again ξ is continuous on a compact set and hence uniformly continuous. Adding all terms up, we have that

$$\begin{aligned}f(X_t) &= f(x_0) + \int_{t_0}^{t} \bar{b}(s, X_s)f'(X_s) + \frac{1}{2}\bar{\sigma}^2(s, X_s)f''(X_s)ds + \int_{t_0}^{t} \bar{\sigma}(s, X_s)f'(X_s)ds \\ &\quad + \int_{t_0}^{t} f'(X_s)d\xi_s.\end{aligned}$$

□

Remark 1.10. *Note that*

$$\begin{aligned}
\int_{t_0}^{t} f'(X_s)\, d\xi_s &= \sum_{k=0}^{n} \int_{t_0}^{t} f'(X_s) dL_{r_k}^{X}(t) - \sum_{k=0}^{n} \int_{t_0}^{t} f'(X_s) dL_{l_k}^{X}(t) \\
&= \sum_{k=0}^{n} \int_{t_0}^{t} f'(X_s) dL_{r_k}^{X}(t) + \sum_{k=0}^{n} \int_{t_0}^{t} -f'(X_s) dL_{l_k}^{X}(t).
\end{aligned}$$

In other words, if we distinguish between reflections to the right, modeled by $\xi_t^{(r)}$ and reflections to the left, modeled by $\xi_t^{(l)}$ such that $\xi_t = \xi_t^{(r)} - \xi_t^{(l)}$, we have that

$$\int_{t_0}^{t} f'(X_s)\, d\xi_s = \int_{t_0}^{t} f'(X_s)\, d\xi_s^{(r)} - \int_{t_0}^{t} f'(X_s)\, d\xi_s^{(r)}.$$

Remark 1.11. *Proposition 1.8 reveals that the generator B of singular behavior given by a reflection ought to be defined by $Bf(x) = \pm f'(x)$.*

B Additional Lemmas Regarding Existence And Uniqueness under a Fixed Control

B.1 Long-term Average Cost Criterion

In Section II.2.1, the existence and uniqueness for the linear constraints stemming from the linear programming formulation under a fixed control were shown in the case of a boundary behavior that is given by a reflection at the left endpoint e_l of the state space and a jump away from the right endpoint e_r of the state space. The analysis remains similar if different boundary behavior is chosen. In this section, we provide the crucial lemmas for the case where the boundary behavior is given by reflections at both e_l and e_r, and the case where the boundary behavior is given by a reflection at e_r and a jump away from e_l. With these results, the analysis from Section II.2.1 can be adapted to show existence and uniqueness. Throughout this section, we will consider a function f_D as referred to in Section II.2.1, which is of the form

$$f_D(x) = \int_{c_2}^{x} \left[\int_{c_1}^{y} \frac{2f_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz + K_1 e^{-\int_{c_1}^{y} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} \right] dy + K_2.$$

As presented before, for the sake of exposition, the second derivative of f_D is to be considered formally, and a mollifying argument is employed to make the derivations rigorous.

First, consider these case of two reflections at both end point of the state space. This means that the integral representing the singular behavior of the process takes the form

$$\int_E Bf(x)\,\bar{\mu}_1(dx) = f'(e_l)\bar{\mu}_1(\{e_l\}) - f'(e_r)\bar{\mu}_1(\{e_r\}).$$

Lemma 2.1. *Let D be an interval in $\mathscr{B}(E)$. There is a choice of c_1, c_2, K_1, K_2 such that $f_D'(e_r) = 0$ and $f_D'(e_l) < 0$.*

Proof. Set $c_1 = e_r$ and $K_1 = 0$. Then

$$f_D'(e_r) = \int_{e_r}^{e_r} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz = 0$$

and

$$f_D'(e_l) = \int_{e_r}^{e_l} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz = -\int_{e_l}^{e_r} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)}dt} dz < 0, \tag{2.2}$$

since the integrand is positive. Note that the negative sign on the right hand side was introduced by changing lower and upper limits of the integration. □

Lemma 2.3. *Let D be an interval in $\mathscr{B}(E)$. There is a choice of c_1, c_2, K_1, K_2 such that $f_D'(e_l) = 0$ and $f_D'(e_r) > 0$.*

Proof. Set $c_1 = e_l$ and $K_1 = 0$. Then

$$f'_D(e_l) = \int_{e_l}^{e_l} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz = 0$$

and

$$f'_D(e_r) = \int_{e_l}^{e_r} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz > 0, \tag{2.4}$$

as the integrand is positive. □

Remark 2.5. *Given $D = (c, d]$ or $[c, d]$ for $e_l \le c < d \le e_r$, note that the value of $f'_D(e_r)$ is increasing in d. Further, the integrand in (2.4) will always be dominated by $\frac{2}{\bar{\sigma}^2(z)} e^{\int_{e_l}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt}$, so by the dominated convergence theorem, the function $d \mapsto f'_D(e_r)$ is continuous. Finally, note that*

$$f'_{(e_l,d]}(e_r) - f'_{(e_l,c]}(e_r) = \int_{e_l}^{e_r} \frac{2I_{(c,d]}(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz.$$

Remark 2.6. *To mimic the proof of Theorem II.2.25 for a reflection at $\{e_r\}$ and a jump from $\{e_l\}$ to s, define $\bar{\mu}_1$ by setting $\bar{\mu}_1(\{e_r\}) = \frac{1}{f'_E(e_r)} > 0$ and $\bar{\mu}_1(\{e_l\}) = e^{-\int_{e_l}^{e_r} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} \bar{\mu}_1(\{e_r\})$. The continuous, increasing function F needed to construct the measure $\bar{\mu}_0$ is given by $F(d) = f'_{(e_l,d]}(e_r)\bar{\mu}_1(\{e_r\})$. In particular,*

$$F(d) = \int_{e_l}^{d} \bar{\mu}_1(\{e_r\}) \cdot \frac{2}{\bar{\sigma}^2(z)} e^{\int_{e_r}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz.$$

and the integrand gives the density of $\bar{\mu}_0$.

Second, consider the case of a jump from e_l to s, and a reflection at e_r. This means that the integral representing the singular behavior of the process takes the form

$$\int_E Bf(x)\, \bar{\mu}_1(dx) = (f(s) - f(e_l))\, \bar{\mu}_1(\{e_l\}) - f'(e_r)\bar{\mu}_1(\{e_r\}).$$

Lemma 2.7. *Let D be an interval in $\mathscr{B}(E)$ and $s \in E$, $e_l < s \le e_r$. There is a choice of c_1, c_2, K_1, K_2 such that $f'_D(e_r) > 0$, $f_D(s) - f_D(e_l) = 0$.*

Proof. Follow the construction of f_{I_1} and f_{I_2} as seen in Lemma II.2.15, and concatenate the two functions to form the function f_D. To have $f_D(s) - f_D(e_l) = 0$, we need

$$\begin{aligned} K_1 &= -\left(\int_s^{e_l} e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy\right)^{-1} \cdot \left(\int_s^{e_l} \int_s^y \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\, dy\right) \\ &= \left(\int_{e_l}^s e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy\right)^{-1} \cdot \left(\int_{e_l}^s \int_y^s \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz\, dy\right) > 0. \end{aligned}$$

This yields

$$f'_D(e_r) = \int_s^{e_r} \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_{e_r}^{z} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^{e_r} \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} > 0. \tag{2.8}$$

$\square$

Remark 2.9. *Given $D = (c, d]$ or $D = [c, d]$ for $e_l \le c < d \le e_r$, note that the value of $f'_D(e_r)$ is increasing in d. Indeed, since in (2.8), both the first term and K_1 are increasing in d. Further, the integrand in (2.8) will always be dominated by $\frac{2}{\bar\sigma^2(z)} e^{\int_{e_r}^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt}$, so by the dominated convergence theorem, the function $g : d \mapsto f'_D(e_r)$ is continuous and, if we set $c = e_l$, $g(e_l) = 0$ holds. Finally, note that*

$$\begin{aligned} & f'_{(e_l,d]}(e_r) - f'_{(e_l,c]}(e_r) \\ & \quad = \int_s^{e_r} \frac{2I_{(c,d]}(z)}{\bar\sigma^2(z)} e^{\int_{e_r}^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dz \\ & \qquad - \left(\int_s^{e_l} e^{-\int_s^y \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dy \right)^{-1} \cdot \left(\int_s^{e_l} \int_s^y \frac{2I_{(c,d]}(z)}{\bar\sigma^2(z)} e^{\int_y^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dz\, dy \right) e^{-\int_s^{e_r} \frac{2\bar b(t)}{\bar\sigma^2(t)} dt}. \end{aligned}$$

Remark 2.10. *To mimic the proof of Theorem II.2.25 for a reflection at $\{e_r\}$ and a jump from $\{e_l\}$ to s, define $\bar\mu_1$ by setting $\bar\mu_1(\{e_r\}) = \frac{1}{f'_E(e_r)} > 0$ and*

$$\bar\mu_1(\{e_l\}) = \bar\mu_1(\{e_r\}) \frac{e^{-\int_s^{e_r} \frac{2\bar b(t)}{\bar\sigma^2(t)}} dt}{\int_s^{e_l} e^{-\int_s^y \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dy}.$$

The continuous, increasing function F needed to construct the measure $\bar\mu_0$ is given by $F(d) = f'_{(e_l,d]}(e_r)\bar\mu_1(\{e_r\})$. In particular,

$$\begin{aligned} F(d) \;\; = \;\; & \int_{e_l}^d \bar\mu_1(\{e_r\}) \left[\frac{2I_{(s,d]}(z)}{\bar\sigma^2(z)} e^{\int_y^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} \right. \\ & \left. + \left(\int_s^{e_l} e^{-\int_s^y \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dy \right)^{-1} \cdot \left(\int_s^y \frac{2I_{(e_l,s]}(z)}{\bar\sigma^2(z)} e^{\int_{e_r}^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dy \right) e^{-\int_s^{e_r} \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} \right] dz. \end{aligned}$$

and the integrand gives the density of $\bar\mu_0$.

Lemma 2.11. *Let D be an interval in $\mathscr{B}(E)$ and $s \in E$, $e_l < s \le e_r$. There is a choice of c_1, c_2, K_1, K_2 such that $f'_D(e_r) = 0$, $f_D(s) - f_D(e_l) < 0$.*

Proof. Follow the construction of f_{I_1} and f_{I_2} as seen in Lemma II.2.15, and concatenate the two functions to form the function f_D. To ensure that $f'_D(e_r) = 0$, we need

$$K_1 = -\left(e^{-\int_s^{e_r} \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} \right)^{-1} \cdot \left(\int_s^{e_r} \frac{2I_D(z)}{\bar\sigma^2(z)} e^{\int_{e_r}^z \frac{2\bar b(t)}{\bar\sigma^2(t)} dt} dz \right).$$

Note that $K_1 < 0$. As $f_D(s) = K_2$ and

$$\begin{aligned} f_D(e_l) &= \int_s^{e_l} \int_s^y \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy + K_2 \\ &= \int_{e_l}^s \int_y^s \frac{2I_D(z)}{\bar{\sigma}^2(z)} e^{\int_y^z \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dz + K_1 e^{-\int_s^y \frac{2\bar{b}(t)}{\bar{\sigma}^2(t)} dt} dy + K_2 > K_2, \end{aligned}$$

again due to the fact that we are integrating backwards twice, we have that $f_D(s) - f_D(e_l) < 0$. □

B.2 Infinite Horizon Discounted Criterion

In Section II.2.2, we derived the existence of solutions to the linear constraints for the discounted infinite horizon criterion, under a fixed control, as an easy implication of the existence from the long-term average case. Additional results were provided to show the uniqueness for the discounted infinite horizon criterion under the assumption that the boundary behavior is given by two reflections at both ends e_l and e_r of the state space. The following results treat the remaining cases where the singular behavior is given by a reflection at the left endpoint e_l, and a jump to $s \in E$ from the right endpoint e_r, and the case that the singular behavior is given by a reflection at the right endpoint e_r and a jump to $s \in E$ from e_l. The derivations of Section II.2.2 can be adapted to show uniqueness in these case, using the results presented here. For the following result, refer to Definition II.2.37 for the specific forms of ϕ and ψ.

Lemma 2.12. *Let D be an interval in $\mathscr{B}(E)$ and $e_l < s \leq e_r$. Then, there is a sequence of functions $\{f_{D,k}\}_{k\in\mathbb{N}}$, such that $\bar{A}_\alpha f_k \to I_D$ boundedly and pointwise with $\lim_{k\to\infty} (f_{D,k}(s) - f_{D,k}(e_l)) = 0$ and $\lim_{k\to\infty} f'_{D,k}(e_r) > 0$.*

Proof. In the definition of ϕ_α and ψ_α (compare Definition II.2.37), choose $y = s$ such that $0 < \phi_\alpha < 1$ and $\psi_\alpha > 1$ on $[e_l, s)$ and $\phi_\alpha(s) = \psi_\alpha(s) = 1$. Consider the sequence of functions given by

$$f_{k,D}(x) = \frac{2}{B_\alpha}\left[\phi_k(x)\int_{e_l}^x \psi_k(y)g_k(y)m_k(y)\,dy - \psi_k(x)\int_{e_l}^x \phi_k(y)g_k(y)m_k(y)\,dy\right] + K\cdot\psi_k(x) \tag{2.13}$$

for $k \in \mathbb{N}$, where g_k and m_k are mollifiers according to Proposition II.2.19, with

$$K = \frac{\frac{2}{B_\alpha}\left[\phi_\alpha(s)\int_{e_l}^s \psi_\alpha(y)g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_l}^s \phi_\alpha(y)g(y)m(y)\,dy\right]}{\psi_\alpha(e_l) - \psi_\alpha(s)}. \tag{2.14}$$

As $0 < \phi_\alpha < 1$ and $\psi_\alpha > 1$ on $[e_l, s)$,

$$\begin{aligned}
\frac{2}{B_\alpha}&\left[\phi_\alpha(s)\int_{e_l}^{s}\psi_\alpha(y)g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_l}^{s}\phi_\alpha(y)g(y)m(y)\,dy\right]\\
&< \frac{2}{B_\alpha}\left[\phi_\alpha(s)\int_{e_l}^{s}g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_l}^{s}g(y)m(y)\,dy\right]\\
&= \frac{2}{B_\alpha}\left[\phi_\alpha(s)-\psi_\alpha(s)\right]\int_{e_l}^{s}g(y)m(y)\,dy = 0
\end{aligned}$$

holds, as $B_\alpha < 0$. As ψ_α is increasing, $K > 0$. Further, $f_{k,D}(e_l) = K \cdot \psi_\alpha(e_l)$ and

$$f_{k,D}(s) = \frac{2}{B_\alpha}\left[\phi_k(x)\int_{e_l}^{s}\psi_k(y)g_k(y)m_k(y)\,dy - \psi_k(x)\int_{e_l}^{s}\phi_k(y)g_k(y)m_k(y)\,dy\right] + K\cdot\psi_k(s)$$

holds and thereby, by the choice of K,

$$\begin{aligned}
&\lim_{k\to\infty}\left(f_{D,k}(s) - f_{D,k}(e_l)\right)\\
&\quad= \lim_{k\to\infty}\Bigg(\frac{2}{B_\alpha}\left[\phi_k(s)\int_{e_l}^{s}\psi_k(y)g_k(y)m_k(y)\,dy - \psi_k(s)\int_{e_l}^{s}\phi_k(y)g_k(y)m_k(y)\,dy\right]\\
&\qquad\qquad + K\cdot(\psi_k(s)-\psi_k(e_l))\Bigg)\\
&\quad= 0.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\lim_{k\to\infty} f'_{D,k}(e_r) = K\cdot\psi'_\alpha(e_r) + \frac{2}{B_\alpha}\Bigg[\phi'_\alpha(e_r)&\int_{e_l}^{e_r}\psi_\alpha(y)g(y)m(y)\,dy\\
&-\psi'_\alpha(e_r)\int_{e_l}^{e_r}\phi_\alpha(y)g(y)m(y)\,dy\Bigg] > 0
\end{aligned}$$

By Theorem II.2.49 and Proposition II.2.47, it therefore follows that $\bar{A}_\alpha f_k \to I_D(E)$. □

Remark 2.15. *If in (2.13), we choose $K > 0$ but different from the choice given in (2.14), we have that $\lim_{k\to\infty}(f_{D,k}(s) - f_{D,k}(e_l)) \neq 0$ and $\lim_{k\to\infty} f'_{D,k}(e_r) > 0$*

Lemma 2.16. *Let D be an interval in $\mathscr{B}(E)$ and $e_l \leq s < e_r$. Then, there is a sequence of functions $\{f_{D,k}\}_{k\in\mathbb{N}}$, such that $\bar{A}_\alpha f_k \to I_D(E)$ boundedly and pointwise with $\lim_{k\to\infty}(f_{D,k}(e_r) - f_{D,k}(s)) = 0$ and $\lim_{k\to\infty} f'_{D,k}(e_l) < 0$.*

Proof. In the definition of ϕ_α and ψ_α, choose $y = s$ such that $\phi_\alpha > 1$ and $0 < \psi_\alpha < 1$ on $(s, e_r]$ and $\phi_\alpha(s) = \psi_\alpha(s) = 1$. Consider the sequence of functions given by

$$f_{k,D}(x) = \frac{2}{B_\alpha}\left[\phi_k(x)\int_{e_r}^{x}\psi_k(y)g_k(y)m_k(y)\,dy + \psi_k(x)\int_{x}^{e_r}\phi_k(y)g_k(y)m_k(y)\,dy\right] + K\cdot\phi_k(x) \tag{2.17}$$

for $k \in \mathbb{N}$, where g_k and m_k are mollifiers according to Proposition II.2.19, with

$$K = \frac{\frac{2}{B_\alpha}\left[\phi_\alpha(s)\int_{e_r}^{s}\psi_\alpha(y)g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_r}^{s}\phi_\alpha(y)g(y)m(y)\,dy\right]}{\psi_\alpha(e_r)-\psi_\alpha(s)}. \tag{2.18}$$

As $\psi_\alpha > 1$ and $0 < \phi_\alpha < 1$ on $(s, e_r]$,

$$\begin{aligned}
&\frac{2}{B_\alpha}\left[\phi_\alpha(s)\int_{e_r}^{s}\psi_\alpha(y)g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_r}^{s}\phi_\alpha(y)g(y)m(y)\,dy\right]\\
&\qquad < \frac{2}{B_\alpha}\left[\phi_\alpha(s)\int_{e_r}^{s}g(y)m(y)\,dy - \psi_\alpha(s)\int_{e_r}^{s}g(y)m(y)\,dy\right]\\
&\qquad = \frac{2}{B_\alpha}\left[\phi_\alpha(s)-\psi_\alpha(s)\right]\int_{e_r}^{s}g(y)m(y)\,dy = 0
\end{aligned}$$

holds, as $B_\alpha < 0$. As ψ_α is increasing, $K < 0$. Further, $f_{k,D}(e_r) = K\cdot\psi_\alpha(e_r)$ and

$$f_{k,D}(s) = \frac{2}{B_\alpha}\left[\phi_k(x)\int_{e_r}^{s}\psi_k(y)g_k(y)m_k(y)\,dy + \psi_k(x)\int_{s}^{e_r}\phi_k(y)g_k(y)m_k(y)\,dy\right] + K\cdot\psi_k(s)$$

holds and thereby, by the choice of K,

$$\begin{aligned}
&\lim_{k\to\infty}\left(f_{D,k}(s) - f_{D,k}(e_l)\right)\\
&\quad = \lim_{k\to\infty}\Bigg(\frac{2}{B_k}\left[\phi_k(s)\int_{e_r}^{s}\psi_k(y)g_k(y)m_k(y)\,dy - \psi_k(s)\int_{e_r}^{s}\phi_k(y)g_k(y)m_k(y)\,dy\right]\\
&\qquad\qquad + K\cdot(\psi_k(s)-\psi_k(e_r))\Bigg)\\
&\quad = 0.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\lim_{k\to\infty} f'_{D,k}(e_r) = K\cdot\psi'_\alpha(e_l) + \frac{2}{B_\alpha}\Bigg[\phi'_\alpha(e_l)\int_{e_r}^{e_l}\psi_\alpha(y)g(y)m(y)\,dy&\\
-\psi'_\alpha(e_l)\int_{e_r}^{e_l}\phi_\alpha(y)g(y)m(y)\,dy\Bigg] &< 0
\end{aligned}$$

By Theorem II.2.49 and Proposition II.2.47, we have that $\bar{A}_\alpha f_k \to I_D(E)$. □

Remark 2.19. *If in (2.17), we choose $K < 0$ but different from the choice given in (2.18), we have that $\lim_{k\to\infty}(f_{D,k}(e_r) - f_{D,k}(s)) \neq 0$ and $\lim_{k\to\infty} f'_{D,k}(e_r) < 0$.*

C Analytic Solutions to Selected Control Problems

C.1 Simple Particle Problem without Cost of Control

In this subsection we show how to find the density p of the state space marginal of expected occupation measures μ_0 for the simple particle problem under the long-term average criterion, as referred to in Section V.1.1. The state space is $E = [-1, 1]$, the control space is $U = [-1, 1]$ and the SDE of interest, in integral form, is

$$X_t = x_0 + \int_0^t u(X_s)ds + \sigma W_t + \xi_t,$$

where the process ξ_t models reflection of the process at both boundaries. In other words, we have that $\xi_t = L^X_{\{-1\}}(t) - L^X_{\{1\}}(t)$ where $L^X_{\{a\}}$ is the local time of X at $a \in E$. The starting point x_0 can be chosen arbitrarily in E. Under the long term average criterion, the position of the starting point is irrelevant. Note that this process does not feature any control on the singular part.

Our objective is to keep the process as close as possible to the origin. For cost criteria that do not charge for the use of the control, like $c_0(x, u) = x^2$ and $c_1(x, u) = c\left(\delta_{\{1\}}(x) + \delta_{\{-1\}}(x)\right)$ for some $c \in \mathbb{R}$, it is obvious to see that the optimal control is given by the following function.

$$u(x) = \begin{cases} +1 & x < 0 \\ 0 & x = 0 \\ -1 & x \geq 0 \end{cases} \tag{3.1}$$

We assume that $\mu_0(dx \times du) = \eta_0(du, x)p(x)dx$ and $\mu_{1,E}(dx) = w_1\delta_{\{-1\}} + w_2\delta_{\{1\}}$, where $\eta_0(\cdot, x)$ is a degenerate probability distribution putting all mass on $u(x)$. The linear programming formulation for this problem is hence

$$\int_{-1}^{1} \left(u(x)f'(x) + \frac{\sigma^2}{2}f''(x) \right) p(x)\,dx + w_1 f'(-1) - w_2 f'(1) = 0 \qquad \forall f \in C_c^2(E). \tag{3.2}$$

Note that since E is compact, we do not have to distinguish between $C^2(E)$ and $C_c^2(E)$. To find p, we employ a mollifying argument on u as follows. Define

$$u_n(x) = \begin{cases} u(x) & x < -\frac{1}{n} \\ u(-\frac{1}{n}) + \frac{u(\frac{1}{n}) - u(-\frac{1}{n})}{\frac{2}{n}}(x + \frac{1}{n})) & -\frac{1}{n} \leq x < \frac{1}{n} \\ u(x) & x \geq \frac{1}{n} \end{cases},$$

which obviously converges to $u(x)$ boundedly and pointwise. We try to solve (3.2) using u_n rather than u. The density in this setting is referred to as p_n. This means we seek to solve the equation

$$\int_{-1}^{1} \left(u_n(x)f'(x) + \frac{\sigma^2}{2}f''(x) \right) p_n(x)\,dx + w_1 f'(-1) - w_2 f'(1) = 0 \qquad \forall f \in C_c^2(E).$$

For $f \in C^2(E)$, assuming that p_n is twice differentiable, integration by parts, applied twice on the terms featuring f'', yields

$$
\begin{aligned}
0 = \quad & [u_n(x)p_n(x)f(x)]_{-1}^{1} - \int_{-1}^{1} [u_n p_n]'(x) f(x)\, dx \\
& + \frac{\sigma^2}{2}\left\{ [p_n(x)f'(x)]_{-1}^{1} - [p_n'(x)f(x)]_{-1}^{1} + \int_{-1}^{1} p_n''(x) f(x)\, dx \right\} + w_{n,1} f'(-1) - w_{n,2} f'(1) \\
= \quad & f(1)\left[u_n(1)p_n(1) - \frac{\sigma^2}{2}p_n'(1)\right] - f(-1)\left[u_n(-1)p_n(-1) - \frac{\sigma^2}{2}p_n'(-1)\right] \\
& + f'(1)\left[\frac{\sigma^2}{2}p_n(1) - w_{n,2}\right] - f'(-1)\left[\frac{\sigma^2}{2}p_n(-1) - w_{n,1}\right] \\
& + \int_{-1}^{1} \left(\frac{\sigma^2}{2}p_n''(x) - [u_n p_n]'(x)\right) f(x)\, dx.
\end{aligned}
$$

As this equation has to hold for all $f \in C^2(E)$, we have in particular that

$$
\begin{aligned}
\frac{\sigma^2}{2}p_n''(x) - [u_n p_n]'(x) &= 0, \quad x \in (-1,1) \qquad (3.3) \\
\left[u_n(1)p_n(1) - \frac{\sigma^2}{2}p_n'(1)\right] - f(-1)\left[u_n(-1)p_n(-1) - \frac{\sigma^2}{2}p_n'(-1)\right] &= 0 \qquad (3.4) \\
f'(1)\left[\frac{\sigma^2}{2}p_n(1) - w_{n,2}\right] - f'(-1)\left[\frac{\sigma^2}{2}p_n(-1) - w_{n,1}\right] &= 0 \qquad (3.5)
\end{aligned}
$$

The differential equation (3.3) can be solved with (3.4) giving the necessary boundary constraints. The values for the weights w_1 and w_2 are then given by (3.5). We solve the differential equation first. Integrating (3.3) yields

$$\frac{\sigma^2}{2}p_n'(x) - u_n(x)p_n(x) = k_1 \Leftrightarrow p_n'(x) - \frac{2}{\sigma^2}u_n(x)p_n(x) = \frac{2}{\sigma^2}k_1.$$

Condition (3.4) in particular claims that $u(-1)p_n(-1) - \frac{\sigma^2}{2}p_n'(-1) = 0$, from which we can conclude that $k_1 = 0$. Define the integrating factor $M(x) = e^{\int_{c_2}^{x} -\frac{2}{\sigma^2}u_n(v)\, dv}$ we see, as $M'(x) = -\frac{2}{\sigma^2}u_n(x)M(x)$, that

$$
\begin{aligned}
M(x)p_n'(x) - \frac{2}{\sigma^2}u_n(x)M(x)p_n(x) &= 0 \\
\Leftrightarrow \Big(p_n(x)M(x)\Big)' &= 0 \\
\Leftrightarrow p_n(x)M(x) &= k_2.
\end{aligned}
$$

Dividing by $M(x)$ and explicitly writing out $M(x)$, we finally receive

$$p_n(x) = k_2 e^{\int_{c_2}^{x} \frac{2}{\sigma^2}u_n(v)\, dv}. \qquad (3.6)$$

Recall that c_2 is an arbitrary constant, so we set $c_2 = -1$. By setting

$$k_2 = \left(\int_{-1}^{1} e^{\int_{-1}^{r} \frac{2}{\sigma^2} u_n(v)\, dv}\, dr \right)^{-1}$$

we ensure that p is actually a probability density. So our solution is

$$p_n(x) = \frac{e^{\int_{-1}^{x} \frac{2}{\sigma^2} u_n(v)\, dv}}{\int_{-1}^{1} e^{\int_{-1}^{r} \frac{2}{\sigma^2} u_n(v)\, dv}\, dr}.$$

We use (3.5) to compute the reflection weights. We deduce that

$$\frac{\sigma^2}{2} p_n(1) = w_{n,2} \quad \text{and} \quad \frac{\sigma^2}{2} p_n(-1) = w_{n,1}.$$

Now we use the mollifying argument. Note that

$$\lim_{n\to\infty} p_n(x) = \frac{e^{\int_{-1}^{x} \frac{2}{\sigma^2} u(v)\, dv}}{\int_{-1}^{1} e^{\int_{-1}^{r} \frac{2}{\sigma^2} u(v)\, dv}\, dr} =: p(x)$$

boundedly and pointwise. Thereby,

$$\lim_{n\to\infty} w_{n,1} = \frac{\sigma^2}{2} p(-1) =: w_1 \quad \text{and} \quad \lim_{n\to\infty} w_{n,2} = \frac{\sigma^2}{2} p(1) =: w_2$$

do exist. An application of the bounded convergence theorem now reveals that

$$\begin{aligned}
&\int_{-1}^{1} \left(u(x) f'(x) + \frac{\sigma^2}{2} f''(x) \right) p(x)\, dx + w_1 f'(-1) - w_2 f'(1) \\
&\quad = \lim_{n\to\infty} \left(\int_{-1}^{1} \left(u_n(x) f'(x) + \frac{\sigma^2}{2} f''(x) \right) p_n(x)\, dx + w_{n,1} f'(-1) - w_{n,2} f'(1) \right) = 0
\end{aligned}$$

which show that p, w_1 and w_2 fulfill the constraints. To evaluate the cost criterion for this particular control u, we simply have to evaluate

$$J(u) \equiv \int_{-1}^{1} x^2 p(x)\, dx + c(w_1 + w_2).$$

C.2 Modified Bounded Follower

In this subsection we illustrate how to find the optimal solution of the modified bounded follower problem under the long-term average criterion, as referred to in Section V.1.3. This problem has the state space $E = [0, 1]$, the process is reflected at the left endpoint of the state space $\{0\}$ and performs a jump from the right endpoint of the state space, $\{1\}$, to the left end, $\{0\}$. The control space is given $U = [-1, 1]$ and the process is governed by the SDE in integral form

$$X_t = x_0 + \int_0^t u(X_s)ds + \sigma W_t + \xi_t.$$

The starting point x_0 can be chosen arbitrarily in E. Under the long term average criterion, the position of the starting point is irrelevant. ξ_t captures the singular behavior of the process. The cost functions are of the form $c_0(x,u) = x^2$ and $c_1(x,u) = c\delta_{\{1\}}(x)$ for some $c \in \mathbb{R}$. Again, this process does not feature control of the singular behavior. We can readily convince ourselves that the optimal control for this problem is a (deterministic) bang-bang control of the form

$$u_a(x) = \begin{cases} -1 & x < a \\ +1 & x \geq a \end{cases} \tag{3.7}$$

where the 'switching point' a depends on the coefficients of the SDE. Indeed, since due to the jump behavior at $\{1\}$, it is beneficial to push the process to the right endpoint as soon as it crosses a certain threshold a if the costs c of a jump are small compared to the costs of being far away from the origin. Obviously, we can stipulate that $a > 0$.
In order to find the occupation measures μ_0 and μ_1 associated with this problem, assume that $\mu_0(dx \times du) = \eta_0(du, x)p(x)dx$ and $\mu_1(dx) = w_1\delta_{\{0\}} + w_2\delta_{\{1\}}$. The linear constraints are given by

$$\begin{aligned} 0 &= \int_0^1 \int_{-1}^1 \left[u(x)f'(x) + \frac{\sigma^2}{2}f''(x)\right] \eta_0(du,x)p(x)\,dx + w_1 f'(0) + w_2\left[f(0) - f(1)\right] && (3.8) \\ &= \int_0^1 \left[u(x)f'(x) + \frac{\sigma^2}{2}f''(x)p(x)\right]\,dx + w_1 f'(0) + w_2\left[f(0) - f(1)\right], && (3.9) \end{aligned}$$

where we used that the proposed control in (3.7) is deterministic. Define

$$u_n(x) = \begin{cases} u(x) & x < a - \frac{1}{n} \\ u(a - \frac{1}{n}) + \frac{u(a+\frac{1}{n}) - u(a-\frac{1}{n})}{\frac{2}{n}}(x - a + \frac{1}{n})) & a - \frac{1}{n} \leq x < a + \frac{1}{n} \\ u(x) & x \geq a + \frac{1}{n} \end{cases},$$

which converges to u_a pointwise and bounded. As seen in Appendix C.1 we solve (3.8) using u_n rather than u. The density in this setting is referred to as p_n. Assume that p_n is twice differentiable. Repeatedly integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_0^1 \left[u_n(x)f'(x) + \frac{\sigma^2}{2}f''(x)\right] p_n(x)\,dx + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right] \\
&= \int_0^1 u_n(x)f'(x)p_n(x)dx + \frac{\sigma^2}{2}\int_0^1 f''(x)p_n(x)\,dx + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right] \\
&= \left[u_n(x)p_n(x)f(x)\right]_0^1 - \int_0^1 \left[u_n p_n\right]'(x)f(x)\,dx + \frac{\sigma^2}{2}\left[\left[p_n(x)f'(x)\right]_0^1 - \int_0^1 p_n'(x)f'(x)\,dx\right] \\
&\qquad + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right] \\
&= \left[u_n(x)p_n(x)f(x)\right]_0^1 - \int_0^1 \left[u_n p_n\right]'(x)f(x)\,dx + \frac{\sigma^2}{2}\left[p_n(x)f'(x)\right]_0^1 \\
&\qquad - \frac{\sigma^2}{2}\left[\left[p_n'(x)f(x)\right]_0^1 - \int_0^1 p_n''(x)f(x)\,dx\right] + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right] \\
&= \int_0^1 f(x)\left[-(u_n p_n)'(x) + \frac{\sigma^2}{2}p_n''(x)\right]dx + \left[u_n(x)p_n(x)f(x)\right]_0^1 \\
&\qquad + \frac{\sigma^2}{2}\left(\left[f'(x)p_n(x)\right]_0^1 - \left[f(x)p_n'(x)\right]_0^1\right) + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right].
\end{aligned}$$

This equation has to hold for all $f \in C^2(E)$. In particular, this means that

$$-(u_n p_n)'(x) + \frac{\sigma^2}{2}p_n''(x) = 0, \quad x \in (0,1) \tag{3.10}$$

$$\left[u_n(x)p_n(x)f(x)\right]_0^1 + \frac{\sigma^2}{2}\left(\left[f'(x)p_n(x)\right]_0^1 - \left[f(x)p_n'(x)\right]_0^1\right) + w_{n,1}f'(0) + w_{n,2}\left[f(0) - f(1)\right] = 0 \tag{3.11}$$

We proceed by solving the differential equation given by (3.10). Equation (3.11) is used to determine constants that appear during the solution process. Observe that we can integrate (3.10) once to deduce

$$-u_n(x)p_n(x) + \frac{\sigma^2}{2}p_n'(x) = k_1 \Leftrightarrow p_n'(x) - \frac{2}{\sigma^2}u_n(x)p_n(x) = \frac{2}{\sigma^2}k_1.$$

Using the integrating factor $M(x) = e^{\int_{c_2}^{x} -\frac{2}{\sigma^2} u_n(v)\, dv}$ we see, since $M'(x) = -\frac{2}{\sigma^2} u_n(x) M(x)$, that

$$\begin{aligned}
M(x)p_n'(x) - \frac{2}{\sigma^2} u_n(x) M(x) p_n(x) &= M(x)\frac{2}{\sigma^2} k_1 \\
\Leftrightarrow \Big(p_n(x)M(x)\Big)' &= M(x)\frac{2}{\sigma^2} k_1 \\
\Leftrightarrow p_n(x)M(x) &= \int_{c_1}^{x} M(y)\frac{2}{\sigma^2} k_1\, dy + k_2.
\end{aligned}$$

Dividing by $M(x)$ and explicitly writing out $M(x)$, we finally receive

$$p_n(x) = \frac{\int_{c_1}^{x} e^{\int_{c_2}^{y} -\frac{2}{\sigma^2} u_n(v)\, dv} \frac{2}{\sigma^2} k_1\, dy + k_2}{e^{\int_{c_2}^{x} -\frac{2}{\sigma^2} u_n(v)\, dv}}$$

At this point, we introduce the constraints given by (3.11). We can rewrite this equation in the form

$$\begin{aligned}
f(1)\left[u_n(1)p_n(1) - \frac{\sigma^2}{2} p_n'(1) - w_{n,2}\right] - f(0)\left[u_n(0)p_n(0) - \frac{\sigma^2}{2} p_n'(0) - w_{n,2}\right] \\
+ f'(1)\left[\frac{\sigma^2}{2} p_n(1)\right] - f'(0)\left[\frac{\sigma^2}{2} p_n(0) - w_{n,1}\right] = 0
\end{aligned}$$

As this identity has to hold for all $f \in C^2(E)$, we can deduce the following boundary conditions.

$$\begin{array}{ll}
\sigma^2 p_n(0) = 2w_{n,1}, & 2u_n(1)p_n(1) - \sigma^2 p_n'(1) = 2w_{n,2} \\
\sigma^2 p_n(1) = 0, & 2u_n(0)p_n(0) - \sigma^2 p_n'(0) = 2w_{n,2}
\end{array} \tag{3.12}$$

In particular, we have $p(1) = 0$, implying that

$$0 = \frac{\int_{c_1}^{1} e^{\int_{c_2}^{y} -\frac{2}{\sigma^2} u_n(v)\, dv} \frac{2}{\sigma^2} k_1\, dy + k_2}{e^{\int_{c_2}^{1} -\frac{2}{\sigma^2} u_n(v)\, dv}}$$

which we can attain by setting $c_1 = 1$ and $k_2 = 0$. Hence, we have

$$\begin{aligned}
p_n(x) = \frac{\int_{1}^{x} e^{\int_{c_2}^{y} -\frac{2}{\sigma^2} u_n(v)\, dv} \frac{2}{\sigma^2} k_1\, dy}{e^{\int_{c_2}^{x} -\frac{2}{\sigma^2} u_n(v)\, dv}} &= k_1 \int_{1}^{x} \frac{2}{\sigma^2} e^{\int_{c_2}^{y} \frac{2}{\sigma^2} u_n(v)\, dv - \int_{c_2}^{x} -\frac{2}{\sigma^2} u_n(v)\, dv}\, dy \\
&= k_1 \int_{1}^{x} \frac{2}{\sigma^2} e^{\int_{x}^{y} -\frac{2}{\sigma^2} u_n(v)\, dv}\, dy.
\end{aligned}$$

Recall that p_n has to be a probability density. Consequently, we set

$$k_1 = \frac{1}{\int_0^1 \int_1^r \frac{2}{\sigma^2} e^{\int_r^y -\frac{2}{\sigma^2} u_n(v)\, dv}\, dy\, dr}.$$

The final form of the density is given by

$$p_n(x) = \frac{\int_1^x e^{\int_x^y -\frac{2}{\sigma^2} u_n(v)\,dv}\, dy}{\int_0^1 \int_1^w e^{\int_w^y -\frac{2}{\sigma^2} u_n(v)\,dv}\, dy\, dw}.$$

As $a \neq 0$, we deduce that

$$w_{n,1} = \frac{1}{2}\sigma^2 p_n(0) = \sigma^2 \frac{\left(2e^{\frac{2a}{\sigma^2}} - e^{\frac{-2+4a}{\sigma^2}} - 1\right)}{2k_1},$$

and $w_{2,n}$ is determined in similar fashion using the formulas displayed in (3.12). Observe that by the bounded convergence theorem, the limit

$$\lim_{n\to\infty} p_n(x) = \frac{\int_1^x e^{\int_x^y -\frac{2}{\sigma^2} u(v)\,dv}\, dy}{\int_0^1 \int_1^w e^{\int_w^y -\frac{2}{\sigma^2} u(v)\,dv}\, dy\, dw} =: p(x), \tag{3.13}$$

is well-defined. Also, set

$$w_1 := \lim_{n\to\infty} w_{n,1} \quad \text{and} \quad w_{n,2} \lim_{n\to\infty} := w_{n,2}.$$

Given the form of the optimal control as in (3.7), an evaluation of the integrals in (3.13) yields

$$p(x) = \begin{cases} \frac{1}{k_0} \cdot \left[e^{\frac{2}{\sigma^2}\cdot(a-x-1)} \cdot \left(e^{\frac{2}{\sigma^2}} - e^{\frac{2\cdot a}{\sigma^2}} \right) - 1 + e^{\frac{2}{\sigma^2}\cdot(a-x)} \right] & x < a \\ \frac{1}{k_0} \cdot \left(1 - e^{\frac{2}{\sigma^2}\cdot(x-1)} \right) & x \geq a \end{cases}$$

with

$$\begin{aligned} k_0 = \quad & \frac{1}{2}\left(2 - 2a + \left(-1 + e^{\frac{2}{\sigma^2}(a-1)}\right) \cdot \sigma^2\right) \\ & + \frac{1}{2}\left(-2a + e^{\frac{-2}{\sigma^2}} \cdot \left(2e^{\frac{2}{\sigma^2}} - e^{\frac{2}{\sigma^2}a}\right) \cdot \left(-1 + e^{\frac{2}{\sigma^2}a}\right) \cdot \sigma^2\right) \end{aligned}$$

Applying the mollifying argument, we discover that

$$\begin{aligned} & \int_0^1 \left[u(x) f'(x) + \frac{\sigma^2}{2} f''(x) p(x) \right] dx + w_1 f'(0) + w_2 \left[f(0) - f(1) \right] \\ & \quad = \lim_{n\to\infty} \left(\int_0^1 \left[u_n(x) f'(x) + \frac{\sigma^2}{2} f''(x) p_n(x) \right] dx + w_{1,n} f'(0) + w_{2,n} \left[f(0) - f(1) \right] \right) = 0, \end{aligned}$$

and hence p, w_1 and w_2 solve the linear constraints posed by (3.8). The cost criterion is given by

$$J(a) \equiv \int_0^1 x^2 p(x)\, dx + c w_2$$

which is equal to

$$
\begin{aligned}
J(a) \quad = \quad & \frac{-6\sigma^2 e^{\frac{2a}{\sigma^2}} \left(2a^2+\sigma^4\right)}{6\left(-2\sigma^2 e^{\frac{2a}{\sigma^2}} + \sigma^2 e^{\frac{4a}{\sigma^2}} - 2\sigma^2 e^{\frac{2(a+1)}{\sigma^2}} + e^{\frac{2}{\sigma^2}}\left(4a+3\sigma^2-2\right)\right)} \\
& \quad + \frac{e^{\frac{2}{\sigma^2}} \left(8a^3+12a^2\sigma^2+12a\sigma^4-12c+9\sigma^6-6\sigma^4+6\sigma^2-4\right)}{6\left(-2\sigma^2 e^{\frac{2a}{\sigma^2}} + \sigma^2 e^{\frac{4a}{\sigma^2}} - 2\sigma^2 e^{\frac{2(a+1)}{\sigma^2}} + e^{\frac{2}{\sigma^2}}\left(4a+3\sigma^2-2\right)\right)} \\
& \quad + \frac{3\sigma^6 e^{\frac{4a}{\sigma^2}} - 6\sigma^6 e^{\frac{2(a+1)}{\sigma^2}}}{6\left(-2\sigma^2 e^{\frac{2a}{\sigma^2}} + \sigma^2 e^{\frac{4a}{\sigma^2}} - 2\sigma^2 e^{\frac{2(a+1)}{\sigma^2}} + e^{\frac{2}{\sigma^2}}\left(4a+3s^2-2\right)\right)} \cdot cw_2.
\end{aligned}
$$

This expression can be obtained with a computer algebra system. The minimum of J, as well as the minimizer a, for specific choices of σ and c, can be found using numerical minimization.

D List of Abbreviation and Symbols

This appendix provides an overview of frequently used abbreviations and symbols for the ease of reading.

$\mathbb{R}_{\geq 0}$	the non-negative real numbers.
E	state space of a stochastic process, either $E = [e_l, e_r]$, $E = (-\infty, e_r]$, $E = [e_l, \infty)$ or $E = (-\infty, \infty)$ for $-\infty < e_l < e_r < \infty$.
U	control space, $U = [u_l, u_r]$ for $-\infty < u_l < u_r < \infty$.
b	drift term of a stochastic process, maps $E \times U$ into $\mathbb{R}$.
σ	diffusion term of a stochastic process, maps $E \times U$ into $\mathbb{R}_{\geq 0}$.
A	the generator of the continuous behavior of a stochastic process.
B	the generator of the singular behavior of a stochastic process.
$\|\cdot\|_\infty$	the supremum norm defined by $\sup\{\|f(x)\| : x \in \text{domain}(f)\}$ when used on a function, or the maximum norm when used on a vector or matrix.
$B(S)$	the set of Borel measurable functions from S to $\mathbb{R}$.
$C(S)$	the set of continuous functions from S to $\mathbb{R}$.
$C_c(S)$	the set of continuous functions from S to $\mathbb{R}$ with compact support.
$C_b(S)$	the set of bounded continuous functions from S to $\mathbb{R}$.
$C_b^u(S)$	the set of bounded, uniformly continuous functions from S to $\mathbb{R}$.
$C^2(S)$	the set of twice continuously differentiable functions from S to $\mathbb{R}$.
$C_c^2(S)$	the set of twice continuously differentiable functions from S to $\mathbb{R}$ with compact support.
$\|\cdot\|_{\mathscr{D}}$	the norm on $C_c^2(S)$ given by $\|f\|_{\mathscr{D}} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$
$\mathscr{D}_\infty(S)$	the separable space of functions given by $(C_c^2(S), \|\cdot\|_{\mathscr{D}})$
$L^1(S)$	the set of Lebesgue integrable functions on S.
$\mathcal{P}(S)$	the space of probability measures on a measurable space S.
$\mathcal{M}(S)$	the space of finite Borel measures on a measurable space S.
$\mathcal{M}^l(S)$	the space of finite Borel measures on a space S with full mass less then or equal to l.

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