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The Strong Law of Large Numbers for U-Statistics under Random Censorship

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by

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Abstract

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We introduce a semi-parametric U-statistics estimator for randomly right censored data. We will study the strong law of large numbers for this estimator under proper assumptions about the conditional expectation of the censoring indicator with respect to the observed life times. Moreover we will conduct simulation studies, where the semi-parametric estimator is compared to a U-statistic based on the Kaplan-Meier product limit estimator in terms of bias, variance and mean squared error, under different censoring models.

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Chapter 1

Introduction

Assume that $X_1, ..., X_n$ are independent and identically distributed (i. i. d.) random variables (r. v.) on $\mathbb{R}$, which are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote their common probability distribution function (d. f.) by F. For some $1 \leq k \leq n$ let $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a symmetric Borel-measurable function. Define the target value

$$\theta^* := \mathbb{E}[\phi] = \int \cdots \int \phi \prod_{j=1}^{k} dF. \tag{1.1}$$

Examples of this kind of parameters include the expected value, variance and any higher moments of X, depending on how ϕ is set. One approach to estimate those integrals is given by the so called U-statistics. To obtain this estimator we need to replace the true d. f. F by the empirical d. f. F_n which is defined by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}.$$

Now plugging F_n into (1.1) yields

$$\int ... \int \phi \prod_{j=1}^{k} dF_n = \frac{1}{n^k} \sum_{i_1=1}^{n} ... \sum_{i_k=1}^{n} \phi(X_{i_1}, ..., X_{i_k})$$

The expression on the right hand side in the equation above is known as V-statistic. It includes repeated observations. An unbiased estimate of θ^*, based on distinct observations only, can be introduced as

$$U_{k,n}(\phi) = \binom{n}{k}^{-1} \sum_{[n,k]} \phi(X_{i_1}, ..., X_{i_k}) \ , \tag{1.2}$$

where the sum iterates over all sets $\{i_1, ..., i_k\}$ s. t. $1 \leq i_1 < i_2 < ... < i_n \leq n$. We call (1.2) U-statistics of order k. In Lee (1990) it was shown that the U-statistics is the unbiased minimum variance estimator for (1.1). Observe that for $k = 2$, equation (1.2) simplifies to

$$U_{2,n}(\phi) = \frac{2}{n(n-1)} \sum\sum_{1 \leq i < j \leq n} \phi(X_i, X_j)$$

and we have

$$\mathbb{E}[U_{2,n}(\phi)] = \int\int \phi dF dF \ .$$

We will call ϕ the kernel of the U-statistics. Consider the following examples for different kernels ϕ.

Example 1.1. Suppose $X \sim F$ s. t. the second moment of X is finite. Moreover let $\phi(x_1, x_2) := 2^{-1} \cdot (x_1 - x_2)^2$. Then we have

$$\begin{aligned} \theta^* &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2}(x_1 - x_2)^2 F(dx_1) F(dx_2) \\ &= Var(X) \ . \end{aligned}$$

The corresponding U-statistics is therefore estimating the variance in this case.

Example 1.2. Suppose $X \sim F$ s. t. the expectation of X is finite. Then the r-th probability weighted moment of X is defined by

$$\beta_r := \int_{-\infty}^{\infty} x(F(x))^r F(dx)$$

for $r \geq 1$. Now consider that the following holds true

$$\beta_{r-1} = \underset{\mathbb{R}^r}{\int \cdots \int} \frac{1}{r} \max(x_1, \ldots, x_r) F(dx_1) \ldots F(dx_r) \ ,$$

compare Lee (1990), page 9. Thus we can estimate β_{r-1} by choosing the kernel

$$\phi(x_1, \dots, x_r) = \frac{1}{r} \max(x_1, \dots, x_r)$$

for the corresponding U-statistics. Now let $r = 2$. Then the U-statistics with kernel $\phi(x_1, x_2) := 2^{-1} \cdot \max(x_1, x_2)$ is an estimator for β_1, the first probability weighted moment.

In lifetime analysis, one often deals with the problem of incomplete observations. The incompleteness is often caused by censoring. In this thesis we are concerned with right censored data. A framework to model this kind of data is provided by the Random Censorship Model (RCM). Here we observe data of the form $(Z_i, \delta_i)_{i \le n}$ where the Z_i are the observed sample values, which might include censoring and the δ_i indicate whether the corresponding Z_i was censored or not. Here the sequence $(Z_i, \delta_i)_{i \le n}$ is assumed to be independent and identically distributed (i. i. d.). Furthermore we can write for $i = 1, ..., n$

$$Z_i = min(X_i, Y_i) \text{ and } \delta_i = I_{X_i \le Y_i}$$

where X_i denotes the true lifetime and Y_i is the so called censoring time. The sequences $(X_i)_{i \le n}$ and $(Y_i)_{i \le n}$ are assumed to be i. i. d.and to be independent of each other. Throughout this work the probability distribution functions (d. f.) of X, Y and Z will be denoted F, G and H respectively. We assume that those d. f.'s are continuous and concentrated on $\mathbb{R}_+ := \mathbb{R} \cap [0, \infty]$.

One way to derive new estimators for θ^*, based on our observations $(Z_i, \delta_i)_{i \le n}$ instead of $(X_i)_{i \le n}$, is to substitute the true d. f. F by an appropriate estimate. Following the calculations in Chapter 7 of Shorack and Wellner (2009), one may find those estimators by considering the cumulative hazard function of F

$$\Lambda(z) = \int_0^z \frac{1}{1-F(t)} F(dt) = \int_0^z \frac{1}{1-F(t)} H^1(dt) \ ,$$

with $H^1(z) = \mathbb{P}(\delta = 1, Z \le z)$. An estimator for the cumulative hazard rate was introduced by Nelson (1972) and Aalen (1978), i. e.

$$\Lambda_n(z) = \int_0^z \frac{1}{1-H_n(t-)} H_n^1(dt) = \sum_{i=1}^n \frac{\delta_i \mathbb{1}_{\{Z_i \le z\}}}{n - R_{i,n} + 1} \ ,$$

where

$$H_n^1(z) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{Z_i \le z\}}$$

is the empirical version of H^1 and $R_{i,n}$ denotes the rank of Z_i in a sample of n. Noting the fact that $1 - F(x) = \exp(-\Lambda(x))$ and using the approximation $\exp(-x) \approx 1 - x$ yields the following estimator

$$1 - F_n^{km}(z) = \prod_{i: Z_i \le z} \left(\frac{n - R_{i,n}}{n - R_{i,n} + 1} \right)^{\delta_i} \approx \exp(-\Lambda_n(z)) \ .$$

The estimator above is the well known Kaplan-Meier product limit estimator (PLE). It was introduced by Kaplan and Meier (1958). If one can not make any further assumptions about the censorship, in addition to the RCM, then the Kaplan-Meier PLE is the commonly used estimator of the true d. f. F. Note that F_n^{km} can be expressed in terms of ordered observations as

$$1 - F_n^{km}(z) = \prod_{i=1}^n \left(1 - \frac{\delta_{[i:n]}}{n - i + 1} \right)^{\mathbb{1}_{\{Z_{i:n} \le z\}}}$$

where $Z_{1:n} \le ... \le Z_{n:n}$ and $\delta_{[i:n]}$ denotes the concomitant of the i-th order statistics, i. e. $\delta_{[i:n]} = \delta_j$ whenever $Z_{i:n} = Z_j$.

Let's go back to our integral equation (1.1) and consider the case $k = 1$. In this case we have

$$\theta^* = \int \phi dF \ . \tag{1.3}$$

Replacing the true F in the integral equation above by F_n^{km} yields

$$S_{1,n}^{km}(\phi) := \int_0^\infty \phi dF_n^{km} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i,n}^{km}$$

where $W_{i,n}^{km}$ denotes the weight placed on $Z_{i:n}$ by F_n^{km}, that is,

$$\begin{aligned} W_{i,n}^{km} &= F_n^{km}(Z_{i:n}) - F_n^{km}(Z_{i-1:n}) \\ &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{[j:n]}} \ . \end{aligned}$$

It is easy to see that the Kaplan-Meier estimator only puts mass at uncensored Z-values, since

$$W_{i,n}^{km} = \begin{cases} 0 & \text{if } \delta_{[i:n]} = 0 \\ \frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1} \right] > 0 & \text{if } \delta_{[i:n]} = 1 \end{cases} .$$

The strong law of large numbers (SLLN) for $S_{1,n}^{km}(\phi)$ has been established by Stute and Wang (1993). Let's now consider the case $k = 2$. Define the following estimator for $n \geq 2$

$$S_{2,n}^{km}(\phi) = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{km} W_{j,n}^{km} .$$

The above estimator will be called Kaplan-Meier U-Statistics of degree 2. The strong law of large numbers for $U_{2,n}^{km}$ has been established by Bose and Sen (1999). The asymptotic distribution of this estimator has been derived in Bose and Sen (2002).

Remark 1.3. In Bose and Sen (1999) the normalized version of $S_{2,n}^{km}(\phi)$ was introduced as

$$\frac{S_{2,n}^{km}(\phi)}{S_{2,n}^{km}(1)} = \frac{\sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{km} W_{j,n}^{km}}{\sum\sum_{1\le i<j\le n} W_{i,n}^{km} W_{j,n}^{km}} \,.$$

The normalizing factor $(S_{2,n}^{km}(1))^{-1}$ was motivated by the fact that the following holds true for uncensored data

$$\frac{W_{i,n}^{km} W_{j,n}^{km}}{\sum\sum_{1\le u<v\le n} W_{u,n}^{km} W_{v,n}^{km}} = \binom{n}{2}^{-1} .$$

This normalization, under proper conditions, leads to a smaller asymptotic bias, as shown in Remark 2 of Bose and Sen (1999).

In addition to the assumptions of the RCM, we make the further assumption that

$$m(z) = \mathbb{P}(\delta = 1 | Z = z) = \mathbb{E}(\delta | Z = z)$$

belongs to some parametric family, i. e.

$$m(z) = m(z, \theta_0)$$

where $\theta_0 = (\theta_{0,1}, ..., \theta_{0,p}) \in \Theta \subset \mathbb{R}^p$. This framework is called the semi-parametric Random Censorship Model (SRCM). Dikta (1998) introduced the following PLE

$$1 - F_n^{se,1}(z) = \prod_{i:Z_i \le z} \left(1 - \frac{1}{n - R_i + 1}\right)^{m(Z_i, \hat{\theta}_n)} .$$

Uniform consistency and a functional CLT result were established for $F_n^{se,1}$ by Dikta (1998). Here $\hat{\theta}_n$ denotes the Maximum Likelihood Estimate (MLE) of θ_0. That is, $\hat{\theta}_n$ is the maximizer of

$$L_n(\theta) = \prod_{i=1}^{n} m(Z_i, \theta)^{\delta_i} (1 - m(Z_i, \theta))^{1-\delta_i}.$$

Later in Dikta (2000) another semi-parametric estimator was introduced, i. e.

$$1 - F_n^{se}(z) = \prod_{i:Z_i \le z} \left(1 - \frac{m(Z_i, \hat{\theta}_n)}{n - R_i + 1}\right) .$$

In this thesis we will consider integrals of measurable functions w. r. t. F_n^{se}. By replacing again the true d. f. F by F_n^{se} in equation (1.3), we obtain the following semi-parametric estimator

$$S_{1,n}^{se}(\phi) = \int_0^\infty \phi dF_n^{se} = \sum_{i=1}^n \phi(Z_{i:n}) W_{i,n}^{se}$$

where

$$W_{i,n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n - i + 1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n - j + 1}\right)$$

is the mass assigned to $Z_{i:n}$ by F_n^{se}. $W_{i,n}^{se}$ will be called i-th semi-parametric weight throughout this document. The SLLN and the CLT for the semi-parametric estimator $S_{1,n}^{se}$ have been established in Dikta (2000) and Dikta et al. (2005) respectively. In Dikta (2014) it is shown that $S_{1,n}^{se}$ is asymptotically efficient. Moreover Dikta et al. (2016) shows a way to derive strongly consistent, asymptotically normal and efficient estimators from solving a Volterra type integral equation by different numeric schemes. One of the estimators derived is

$$S_{1,n}^{se,2}(\phi) = \int_0^\infty \phi dF_n^{se,2} = \sum_{i=1}^n \phi(Z_{i:n}) W_{i,n}^{se,2}$$

where

$$W_{i,n}^{se,2} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n - i + 1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n - j + m(Z_{j:n})}\right) .$$

This estimator is a proper distribution function, while $S_{1,n}^{se}$ and $S_{1,n}^{km}$ are sub-distribution functions if the largest observation is censored.

During this thesis we will establish the strong law of large numbers, under proper conditions, for the following estimator

$$S^{se}_{2,n}(\phi) := \sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W^{se}_{i,n} W^{se}_{j,n} \ .$$

We will call $S^{se}_{2,n}$ semi-parametric U-Statistic or semi-parametric estimator throughout this work.

The main result of this thesis is stated in the following theorem.

Theorem 1.4. *Suppose that conditions (A1) through (A4), (M1) and (M2) hold (see Chapter 2). Then the following statement holds with probability one*

$$\lim_{n\to\infty} S^{se}_{2,n}(\phi) = \frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} \phi(s,t)F(ds)F(dt) \ .$$

In the remark below, we will compare the limit above to the target value $\mathbb{E}[\phi]$.

Remark 1.5. Suppose the conditions in Theorem 1.4 holds. Recall the target value from Chapter 1

$$\mathbb{E}[\phi] = \int_0^{\infty}\int_0^{\infty} \phi(s,t)F(ds)F(dt) \ .$$

Now let's compare the limit in Theorem 1.4. Since ϕ is non-negative by condition (A1), we have

$$\frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} \phi(s,t)F(ds)F(dt) \le \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \phi(s,t)F(ds)F(dt) = \frac{1}{2}\cdot\mathbb{E}[\phi] \ .$$

Therefore the following holds

$$2\cdot S^{se}_{2,n}(\phi) \to \int_0^{\tau_H}\int_0^{\tau_H} \phi(s,t)F(ds)F(dt) \le \mathbb{E}[\phi] \ .$$

Remark 1.3 shows a normalized version of $S_{2,n}^{km}$, which was discussed in Bose and Sen (1999), Remark 2. Similarly we will extend the result of Theorem 1.4 to the normalized version of the semi-parametric estimator, in the following remark.

Remark 1.6. Assume conditions (A1) through (A4), (M1) and (M2) are satisfied. Consider that, according to Theorem 1.4, we have

$$S_{2,n}^{se}(1) = \sum\sum_{1\leq i<j\leq n} W_{i,n}W_{j,n} \rightarrow \frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} F(ds)F(dt) = \frac{1}{2}F^2(\tau_H)\ .$$

almost surely. Therefore the following statement holds true

$$\lim_{n\rightarrow\infty}\frac{S_n(\phi)}{S_n(1)} = F^{-2}(\tau_H)\int_0^{\tau_H}\int_0^{\tau_H}\phi(s,t)F(ds)F(dt)$$

almost surely.

Chapter 2

Notation and assumptions

In this chapter we will state the main definitions and assumptions used throughout this work. We will start by defining the estimator to be considered and introduce all necessary notation for the remaining chapters.

2.1 Definitions and notation

Define for $n \geq 2$

$$W_{i,n}^{se} := \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n-j+1}\right)$$

and

$$S_{2,n}^{se}(\phi) := \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{se} W_{j,n}^{se} \; .$$

Furthermore let

$$W_{i,n}(q) := \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

and

$$S_n(q) := \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}(q) W_{j,n}(q)$$

for some measurable function q s. t. $q(t) \in [0, 1]$ for all $t \in \mathbb{R}_+$. Next define

$$\mathcal{F}_n := \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\} \; .$$

The following quantities will be needed in section 4.1. Define for $n \geq 2$ and $s < t$

$$\begin{aligned}
B_n(s,q) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < s\}}} \\
C_n(s,q) &:= \sum_{i=1}^{n+1} \left[\frac{1-q(s)}{n-i+2}\right] \mathbb{1}_{\{Z_{i-1:n} < s \leq Z_{i:n}\}} \\
D_n(s,t,q) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{1}_{\{Z_k < s\}}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{1}_{\{s < Z_k < t\}}} \\
\Delta_n(s,t,q) &:= \mathbb{E}\left[D_n(s,t,q)\right] \\
\bar{\Delta}_n(s,t,q) &:= \mathbb{E}\left[C_n(s,q) D_n(s,t,q)\right]
\end{aligned}$$

and

$$D(s,t,q) := \exp\left(2\int_0^s \frac{1-q(x)}{1-H(x)} H(dx) + \int_s^t \frac{1-q(x)}{1-H(x)} H(dx)\right) .$$

We will write $B_n(s) \equiv B_n(s,q)$, $C_n(s) \equiv C_n(s,q)$, $D_n(s,t) \equiv D_n(s,t,q)$, $\Delta_n(s,t) \equiv \Delta_n(s,t,q)$, $\bar{\Delta}_n(s,t) \equiv \bar{\Delta}_n(s,t,q)$ and $D(s,t) \equiv D(s,t,q)$. Next let

$$\bar{S}_n(q) := \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) \bar{W}_{i:n}(q) \bar{W}_{j:n}(q)$$

where

$$\bar{W}_{i:n}(q) := \frac{1}{n-i+1} \prod_{k=1}^{n} \left(1 - \frac{q(Z_{k:n})}{n-k+1}\right) .$$

Moreover define for $s < t$

$$\begin{aligned}
S(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) q(s) q(t) \exp\left(\int_0^s \frac{1-q(x)}{1-H(x)} H(dx)\right) \\
\times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)} H(dx)\right) H(ds) H(dt)
\end{aligned}$$

and

$$\bar{S}(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) \exp\left(\int_0^s \frac{1-q(x)}{1-H(x)} H(dx)\right)$$

$$\times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)} H(dx)\right) H(ds)H(dt)\ .$$

We will write $S_n \equiv S_n(q)$, $W_{i,n} \equiv W_{i,n}(q)$, $S \equiv S(q)$ and $\bar{S} \equiv \bar{S}(q)$ throughout this thesis. Moreover we define $\tau_H = \inf\{z|H(z) = 1\}$.

2.2 Assumptions

The following assumptions will be needed, in order to establish Theorem 1.4:

(A1) The kernel $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable, non-negative and symmetric in its arguments. In effect $\phi(s,t) = \phi(t,s)$ for all $s,t \in \mathbb{R}_+$.

(A2) H is continuous and concentrated on the non-negative real line.

(A3) The following statement holds true

$$\int_0^{\tau_H}\int_0^{\tau_H} \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^{\epsilon}(1-H(t))^{\epsilon}} F(ds)F(dt) < \infty$$

for some $0 < \epsilon \leq 1$.

(A4) $m(z,\theta)$ is non-decreasing in z.

Here condition (A1) is a standard assumption for U-Statistics (c. f. Lee (1990)). Assumptions (A2) is the same as in Dikta (2000). (A3) is here the 2-dimensional equivalent to the condition in Theorem 1.1 of Dikta (2000). Condition (A4) poses an additional restriction on the censoring model m here. We will discuss the restrictions imposed by (A4) and see examples of different models for m, which satisfy this condition in Chapter 5. Moreover, Chapter 6 shows simulation studies under different choices for m.

We will need the following assumptions about the Censoring Model m and the Maximum Likelihood estimate $\hat{\theta}_n$:

(M1) $\hat{\theta}_n$ is measurable and tends to θ_0 almost surely.

(M2) For any $\epsilon > 0$ there exists a neighborhood $V(\epsilon, \theta_0) \subset \Theta$ of θ_0 s. t. for all $\theta \in V(\epsilon, \theta_0)$

$$\sup_{z \geq 0} |m(z, \theta) - m(z, \theta_0)| < \epsilon \; .$$

Condition (M1) above guarantees the strong consistency of the MLE. (M1) and (M2) are identical to (A1) and (A2) in Dikta (2000).

Chapter 3

Existence of the limit

In this chapter we will establish basic properties of $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$. A representation for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the result established in Bose and Sen (1999), Lemma 1, is derived in Section 3.1. In Stute and Wang (1993) the proof of existence of the limit of the considered estimator was based on the fact that the conditional expectation above was a reverse supermartingale in their case. Later in Dikta (2000) and in Bose and Sen (1999) the same type of argument was used for the estimators they considered. We will not be able to establish the reverse supermartingale property for $S_{2,n}^{se}$ in general. But we will be able to state a condition on q, s. t. $S_n(q)$ is indeed a supermartingale. This will be discussed in more detail in Section 3.2. Section 3.3 will show how this implies the almost sure existence by the same argument as in Stute and Wang (1993).

3.1 Preliminary Considerations

We will first derive an explicit representation for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the one established in the proof of Bose and Sen (1999), Lemma 1.

Lemma 3.1. *Define for* $1 \leq i < j \leq n$

$$Q_{ij}^{n+1} = \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i \pi_n (1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

where

$$Q_i^{n+1} = (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\} \tag{3.1}$$

and

$$\pi_i = \prod_{k=1}^{i-1}\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\ .$$

Then we have

$$\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\le i<j\le n+1}\phi(Z_{i:n+1},Z_{j:n+1})W_{i,n+1}W_{j,n+1}Q_{ij}^{n+1}\ .$$

Proof. We will need the following result for the proof of lemma 3.1. Let

$$A_i = \pi_i + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]$$

for $1\le i\le n$ with π_i as defined above. Note that $\pi_1 = 1$, since the product is empty and hence taken as 1. Therefore we have $A_1 = \pi_1 = 1$. Moreover the following holds true for any $1\le i\le n-1$

$$\begin{aligned}
A_{i+1} &= \pi_{i+1} + \sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\\
&= \pi_i\left[\frac{n-i+1-q(Z_{i:n+1})}{n-i+2-q(Z_{i:n+1})}\right] + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] + \left[\frac{\pi_i}{n-i+2-q(Z_{i:n+1})}\right]\\
&= \pi_i + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\\
&= A_i\ .
\end{aligned}$$

And therefore

$$1 = A_1 = A_2 = \cdots = A_{n-1} = A_n\ . \tag{3.2}$$

Now let's establish the statement of Lemma 3.1. Let F_n^q denote the measure that assigns mass to $Z_{1:n}, \ldots, Z_{n:n}$, then

$$
\begin{aligned}
\mathbb{E}[S_n|\mathcal{F}_{n+1}] &= \mathbb{E}[\sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n} W_{j,n} | \mathcal{F}_{n+1}] \\
&= \mathbb{E}[\sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} | \mathcal{F}_{n+1}] \\
&= \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) \mathbb{E}[F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} | \mathcal{F}_{n+1}] \ .
\end{aligned}
$$

Consider for $1 \le i < j \le n$

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} | \mathcal{F}_{n+1}] \\
&= \mathbb{E}\left[\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{\{Z_{n+1}=Z_{r:n+1}\}} | \mathcal{F}_{n+1}\right] \ .
\end{aligned}
$$

Define the set $A_{rn} := \{Z_{n+1} = Z_{r:n+1}\}$. Note that on A_{rn} we have for $1 \le l \le n+1$

$$
Z_{l:n+1} = \begin{cases} Z_{l:n} & l < r \\ Z_{l-1:n} & l > r \end{cases} \tag{3.3}
$$

and therefore

$$
F_n^q\{Z_{l:n+1}\} = \begin{cases} W_{l:n} & l < r \\ 0 & l = r \\ W_{l-1:n} & l > r \end{cases} \ . \tag{3.4}
$$

We have

$$
\begin{aligned}
&\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{\{Z_{n+1}=Z_{r:n+1}\}} \\
&= \sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} W_{i-1,n} W_{j-1,n} I_{A_{rn}} + \sum_{r=i+1}^{j-1} W_{i,n} W_{j-1,n} I_{A_{rn}} + \sum_{r=j+1}^{n+1} W_{i,n} W_{j,n} I_{A_{rn}}
\end{aligned}
$$

$$=: T_1 + T_2 + T_3 \ . \tag{3.5}$$

Let's now consider each of the sums T_1, T_2, and T_3 in the above equation individually. First consider T_1. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i-1:n})}{n-i+2} \prod_{k=1}^{i-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}}
\end{aligned}$$

using (3.3). Next we will continue to find an expression for T_1 in terms of $W_{i,n+1}$ and $W_{j,n+1}$. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&\quad \times \left[\frac{\prod_{k=1}^{r-1}\left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right]}{\prod_{k=1}^{r-1}\left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right]}\right]^2 \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}}
\end{aligned}$$

$$\times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right]^2 .$$

Using index transformation on the products $\prod_{k=r}^{i-2}[\ldots]$ and $\prod_{k=r}^{j-2}[\ldots]$ yields

$$\begin{aligned}
T_1 = {} & \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{i-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{j-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] I_{A_{rn}} \\
& \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right]^2 \\
= {} & \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{i-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{j-1} \left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} I_{A_{rn}} \\
& \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right]^2 \\
= {} & W_{i,n+1} W_{j,n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right]^2 \\
& \times \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})}\right]^2 I_{A_{rn}} .
\end{aligned}$$

Note that

$$\begin{aligned}
\prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right] &= \frac{n+1}{n} \cdot \frac{n}{n-1} \cdots \frac{n-r+4}{n-r+3} \cdot \frac{n-r+3}{n-r+2} \\
&= \frac{n+1}{n-r+2} . \qquad (3.6)
\end{aligned}$$

and recall the following definition

$$\pi_r = \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] .$$

Now we finally get

$$
\begin{aligned}
T_1 = W_{i,n+1} W_{j,n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1} & \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]^2 \\
& \times \left[\frac{n+1}{n-r+2} \right]^2 \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}} \\
= W_{i,n+1} W_{j,n+1} \sum_{r=1}^{i-1} \pi_r^2 & \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}} \ .
\end{aligned}
$$

Next consider T_2. We will, again, firstly express T_2 completely in terms of the ordered Z values w. r. t. order $n+1$ using (3.3). Consider

$$
\begin{aligned}
T_2 = & \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] \\
& \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] I_{A_{rn}} \\
= & \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n+1})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}} \ .
\end{aligned}
$$

Now let's find a representation of T_2 which relies on $W_{i,n+1}$ and $W_{j,n+1}$ only. Consider

$$
\begin{aligned}
T_2 = & \sum_{r=i+1}^{j-1} \left[\frac{n-i+2}{n-i+1} \right] \left[\frac{q(Z_{i:n+1})}{n-i+2} \right] \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}} \\
& \times \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1} \right] \\
& \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1} \right] \\
= & \left[\frac{n-i+2}{n-i+1} \right] \left[\frac{q(Z_{i:n+1})}{n-i+2} \right] \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]\\
&\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] .
\end{aligned}
$$

Applying (3.6) to $\prod_{k=1}^{i-1}[\ldots]$ yields

$$
\begin{aligned}
T_2 &= \left[\frac{n+1}{n-i+1}\right]\left[\frac{q(Z_{i:n+1})}{n-i+2}\right]\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\\
&\quad\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]\\
&= \left[\frac{n+1}{n-i+1}\right]W_{i,n+1}\pi_i\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] .
\end{aligned}
$$

Again doing an index transformation on $\prod_{k=r}^{j-2}[\ldots]$ yields

$$
\begin{aligned}
&= \left[\frac{n+1}{n-i+1}\right]W_{i,n+1}\pi_i\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r+1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]I_{A_{rn}}\\
&= W_{i,n+1}\pi_i\frac{n+1}{n-i+1}\sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] I_{A_{rn}} \\
= {} & W_{i,n+1}W_{j,n+1}\pi_i\frac{n+1}{n-i+1} \\
&\times \sum_{r=i+1}^{j-1}\prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] \\
&\times \frac{n-r+2}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}} \ .
\end{aligned}
$$

Now applying (3.6) to the latter product yields

$$
T_2 = W_{i,n+1}W_{j,n+1}\pi_i\frac{n+1}{n-i+1}\sum_{r=i+1}^{j-1}\pi_r\frac{n+1}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}} \ .
$$

We will proceed similarly for T_3. Consider

$$
T_3 = \sum_{r=j+1}^{n+1} W_{i,n}W_{j,n}\mathbb{1}_{\{A_{rn}\}} \ .
$$

Note that for $j = n+1$ the sum above is empty and hence zero. Consider for $j \leq n$

$$
\begin{aligned}
T_3 = {} & \sum_{r=j+1}^{n+1}\frac{q(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right] \\
&\times \frac{q(Z_{j:n})}{n-j+1}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}} \\
= {} & \sum_{r=j+1}^{n+1}\frac{q(Z_{i:n+1})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+1}\right] \\
&\times \frac{q(Z_{j:n+1})}{n-j+1}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}} \\
= {} & \sum_{r=j+1}^{n+1}\frac{n-i+2}{n-i+1}\frac{q(Z_{i:n+1})}{n-i+2}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\times \frac{n-j+2}{n-j+1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]
\end{aligned}
$$

$$\times \prod_{k=1}^{j-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{j-1}\left[\frac{n-k+2}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}}$$

$$= \sum_{r=j+1}^{n+1}\frac{n-i+2}{n-i+1}\frac{n-j+2}{n-j+1}\pi_i\pi_j W_{i,n+1}W_{j,n+1}$$

$$\times \prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]\prod_{k=1}^{j-1}\left[\frac{n-k+2}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}}\ .$$

Again, by (3.6), we have

$$T_3 = \sum_{r=j+1}^{n+1}\frac{(n+1)^2\pi_i\pi_j}{(n-i+1)(n-j+1)}W_{i,n+1}W_{j,n+1}\mathbb{1}_{\{A_{rn}\}}\ .$$

Therefore

$$T_3 = \begin{cases} W_{i,n+1}W_{j,n+1}\pi_i\pi_j\left[\frac{(n+1)^2}{(n-i+1)(n-j+1)}\right]\sum\limits_{r=j+1}^{n+1}\mathbb{1}_{\{A_{rn}\}} & j \le n \\ 0 & j = n+1 \end{cases}$$

for $1 \le i < j \le n$. Next, substituting the expressions for T_1, T_2 and T_3 in equation (3.5) together with the fact that

$$\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] = \frac{1}{n+1}$$

yields

$$\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= \mathbb{E}[T_1+T_2+T_3|\mathcal{F}_{n+1}] \\
&= W_{i,n+1}W_{j,n+1} \times \Bigg\{\sum_{r=1}^{i-1}\pi_r^2\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]^2\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] \\
&\quad + \sum_{r=i+1}^{j-1}\pi_i\pi_r\left[\frac{n+1}{n-i+1}\right]\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}]
\end{aligned}$$

$$
\begin{aligned}
&+\pi_i\pi_j\frac{(n+1)^2}{(n-i+1)(n-j+1)}[1-I_{\{j=n+1\}}]\sum_{i=j+1}^{n+1}\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}]\Bigg\}\\
&= W_{i,n+1}W_{j,n+1}\left[\frac{1}{n+1}\right]\times\left\{\sum_{r=1}^{i-1}\pi_r^2\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&+\sum_{r=i+1}^{j-1}\pi_i\pi_r\left[\frac{n+1}{n-i+1}\right]\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]\\
&\left.+\pi_i\pi_j\frac{(n+1)^2}{n-i+1}[1-I_{\{j=n+1\}}]\right\}\ .
\end{aligned}
$$

Next consider that we have

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]\\
&= W_{i,n+1}W_{j,n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=i+1}^{j-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]+\pi_j\right]\right\}\ .
\end{aligned}
$$

for $1\le i<j\le n$. Applying (3.2) yields

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]\\
&= W_{i,n+1}W_{j,n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}(A_j-A_{i+1}+\pi_{i+1})\right\}\\
&= W_{i,n+1}W_{j,n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\qquad\left.+\frac{\pi_i\pi_{i+1}}{n-i+1}\right\}\\
&= W_{i,n+1}W_{j,n+1}Q_i^{n+1}\ .
\end{aligned}
$$

It remains to consider the case $j=n+1$. We have

$$\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]$$

$$
\begin{aligned}
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\sum_{r=i+1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right\}\\
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]-\sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\}\\
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]-\sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\}\ .
\end{aligned}
$$

Now using (3.2) again yields

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]\\
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\left[A_{n+1}-\pi_{n+1}-(A_{i+1}-\pi_{i+1})\right]\right\}\\
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.\\
&\qquad\left.+\frac{\pi_i}{n-i+1}\left[\pi_{i+1}-\pi_{n+1}\right]\right\}\ .
\end{aligned}
$$

Note that for $1 \le i < n$ we have

$$
\pi_{i+1} = \frac{\pi_i(1-q(Z_{i:n+1}))}{2-q(Z_{i:n+1})}\ .
$$

Thus we obtain

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]\\
&= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.
\end{aligned}
$$

$$+\frac{\pi_i}{n-i+1}\left[\pi_{i+1}-\frac{\pi_n(1-q(Z_{n:n+1}))}{2-q(z_{n:n+1})}\right]\Bigg\}$$

$$= W_{i,n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\}$$

$$= W_{i,n+1}W_{n+1:n+1}\left\{Q_i^{n+1}-\frac{\pi_i\pi_n(n+1)(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\} .$$

□

The following lemma contains a result on the increases of Q_i^{n+1} w. r. t. i. It is especially useful, since we can express Q_i^{n+1} as follows

$$Q_i^{n+1} = Q_1^{n+1} + \sum_{k=1}^{i-1}(Q_{k+1}^{n+1} - Q_k^{n+1}) .$$

The result will be used to establish the reverse supermartingale property for S_n in Lemma 3.3.

Lemma 3.2. *Let Q_i^{n+1} be defined as in Lemma 3.1 for $1 \leq i \leq n$. Moreover define*

$$\tilde{\pi}_i := \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right] .$$

Then we have

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \frac{(q_i-q_{i+1})(n-i)(n-i+1)-q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \times \frac{\tilde{\pi}_i(n-i+2)^2}{n+1} .$$

Proof. For the sake of simplicity we will write $q_i \equiv q(Z_{i:n+1})$ during this proof. From equation (3.1) we get

$$\frac{Q_{i+1}^{n+1}-Q_i^{n+1}}{n+1} = \left\{\sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q_r}\right]^2 + \frac{\pi_{i+1}\pi_{i+2}}{n-i}\right\}$$

$$
\begin{aligned}
&-\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q_r}\right]^2+\frac{\pi_i\pi_{i+1}}{n-i+1}\right\} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2}+\frac{\pi_{i+1}\pi_{i+2}}{n-i}-\frac{\pi_i\pi_{i+1}}{n-i+1} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2}+\frac{\pi_i^2(n-i+1-q_i)^2(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&- \frac{\pi_i^2(n-i+1-q_i)}{(n-i+1)(n-i+2-q_i)} \\
&= \pi_i^2\left\{\frac{1}{(n-i+2-q_i)^2}+\frac{(n-i+1-q_i)^2(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right. \\
&\qquad \left. -\frac{n-i+1-q_i}{(n-i+1)(n-i+2-q_i)}\right\} \\
&=: \pi_i^2\left\{a(n,i)+b(n,i)-c(n,i)\right\} \ . \qquad (3.7)
\end{aligned}
$$

Next consider

$$
\begin{aligned}
&b(n,i)-c(n,i) \\
&= (n-i+1-q_i)\left[\frac{(n-i+1-q_i)(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right. \\
&\qquad \left. -\frac{1}{(n-i+1)(n-i+2-q_i)}\right] \\
&= (n-i+1-q_i)\left[\frac{(n-i+1-q_i)(n-i-q_{i+1})(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right. \\
&\qquad \left. -\frac{(n-i+2-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right] \ . \qquad (3.8)
\end{aligned}
$$

Next we will simplify the difference of the numerators above. We have

$$
\begin{aligned}
&(n-i+1-q_i)(n-i-q_{i+1})(n-i+1) \\
&\quad -(n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1)-q_{i+1}(n-i+1-q_i)(n-i+1) \\
&\quad -(n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1)-q_{i+1}(n-i+1-q_i)(n-i+1)
\end{aligned}
$$

$$
\begin{aligned}
&- (n-i+1-q_i)(n-i+1-q_{i+1})(n-i) - (n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&- (n-i+1-q_i)(n-i+1)(n-i) + q_{i+1}(n-i+1-q_i)(n-i) \\
&- (n-i+1-q_{i+1})(n-i) \\
&= -q_{i+1}(n-i+1-q_i) - (n-i+1-q_{i+1})(n-i) \ .
\end{aligned}
$$

Hence we get, according to (3.8)

$$
\begin{aligned}
& b(n,i) - c(n,i) \\
&= -(n-i+1-q_i)\left[\frac{q_{i+1}(n-i+1-q_i) + (n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right] \ .
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
& a(n,i) + b(n,i) - c(n,i) \\
&= \frac{1}{(n-i+2-q_i)^2} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(n-i)(n-i+1)(n-i+1-q_{i+1})}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \ .
\end{aligned}
$$

Consider again the numerator of the latter expression. We have

$$
\begin{aligned}
&= (n-i)(n-i+1)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2 \\
&\quad - (n-i)(n-i+1-q_i)(n-i+1-q_{i+1}) \\
&= q_i(n-i)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2 \\
&= q_i(n-i)^2 + q_i(1-q_{i+1})(n-i) - q_{i+1}(n-i)^2 \\
&\quad - 2q_{i+1}(1-q_i)(n-i) - q_{i+1}(1-q_i)^2
\end{aligned}
$$

$$
\begin{aligned}
&= (q_i - q_{i+1})(n-i)^2 + q_i(n-i) - q_i q_{i+1}(n-i) \\
&\quad - 2q_{i+1}(n-i) + 2q_i q_{i+1}(n-i) - q_{i+1}(1-q_i)^2 \\
&= (q_i - q_{i+1})(n-i)^2 + (q_i + q_i q_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2 \ .
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&a(n,i) + b(n,i) - c(n,i) \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + (q_i + q_i q_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + [(q_i - q_{i+1}) - q_{i+1}(1-q_i))(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \ . \qquad (3.9)
\end{aligned}
$$

Finally note that

$$
\begin{aligned}
\tilde{\pi}_i &= \frac{n+1}{n-i+2} \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \\
&= \pi_i \cdot \frac{n+1}{n-i+2} \qquad (3.10)
\end{aligned}
$$

with π_i as defined in Lemma 3.1. Now the statement of the lemma follows directly by combining (3.7), (3.9) and (3.10) □

3.2 S_n is not a reverse supermartingale in general

As discussed in Chapter 1, the strong law of large numbers for Kaplan-Meier U-statistics was established by Bose and Sen (1999). Recall the definition of the estimator they considered:

$$
S_{2,n}^{km} = \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{km} W_{j,n}^{km}
$$

with

$$W_{i,n}^{km} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right] .$$

The proof of existence of the almost sure limit $S = \lim_{n\to\infty} S_n^{km}$ was here essentially based upon a reverse supermartingale argument together with Neveu (1975), proposition V-3-11. In Lemma 1 of Bose and Sen (1999) a representation for $\mathbb{E}[S_{2,n}^{km}|\mathcal{F}_{n+1}]$ was derived, which is similar to Lemma 3.1 in this thesis. It was shown that

$$\mathbb{E}[S_{2,n}^{km}\Big|\mathcal{F}_{n+1}] = \sum_{1\le i<j\le n+1}\sum \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i,n+1}^{km} W_{j,n+1}^{km} Q_{ij}^{km} ,$$

for $1 \le i < j \le n$. Here Q_{ij}^{km} is defined as follows

$$Q_{ij}^{km} = \begin{cases} Q_i^{km} & \text{if } j \le n \\ Q_i^{km} - \pi_i \pi_n (1 - \delta_{[n:n+1]})) \frac{n-i+2}{(n+1)(n-i+1)} & \text{if } j = n+1 \end{cases} ,$$

with

$$Q_i^{km} = \frac{1}{n+1} \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n-r+2}{n-r+1}\right]^{2\delta_{[r:n+1]}} \right.$$
$$\left. + \pi_i^2 (n-i+2) \left[\frac{(n-i)(n-i+2)}{(n-i+1)^2}\right]^{\delta_{[i:n+1]}} \right\} .$$

Next Bose and Sen (1999) show that $Q_{ij}^{km} \le 1$ for $1 \le i < j \le n$, in order to establish the reverse supermartingale property for $(S_n^{km}, \mathcal{F}_n)$. However their proof relies on the fact that

$$\begin{aligned} W_{i,n}^{km} &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right] \\ &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1}\right]^{\delta_{[k:n]}} . \end{aligned}$$

But the corresponding statement is not true for $W_{i:n}(q)$, since we have in general that

$$\begin{aligned} W_{i,n}(q) &= \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\ &\neq \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1}\right]^{q(Z_{k:n})} . \end{aligned}$$

In Dikta (2000), the following estimator was considered

$$S_{1,n}^{se} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{se} .$$

The proof of existence shows here a similar structure, as the one by Bose and Sen (1999). In Lemma 2.1 of Dikta (2000), it was shown that $\mathbb{E}[\mu_n\{Z_{1:n+1}\}|\mathcal{F}_{n+1}] = W_{1:n}^{se}$ and for $2 \leq i \leq n$

$$\mathbb{E}[\mu_n\{Z_{i:n+1}\} \Big| \mathcal{F}_{n+1}] = W_{i:n}^{se} Q_i^{se} ,$$

where μ_n is the measure assigning mass $W_{i:n}$ to $Z_{i:n}$ and

$$Q_i^{se} = \pi_i + \sum_{k=1}^{i-1} \frac{\pi_k}{n-k+2-q(Z_{k:n+2})} .$$

Here π_i is defined as in Lemma 3.1. Furthermore it was shown that $Q_i^{se} = Q_{i+1}^{se} = 1$ for all $2 \leq i \leq n$, which, among other arguments, implies the reverse supermartingale property for S_n^{se}.

The discussion above shows that we can not establish the supermartingale property for S_n without further restrictions, by the same arguments as were presented in Bose and Sen (1999) and Dikta (2000).

In the following Lemma we will establish the supermartingale property for S_n under the additional assumption that q is non-decreasing.

Lemma 3.3. *Let $q(z)$ be non-decreasing for all $z \in \mathbb{R}_+$. Then $S_n(q)$ is a non-negative reverse supermartingale.*

Proof. First note that

$$Q_1^{n+1} = (n+1)\frac{\pi_1\pi_2}{n} = \frac{(n+1)(n-q_1)}{n(n+1-q_1)} = \frac{n(n+1)-q_1(n+1)}{n(n+1)-q_1 n} \leq 1 \tag{3.11}$$

Recall that we have

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} = & \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\ & \times \frac{\tilde{\pi}_i(n-i+2)^2}{n+1} \ . \end{aligned} \tag{3.12}$$

according to Lemma 3.2. Next consider that we have

$$q_i - q_{i+1} \leq 0 \quad \text{ and } \quad q_{i+1}(1-q_i) \geq 0 \ ,$$

since $q(z)$ is non-decreasing in z. Combining the latter with equation (3.12) yields

$$Q_{i+1}^{n+1} - Q_i^{n+1} \leq 0 \quad \text{ for all } \quad t \in [0, \infty) \ . \tag{3.13}$$

Consider that we can write Q_i^{n+1} as

$$Q_i^{n+1} = Q_1^{n+1} + \sum_{k=1}^{i-1} \left(Q_{k+1}^{n+1} - Q_i^{n+1} \right) \ .$$

Applying inequalities (3.11) and (3.13) to the above equation yields $Q_i^{n+1} \leq 1$ for all $i \leq n$. Next recall from Lemma 3.1 that

$$Q_{ij}^{n+1} = \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i \pi_n (1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

Thus $Q_{ij}^{n+1} \leq Q_i^{n+1} \leq 1$ for all $1 \leq i < j \leq n+1$. Now the latter together with Lemma 3.1 imply the statement of the Lemma. □

The assumption that q is monotone non-decreasing in Lemma 3.3, is transfered to the censoring model m by (A4). This restricts the choices of censoring models m. Examples for non-decreasing m include the proportional hazards model (see Example 5.1). We will discuss the above mentioned restriction and give examples of different censoring models in Chapter 5.

3.3 Existence of the limit

During the preceding section we have seen that $S_n(q)$ is a reverse supermartingale, whenever q is monotone non-decreasing. We will now show how this implies the almost sure existence of $\lim_{n\to\infty} S_n(q)$, by a standard argument.

Let $\mathcal{F}_\infty = \bigcap_{n\geq 2} \mathcal{F}_n$. The following result applies the Hewitt-Savage zero-one law, in order to show that $\mathcal{F}_\infty$ is trivial. It will be useful in order to prove Theorem 3.5, because it implies that $\mathbb{E}[S_n|\mathcal{F}_\infty] = \mathbb{E}[S_n]$.

Lemma 3.4. *For each $A \in \mathcal{F}_\infty$ we have $\mathbb{P}(A) \in \{0, 1\}$.*

Proof. Denote $\tilde{Z} := (Z_1, Z_2, \dots) \in \mathbb{R}^\infty$ and let $1 \leq n < \infty$ be fixed but arbitrary. We will use the Hewitt-Savage zero-one law to prove the statement of this lemma. Let π be a map

$$\pi : (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) \longrightarrow (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$$

$$(Z_1, Z_2, \dots, Z_n, Z_{n+1}, \dots) \longmapsto (Z_{\tilde{\pi}(1)}, Z_{\tilde{\pi}(2)}, \dots, Z_{\tilde{\pi}(n)}, Z_{n+1}, \dots) \ .$$

where $\tilde{\pi}$ is some permutation of $\{1, \dots, n\}$. Denote by Π_n the set of all $n!$ of such maps. We need to show that for all $A \in \mathcal{F}_\infty$ and for all $\pi_0 \in \Pi$ there exists $B \in \mathcal{B}(\mathbb{R}^\infty)$ s. t.

$$A = \{\omega | \tilde{Z}(\omega) \in B\} = \{\omega | \pi_0(\tilde{Z}(\omega)) \in B\} \ . \tag{3.14}$$

Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Note that each of the maps $\pi \in \Pi_n$ is measurable. Hence the map

$$(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) \longrightarrow (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$$
$$(Z_1, Z_2, \dots, Z_n, Z_{n+1}, \dots) \longmapsto (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots) \ .$$

is measurable. Therefore there must exist $\tilde{B} \in \mathcal{B}(\mathbb{R}^\infty)$ such that

$$A = \{\omega | (Z_{1:n}(\omega), \dots, Z_{n:n}(\omega), Z_{n+1}(\omega), Z_{n+2}(\omega), \dots) \in \tilde{B}\} \ .$$

Thus we can write A as

$$\begin{aligned} A &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi(\tilde{Z}) \in \tilde{B} \right\} \\ &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in \pi^{-1}(\tilde{B}) \right\} \\ &= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \right\} \ . \end{aligned}$$

Consider that

$$\bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \in \mathcal{B}(\mathbb{R}^\infty) \ ,$$

as a countable union of sets in $\mathcal{B}(\mathbb{R}^\infty)$. Moreover note that

$$\bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) = \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \ ,$$

since the union is iterating over all $\pi \in \Pi_n$. Thus we obtain

$$\begin{aligned}
A &= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\
&= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\
&= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi_0(\tilde{Z}) \in \pi^{-1}(\tilde{B}) \right\} \\
&= \left\{ \omega | \pi_0(\tilde{Z}) \in B \right\} \ .
\end{aligned}$$

Whence establishing (3.14). □

Theorem 3.5. *Let $q(z)$ be non-decreasing for all $z \in \mathbb{R}_+$. Then $S_n(q)$ converges almost surely to some limit S_∞ and the following holds almost surely*

$$S_\infty = \lim_{n \to \infty} \mathbb{E}[S_n] \ .$$

Proof. According to Lemma 3.3, $(S_n, \mathcal{F}_n)_{n \geq 2}$ is a non-negative supermartingale. Hence S_n converges almost surely to a limit S_∞ according to Neveu (1975), Lemma V-3-11. Moreover we have

$$S_\infty = \lim_{n \to \infty} \mathbb{E}[S_n | \mathcal{F}_\infty] \tag{3.15}$$

almost surely, according to Lemma V-3-11. But now Lemma 3.4 implies that the limit on the right hand side of (3.15) is almost surely constant, in particular

$$S_\infty = \lim_{n \to \infty} \mathbb{E}[S_n] \ .$$

□

Chapter 4

Identifying the limit

In the previous chapter we established the existence of the limit $\lim_{n\to\infty} S_n(q)$. We will now continue to identify the limit $\lim_{n\to\infty} S_{2,n}^{se} = \lim_{n\to\infty} S_n(m(\cdot, \hat{\theta}_n))$ throughout this chapter. The interdependence structure of the proofs within this chapter is shown in figure 4.1 below.

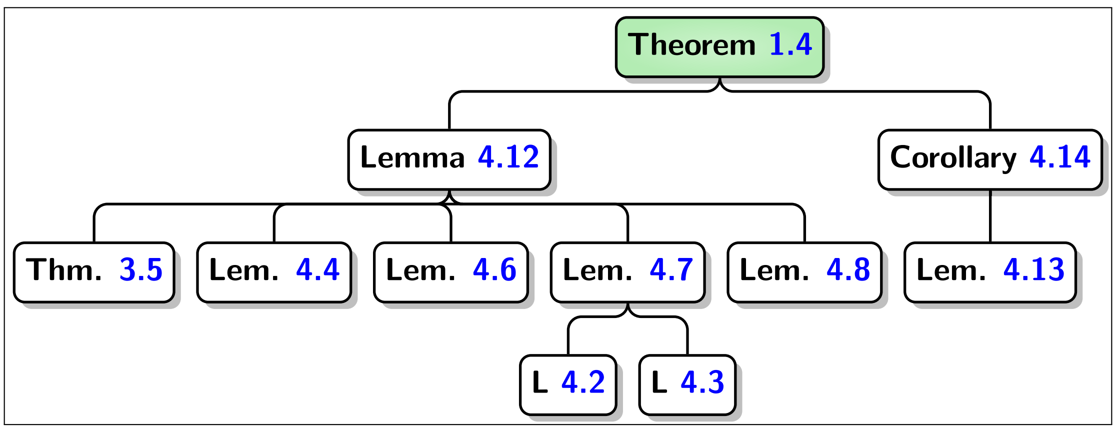

Figure 4.1: Interdependence Structure of the lemmas and theorems within this chapter.

4.1 The reverse supermartingale D_n

During this chapter, we will closely follow the calculations of Bose and Sen (1999). They considered the process $D_n(s, t, \tilde{m})$, where $\tilde{m}(z) = \mathbb{E}[\delta | Z = z]$ does not necessarily belong to a parametric family, while we will be considering $D_n(s, t, q)$ for some measurable function q with values in $[0, 1]$. Since it was not entirely clear, if the special representation of $\tilde{m}$ as conditional expectation was used in the proofs of lemmas 2, 3 and 4 in Bose and Sen (1999), we conducted a detailed investigation. It will turn out that the proofs work in the same way for $D_n(s,t,q)$. For the sake of completeness, we will show the detailed proofs for $D_n(s,t,q)$ in this chapter.

First recall the following quantities from Chapter 2. We have

$$\begin{aligned}
B_n(s) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}_{\{Z_k<s\}}} \\
C_n(s) &:= \sum_{i=1}^{n+1} \left[\frac{1-q(s)}{n-i+2}\right] \mathbb{1}_{\{Z_{i-1:n}<s\le Z_{i:n}\}} \\
D_n(s,t) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+2}\right]^{2\mathbb{1}_{\{Z_k<s\}}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+1}\right]^{\mathbb{1}_{\{s<Z_k<t\}}} \\
\Delta_n(s,t) &:= \mathbb{E}\left[D_n(s,t)\right] \\
\bar{\Delta}_n(s,t) &:= \mathbb{E}\left[C_n(s)D_n(s,t)\right] \ .
\end{aligned}$$

for $n \ge 2$ and $s < t$. Here $Z_{0:n} := 0$ and $Z_{n+1:n} := \infty$.

During this section, we will first derive a representation of $\mathbb{E}[S_n]$ which involves the process D_n. This will be done in Lemma 4.2 and Lemma 4.3. We will then show that $\{D_n, \mathcal{F}_n\}$ is a reverse supermartingale in Lemma 4.5 and finally identify the limit of D_n in Lemma 4.4.

The lemma below contains a basic result needed to prove Lemma 4.3.

Lemma 4.1. *Let $i \neq j$. Then the conditional expectation*

$$\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t]$$

is independent of i, j *and hence*

$$\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t] = \mathbb{E}[B_n(s)B_n(t)|Z_1 = s, Z_2 = t]$$

holds almost surely.

Proof. For the sake of notational simplicity denote for $s < t$ $s_k^n := \mathbb{1}_{\{Z_{k:n} < s\}}$ and $t_k^n := \mathbb{1}_{\{s \le Z_{k:n} < t\}}$. Note that $i \neq j$ implies $s \neq t$, since the $(Z_i)_{i \le n}$ are pairwise distinct. Now consider on $\{s < t\}$

$$\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\left(1 + \frac{1 - q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n} \Big| Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=s\}}\mathbb{1}_{\{Z_{k_2:n}=t\}}\left(1 + \frac{1 - q(s)}{n-k_1}\right)\right. \\
&\qquad \times \prod_{k=1}^{k_1-1}\left(1 + \frac{1 - q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n} \\
&\qquad \times \prod_{k=k_1+1}^{k_2-1}\left(1 + \frac{1 - q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n} \\
&\qquad \times \left.\prod_{k=k_2+1}^{n}\left(1 + \frac{1 - q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n} \Big| Z_i = s, Z_j = t\right]
\end{aligned}$$

since $s_{k_1}^n = 0$, $t_{k_1}^n = 1$, $s_{k_2}^n = 0$ and $t_{k_2}^n = 0$. Moreover we have

$$\begin{cases}
s_k^n = 1 \text{ and } t_k^n = 0 & \text{if } k < k_1 \\
s_k^n = 0 \text{ and } t_k^n = 1 & \text{if } k_1 < k < k_2 \\
s_k^n = 0 \text{ and } t_k^n = 0 & \text{if } k_2 < k
\end{cases}.$$

Therefore we obtain

$$\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right]$$

$$= \mathbb{E}\left[\sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=s\}}\mathbb{1}_{\{Z_{k_2:n}=t\}}\left(1+\frac{1-q(s)}{n-k_1}\right)\right.$$
$$\times \prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n}$$
$$\left.\times \prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{t_k^n}\Bigg|Z_i=s, Z_j=t\right].$$

Next we need to introduce some more notation. For $1 \le i, j \le n$ and $n \ge 2$, let $\{Z_{k:n-2}\}_{k\le n-2}$ denote the ordered Z-values among $Z_1, \ldots, Z_n$ with Z_i and Z_j removed from the sample. Note that

$$Z_{k:n} = \begin{cases} Z_{k:n-2} & k < k_1 \\ Z_{k-1:n-2} & k_1 < k < k_2 \end{cases}. \tag{4.1}$$

Thus we have

$$\mathbb{E}\left[B_n(s)B_n(t)|Z_i=s, Z_j=t\right]$$
$$= \mathbb{E}\left[\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\mathbb{1}_{\{Z_{k_2-2:n-2}<t\le Z_{k_2-1:n-2}\}}\right.$$
$$\times\left(1+\frac{1-q(s)}{n-k_1}\right)\prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}}$$
$$\left.\times \prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k-1:n-2})}{n-k}\right)^{t_{k-1}^{n-2}}|Z_i=s, Z_j=t\right]$$
$$= \mathbb{E}\left[\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\mathbb{1}_{\{Z_{k_2-2:n-2}<t\le Z_{k_2-1:n-2}\}}\right.$$
$$\times\left(1+\frac{1-q(s)}{n-k_1}\right)\prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}}$$
$$\left.\times \prod_{k=k_1}^{k_2-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}}\right]$$
$$= \mathbb{E}\left[\sum_{k_1=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\left(1+\frac{1-q(s)}{n-k_1}\right)\right.$$

$$\times \prod_{k=1}^{n-2} \left(1 + \frac{1 - q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}}$$

$$\times \prod_{k=k_1}^{n-2} \left(1 + \frac{1 - q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}} \Bigg]$$

which is independent of i, j.

Next consider the case $t < s$. Define $\tilde{t}_k^n := \mathbb{1}_{\{Z_{k:n} < t\}}$ and $\tilde{s}_k^n := \mathbb{1}_{\{t \le Z_{k:n} < s\}}$. Using similar arguments we can show that in this case

$$\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t) | Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\Bigg[\sum_{k_1=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2} < t \le Z_{k_1:n-2}\}} \left(1 + \frac{1 - q(t)}{n - k_1}\right) \\
&\qquad \times \prod_{k=1}^{n-2} \left(1 + \frac{1 - q(Z_{k:n-2})}{n-k}\right)^{2\tilde{t}_k^{n-2}} \\
&\qquad \times \prod_{k=k_1}^{n-2} \left(1 + \frac{1 - q(Z_{k:n-2})}{n-k-1}\right)^{\tilde{s}_k^{n-2}} \Bigg]
\end{aligned}$$

which is independent of i, j as well. Thus we have on $\{s \neq t\}$ that $\mathbb{E}\left[B_n(s)B_n(t) | Z_i = s, Z_j = t\right]$ is independent of i, j and hence

$$\mathbb{E}\left[B_n(s)B_n(t) | Z_i = s, Z_j = t\right] = \mathbb{E}\left[B_n(s)B_n(t) | Z_1 = s, Z_2 = t\right] \; .$$

$\square$

Lemma 4.2. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $n \ge 2$ and $1 \le i, j \le n$*

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_i, Z_j) B_n(Z_i) B_n(Z_j)] \\
&= \mathbb{E}[\tilde{\phi}(Z_1, Z_2) B_n(Z_1) B_n(Z_2)] \; .
\end{aligned}$$

Proof. Consider that $\{Z_i = Z_j\}$ is a measure zero set, since H is continuous. Therefore the following holds for $1 \leq i, j \leq n$

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j) B_n(Z_i) B_n(Z_j)\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{Z_i \neq Z_j\}} \tilde{\phi}(Z_i, Z_j) \mathbb{E}\left[B_n(Z_i) B_n(Z_j) | Z_i, Z_j\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{i \neq j\}} \tilde{\phi}(Z_i, Z_j) \mathbb{E}\left[B_n(Z_i) B_n(Z_j) | Z_i, Z_j\right]\right] \\
&= \int_0^\infty \int_0^\infty \mathbb{1}_{\{i \neq j\}} \tilde{\phi}(s, t) \mathbb{E}\left[B_n(s) B_n(t) | Z_i = s, Z_j = t\right] H(ds) H(dt) \ . \qquad (4.2)
\end{aligned}
$$

According to Lemma 4.1 we have for $1 \leq i \neq j \leq n$

$$\mathbb{E}[B_n(s) B_n(t) | Z_i = s, Z_j = t] = \mathbb{E}[B_n(s) B_n(t) | Z_1 = s, Z_2 = t]$$

Therefore we obtain, according to (4.2) that

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j) B_n(Z_i) B_n(Z_j)\right] &= \mathbb{E}\left[\tilde{\phi}(Z_i, Z_j) \mathbb{E}\left[B_n(Z_i) B_n(Z_j) | Z_i, Z_j\right]\right] \\
&= \mathbb{E}\left[\tilde{\phi}(Z_1, Z_2) B_n(Z_1) B_n(Z_2)\right] \ .
\end{aligned}
$$

□

Lemma 4.3. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a measurable function. Then we have for $n \geq 2$*

$$
\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1, Z_2) B_n(Z_1) B_n(Z_2)] \\
&= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\} \mathbb{1}_{\{Z_1 < Z_2\}}] \ .
\end{aligned}
$$

Proof. Note that w.l.o.g. we can assume that the $(Z_i)_{i \leq n}$ are pairwise distinct, since H is continuous. Consider the following

$$B_n(Z_1) B_n(Z_2) = \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < Z_1\}} + \mathbb{1}_{\{Z_k < Z_2\}}}$$

$$
\begin{aligned}
&= \left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right]^{\mathbb{1}_{\{Z_1 < Z_2\}}} \left[1 + \frac{1 - q(Z_2)}{n - R_{2,n}}\right]^{\mathbb{1}_{\{Z_2 < Z_1\}}} \\
&\qquad \times \prod_{k=3}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < Z_1\}} + \mathbb{1}_{\{Z_k < Z_2\}}} \\
&= \mathbb{1}_{\{Z_1 < Z_2\}} \left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right] \\
&\qquad\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2} < Z_1\}} + \mathbb{1}_{\{Z_{k+2} < Z_2\}}} \\
&\quad + \mathbb{1}_{\{Z_1 > Z_2\}} \left[1 + \frac{1 - q(Z_2)}{n - R_{2,n}}\right] \\
&\qquad\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2} < Z_1\}} + \mathbb{1}_{\{Z_{k+2} < Z_2\}}} \\
&\quad + \mathbb{1}_{\{Z_1 = Z_2\}} \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - R_{k+2,n}}\right]^{2\mathbb{1}_{\{Z_{k+2} < Z_1\}}} .
\end{aligned} \tag{4.3}
$$

On $\{Z_1 < Z_2\}$ we have

$$
\begin{aligned}
\prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2} < Z_2\}}} &= \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{\mathbb{1}_{\{Z_{k+2} < Z_1\}}} \\
&\quad \times \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_1 < Z_{k+2} < Z_2\}}}
\end{aligned}
$$

where $\tilde{R}_{k,n-2}$ denotes the rank of the Z_k, $k = 3, \ldots, n$ among themselves. The above holds since

$$
R_{k+2,n} = \begin{cases} \tilde{R}_{k,n-2} & \text{if } Z_{k+2} < Z_1 \\ \tilde{R}_{k,n-2} + 1 & \text{if } Z_1 < Z_{k+2} < Z_2 \end{cases}
$$

for $k = 1, \ldots, n-2$. Therefore (4.3) yields

$$
\begin{aligned}
B_n(Z_1)B_n(Z_2) = \mathbb{1}_{\{Z_1 < Z_2\}} &\left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right] \\
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1 - q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2} < Z_1\}}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_1 < Z_{k+2} < Z_2\}}} \\
&+ \mathbb{1}_{\{Z_2 < Z_1\}} \left[1 + \frac{1-q(Z_2)}{n - R_{2,n}}\right] \\
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2} < Z_2\}}} \\
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_2 < Z_{k+2} < Z_1\}}} \\
&+ \mathbb{1}_{\{Z_1 = Z_2\}} \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2} < Z_1\}}} .
\end{aligned}
\tag{4.4}
$$

Now let's denote $Z_{k:n-2}$ the ordered Z-values among $Z_3, \ldots, Z_n$ for $k = 1, \ldots, n-2$. Consider that we can write

$$
\left[1 + \frac{1-q(Z_1)}{n - R_{1,n}}\right] = \sum_{i=1}^{n-1} \left[1 + \frac{1-q(s)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2} < Z_1 \le Z_{i:n-2}\}} .
$$

Recall that we set $Z_{0:n} = 0$ and $Z_{n-1:n-2} = \infty$. Now note that $Z_{k:n-2}$ is independent of Z_1 and Z_2 for $k = 1, \ldots, n-2$. Therefore we obtain the following, by conditioning (4.4) on Z_1, Z_2:

$$
\begin{aligned}
&\mathbb{E}[B_n(Z_1)B_n(Z_2)|Z_1 = s, Z_2 = t] \\
&= \mathbb{1}_{\{s<t\}} \mathbb{E}\left[\left(\sum_{i=1}^{n-1} \left[1 + \frac{1-q(s)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2} < s \le Z_{i:n-2}\}}\right)\right. \\
&\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2} < s\}}} \\
&\qquad \left. \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s < Z_{k:n-2} < t\}}} \right] \\
&\quad + \mathbb{1}_{\{t<s\}} \mathbb{E}\left[\left(\sum_{i=1}^{n-1} \left[1 + \frac{1-q(t)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2} < t \le Z_{i:n-2}\}}\right)\right. \\
&\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2} < t\}}}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{t<Z_{k:n-2}<s\}}}\Bigg] \\
&+ \mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] \\
&= \alpha(s,t)+\alpha(t,s)+\beta(s,t)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(s,t) := \mathbb{1}_{\{s<t\}}\mathbb{E}\Bigg[&\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}}\right) \\
&\times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}} \\
&\times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}\Bigg]
\end{aligned}
$$

and

$$
\beta(s,t) := \mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] .
$$

Consider that we have

$$
\mathbb{E}[\alpha(Z_1,Z_2)] = \mathbb{E}[\alpha(Z_2,Z_1)] ,
$$

because Z_1 and Z_2 are i. i. d. and α is symmetric in its arguments. Moreover

$$
\mathbb{E}[\beta(Z_1,Z_2)] = 0
$$

since H is continuous. Therefore we get

$$
\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1,Z_2)B_n(Z_1)B_n(Z_2)] \\
&= \mathbb{E}[\tilde{\phi}(Z_1,Z_2)(\alpha(Z_1,Z_2)+\alpha(Z_2,Z_1)+\beta(Z_1,Z_2))] \\
&= \mathbb{E}[2\tilde{\phi}(Z_1,Z_2)\alpha(Z_1,Z_2)] . \qquad (4.5)
\end{aligned}
$$

under (A1). Next consider that

$$\alpha(s,t) = \mathbb{1}_{\{s<t\}}\mathbb{E}\left[(1+C_{n-2}(s))D_{n-2}(s,t)\right]$$
$$= \mathbb{1}_{\{s<t\}}(\Delta_{n-2}(s,t) + \bar{\Delta}_{n-2}(s,t)) \ .$$

The latter equality holds, since

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}} \\
&= \sum_{i=1}^{n-1}\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}} + \sum_{i=1}^{n-1}\left[\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}} \\
&= 1 + C_{n-2}(s) \ .
\end{aligned}$$

Now the statement of the lemma follows directly from (4.5). □

Next recall the following definition for $s<t$ from Chapter 2:

$$D(s,t) := \exp\left(2\int_0^s \frac{1-q(x)}{1-H(x)}H(dx) + \int_s^t \frac{1-q(x)}{1-H(x)}H(dx)\right) \ .$$

The next lemma identifies the almost sure limit of D_n.

Lemma 4.4. *For any $s<t\le T$ s. t. $H(T)<1$, we have*

$$\lim_{n\to\infty} D_n(s,t) = D(s,t)$$

almost surely.

Proof. First define the following quantities for for $s<t$ and $k=1,\ldots,n$

$$x_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)}$$
$$y_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}$$

$$s_k := \mathbb{1}_{\{Z_k < s\}}$$

$$t_k := \mathbb{1}_{\{s < Z_k < t\}} \ .$$

Now consider that we have

$$\begin{aligned}
D_n(s,t) &= \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n(1 - H_n(Z_k) + 2/n)} \mathbb{1}_{\{Z_k < s\}} \right]^2 \\
&\quad \times \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n(1 - H_n(Z_k) + 1/n)} \mathbb{1}_{\{s < Z_k < t\}} \right] \\
&= \prod_{k=1}^{n} [1 + x_k s_k]^2 \prod_{k=1}^{n} [1 + y_k t_k] \\
&= \exp\left(2 \sum_{k=1}^{n} \ln [1 + x_k s_k] + \sum_{k=1}^{n} \ln [1 + y_k t_k] \right) \ .
\end{aligned}$$

Note that $0 \leq x_k s_k \leq 1$ and $0 \leq y_k t_k \leq 1$. Consider that the following inequality holds

$$-\frac{x^2}{2} \leq \ln(1+x) - x \leq 0$$

for any $x \geq 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$-\frac{1}{2} \sum_{k=1}^{n} x_k^2 s_k \leq \sum_{k=1}^{n} \ln(1 + x_k s_k) - \sum_{k=1}^{n} x_k s_k \leq 0 \ .$$

But now

$$\begin{aligned}
\sum_{k=1}^{n} x_k^2 s_k &= \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1 - q(Z_k)}{1 - H_n(Z_k) + \frac{2}{n}} \right)^2 \mathbb{1}_{\{Z_k < s\}} \\
&\leq \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1}{1 - H_n(s) + \frac{1}{n}} \right)^2 \\
&= \frac{1}{n(1 - H_n(s) + n^{-1})^2} \longrightarrow 0
\end{aligned}$$

almost surely as $n \to \infty$, since $H(s) < H(t) < 1$ (c.f. Stute and Wang (1993), p.

1603). Therefore we have

$$|\sum_{k=1}^{n} \ln(1+x_k s_k) - \sum_{k=1}^{n} x_k s_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$|\sum_{k=1}^{n} \ln(1+y_k t_k) - \sum_{k=1}^{n} y_k t_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s,t) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k\right) .$$

Now consider the following

$$\begin{aligned}
\sum_{k=1}^{n} x_k s_k &= \frac{1}{n}\sum_{k=1}^{n} \frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}} \mathbb{1}_{\{Z_k<s\}} \\
&= \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} - \frac{1-q(z)}{1-H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz) .
\end{aligned} \tag{4.6}$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$\begin{aligned}
&\left|\int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz)\right| \\
&\leq \frac{\sup_{z\leq T}|H_n(z)-H(z)-\frac{2}{n}|}{1-H(T)} \int_0^{T-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned} \tag{4.7}$$

almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(T) < 1$.

Moreover we have

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^s \frac{1-q(z)}{1-H(z)} H(dz)$$

almost surely by the strong law of large numbers. Therefore we obtain

$$\lim_{n\to\infty} \sum_{k=1}^n x_k s_k = \int_0^s \frac{1-q(z)}{1-H(z)} H(dz) \; .$$

By the same arguments, we can show that

$$\lim_{n\to\infty} \sum_{k=1}^n y_k t_k = \int_s^t \frac{1-q(z)}{1-H(z)} H(dz) \; .$$

almost surely. Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz) \right)$$

almost surely. □

Lemma 4.5. $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ *is a non-negative reverse supermartingale.*

Proof. Consider that for $s < t$ and $n \geq 1$, we have

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] &= \mathbb{E}\left[\prod_{k=1}^n \left(1 + \frac{1-q(Z_{k:n})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \right. \\
&\qquad \left. \times \prod_{k=1}^n \left(1 + \frac{1-q(Z_{k:n})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} |\mathcal{F}_{n+1} \right] \\
&= \sum_{i=1}^{n+1} \mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}} \prod_{k=1}^n \ldots |\mathcal{F}_{n+1} \right] \\
&= \sum_{i=1}^{n+1} \mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}} \prod_{k=1}^{i-1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \right. \\
&\qquad \times \prod_{k=i}^{n} \left(1 + \frac{1-q(Z_{k+1:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+1}<s\}}}
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\
&\times \prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+1}<t\}}} |\mathcal{F}_{n+1}\Bigg] \\
= \sum_{i=1}^{n+1}\mathbb{E}\Bigg[&\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
&\times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
&\times \prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\
&\times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} |\mathcal{F}_{n+1}\Bigg] \; .
\end{aligned}
$$

Note that each product within the conditional expectation is measurable w. r. t. $\mathcal{F}_{n+1}$. Moreover we have for $i = 1, \dots, n$

$$
\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}|\mathcal{F}_{n+1}] &= \mathbb{P}(Z_{n+1} = Z_{i:n+1}|\mathcal{F}_{n+1}) \\
&= \mathbb{P}(R_{n+1,n+1} = i) \\
&= \frac{1}{n+1} \; .
\end{aligned}
$$

Therefore we obtain the following

$$
\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] = \frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}&\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
\times &\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\
\times \prod_{k=i+1}^{n+1}&\left(1+\frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
\times &\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \; .
\end{aligned}
\tag{4.8}
$$

We will now proceed by induction on n. First let

$$x_k := 1 - q(Z_{k:2}),\ s_k := \mathbb{1}_{\{Z_{k:2}<s\}} \text{ and } t_k := \mathbb{1}_{\{s<Z_{k:2}<t\}}$$

for $k = 1, 2$. Note that that x_k and y_k are different, compared to the corresponding definitions in lemma 4.4, as they involve the ordered Z-values here. Next consider

$$\begin{aligned}\mathbb{E}[D_1(s,t)|\mathcal{F}_2] = \frac{1}{2}\Bigg[&\left(1 + \frac{1-q(Z_{2:2})}{2}\right)^{2\mathbb{1}_{\{Z_{2:2}<s\}}} \times (1 + (1 - q(Z_{2:2})))^{\mathbb{1}_{\{s<Z_{2:2}<t\}}} \\ &+ \left(1 + \frac{1-q(Z_{1:2})}{2}\right)^{2\mathbb{1}_{\{Z_{1:2}<s\}}} \times (1 + (1 - q(Z_{1:2})))^{\mathbb{1}_{\{s<Z_{1:2}<t\}}}\Bigg] \\ = \frac{1}{2}\Bigg[&\left(1 + \frac{x_2}{2}s_2\right)^2 \times (1 + x_2 t_2) + \left(1 + \frac{x_1}{2}s_1\right)^2 \times (1 + x_1 t_1)\Bigg] \ .\end{aligned}$$

Moreover we have

$$\begin{aligned}D_2(s,t) &= \prod_{k=1}^{2}\left[1 + \frac{1-q(Z_{k:2})}{4-k}\right]^{2\mathbb{1}_{\{Z_{k:2}<s\}}} \prod_{k=1}^{2}\left[1 + \frac{1-q(Z_{k:2})}{3-k}\right]^{\mathbb{1}_{\{s<Z_{k:2}<t\}}} \\ &= \left[1 + \frac{x_1}{3}s_1\right]^2 \times \left[1 + \frac{x_1}{2}t_1\right] \times \left[1 + \frac{x_2}{2}s_2\right]^2 \times [1 + x_2 t_2] \\ &= \left[1 + \frac{x_1}{2}t_1 + \left(\frac{x_1^2}{9} + \frac{2}{3}x_1\right)s_1\right] \times \left[1 + x_2 t_2 + \left(\frac{x_2^2}{4} + x_2\right)s_2\right] \ .\end{aligned}$$

Therefore we obtain

$$\mathbb{E}[D_1(s,t)|\mathcal{F}_2] - D_2(s,t) \leq \frac{x_1^2}{72} - \frac{x_1}{6} \leq 0 \ ,$$

since $0 \leq x_1 \leq 1$. Thus $\mathbb{E}[D_1(s,t)|\mathcal{F}_2] \leq D_2(s,t)$ for any $s < t$, as needed. Now assume that

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] \leq D_{n+1}(s,t)$$

holds for any $n \geq 1$. Note that the latter is equivalent to assuming

$$\begin{aligned}
&\frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+1}\right)^{\mathbb{1}_{\{s<y_k<t\}}}\\
&\qquad\times\prod_{k=i+1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}}\\
&\leq\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}}
\end{aligned}\tag{4.9}$$

holds for arbitrary $y_k \geq 0$. Next define

$$A_{n+2}(s,t) := \prod_{k=2}^{n+2}\left[1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right]^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}}\times\left[1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right]^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}$$

for $s < t$ and $n \geq 1$. According to (4.8), we have

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}]\\
&=\frac{1}{n+2}\sum_{i=1}^{n+2}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}\\
&\qquad\times\prod_{k=i+1}^{n+2}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}\\
&=\frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=2}^{n+2}\prod_{k=1}^{i-1}\cdots\times\prod_{k=i+1}^{n+2}\cdots\\
&=\frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=1}^{n+1}\prod_{k=1}^{i}\cdots\times\prod_{k=i+2}^{n+2}\cdots\\
&=\frac{A_{n+2}}{n+2}+\frac{1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\\
&\qquad\times\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}\\
&\qquad\times\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}}\\
&\qquad\times\prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}
\end{aligned}$$

$$\times \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}} .$$

Using (4.9) on the right hand side of the equation above yields

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&\leq \frac{A_{n+2}}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}} \left(1 + \frac{1 - q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}} \\
&\qquad \times \prod_{k=1}^{n+1}\left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}} \\
&\qquad\qquad \times \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}} \\
&= A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\right. \\
&\qquad\qquad \left. \times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\right] .
\end{aligned}$$

For the moment, let

$$x_1 := 1 - q(Z_{1:n+2}),\ s_1 := \mathbb{1}_{\{Z_{1:n+2}<s\}} \text{ and } t_1 := \mathbb{1}_{\{s<Z_{1:n+2}<t\}} .$$

Now we can rewrite the conditional expectation above as

$$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+1}\right)\right] . \quad (4.10)$$

Next consider that we have

$$\begin{aligned}
\left(1 + \frac{x_1 t_1}{n+1}\right) &= \left(1 + \frac{x_1 t_1}{n+2} - \frac{1}{n+2}\right)\left(1 + \frac{1}{n+1}\right) \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{1}{n+1}\left(1 + \frac{x_1 t_1}{(n+2)}\right) - \frac{1}{n+1} \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{x_1 t_1}{(n+1)(n+2)} .
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
&\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+1}\right) \\
&= \frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+2}\right) + \left(1+\frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2}\ .
\end{aligned}
$$

But now

$$
\begin{aligned}
\left(1+\frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2} &= \left(1+2\frac{x_1 s_1}{n+2}+\frac{x_1^2 s_1}{(n+2)^2}\right)\frac{x_1 t_1}{(n+2)^2} \\
&= \frac{x_1 t_1}{(n+2)^2}
\end{aligned}
$$

since $s_1 \cdot t_1 = 0$ for all $s < t$. Hence we can rewrite the term in brackets in (4.10) as

$$
\begin{aligned}
&\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+1}\right) \\
&= \frac{1}{n+2}+\frac{x_1 t_1}{(n+2)^2}+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+2}\right) \\
&= \frac{1}{n+2}\left(1+\frac{x_1 t_1}{n+2}\right)+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+2}\right) \\
&= \left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x_1}{n+2}\right)^{2s_1}\right]\left(1+\frac{x_1}{n+2}\right)^{t_1} \\
&\leq \left(1+\frac{x_1}{n+3}\right)^{2s_1}\left(1+\frac{x_1}{n+2}\right)^{t_1}\ .
\end{aligned}
$$

The latter inequality above holds, since

$$
\left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x}{n+2}\right)^2\right] \leq \left(1+\frac{x}{n+3}\right)^2
$$

for any $0 \leq x \leq 1$. (c. f. Bose and Sen (1999), page 197). Therefore we can rewrite (4.10) as

$$
\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+3}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}
$$

$$\times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}$$

$$= D_{n+2}(s,t) \ .$$

This concludes the proof. $\square$

Lemma 4.6. *Let $s < t$ s. t. $H(t) < 1$. Then $\Delta_n(s,t) \nearrow D(s,t)$.*

Proof. Consider that we have for $n \geq 2$

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty]$$

by definition of $\Delta_n(s,t)$ and Lemma 3.4. Next note that we have $D_n(s,t) \to D(s,t)$ almost surely, according to Lemma 4.4. Moreover we get from Lemma 4.5 that $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ is a reverse supermartingale. Now this together with Proposition V-3-11 of Neveu (1975) yields

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t) \ .$$

$\square$

We will now proceed to find an explicit representation for $\mathbb{E}[S_n]$ in terms of the reverse supermartingale D_n, in order to identify the limit $S(q)$. Consider the following lemma.

Lemma 4.7. *For continuous $H(\cdot)$, we have*

$$\mathbb{E}[S_n(q)] = \frac{n-1}{n}\mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \ .$$

Proof. Consider the following

$$\begin{aligned}
\mathbb{E}[S_n(q)] &= \sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})\frac{q(Z_{i:n})}{n-i+1)}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]\right.\\
&\qquad\left.\times\frac{q(Z_{j:n})}{n-j+1}\prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]\right]\\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k+1}\right]\right.\\
&\qquad\left.\times q(Z_{j:n})\prod_{l=1}^{j-1}\left[1+\frac{1-q(Z_{l:n})}{n-l+1}\right]\right]\\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})q(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]\\
&= \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\mathbb{1}_{\{i\neq j\}}\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})q(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]\\
&= \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\mathbb{1}_{\{i\neq j\}}\phi(Z_i, Z_j)q(Z_i)q(Z_j)B_n(Z_i)B_n(Z_j)\right] . \qquad (4.11)
\end{aligned}$$

According to Lemma 4.2 we obtain

$$\mathbb{E}[S_n(q)] = \frac{n-1}{2n}\mathbb{E}\left[\phi(Z_1, Z_2)q(Z_1)q(Z_2)B_n(Z_1)B_n(Z_2)\right] .$$

Now, since ϕ and q are measurable, we can apply Lemma 4.3 to obtain the result. □

The result of the following lemma will be extended to uniform convergence in Lemma 4.10.

Lemma 4.8. *For continuous H and $t \le T < \tau_H$, we have $C_n(t) \to 0$ as $n \to \infty$ w. p. 1, and $C_n(t) \in [0, 1]$ for all $n \ge 1$ and $t \ge 0$.*

Proof. It is easy to see that $0 \le C_n(t) \le 1$ for any $t \ge 0$ and $n \ge 2$, since $0 \le q(t) \le 1$ and $\mathbb{1}_{\{Z_{i-1:n}<t\le Z_{i:n}\}} = 1$ for exactly one $i \in \{1, \dots, n+1\}$. Let's now consider

$$C_n(t) = \sum_{i=1}^{n+1}\frac{1-q(t)}{n-i+2}[\mathbb{1}_{\{Z_{i-1:n}<t\}} - \mathbb{1}_{\{Z_{i:n}<t\}}]$$

$$
\begin{aligned}
&= \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2} \mathbb{1}_{\{Z_{i-1:n}<t\}} - \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2} \mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=0}^{n} \frac{1-q(t)}{n-i+1} \mathbb{1}_{\{Z_{i:n}<t\}} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2} \mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=1}^{n} \frac{1-q(t)}{n-i+1} \mathbb{1}_{\{Z_{i:n}<t\}} + \frac{(1-q(t))}{n+1} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2} \mathbb{1}_{\{Z_{i:n}<t\}} \\
&= (1-q(t)) \left\{ \frac{1}{n+1} + \sum_{i=1}^{n} \left[\frac{1}{n-i+1} - \frac{1}{n-i+2} \right] \mathbb{1}_{\{Z_{i:n}<t\}} \right\} \\
&= (1-q(t)) \sum_{i=1}^{n} \left[\frac{1}{n-nH_n(Z_{i:n})+1} \frac{1}{n-nH_n(Z_{i:n})+2} \right] \mathbb{1}_{\{Z_{i:n}<t\}} \\
&\quad + \frac{1-q(t)}{n+1} \\
&= (1-q(t)) \int_0^t \left[\frac{1}{1-H_n(x)+\frac{1}{n}} - \frac{1}{1-H_n(x)+\frac{2}{n}} \right] H_n(dx) \\
&\quad + \frac{1-q(t)}{n+1} \ . \qquad (4.12)
\end{aligned}
$$

In Lemma 4.4 we have seen that

$$
\int_0^t \frac{1}{1-H_n(x)+\frac{2}{n}} H_n(dx) \to \int_0^t \frac{1}{1-H(x)} H(dx) \ .
$$

By the same arguments we obtain

$$
\int_0^t \frac{1}{1-H_n(x)+\frac{1}{n}} H_n(dx) \to \int_0^t \frac{1}{1-H(x)} H(dx) \ .
$$

Therefore the right hand side of (4.12) converges to zero. □

The following lemma contains an integration by parts result, which will be useful in order to prove Lemma 4.10. Recall the following quantities from chapter 2:

$$
H^1(x) = \int_0^x m(z, \theta_0) H(dz)
$$

and

$$H_n^1(x) = \int_0^x m(z,\theta_0) H_n(dz) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{Z_{i:n}\le x\}} m(Z_{i:n},\theta_0)\ ,$$

c. f. Dikta (1998), Lemma 3.12.

Lemma 4.9. *For any $0 \le s < t \le T$ we have*

$$\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^t \frac{1}{1-H(z)} H(dz) \\
&= \frac{H_n(t)-H(t)}{1-H(t)} - \frac{H_n(s-)-H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2} H(dz) - \gamma_n(t)
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) \\
&= \frac{H_n^1(t)-H^1(t)}{1-H(t)} - \frac{H_n^1(s-)-H^1(s)}{1-H(s)} - \int_s^t \frac{H_n^1(z-)-H^1(z)}{(1-H(z))^2} H(dz) - \gamma_n^1(t)\ ,
\end{aligned} \tag{4.14}$$

where

$$\gamma_n(t) = \frac{H_n(t)-H_n(t-)}{1-H(t)} \quad \text{and} \quad \gamma_n^1(t) = \frac{H_n^1(t)-H_n^1(t-)}{1-H(t)}\ .$$

Proof. First consider that we can write

$$\int_s^t \frac{1}{1-H(z)} H_n(dz) = \int_s^{t-} \frac{1}{1-H(z)} H_n(dz) + \gamma_n(s)\ .$$

Thus we have

$$\begin{aligned}
\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) &= \int_s^t \frac{1}{1-H(z)} H_n(dz) - \gamma_n(s) \\
&= \int_s^t \left(\frac{1}{1-H(z)} - 1\right) H_n(dz) + \int_s^t 1 H_n(dz) - \gamma_n(s) \\
&= \int_s^t \frac{H(z)}{1-H(z)} H_n(dz) + H_n(t) - H_n(s-) - \gamma_n(s)\ ,
\end{aligned}$$

since the following statement holds

$$\int_s^t 1 H_n(dz) = \int_0^t 1 H_n(dz) - \int_0^{s-} 1 H_n(dz) = H_n(t) - H_n(s-) \ .$$

We will now use a version of integration by parts (see Cohn (2013), p. 164) to show

$$\begin{aligned}
&\int_s^t \frac{H(z)}{1-H(z)} H_n(dz) + H_n(t) - H_n(s-) \\
&= \frac{H_n(t)}{1-H(t)} - \frac{H_n(s-)}{1-H(s)} - \int_s^t \frac{H_n(z)}{(1-H(z))^2} H(dz)
\end{aligned}$$

First let's define $\tilde{G}(x) := H_n(x)$ and

$$\tilde{F}(x) := \frac{H(x)}{1-H(x)} \ .$$

Moreover denote $\mu_{\tilde{F}}$ and $\mu_{\tilde{G}}$ the measures induced by $\tilde{F}$ and $\tilde{G}$ respectively. Note that we have

$$\mu_{\tilde{F}}(]s,t]) = \tilde{F}(t) - \tilde{F}(s) \ . \tag{4.15}$$

Next consider that we can write

$$\tilde{F}(x) = \int_0^x \frac{1}{(1-H(z))^2} H(dz) \ ,$$

since we have

$$\begin{aligned}
\int_0^x \frac{1}{(1-H(z))^2} H(dz) &= \int_0^{H(x)} \frac{1}{(1-u)^2} du \\
&= \int_0^{H(x)} \frac{1}{(1-u)^2} du \\
&= \frac{1}{1-H(x)} - 1 \\
&= \frac{H(x)}{1-H(x)} \ .
\end{aligned}$$

Now combining the above with (4.15) yields

$$\mu_{\tilde{F}}(]s,t]) = \tilde{F}(t) - \tilde{F}(s) = \int_s^t \frac{1}{(1-H(z))^2} H(dz) \ .$$

Therefore the Radon Nikodym derivative of $\mu_{\tilde{F}}$ w. r. t. H is given by

$$\frac{\mu_{\tilde{F}}(dx)}{H(dx)} = \frac{1}{(1-H(x))^2} \ . \tag{4.16}$$

Note that $\tilde{F}$ and $\tilde{G}$ are bounded, right-continuous and vanish at $-\infty$. Thus we can apply Cohn (2013), p. 164, to obtain

$$\int_s^t \tilde{F}(z)\mu_{\tilde{G}}(dz) = \tilde{F}(t)\tilde{G}(t) - \tilde{F}(s-)\tilde{G}(s-) - \int_s^t \tilde{G}(z-)\mu_{\tilde{F}}(dz) \ .$$

Now we get by (4.16) and by definition of $\tilde{F}$ and $\tilde{G}$ that

$$\begin{aligned}\int_0^s \frac{H(z)}{1-H(z)} H_n(dz) &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_s^t H_n(z-)\mu_{\tilde{F}}(dz) \\ &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-)}{(1-H(z))^2} H(dz) \ .\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) &= \int_s^t \frac{H(z)}{1-H(z)} H_n(dz) + H_n(t) - H_n(s-) - \gamma_n(s) \\ &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_0^s \frac{H_n(z-)}{(1-H(z))^2} H(dz) \\ &\quad + H_n(t) - H_n(s-) - \gamma_n(s) \\ &= \frac{H_n(t)}{1-H(t)} - \frac{H_n(s-)}{1-H(s)} - \int_0^s \frac{H_n(z-)}{(1-H(z))^2} H(dz) \\ &\quad - \gamma_n(s) \ . \end{aligned} \tag{4.17}$$

The latter equality holds, since

$$\frac{H_n(t)H(t)}{1-H(t)} + H_n(t) = \frac{H_n(t)}{1-H(t)}$$

and

$$\frac{H_n(s-)H(s)}{1-H(s)} + H_n(s-) = \frac{H_n(s-)}{1-H(s)} \ .$$

Now consider the following

$$\int_s^t \frac{1}{1-H(z)} H(dz) = \int_s^t \frac{H(z)}{1-H(z)} H(dz) + H(t) - H(s) \ .$$

Define $\bar{G}(x) := H(x)$ and note that $\bar{G}(x)$ is bounded, right-continuous and vanishes at $-\infty$. Therefore applying Cohn (2013), p. 164, to $\tilde{F}$ and $\bar{G}$ yields

$$\int_s^t \frac{H(z)}{1-H(z)} H(dz) = \frac{H^2(t)}{1-H(t)} - \frac{H^2(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz) \ .$$

Hence we have

$$\begin{aligned}\int_s^t \frac{1}{1-H(z)} H(dz) &= \frac{H^2(t)}{1-H(t)} - \frac{H^2(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz) \\ &\quad + H(t) - H(s) \\ &= \frac{H(t)}{1-H(t)} - \frac{H(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz) \ . \end{aligned} \tag{4.18}$$

Now combining (4.17) and (4.18) yields

$$\begin{aligned}&\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^t \frac{1}{1-H(z)} H(dz) \\ &= \frac{H_n(t)-H(t)}{1-H(t)} - \frac{H_n(s-)-H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-)-H(z)}{1-H(z)} H(dz) - \gamma_n(t) \ .\end{aligned}$$

Thus equation (4.13) from the statement of the lemma has been established. Next define $\tilde{G}^1(x) := H_n^1(x)$ and apply Cohn (2013), p. 164, to $\tilde{F}$ and $\tilde{G}^1$ to obtain

$$\int_s^t \frac{H(z)}{1-H(z)} H_n^1(dz) = \frac{H_n^1(t)H(t)}{1-H(t)} - \frac{H_n^1(s-)H(s)}{1-H(s)} - \int_s^t \frac{H_n^1(z)}{(1-H(z))^2} H(dz) \quad (4.19)$$

Next define $\bar{G}^1(x) := H^1(x)$ and apply Cohn (2013), p. 164, to $\tilde{F}$ and $\bar{G}^1$ to obtain

$$\int_s^t \frac{H(z)}{1-H(z)} H^1(dz) = \frac{H^1(t)H(t)}{1-H(t)} - \frac{H^1(s-)H(s)}{1-H(s)} - \int_s^t \frac{H^1(z)}{(1-H(z))^2} H(dz) \quad (4.20)$$

Finally consider the following

$$\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) \\
&= \int_s^t \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) - \gamma_n^1(t) \\
&= \int_s^t \frac{H(z)}{1-H(z)} H_n^1(dz) + H_n^1(t) - H_n^1(s-) \\
&\quad - \int_s^t \frac{1}{1-H(z)} H(dz) + H^1(t) - H^1(s-) - \gamma_n^1(t)\ .
\end{aligned}$$

Now combining the above with equations (4.19) and (4.20) yields the second part of the lemma. □

The lemma below contains a statement about uniform convergence of processes considered in the proof of Lemma 4.4. It will be used to establish Corollary 4.11.

Lemma 4.10. *The following holds for any $T < \tau_H$.*

$$\sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_s^t \frac{1-m(z,\theta_0)}{1-H(z)} H(dz) \right| \to 0$$

almost surely as $n \to \infty$.

Proof. First consider the following

$$\sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_s^t \frac{1-m(z,\theta_0)}{1-H(z)} H(dz) \right|$$

$$
\begin{aligned}
&= \sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^{t-} \frac{1}{1-H(z)} H(dz) \right. \\
&\qquad \left. + \int_s^{t-} \frac{m(z,\theta_0)}{1-H(z)} H(dz) - \int_s^{t-} \frac{m(z,\theta_0)}{1-H(z)} H_n(dz) \right| \\
&= \sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^{t-} \frac{1}{1-H(z)} H(dz) \right. \\
&\qquad \left. + \int_s^{t-} \frac{1}{1-H(z)} H^1(dz) - \int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) \right| \\
&\le \sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^{t-} \frac{1}{1-H(z)} H(dz) \right| \\
&\quad + \sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1}{1-H(z)} H^1(dz) - \int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) \right| . \qquad (4.21)
\end{aligned}
$$

Applying Lemma 4.9 equation (4.13) to the first term above yields

$$
\begin{aligned}
&\sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^{t-} \frac{1}{1-H(z)} H(dz) \right| \\
&= \sup_{0\le s<t\le T} \left| \frac{H_n(t)-H(t)}{1-H(t)} - \frac{H_n(s-)-H(s)}{1-H(s)} \right. \\
&\qquad \left. - \int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2} H(dz) - \frac{H_n(t-)-H_n(t)}{1-H(t)} \right| \\
&\le \sup_{0\le s<t\le T} \left| \frac{H_n(t)-H(t)}{1-H(t)} \right| + \sup_{0\le s<t\le T} \left| \frac{H_n(s-)-H(s)}{1-H(s)} \right| \\
&\quad + \sup_{0\le s<t\le T} \left| \int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2} H(dz) \right| + \sup_{0\le s<t\le T} \left| \frac{H_n(t-)-H_n(t)}{1-H(t)} \right| .
\end{aligned}
$$

Next consider that we have

$$
\sup_{0\le s<t\le T} \left| \frac{H_n(t)-H(t)}{1-H(t)} \right| \le \frac{\sup_{x\le T} |H_n(x)-H(x)|}{1-H(T)}
$$

and

$$
\sup_{0\le s<t\le T} \left| \frac{H_n(s-)-H(s)}{1-H(s)} \right| \le \frac{\sup_{x\le T} |H_n(x)-H(x)| + \frac{1}{n}}{1-H(T)} .
$$

Furthermore consider that the following holds

$$\begin{aligned}\sup_{0\le s<t\le T}\left|\int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| &\le \sup_{0\le s<t\le T}\left|\int_0^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \\ &\quad + \sup_{0\le s<t\le T}\left|\int_0^s \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \\ &\le 2\cdot \frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{(1-H(T))^2} \ .\end{aligned}$$

The latter inequality holds, since we have for any $t\le T$

$$\left|\int_0^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \le \int_0^t \frac{|H_n(z-)-H(z)|}{(1-H(T))^2}H(dz) \le \frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{(1-H(T))^2} \ ,$$

using Jensen's inequality. Moreover note that $H_n(s)-H_n(s-)\le n^{-1}$ for any $0\le s\le T$ and hence

$$\sup_{0\le s<t\le T}\left|\frac{H_n(s-)-H_n(s)}{1-H(s)}\right| \le \frac{1}{n(1-H(T))} \ .$$

Therefore we obtain

$$\begin{aligned}&\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right| \\ &\le \frac{\sup_{x\le T}|H_n(x)-H(x)|}{1-H(T)}+\frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{1-H(T)} \\ &\qquad + 2\cdot\frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{(1-H(T))^2}+\frac{1}{n(1-H(T))} \\ &\to 0\end{aligned}$$

almost surely as $n\to\infty$ by the Glivenko-Cantelli Theorem and since $H(T)<1$. Now let's consider the latter term in (4.21). Applying Lemma 4.9 equation (4.14) yields

$$
\begin{aligned}
&\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n^1(dz)-\int_s^{t-}\frac{1}{1-H(z)}H^1(dz)\right| \\
&=\sup_{0\le s<t\le T}\left|\frac{H_n^1(t)-H^1(t)}{1-H(t)}-\frac{H_n^1(s-)-H^1(s)}{1-H(s)}\right. \\
&\qquad\qquad\left.-\int_s^t\frac{H_n^1(z-)-H^1(z)}{(1-H(z))^2}H(dz)-\frac{H_n^1(t-)-H_n^1(t)}{1-H(t)}\right| \\
&\le \sup_{0\le s<t\le T}\left|\frac{H_n^1(t)-H^1(t)}{1-H(t)}\right|+\sup_{0\le s<t\le T}\left|\frac{H_n^1(s-)-H^1(s)}{1-H(s)}\right| \\
&\quad+\sup_{0\le s<t\le T}\left|\int_s^t\frac{H_n^1(z-)-H^1(z)}{(1-H(z))^2}H(dz)\right|+\sup_{0\le s<t\le T}\left|\frac{H_n^1(t-)-H_n^1(t)}{1-H(t)}\right| \\
&\le \frac{\sup\limits_{x\le T}|H_n^1(x)-H^1(x)|}{1-H(T)}+\frac{\sup\limits_{x\le T}|H_n^1(x)-H^1(x)|+\frac{1}{n}}{1-H(T)} \\
&\quad+2\cdot\frac{\sup\limits_{x\le T}|H_n^1(x)-H^1(x)|+\frac{1}{n}}{(1-H(T))^2}+\frac{1}{n(1-H(T))} \\
&\to 0
\end{aligned}
$$

almost surely as $n\to\infty$ by the Glivenko Cantelli Theorem and since $H(T)<1$. □

The following Corollary is important for the proof of Theorem 1.4.

Corollary 4.11. *The measure zero sets* $\{\omega|C_n(s,m;\omega)\nrightarrow C(s,m)$ *as* $n\to\infty\}$ *and* $\{\omega|D_n(s,t,m;\omega)\nrightarrow D(s,t,m)$ *as* $n\to\infty\}$ *are independent of* s *and* t.

Proof. In Lemma 4.4 we have seen that $D_n(s,t,q)$ converges almost surely to $D(s,t,q)$ by Glivenko Cantelli and the SLLN. In order to establish the statement of the corollary, we need to show that this convergence is uniform in s and t. Let $q\equiv m(\cdot,\theta_0)$ and recall from the proof of Lemma 4.4 that we have

$$
\begin{aligned}
&\left|\int_0^{s-}\frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))}H_n(dz)\right| \\
&\le\frac{\sup_{z\le T}|H_n(z)-H(z)-\frac{2}{n}|}{1-H(T)}\int_0^{T-}\frac{1}{1-H_n(z)}H_n(dz)\longrightarrow 0
\end{aligned}
$$

almost surely as $n \to \infty$. Note that the right hand side above converges to zero independent of s and t. Next recall that

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^s \frac{1-q(z)}{1-H(z)} H(dz) \tag{4.22}$$

by the SLLN. Note that this means pointwise convergence. But according to Lemma 4.10 we also have

$$\sup_{0\le s\le T} \left| \int_0^{s-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_0^s \frac{1-m(z,\theta_0)}{1-H(z)} H(dz) \right| \to 0$$

almost surely as $n \to \infty$. Thus we can show that the convergence in (4.22) is indeed uniform in s and t. For the last part of the proof, we need

$$\sup_{0\le s<t\le T} \left| \int_s^{t-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_s^t \frac{1-m(z,\theta_0)}{1-H(z)} H(dz) \right| \to 0$$

almost surely as $n \to \infty$, which is provided by Lemma 4.10 as well. Hence $D_n(s,t,m) \to D(s,t,m)$ almost surely, uniformly in s and t as $n \to \infty$. By similar arguments we get that $C_n(s,m) \to C(s,m)$ almost surely, uniformly in s as $n \to \infty$, considering the proof of Lemma 4.8. $\square$

We will now identify the almost sure limits of $S_n(q)$ and $\bar{S}_n(q)$ in Lemma 4.12. Recall the following definitions from Chapter 2:

$$\bar{S}_n(q) := \sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) \bar{W}_{i,n}(q) \bar{W}_{j,n}(q)$$

where

$$\bar{W}_{i,n}(q) := \prod_{k=1}^{n} \left(1 - \frac{q(Z_{k:n})}{n-k+1}\right) .$$

Furthermore recall that we set

$$S(q) := \frac{1}{2}\int_0^\infty\int_0^\infty \phi(s,t)q(s)q(t)\exp\left(\int_0^s \frac{1-q(x)}{1-H(x)}H(dx)\right) \times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)}H(dx)\right)H(ds)H(dt)$$

and

$$\bar{S}(q) := \frac{1}{2}\int_0^\infty\int_0^\infty \phi(s,t)\exp\left(\int_0^s \frac{1-q(x)}{1-H(x)}H(dx)\right) \times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)}H(dx)\right)H(ds)H(dt)\ .$$

Lemma 4.12. *Let H be continuous and let $q(z)$ be non-decreasing for all $z \in \mathbb{R}_+$. Then the following statements hold true:*

$$\lim_{n\to\infty} S_n(q) = S(q)$$

and

$$\lim_{n\to\infty} \bar{S}_n(q) = \bar{S}(q)$$

with probability one, if the limit on the right hand side exists.

Proof. Suppose H is continuous and q is monotone non-decreasing. First consider that S_n converges almost surely to some limit S_∞ and we have

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \mathbb{E}[S_n]\ ,$$

according to Theorem 3.5. Next consider that we have

$$\begin{aligned}\mathbb{E}[S_n(q)] &= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1,Z_2)+\bar{\Delta}_{n-2}(Z_1,Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \\ &= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}]\end{aligned}$$

$$+\frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\bar{\Delta}_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \tag{4.23}$$

by Lemma 4.7. We will now consider the two terms on the right hand side above individually, starting with the second term above. Consider that for $s<t$

$$\lim_{n\to\infty} C_n(s)D_n(s,t) \le \lim_{n\to\infty} C_n(s)D(s,t) = 0$$

almost surely as $n \to \infty$, since $0 \le C_n(s) \le 1$ and by Corollary 4.11. Also $C_n(s)D_n(s,t) \ge 0$ for all $n \ge 2$ and $s<t$. Thus $C_n(s)D_n(s,t) \to 0$ almost surely as $n \to \infty$ if $s<t$. Furthermore note that $C_n(s)D_n(s,t) \le D(s,t)$ almost surely, for all $n \ge 2$ and $s<t$ by Lemma 4.6. Moreover note that $D(s,t)$ is integrable, since on $\{Z_1 < Z_2\}$ we have

$$\begin{aligned}
\mathbb{E}[D(Z_1,Z_2)] &= \mathbb{E}\left[\int_0^{Z_1}\frac{1-q(x)}{1-H(x)}H(dx) + \int_0^{Z_2}\frac{1-q(x)}{1-H(x)}H(dx)\right] \\
&\le \mathbb{E}\left[\int_0^{Z_{n:n}}\frac{1}{1-H(x)}H(dx) + \int_0^{Z_{n:n}}\frac{1}{1-H(x)}H(dx)\right] \\
&\le \mathbb{E}\left[-2\ln(1-H(Z_{n:n}))\right] \\
&< \infty \,.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{1}_{\{Z_1<Z_2\}}\bar{\Delta}_{n-2}(Z_1,Z_2) &= \lim_{n\to\infty}\mathbb{1}_{\{Z_1<Z_2\}}\mathbb{E}\left[C_{n-2}(Z_1)D_{n-2}(Z_1,Z_2)\right] \\
&= \mathbb{1}_{\{Z_1<Z_2\}}\mathbb{E}\left[\lim_{n\to\infty} C_n(Z_1)D_n(Z_1,Z_2)\right] \\
&= 0
\end{aligned}$$

according to the Dominated Convergence Theorem. Thus

$$\phi(Z_1,Z_2)q(Z_1)q(Z_2)\mathbb{1}_{\{Z_1<Z_2\}}\bar{\Delta}_{n-2}(Z_1,Z_2) \to 0$$

almost surely as $n \to \infty$. Furthermore note that we have

$$\bar{\Delta}_{n-2}(Z_1, Z_2) \leq \Delta_{n-2}(Z_1, Z_2) \leq D(Z_1, Z_2)$$

almost surely for all $n \geq 2$ by Lemma 4.6. Hence we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\mathbb{1}_{\{Z_1<Z_2\}}\bar{\Delta}_{n-2}(Z_1, Z_2)] \\
&= \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\mathbb{1}_{\{Z_1<Z_2\}} \lim_{n\to\infty} \bar{\Delta}_{n-2}(Z_1, Z_2)] \\
&= 0
\end{aligned}$$

almost surely, by virtue of the Dominated Convergence Theorem. It remains to consider the first term in (4.23). According to Lemma 4.6, we have $\Delta_n(s,t) \nearrow D(s,t)$ for $s < t$ and $H(t) < 1$. Thus, applying the Dominated Convergence Theorem again, yields

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)D(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \ .
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[S_n(q)] &= \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)D(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \int_0^\infty \int_0^\infty \mathbb{1}_{\{s<t\}}\phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)}H(dz)\right) \\
&\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)}H(dz)\right) H(ds)H(dt) \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)}H(dz)\right) \\
&\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)}H(dz)\right) H(ds)H(dt)
\end{aligned}$$

almost surely, since $\phi(s,t)q(s)q(t)D(s,t)$ is symmetric by (A1), and Z_1 and Z_2 are

i. i. d.. This concludes the argument for S_n. By similar arguments, we obtain $\bar{S}_n \to \bar{S}$ w. p. 1. $\square$

4.2 Calculating the limit

In order to identify the limit of $S_{2,n}^{se} = S_n(m(\cdot, \hat{\theta}_n))$ we need the statement of Corollary 4.14, which is based upon the following lemma. Define for any $\epsilon > 0$

$$M_{1,\epsilon}(x) := \max(0, m(x, \theta_0) - \epsilon)) \text{ and } M_{2,\epsilon}(x) := \min(1, m(x, \theta_0) + \epsilon)) \ .$$

Lemma 4.13. *Suppose (M1) and (M2) hold. Then the following statements hold for each $0 < \epsilon \leq 1$ and n large enough*

(i) $M_{1,\epsilon}(x) \leq m(x, \hat{\theta}_n) \leq M_{2,\epsilon}(x)$

(ii) $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \leq m(x, \hat{\theta}_n)m(y, \hat{\theta}_n) \leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon$.

Proof. For the sake of simpler notation, we will write $m_n(x) := m(x, \theta_n)$ and $m(x) := m(x, \theta_0)$. Let's start with part *(i)*. Suppose $M_{1,\epsilon}(x) = 0$, then the condition above is trivially satisfied since $m_n(x) \geq 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$. Then

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\geq m(x) - |m_n(x) - m(x)| \ . \end{aligned}$$

But under condition (M1), we have for n large enough that for some $\epsilon > 0$ $\theta_n \in V(\epsilon, \theta_0)$. Now we get, according to (M2) that

$$\sup_{x \geq 0} |m_n(x) - m(x)| < \epsilon \ .$$

Therefore we obtain $m_n(x) \geq m(x) - \epsilon = M_{1,\epsilon}(x)$. Let's now consider $M_{2,\epsilon}$. The case $M_{2,\epsilon} = 1$ is trivial again, since $m_n(x) \leq 1$. Now suppose $M_{2,\epsilon} = m(x) + \epsilon$. Then we obtain, for n large enough

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq m(x) + |m_n(x) - m(x)| \\ &\leq m(x) + \epsilon \\ &= M_{2,\epsilon}(x) \ . \end{aligned}$$

This concludes the proof of part *(i)*. Now note that, according to (M1) and (M2), the following holds for n large enough and some $\epsilon > 0$

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq |m_n(x) - m(x)| + m(x) \\ &\leq m(x) + \epsilon \ . \end{aligned} \tag{4.24}$$

Moreover consider that we have

$$\begin{aligned} m_n(x)m_n(y) &= (m_n(x) - m(x))(m_n(y) - m(y)) \\ &\quad + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \\ &\leq \epsilon^2 + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \ . \end{aligned}$$

Applying the latter inequality to (4.24) yields

$$\begin{aligned} m_n(x)m_n(y) &\leq \epsilon^2 + m(x)(m(y) + \epsilon) + (m(x) + \epsilon)m(y) - m(x)m(y) \\ &= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \ . \end{aligned} \tag{4.25}$$

Now suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = 0$ for $x, y \in \mathbb{R}_+$. Then $m(x) \leq \epsilon$ and

$m(y) \leq \epsilon$. Hence, using (4.25) yields

$$m_n(x)m_n(y) \leq 4\epsilon^2 \ .$$

Next suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$. Using (4.25) again, we obtain

$$\begin{aligned} m_n(x)m_n(y) &\leq m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \\ &\leq \epsilon + \epsilon(1+\epsilon) + \epsilon^2 \\ &= 2\epsilon(1+\epsilon) \ , \end{aligned}$$

since $m(x) \leq \epsilon$ and $m(y) \leq 1$. By similar calculations, we obtain the exact same result for the case $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$, and note that

$$\begin{aligned} M_{1,\epsilon}(x)M_{1,\epsilon}(y) &= (m(x) - \epsilon)(m(y) - \epsilon) \\ &= m(x)m(y) - \epsilon(m(x) + m(y)) + \epsilon^2 \ . \end{aligned}$$

Now (4.25) implies the following

$$\begin{aligned} m_n(x)m_n(y) &\leq m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \\ &= M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 2\epsilon(m(x) + m(y)) \\ &\leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon \ . \end{aligned}$$

Thus we have for $0 \leq \epsilon \leq 1$ that

$$m_n(x)m_n(y) \leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon \ ,$$

as claimed in the statement of this lemma. It remains to show that $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \leq m_n(x)m_n(y)$. By calculations similar to those that lead to (4.24) and (4.25) we obtain

$$m_n(x) \geq m(x) - \epsilon$$

and

$$m_n(x)m_n(y) \geq m(x)m(y) - \epsilon(m(x) + m(y)) - \epsilon^2 \ . \tag{4.26}$$

Now we will consider $M_{2,\epsilon}$ case by case. Suppose $M_{2,\epsilon}(x) = 1$ and $M_{2,\epsilon}(y) = 1$. This is equivalent to $m(x) \geq 1 - \epsilon$ and $m(y) \geq 1 - \epsilon$. Therefore (4.26) implies

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)^2 - 2\epsilon - \epsilon^2 \\ &= 1 - 4\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ . \end{aligned}$$

Next consider the case $M_{2,\epsilon}(x) = 1$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \geq 1 - \epsilon$ and $m(y) \leq 1 - \epsilon$. Moreover we have $M_{2,\epsilon}(x)M_{2,\epsilon}(y) = m(y) + \epsilon$. Hence we obtain the following, according to (4.26)

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)m(y) - \epsilon((1 + (1-\epsilon)) - \epsilon^2 \\ &= m(y) - \epsilon m(y) - 2\epsilon \\ &\geq m(y) - \epsilon(1-\epsilon) - 2\epsilon \\ &\geq m(y) - 3\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ . \end{aligned}$$

By similar calculations we obtain the same result, if $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = 1$. Finally consider the case $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \leq 1 - \epsilon$ and $m(y) \leq 1 - \epsilon$. Furthermore we have

$$M_{2,\epsilon}(x)M_{2,\epsilon}(y) = (m(x) + \epsilon)(m(y) + \epsilon)$$

$$= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \ .$$

Therefore, applying (4.26) again, yields

$$\begin{aligned} m_n(x)m_n(y) &\geq m(x)m(y) - \epsilon(m(x) + m(y)) - \epsilon^2 \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 2\epsilon(m(x) + m(y)) - 2\epsilon^2 \\ &\geq M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon(1-\epsilon) - 2\epsilon^2 \\ &\geq M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ . \end{aligned}$$

This concludes the proof. □

Corollary 4.14. *Suppose conditions (A2), (M1) and (M2) are satisfied. Then the following holds for each $0 < \epsilon \leq 1$ and n large enough*

$$S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot,\hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon\bar{S}_n(M_{1,\epsilon}).$$

Proof. Consider that we have the following for any $n \geq 1$

$$\begin{aligned} S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) = \sum\sum_{1\leq i<j\leq n} &\phi(Z_{i:n}, Z_{j:n})(M_{2,\epsilon}(Z_{i:n})M_{2,\epsilon}(Z_{j:n}) - 4\epsilon) \\ &\times \prod_{k=1}^{i-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right]\prod_{k=1}^{j-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right] \ . \end{aligned}$$

But according to Lemma 4.13 we have

$$m(x,\hat{\theta}_n) \leq M_{2,\epsilon}(x) \text{ and } M_{2,\epsilon}(x)M_{2,\epsilon}(y) \leq m(x,\hat{\theta}_n)m(y,\hat{\theta}_n)$$

for all $x, y \in \mathbb{R}_+$. Hence we obtain

$$\begin{aligned} S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \leq & \sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n}) m(Z_{i:n}, \hat{\theta}_n) m(Z_{j:n}, \hat{\theta}_n) \\ & \times \prod_{k=1}^{i-1}\left[1 - \frac{m(Z_{k:n}, \hat{\theta}_n)}{n-k+1}\right] \prod_{k=1}^{j-1}\left[1 - \frac{m(Z_{k:n}, \hat{\theta}_n)}{n-k+1}\right] \\ = & S_n(m(\cdot, \hat{\theta}_n)). \end{aligned}$$

Similarly we obtain

$$S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon}) \geq S_n(m(\cdot, \hat{\theta}_n)).$$

□

Now we are in a position, to identify $S = \lim_{n\to\infty} S_{2,n}^{se}$. The proof of the main theorem follows.

Proof of Theorem 1.4. Assume that conditions (A1) through (A4), (M1) and (M2) hold. Consider that we have

$$S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot, \hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon})$$

according to Corollary 4.14 under (M1) and (M2). Next take note of the Radon-Nikodym derivatives (c. f. Dikta (2000), page 8)

$$m(s, \theta_0) = \frac{H^1(ds)}{H(ds)} \text{ and } (1 - G(s)) = \frac{H^1(ds)}{F(ds)} .$$

Moreover consider that we have

$$\int_0^s \frac{1 - m(x, \theta_0)}{1 - H(x)} H(dx) = -\ln(1 - G(s))$$

and

$$\int_0^s \frac{\epsilon}{1-H(x)} H(dx) = -\ln((1-H(s))^\epsilon)$$

according to Dikta (2000). Now note that

$$\begin{aligned} M_{1,\epsilon}(x) &= \mathbb{1}_{\{m(x,\theta_0)>\epsilon\}}(m(x,\theta_0)-\epsilon) \\ &\leq m(x,\theta_0)-\epsilon \; . \end{aligned}$$

Therefore, we obtain the following

$$\begin{aligned} \bar{S}(M_{1,\epsilon}) &\leq \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t) \exp\left(\int_0^s \frac{1-m(x,\theta_0)}{1-H(x)} + \frac{\epsilon}{1-H(x)} H(dx)\right) \\ &\qquad\qquad \times \exp\left(\int_0^t \frac{1-m(x,\theta_0)}{1-H(x)} + \frac{\epsilon}{1-H(x)} H(dx)\right) H(ds)H(dt) \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))(1-H(s))^\epsilon(1-H(t))^\epsilon} H(ds)H(dt) \\ &= \frac{1}{2}\int_0^{\tau_H} \int_0^{\tau_H} \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^\epsilon(1-H(t))^\epsilon} F(ds)F(dt) \; . \end{aligned}$$

But by condition (A3), the integral above is finite. Moreover $M_{1,\epsilon}(x)$ is non-decreasing in x, since m is non-decreasing under (A4). Therefore $S(M_{1,\epsilon})$ exists almost surely under (A1) through (A4), by Theorem 3.5. Hence we have that for each $0 < \epsilon \leq 1$ we have $S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon}) \to S(M_{1,\epsilon}) + 4\epsilon \bar{S}(M_{1,\epsilon})$ w. p. 1 as $n \to \infty$, according to Lemma 4.12 . Next consider that

$$\begin{aligned} S(M_{1,\epsilon}) + 4\epsilon \bar{S}(M_{1,\epsilon}) \leq \frac{1}{2}\int_0^\infty \int_0^\infty & \frac{\phi(s,t)}{(1-H(s))^\epsilon(1-H(t))^\epsilon} \\ & \times \frac{m(s,\theta_0)m(t,\theta_0)+4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \; . \end{aligned}$$

By similar arguments we can show that $S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \to S(M_{2,\epsilon}) - 4\epsilon \bar{S}(M_{2,\epsilon})$

w. p. 1 as $n \to \infty$ and

$$S(M_{2,\epsilon}) - 4\epsilon \bar{S}(M_{2,\epsilon}) \geq \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t)(1-H(s))^\epsilon (1-H(t))^\epsilon \\ \times \frac{m(s,\theta_0)m(t,\theta_0) - 4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \; .$$

We have seen so far that for $0 < \epsilon \leq 1$ small enough

$$\begin{aligned}
&\frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t)(1-H(s))^\epsilon (1-H(t))^\epsilon \\
&\qquad \times \frac{m(s,\theta_0)m(t,\theta_0) - 4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&\leq \liminf_{n\to\infty} S_n(m(\cdot,\hat{\theta}_n)) \\
&\leq \limsup_{n\to\infty} S_n(m(\cdot,\hat{\theta}_n)) \\
&\leq \frac{1}{2} \int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-H(s))^\epsilon (1-H(t))^\epsilon} \\
&\qquad \times \frac{m(s,\theta_0)m(t,\theta_0) + 4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \; .
\end{aligned}$$

Finally let $\epsilon \searrow 0$ and apply the Monotone Convergence Theorem to obtain that the upper and lower bound converge both to the same limit. In effect, we have

$$\begin{aligned}
&\lim_{\epsilon\searrow 0} \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t)(1-H(s))^\epsilon (1-H(t))^\epsilon \\
&\qquad \times \frac{m(s,\theta_0)m(t,\theta_0) - 4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty \frac{\phi(s,t)m(s,\theta_0)m(t,\theta_0)}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&= \frac{1}{2} \int_0^{\tau_H} \int_0^{\tau_H} \phi(s,t)F(ds)F(dt) \\
&= \lim_{\epsilon\searrow 0} \frac{1}{2} \int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))} \\
&\qquad \times \frac{m(s,\theta_0)m(t,\theta_0) + 4\epsilon}{(1-H(s))^\epsilon (1-H(t))^\epsilon} H(ds)H(dt) \; .
\end{aligned}$$

Hereby the proof of Theorem 1.4 is concluded. □

Chapter 5

The censoring model

During this chapter we will consider the censoring model m more closely. Recall from the assumptions of SRCM (see Chapter 1) that we have $X \sim F$, $Y \sim G$ and $Z \sim H$, where $Z = \min(X, Y)$. We observe $(Z_i, \delta_i)_{i \le n}$. In the following, we will first see an expression for m in terms of the hazard rates λ_F and λ_G, which was derived in Dikta (1998). Later we will see examples of different configurations for λ_F, λ_G and m, and how assumption (A4) restricts their use in practice.

First recall from Chapter 1 that the cumulative hazard rate corresponding to F is defined as

$$\Lambda_F(z) = \int_0^z \frac{1}{1 - F(t)} F(dt) = \int_0^z \lambda_F(t) dt \ . \tag{5.1}$$

with

$$\lambda_F(z) = \frac{f(z)}{1 - F(z)} \ .$$

Now recall that

$$m(z, \theta) = \mathbb{P}(\delta = 1 | Z = z) = \mathbb{E}(\mathbb{1}_{\{\delta = 1\}} | Z = z) \ .$$

according to Dikta (1998), page 254. Next consider that we have (c. f. Shorack and Wellner (2009), page 294)

$$H_1(z) = P(\delta = 1, Z \le z) = \mathbb{E}(I(X \le Y) I(X \le z))$$

$$= \mathbb{E}(I(X \leq z)\mathbb{E}(I(X \leq Y)|X)) \ .$$

Hence we obtain

$$\begin{aligned}
H_1(z) &= \int_0^z \mathbb{E}(I(X \leq Y)|X = t)F(dt) \\
&= \int_0^z \mathbb{E}(I(Y > t))F(dt) \\
&= \int_0^z \mathbb{P}(Y > t)F(dt) \\
&= \int_0^z 1 - G(t)F(dt) \ .
\end{aligned}$$

Thus $dH_1 = (1-G)dF$. Moreover we have $dH_1 = m \cdot dH$. Therefore we can rewrite Λ_F as

$$\begin{aligned}
\Lambda_F(z) &= \int_0^z \frac{1 - G(t)}{(1 - F(t))(1 - G(t))}F(dt) \\
&= \int_0^z \frac{1}{(1 - F(t))(1 - G(t))}H_1(dt) \\
&= \int_0^z \frac{1}{1 - H(t)}H_1(dt) \\
&= \int_0^z \frac{m(t,\theta)}{1 - H(t)}H(dt) \qquad (5.2)
\end{aligned}$$

Note that combining (5.1) and (5.2) yields

$$\int_0^z \lambda_F(t)dt = \int_0^z \frac{f(t)}{1 - F(t)}dt = \int_0^z \frac{m(t,\theta)h(t)}{1 - H(t)}dt = \int_0^z m(t,\theta)\lambda_H(t)dt$$

Now this implies

$$m(z,\theta_0) = \frac{\lambda_F(z)}{\lambda_H(z)} = \frac{\lambda_F(z)}{\lambda_F(z) + \lambda_G(z)} \ , \qquad (5.3)$$

c. f. Dikta (1998), page 255. Parametric models for m can be found in Cox (1970) and Collett (2014).

We will now see different examples for censoring models in different settings, and how condition (A4) restricts their application in practice. Consider the following examples.

Example 5.1. Suppose that F and G satisfy

$$1 - G(z) = (1 - F(z))^{\beta} \quad \text{for some} \quad \beta > 0 \ ,$$

in addition to the assumptions of semi-parametric RCM. This model is called proportional hazards model. In this case the censoring model $m(\cdot, \theta)$ is independent of Z. Hence we have

$$m(z, \theta) = \mathbb{E}[\delta] = \frac{1}{1+\beta} = \theta \tag{5.4}$$

according to Dikta (1995), p. 1538. Note that m is constant and therefore satisfies condition (A4). The proportional hazards model was discussed in detail by Koziol and Green (1976). Breslow and Crowley (1974) established a CLT result about the Kaplan-Meier PLE under the proportional hazards model. Now

One straight forward approach to obtain a non-parametric estimate of (5.4) is given by

$$\bar{c}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_i \approx \mathbb{E}[\delta] \ .$$

The above quantity was used by Cheng and Lin (1987) to introduce the following estimator

$$1 - F_n^{cl}(z) = \prod_{Z_k : Z_k \le z} \left[\frac{n - R_{k,n}}{n - R_{k,n} + 1} \right]^{\bar{c}_n} \ .$$

It was also shown in Cheng and Lin (1987) that F_n^{cl} is more efficient than F_n^{km}. For integrals of measurable functions w. r. t. F_n^{cl}, strong consistency was established by Stute (1992). In Dikta (1995) it was shown that the limiting distribution is normal under proper conditions.

Next consider that, if the condition of PHM is satisfied, we have

$$m(z, \hat{\theta}_n) = \hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_i = \bar{c}_n$$

according to Dikta (1998), Example 2.8. Therefore $F_n^{se,1}$ is identical to F_n^{cl}. In Dikta (2000), page 3, it was pointed out that $F_n^{se,1}$ and F_n^{se} will show the same gain in efficiency, compared to the Kaplan-Meier PLE.

Section 6.2 shows a simulation study under of a semi-parametric U-statistics estimator based on F_n^{se} under the proportional hazards model.

During the next example we will examine the Weibull distribution. We will write $X \sim Wei(\alpha, \beta)$ if the r. v. X follows a Weibull distribution with parameters α and β. In this case the hazard rate is given by $\lambda(z) = \alpha^{\beta}\beta z^{\beta-1}$.

Example 5.2. Let $X \sim Weibull(\alpha_1, \beta_1)$ and $Y \sim Weibull(\alpha_2, \beta_2)$. Then their respective hazard rates are given by

$$\lambda_F(z) = \alpha_1^{\beta_1}\beta_1 z^{\beta_1 - 1} \text{ and } \lambda_G(z) = \alpha_2^{\beta_2}\beta_2 z^{\beta_2 - 1} \; .$$

According to (5.3), we can now write our censoring model m as

$$m(z, \theta) = \frac{1}{1 + \lambda_G(z)/\lambda_F(z)} = \left(1 + \frac{\alpha_2^{\beta_2}\beta_2}{\alpha_1^{\beta_1}\beta_1} z^{\beta_2 - \beta_1}\right)^{-1} = \frac{1}{1 + \theta_1 z^{\theta_2}}$$

with

$$\theta = (\theta_1, \theta_2) = \left(\frac{\alpha_2^{\beta_2}\beta_2}{\alpha_1^{\beta_1}\beta_1}, \beta_2 - \beta_1\right).$$

The setup described above is called the generalized hazards model (see Dikta (1998), Example 2.9). Note that condition (A4) poses a restriction on this model, since we need $\beta_2 < \beta_1$ s. t. $\theta_2 < 0$ and hence $m(z, \theta_0)$ is non-decreasing in z. In section 6.3, a simulation study of the setup above is shown.

Let's introduce the Pareto (type I) distribution $Par(\alpha, \beta)$ for the next example. If $X \sim Par(\alpha, \beta)$, we have

$$\lambda_F(z) = \left[\frac{\beta}{z}\right]^{\alpha} \mathbb{1}_{\{z \geq \beta\}} \ .$$

Example 5.3. Suppose $X \sim Exp(\alpha)$ and $Y \sim Par(1, \beta)$. Then the censoring model is given by

$$m(z, \theta) = \frac{\alpha}{\alpha + \frac{\beta}{z}\mathbb{1}_{\{z \geq \beta\}}} \quad \text{with} \quad \theta = (\alpha, \beta) \ .$$

Note that $m(z, \theta)$ is monotone non-decreasing if $\beta > 0$ and $z \geq \beta$. But if $z < \beta$, we have $m(z, \theta) = 1$. At $z = \beta$, m has a discontinuity and $m(\beta, \theta) = \alpha(\alpha + 1)^{-1} < 1$. Therefore conditions (A4) is violated in this case. However, we will see a simulation study for this setup in Section 6.4. The results of this study indicate that the considered semi-parametric estimator might still be consistent under this setup.

The following example will involve the Gompertz distribution. If X follows a Gompertz distribution with parameters α and β we will write $X \sim Gom(\alpha, \beta)$. In this case the hazard rate is given by $\lambda_F(z) = \exp(\alpha + \beta z)$.

Example 5.4. Suppose $X \sim Gom(\alpha, \beta)$ and $Y \sim Exp(\gamma)$. Then the censoring model is given by

$$m(z, \theta) = \frac{1}{1 + \gamma \exp(-\alpha - \beta z)} \ .$$

for $\beta > 0$ and $\gamma > 0$. Now $m(z, \theta)$ is non-decreasing in z, since $\beta > 0$.

Example 5.5. Suppose λ_F is known and m is defined as follows

$$m(z, \theta) = \frac{\exp(\theta z)}{1 + \exp(\theta z)} = \frac{1}{1 + \exp(-\theta z)}$$

for $\theta < 0$. We will call the model above logit model.

Remark 5.6. Consider that equation (5.3) implies

$$\lambda_G(z) = \lambda_F(z)\exp(-\theta z).$$

The cumulative hazard function of G is now of the form

$$\Lambda_G(z) = \int\limits_0^z \lambda_F(t)\exp(-\theta t)dt$$

Suppose e. g. λ_F is bounded above, i. e. $\lambda_F(z) \leq c$ for some constant $c < \infty$ and all $z \in \mathbb{R}_+$. Then

$$\Lambda_G(z) \leq c \cdot \int\limits_0^z \exp(-\theta t)dt = c\left(1 - \theta^{-1}\exp(-\theta z)\right) \ .$$

Note that the right hand side above converges to $c < \infty$ as $z \to \infty$, if $\theta > 0$. But this means G is not a proper distribution function, since

$$\lim_{z\to\infty} G(z) = \lim_{z\to\infty} 1 - \exp(-\Lambda_G(z)) < 1 \ .$$

Hence we must have $\theta < 0$, s. t. $\Lambda_G(z) \to \infty$ as $z \to \infty$. Next consider that $m(z, \theta_0)$ is non-decreasing, whenever $\theta > 0$. Thus we can not use the logit model under restriction (A4).

Example 5.7. Suppose the censoring model is given by

$$m(z, \theta) = 1 - \exp(-\exp(\theta z)) \ .$$

This model will be called complementary log-log model.

The following remark shows that condition (A4) makes the complementary log-

log model inapplicable under this setup.

Remark 5.8. Let $m(z, \theta) = 1 - \exp(-\exp(\theta z))$ and let λ_F be known. Now consider

$$\Lambda_G(z) = \int_0^z \frac{\lambda_F(t) \exp(-\exp(\theta t))}{1 - \exp(-\exp(\theta t))} dt$$

Now suppose λ_F is, e. g. either non-increasing or bounded above. In both cases we need $\theta < 0$ to obtain

$$\lim_{z \to \infty} \Lambda(z) = \infty \ .$$

On the other hand, $m(\cdot, \theta)$ is non-decreasing whenever $\theta \geq 0$. Therefore the model is not applicable under condition (A4).

Chapter 6

Simulations

In Chapter 5 we discussed different configurations of our pdf's f and g, and the censoring model m. We will now see simulation studies corresponding to some of those setups. In Section 6.1 we will detail, how those simulations are calculated. The remaining sections of this chapter will show simulations for different setups of f, g and m.

6.1 Computational Aspects

Assume that we have $(Z_i, \delta_i)_{i \leq n}$ is a sample in the sense of RCM. Recall the target value from Chapter 1

$$\theta^* = \mathbb{E}[\phi] = \int_0^\infty \int_0^\infty \phi(s,t) F(ds) F(dt) \ .$$

In the following, we will estimate the integral above under different setups. For the simulations, one chooses first an appropriate censoring model m in connection with the compatible distribution for X and/or Y. The kernel ϕ can be chosen separately. Then the Maximum Likelihood estimate for $\hat{\theta}_n$ is calculated. Afterwards, the semi-parametric and the Kaplan-Meier weights are calculated, using the following formulas

$$W_{i,n}^{se} = F_n^{se}(Z_{i:n}) - F_n^{se}(Z_{i-1:n}) = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{m(Z_{k:n}, \hat{\theta}_n)}{n-k+1} \right]$$

and

$$W_{i,n}^{km} = F_n^{km}(Z_{i:n}) - F_n^{km}(Z_{i-1:n}) = \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1} \right]$$

respectively. Now the the semi-parametric and the Kaplan-Meier U-statistics can be calculated as

$$U_n^{se} = 2 \cdot \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{se} W_{j,n}^{se}$$

and

$$U_n^{km} = 2 \cdot \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{km} W_{j,n}^{km} \ .$$

Note that $U_n^{se} = 2 \cdot S_{2,n}^{se}$ and $U_n^{km} = 2 \cdot S_{2,n}^{km}$. The factor 2 is motivated by Remark 1.5.

As kernel for the following simulation studies, we choose

$$\phi(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2 \ .$$

Hence we are estimating the sample variance, as pointed out in Example 1.1. The semi-parametric and the Kaplan-Meier estimates of θ^* will be denoted as σ_n^{se} and σ_n^{km} respectively. Each simulation is repeated $M = 100$ times for different samples of size n. Let $(Z_i, \delta_i)_{i \le n}^j$ be the sample of generated in the j-th repetition for $j = 1, \ldots, M$, and let $\sigma_n \in \{\sigma_n^{se}, \sigma_n^{km}\}$. We will denote by $\sigma_{n,j}$ the estimate of θ^* based on sample $(Z_i, \delta_i)_{i \le n}^j$ for $j = 1, \ldots, M$. The Bias of σ_n will be calculated by the following formula

$$Bias(\sigma_n) = \frac{1}{M} \sum_{j=1}^{M} (\sigma_{n,j} - \theta^*) \ .$$

For the Variance of σ_n we use

$$Var(\sigma_n) = \frac{1}{M-1} \sum_{j=1}^{M} (\sigma_{n,j} - \bar{\sigma}_M)^2 \quad \text{with} \quad \bar{\sigma}_M = \frac{1}{M} \sum_{j=1}^{M} \sigma_{n,j} \ .$$

The mean squared error (MSE) will be calculated as

$$MSE(\sigma_n) = \frac{1}{M}\sum_{j=1}^{M}(\sigma_{n,j} - \theta^*)^2 \ .$$

Additionally, we will calculate the average proportion of uncensored observations by

$$\bar{c} = \frac{1}{M}\sum_{j=1}^{M} c_{n,j} \quad \text{with} \quad c_{n,j} = \frac{1}{n}\sum_{i=1}^{n}\delta_i \ .$$

Furthermore we will calculate quantiles of F_n^{km} and F_n^{se}, by

$$q_n^{se}(p) = \inf\{t \in \mathbb{R}_+ | F_n^{se}(t) \geq p\}$$

and

$$q_n^{se}(p) = \inf\{t \in \mathbb{R}_+ | F_n^{se}(t) \geq p\} \ ,$$

respectively. In order to get information about the underlying estimates F_n^{se} and F_n^{km} of the true d. f. F, we will calculate the Bias, variance and MSE for $q_n^{se}(p)$ and $q_n^{km}(p)$ for $p \in \{0.25, 0.5, 0.75\}$ as well. The simulation results will be displayed in two tables. One table contains bias, variance and MSE of σ_n^{se} and σ_n^{km}. The other table shows the bias and MSE of q_n^{se} and q_n^{km}. The results are also illustrated by a figure at the end of each section. The left image shows the **squared** Bias, variance and MSE for σ_n^{se} and σ_n^{km}. The right image displays the MSE of $q_n^{se}(p)$ and $q_n^{km}(p)$ for $p \in \{0.25, 0.5, 0.75\}$.

6.2 Simulation 1

Suppose $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta)$. Then we have

$$m(z, \theta) = \frac{\alpha}{\alpha + \beta} = \theta$$

is constant in this case. Hence we are in the situation of proportional hazards model, as described in Example 5.1.

For this simulation, we chose $\alpha = 2$ and $\beta = 1$. The target value was here

$$Var(X) = \frac{1}{\alpha^2} = \frac{1}{4} \; .$$

For this simulation we will calculate the Cheng-Lin estimate (see Example 5.1) of $Var(X)$, namely σ_n^{cl}, additionally to σ_n^{se} and σ_n^{km}. We calculate σ_n^{cl} as

$$\sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i,n}^{cl} W_{j,n}^{cl}$$

where

$$W_{i,n}^{cl} = \left[1 - \left(\frac{n-i}{n-i+1}\right)^{c_n}\right] \times \prod_{k=1}^{i-1} \left[\frac{n-k}{n-k+1}\right]^{c_n} \; .$$

Bias, variance, MSE and quantiles will be calculated and displayed for σ_n^{cl} in Table 6.1 and Table 6.2, in addition to with corresponding values for σ_n^{se} and σ_n^{km}, in order to compare them. We expect that σ_n^{se} and σ_n^{cl} will show similar results, because of Dikta (2000), page 3.

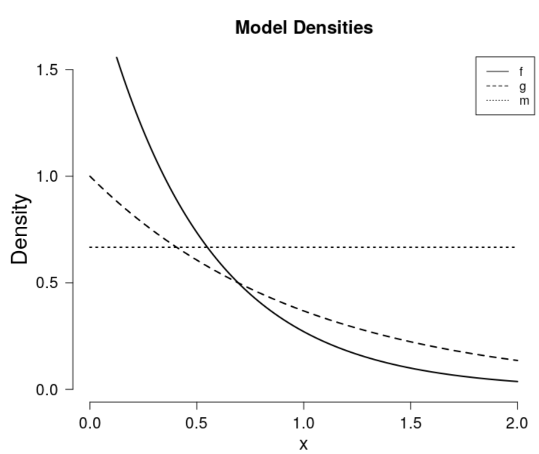

Figure 6.1: Probability density functions f, g and censoring model m for Sim. 1.

Figure 6.1 shows the pdf's f and g, as well as the censoring model. Under this setup we have $m(\cdot, \theta) = 2/3$. Since the censoring model is constant, we can expect that censoring will be occurring at the same rate over the whole domain.

	$n = 100$	$n = 500$	$n = 1000$
$Bias(\sigma_n^{se})$	-0.0581	-0.0317	-0.0203
$Bias(\sigma_n^{km})$	-0.0691	-0.0361	-0.0268
$Bias(\sigma_n^{cl})$	-0.0307	-0.0179	-0.0087
$Var(\sigma_n^{se})$	0.0054	0.0020	0.0013
$Var(\sigma_n^{km})$	0.0091	0.0028	0.0017
$Var(\sigma_n^{cl})$	0.0080	0.0027	0.0016
$MSE(\sigma_n^{se})$	0.0087	0.0030	0.0017
$MSE(\sigma_n^{km})$	0.0138	0.0041	0.0025
$MSE(\sigma_n^{cl})$	0.0089	0.0030	0.0017
$\bar{c}$	0.6646	0.66456	0.66831

Table 6.1: Results for Simulation 1.

Table 6.1 shows that bias, variance and MSE are decreasing to zero for all three estimators. σ_n^{se} and σ_n^{cl} are performing clearly better than σ_n^{km} under this setup, while σ_n^{se} and σ_n^{cl} show roughly the same behavior, as we expected in the beginning of this section.

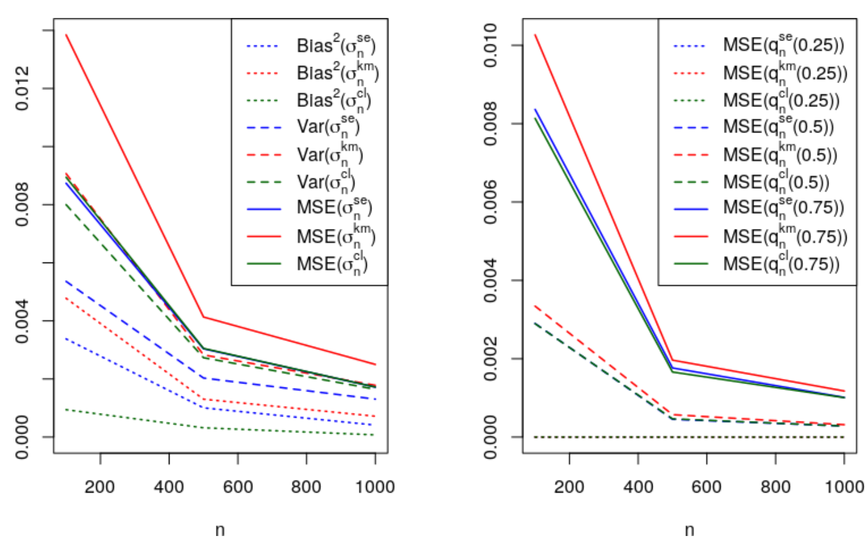

Figure 6.2: Results for Simulation 1. left: bias, variance and MSE for σ_n^{se} and σ_n^{km}. right: MSE for q_n^{se} and q_n^{km}.

Figure 6.2 indicates that the gain in efficiency of σ_n^{se} and σ_n^{cl} versus σ_n^{km} is greater for smaller sample sizes. Moreover we can see that the gain in efficiency for σ_n^{se} and σ_n^{cl} is more related to the variance, than to the bias.

The Quantiles are estimated quite well under this setup, although both estimators mainly underestimated the true quantiles by a small amount.

	$n = 100$	$n = 500$	$n = 1000$	$n = 100$	$n = 500$	$n = 1000$
	Bias			*MSE*		
$q_n^{se}(0.25)$	-0.0105	-0.003	-0.0031	0.0007	0.0001	0.0001
$q_n^{km}(0.25)$	-0.0038	-0.0017	-0.0023	0.0010	0.0002	0.0001
$q_n^{cl}(0.25)$	-0.0067	-0.0012	-0.0019	0.0007	0.0001	0.0001
$q_n^{se}(0.5)$	-0.0109	-0.0010	-0.0032	0.0029	0.0005	0.0003
$q_n^{km}(0.5)$	-0.0046	-0.0001	-0.0017	0.0033	0.0006	0.0003
$q_n^{cl}(0.5)$	-0.0088	0.0006	-0.0024	0.0029	0.0005	0.0003
$q_n^{se}(0.75)$	-0.0123	0.0074	-0.0032	0.0084	0.0018	0.0010
$q_n^{km}(0.75)$	-0.0143	0.0077	-0.0030	0.0103	0.0020	0.0012
$q_n^{cl}(0.75)$	-0.0190	0.0039	-0.0048	0.0081	0.0017	0.0010

Table 6.2: Results for estimated quantiles of Simulation 1.

6.3 Simulation 2

Let $X \sim Weibull(\alpha_1, \beta_1)$ and $X \sim Weibull(\alpha_2, \beta_2)$. Then we obtain for the censoring model

$$m(z, \theta) = \frac{1}{1 + \theta_1 z^{\theta_2}} \text{ with } \theta = \left(\frac{\alpha_2^{\beta_2} \beta_2}{\alpha_1^{\beta_1} \beta_1}, \beta_2 - \beta_1 \right) .$$

For the simulation below we chose $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1.2$ and $\beta_2 = 1$. The target value was here

$$Var(X) = 0.154936 .$$

Figure 6.3 indicates that smaller values are censored rather than larger ones under this setup. This is due to the increasing nature of the censoring model m.

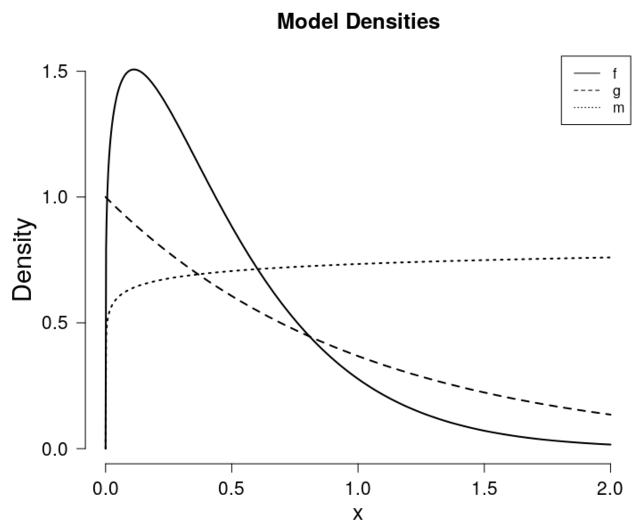

Figure 6.3: Probability density functions f, g and censoring model m for Sim. 2.

Table 6.3 shows that bias, variance and MSE are converging to zero for both estimators, as well under this setup. The semi-parametric estimator is clearly more efficient than the Kaplan-Meier estimate w. r. t. the MSE. Here again, the difference in variance between the semi-parametric and the Kaplan-Meier based estimator is much larger than the difference in squared bias.

	$n = 100$	$n = 500$	$n = 1000$
$Bias(\sigma_n^{se})$	-0.0196	-0.0002	0.0038
$Bias(\sigma_n^{km})$	-0.0201	-0.0114	-0.0114
$Var(\sigma_n^{se})$	0.0017	0.0007	0.0003
$Var(\sigma_n^{km})$	0.0029	0.0008	0.0003
$MSE(\sigma_n^{se})$	0.0020	0.0007	0.0003
$MSE(\sigma_n^{km})$	0.0033	0.0009	0.0004
$\bar{c}$	0.6705	0.6678	0.66538

Table 6.3: Results for Simulation 2.

Figure 6.4 shows, as before that the gain in efficiency is greater for smaller sample sizes n. Again, the gain in efficiency is more severe for smaller n in this simulation.

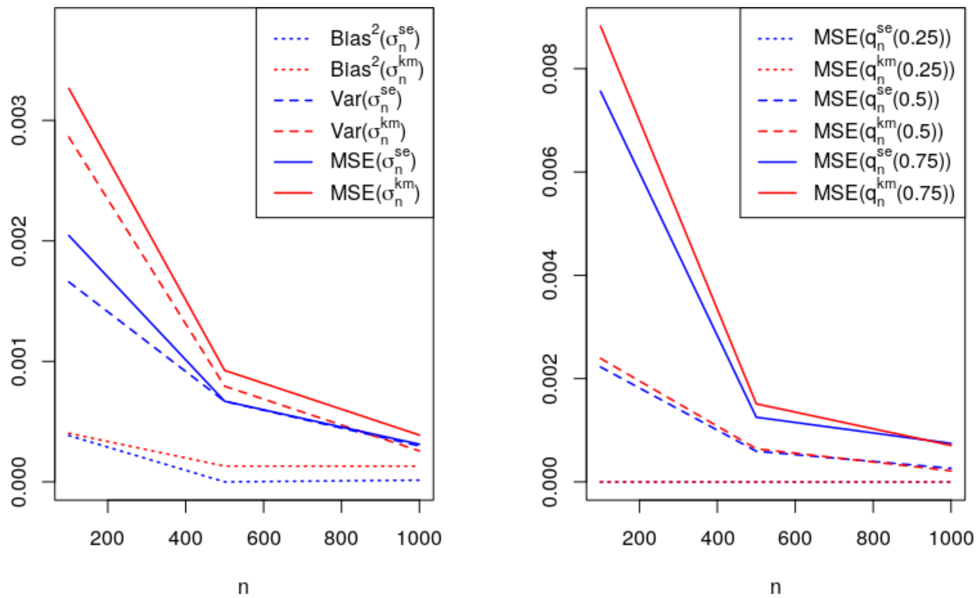

Figure 6.4: Results for Simulation 2. left: bias, variance and MSE for σ_n^{se} and σ_n^{km}. right: MSE for q_n^{se} and q_n^{km}.

Both estimators are estimating the true quantiles well under this setup, as we can see from Table 6.4. As before, the quantiles are, for the most part, slightly underestimated by both estimators. Figure 6.4 shows that q_n^{se} is performing slightly better q_n^{km} in this situation.

	$n=100$	$n=500$	$n=1000$	$n=100$	$n=500$	$n=1000$
	Bias			*MSE*		
$q_n^{se}(0.25)$	-0.0183	-0.0114	-0.0119	0.0009	0.0002	0.0002
$q_n^{km}(0.25)$	-0.0074	-0.0003	-0.0009	0.0006	0.0002	0.0001
$q_n^{se}(0.5)$	-0.0123	-0.0113	-0.008	0.0022	0.0006	0.0003
$q_n^{km}(0.5)$	-0.0068	-0.0058	-0.0021	0.0024	0.0006	0.0002
$q_n^{se}(0.75)$	-0.0092	0.0004	0.0074	0.0076	0.0013	0.0007
$q_n^{km}(0.75)$	-0.0158	-0.0104	-0.0025	0.0088	0.0015	0.0007

Table 6.4: Results for estimated quantiles of Simulation 2.

6.4 Simulation 3

Let $X \sim Exp(\alpha)$ and $Y \sim Par(1, \beta)$. For our model m we obtain in this case

$$m(z, \theta) = \frac{\alpha}{\alpha + \frac{\beta}{z}\mathbb{1}_{\{z \geq \beta\}}} \ .$$

Note that m is **not** non-decreasing over the whole domain in this case (c. f. Example 5.3). For the following simulation we chose $\alpha = 0.5$ and $\beta = 1.2$. The target value was here

$$Var(X) = 4 \ .$$

Considering Figure 6.5, we can not expect any censored observations on $[0, \beta]$.

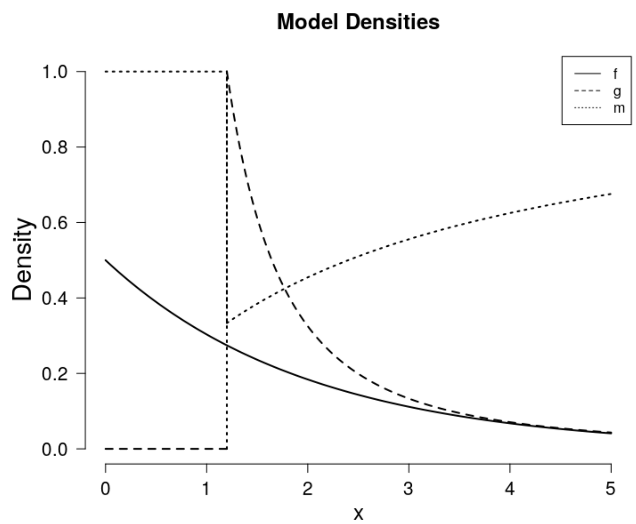

Figure 6.5: Probability density functions f, g and censoring model m for Sim. 3.

Moreover the plot indicates that values in $[\beta, 3]$ are more likely to be censored. On $[\beta, \infty)$, the censoring model is monotone increasing. This implies that smaller values are more likely to be censored than larger values.

	$n = 100$	$n = 500$	$n = 1000$
$Bias(\sigma_n^{se})$	-1.0616	-0.4255	-0.2735
$Bias(\sigma_n^{km})$	-1.0972	-0.5142	-0.3189
$Var(\sigma_n^{se})$	2.8281	0.8522	0.3623
$Var(\sigma_n^{km})$	2.9919	1.2895	0.5611
$MSE(\sigma_n^{se})$	3.9553	1.0333	0.4370
$MSE(\sigma_n^{km})$	4.1957	1.5539	0.6628
$\bar{c}$	0.6971	0.6970	0.6962

Table 6.5: Results for simulation 3.

From Table 6.5, we see that the MSE values of both estimators, σ_n^{se} and σ_n^{km}, are substantially larger than in the previous examples, especially for $n = 100$. However, the MSE values decrease considerably as n increases. Figure 6.6, shows that the semi-parametric estimator is performing better than the Kaplan-Meier estimate again, with a larger gain in efficiency for small n .

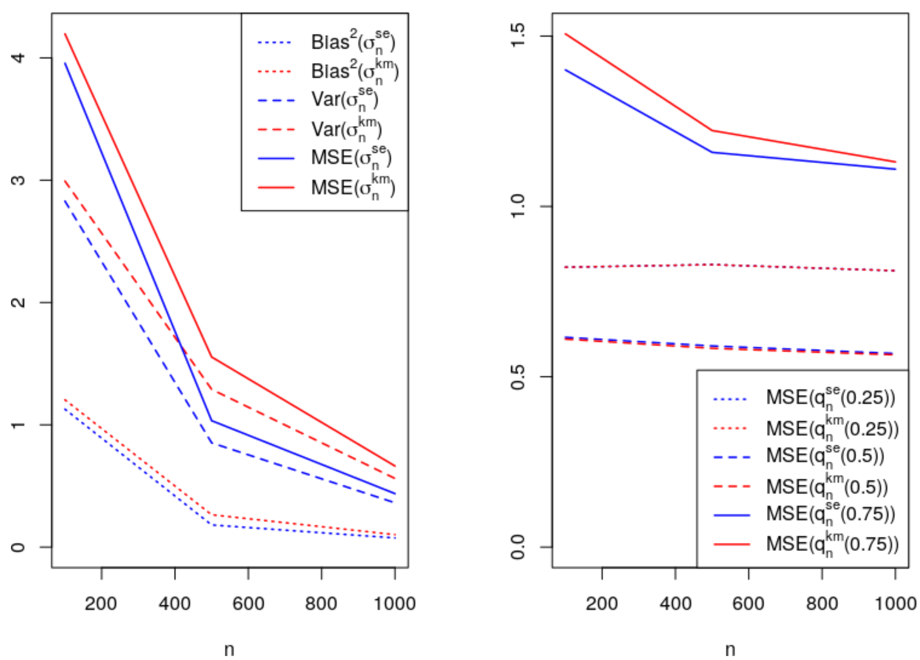

Figure 6.6: Results for Simulation 3. left: bias, variance and MSE for σ_n^{se} and σ_n^{km}. right: MSE for q_n^{se} and q_n^{km}.

Table 6.6 shows that the quantiles are considerably underestimated by both estimators in this case. This might be a consequence of the fact that m violates condition (A4) under this setup. The large MSE values for the quantile estimates are likely to cause the much larger MSE scores of σ_n^{se} and σ_n^{km} in this simulation.

	$n = 100$	$n = 500$	$n = 1000$	$n = 100$	$n = 500$	$n = 1000$
	Bias			*MSE*		
$q_n^{se}(0.25)$	-0.9461	-0.9531	-0.9482	0.9064	0.9105	0.9007
$q_n^{km}(0.25)$	-0.9461	-0.9531	-0.9482	0.9064	0.9105	0.9007
$q_n^{se}(0.5)$	-0.7617	-0.7637	-0.7513	0.6157	0.5904	0.5682
$q_n^{km}(0.5)$	-0.7565	-0.7589	-0.7484	0.6106	0.5835	0.5644
$q_n^{se}(0.75)$	-1.1444	-1.0630	-1.0461	1.4006	1.1587	1.1093
$q_n^{km}(0.75)$	-1.1641	-1.0890	-1.0535	1.5064	1.2227	1.1306

Table 6.6: Results for estimated quantiles of Simulation 3.

Chapter 7

Discussion

The strong law of large numbers for the semiparametric U-statistics estimator $S^{se}_{2,n}$, under proper conditions, has been established in Theorem 1.4. In addition to the assumptions made in Dikta (2000) and Bose and Sen (1999), we assumed that the censoring model, i. e. conditional expectation of the censoring indicator given the observation, is a monotone non-decreasing function. However Chapter 5 shows a variety of examples, which are relevant in the field of survival analysis, for which this additional condition is satisfied. These examples include, among others, the proportional hazards model. The product limit estimator, upon which the semiparametric U-Statistics is based in this example, has the same asymptotic properties as the Cheng and Lin (1987) estimator (c. f. Dikta (2000), page 3). In Chapter 6, we conducted simulation studies for different scenarios. The simulation studies verify the SLLN result in Theorem 1.4. Moreover the studies show that the semiparametric estimator outperforms the Kaplan-Meier estimate, especially in terms of variance, in most cases. This was expected because of the results established by Dikta et al. (2005) and Dikta (2014). The gain in efficiency was especially large for smaller sample sizes. The results of Section 6.4 indicate, that the semiparametric estimator might still be consistent, even if the censoring model is not non-decreasing.

There are some obvious options to extend the results of this thesis in the future. Firstly, one could try to establish the SLLN for the semiparametric estimator under weaker assumptions. In the appendix section, the interested reader may find thoughts on how to work around the additional restriction for the censoring model by modifying Doob's Upcrossing Theorem. Furthermore a CLT statement for the the semiparametric estimator could possibly be derived from Dikta et al. (2005) and Bose and Sen (2002). As another option for future work, based on this thesis, one could transfer the result of Theorem 1.4 to the estimator derived in Dikta et al. (2016), using stochastic equivalence.

Bibliography

Aalen, O., 1978: Nonparametric inference for a family of counting processes. *The Annals of Statistics*, **6 (4)**, 701–726.

Bose, A. and A. Sen, 1999: The strong law of large numbers for Kaplan-Meier U-statistics. *Journal of Theoretical Probability*, **12 (1)**, 181–200.

Bose, A. and A. Sen, 2002: Asymptotic distribution of the Kaplan–Meier U-statistics. *Journal of Multivariate Analysis*, **83 (1)**, 84–123.

Breslow, N. and J. Crowley, 1974: A large sample study of the life table and product limit estimates under random censorship. *The Annals of Statistics*, **2 (3)**, 437–453.

Cheng, P. E. and G. D. Lin, 1987: Maximum likelihood estimation of a survival function under the Koziol-Green proportional hazards model. *Statistics & Probability Letters*, **5 (1)**, 75 – 80, doi:https://doi.org/10.1016/0167-7152(87)90030-7.

Cohn, D., 2013: *Measure Theory*. Birkhäuser Boston.

Collett, D., 2014: *Modelling Survival Data in Medical Research*. Chapman & Hall.

Cox, D., 1970: *The Analysis of Binary Data*. Chapman & Hall.

Dikta, G., 1995: Asymptotic normality under the Koziol-Green model. *Communications in Statistics - Theory and Methods*, **24 (6)**, 1537–1549, doi:10.1080/03610929508831570.

Dikta, G., 1998: On semiparametric random censorship models. *Journal of Statistical Planning and Inference*, **66 (2)**, 253–279.

Dikta, G., 2000: The strong law under semiparametric random censorship models. *Journal of Statistical Planning and Inference*, **83 (1)**, 1–10.

Dikta, G., 2014: Asymptotically efficient estimation under semi-parametric random censorship models. *Journal of Multivariate Analysis*, **124**, 10 – 24, doi:https://doi.org/10.1016/j.jmva.2013.10.002.

Dikta, G., J. Ghorai, and C. Schmidt, 2005: The central limit theorem under semiparametric random censorship models. *Journal of Statistical Planning and Inference*, **127 (1)**, 23–51.

Dikta, G., M. Reißel, and C. Harlaß, 2016: Semi-parametric survival function estimators deduced from an identifying Volterra type integral equation. *Journal of Multivariate Analysis*, **147**, 273 – 284, doi:https://doi.org/10.1016/j.jmva.2016.02.008.

Kaplan, E. L. and P. Meier, 1958: Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association*, **53 (282)**, 457–481.

Koziol, J. A. and S. B. Green, 1976: A cramr-von mises statistic for randomly censored data. *Biometrika*, **63 (3)**, 465–474, doi:10.1093/biomet/63.3.465.

Lee, J., 1990: *U-Statistics: Theory and Practice*. Statistics: A Series of Textbooks and Monographs, Taylor & Francis.

Nelson, W., 1972: Theory and applications of hazard plotting for censored failure data. *Technometrics*, **14 (4)**, 945–966.

Neveu, J., 1975: *Discrete-Parameter Martingales*, Vol. 10. Elsevier.

Shorack, G. and J. Wellner, 2009: *Empirical Processes with Applications to Statistics*. Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104).

Stute, W., 1992: Strong consistency under the Koziol-Green model. *Statistics & Probability Letters*, **14 (4)**, 313 – 320, doi:https://doi.org/10.1016/0167-7152(92)90064-C.

Stute, W. and J.-L. Wang, 1993: The strong law under random censorship. *The Annals of Statistics*, 1591–1607.

van der Vaart, A., 2000: *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press.

Appendix: Thoughts on finding weaker assumptions

In Section 3.2, we were able to show that $S_n(q)$ is a reverse supermartingale under the assumptions of Lemma 3.3. To establish the almost sure existence of limits of supermartingale processes, one considers the number of upcrossings of an interval $[a, b]$ by the process. This was done in the famous Upcrossing Theorem by Doob. During this section we will generalize Doob's Upcrossing Theorem to our framework in order to explore ways to establish weaker assumptions. To get closer to the situation of Doob's Upcrossing Theorem, we define the following quantities. Let $N < \infty$ and define for $1 \leq n \leq N$

$$\tilde{S}_n^N := S_{N-n+1}, \, \tilde{\mathcal{F}}_n^N := \mathcal{F}_{N-n+1} \quad \text{and} \quad \tilde{\xi}_n^N := \xi_{N-n+1} \; .$$

Note that $\{\tilde{\mathcal{F}}_n^N\}_{1 \leq n \leq N}$ is now an increasing σ-field in n. Below we will define everything needed, in order to generalize Doob's Upcrossing Theorem.

Definition A.1. Let $N \geq 2$. For $1 \leq n \leq N$ and $a, b \in \mathbb{R}$ with $a < b$, let

$$\begin{aligned}
T_0 &:= 0 \\
T_1 &:= \begin{cases} \min\{1 \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases} \\
T_2 &:= \begin{cases} \min\{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases} \\
\vdots & \;\; \vdots \;\; \vdots
\end{aligned}$$

$$T_{2m-1} := \begin{cases} \min\{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases}$$

$$T_{2m} := \begin{cases} \min\{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases} .$$

Now we can define the number of upcrossings of $[a, b]$ by $\tilde{S}_1^N, ..., \tilde{S}_N^N$ as follows:

$$U_N^N[a, b] := \begin{cases} \max\{1 \leq m \leq N | T_{2m} < N\} & \text{if } \{1 \leq m \leq N | T_{2m} < N\} \neq \emptyset \\ 0 & \text{if } \{1 \leq m \leq N | T_{2m} < N\} = \emptyset \end{cases}$$

Furthermore let for $1 \leq k \leq n-1$

$$\epsilon_k := \begin{cases} 0 & \text{if } k < T_1 \\ 1 & \text{if } T_1 \leq k < T_2 \\ 0 & \text{if } T_2 \leq k < T_3 \\ 1 & \text{if } T_3 \leq k < T_4 \\ \dots & \text{if } \dots \end{cases}$$

and define

$$Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)$$

for $1 \leq n \leq N$.

Let's now explore how $\lim_{N\to\infty} U_N^N[a, b] < \infty$ implies that S must exist almost surely. Suppose for now that $\lim_{N\to\infty} U_N^N[a, b] < \infty$ and define the set of all ω for which S_n does not converge as

$$\Lambda := \{\omega | S_n(\omega) \text{ does not converge}\} .$$

Consider that can write

$$\begin{aligned}\Lambda &= \{\omega | \liminf_n S_n(\omega) < \limsup_n S_n(\omega)\} \\ &= \bigcup_{a,b\in\mathbb{Q}} \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\} \ .\end{aligned}$$

Recall that we have $U_N^N[a,b]$, the number of upcrossings of $[a,b]$ by $\tilde{S}_1^N, \ldots, \tilde{S}_N^N$. But this is equal to the number of upcrossings of $[a,b]$ by $S_N, \ldots, S_1$. Furthermore recall that

$$U_\infty[a,b] = \lim_{N\to\infty} U_N^N[a,b] \ .$$

Consider that for each $\omega \in \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\}$ we must have $U_\infty[a,b](\omega) = \infty$. This follows directly from the definitions of $\liminf$ and $\limsup$. Thus we can write

$$\Lambda = \bigcup_{a,b\in\mathbb{Q}} \{\omega | U_\infty[a,b](\omega) = \infty\} = \bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}$$

where $\Lambda_{a,b} := \{\omega | U_\infty[a,b](\omega) = \infty\}$. Consequently we get that

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] = \begin{cases} \infty & \text{if } \mathbb{P}(\Lambda_{a,b}) > 0 \\ 0 & \text{if } \mathbb{P}(\Lambda_{a,b}) = 0 \end{cases} \ . \tag{A1}$$

Note that $U_N^N[a,b]$ is clearly non-decreasing in N. Now if $\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] < \infty$, we can apply the Monotone Convergence Theorem to obtain

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] = \mathbb{E}[U_\infty[a,b]] < \infty$$

and hence that

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] \leq \mathbb{E}[U_\infty[a,b]] < \infty \ .$$

Now the latter together with (A1) implies that $\mathbb{P}(\Lambda_{a,b}) = 0$. Therefore we have

$$\mathbb{P}(\Lambda) = \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}\right) = \sum_{a,b\in\mathbb{Q}} \mathbb{P}(\Lambda_{a,b}) = 0 \ .$$

The following Lemmas show how Doob's Upcrossing Theorem can be adapted to our framework. We will show that $\mathbb{E}[U_n^N[a,b]]$ is bounded above by $\mathbb{E}[Y_n^N]/(b-a)$.

Lemma A.2. *For $1 \le n \le N$ we have*

$$\mathbb{E}[U_n^N[a,b]] \le \frac{\mathbb{E}[Y_n^N]}{b-a} \ .$$

Proof. Consider for $1 \le n \le N$ and $N \ge 2$

$$\begin{aligned} Y_n^N &= \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N) \\ &= \tilde{S}_1^N + \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \\ &\ge \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \end{aligned}$$

by definition of ϵ_k. The latter inequality above holds, since $\tilde{S}_1^N \ge 0$. Note that by definition of $T_1, T_2, \ldots$ we have

$$\sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \ge (b-a) U_n^N[a,b] \ .$$

From here the assertion follows directly. □

The following lemma provides a useful representation for the expectation of the process Y_N^n.

Lemma A.3. *For $1 \leq n \leq N$ let*

$$Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)$$

with

$$\epsilon_k := \begin{cases} 1 & (\tilde{S}_1^N, \ldots, \tilde{S}_k^N) \in B_k \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, \ldots, n-1$. Here B_k is an arbitrary set in $\mathfrak{B}(\mathbb{R}^k)$. Then we have

$$\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N | \tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] \ . \tag{A2}$$

Proof. Consider for $1 \leq n \leq N$ and $N \geq 2$

$$\begin{aligned}
&\tilde{S}_{n+1}^N - Y_{n+1}^N \\
&= (1-\epsilon_1)(\tilde{S}_2^N - \tilde{S}_1^N) + (1-\epsilon_2)(\tilde{S}_3^N - \tilde{S}_2^N) + \ldots + (1-\epsilon_k)(\tilde{S}_{n+1}^N - \tilde{S}_n^N) \\
&= (\tilde{S}_n^N - Y_n^N) + (1-\epsilon_n)(\tilde{S}_{n+1}^N - \tilde{S}_n^N) \ .
\end{aligned}$$

Conditioning on $\tilde{\mathcal{F}}_n^N$ on both sides yields

$$\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N | \tilde{\mathcal{F}}_n^N] = \tilde{S}_n^N - Y_n^N + (1-\epsilon_n)\left(\mathbb{E}[(\tilde{S}_{n+1}^N) | \tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right) \ .$$

Now taking expectations on both sides yields

$$\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N] \geq \mathbb{E}[\tilde{S}_n^N - Y_n^N] + \mathbb{E}\left[(1-\epsilon_n)\left(\mathbb{E}[\tilde{S}_{n+1}^N | \tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right)\right] \ .$$

Note that

$$\mathbb{E}[\tilde{S}_2^N - Y_2^N] = \mathbb{E}[\tilde{S}_1^N - Y_1^N] + \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N | \tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right]$$

$$= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right]$$

since $Y_1^N = \tilde{S}_1^N$. Moreover we have

$$\begin{aligned}
\mathbb{E}[\tilde{S}_3^N - Y_3^N] &= \mathbb{E}[\tilde{S}_2^N - Y_2^N] + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right] \\
&\quad + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&\dots \\
\mathbb{E}[\tilde{S}_n^N - Y_n^N] &= \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] .
\end{aligned}$$

Hence we get

$$\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] .$$

□

Remark A.4. Note that we have $Y_1^N = \tilde{S}_1^N$, as the sum in the definition above is in this case empty and hence treated as zero. Moreover note that we have $Y_{n+1}^N = \tilde{S}_{n+1}^N$ if $\epsilon_k = 1$ for all $1 \leq k \leq n$.

The Lemma below establishes an upper bound for $\mathbb{E}[Y_N^N]$ in terms of Q_{ij}^{N-k+1}, as defined in Lemma 3.1.

Lemma A.5. *We have for $N \geq 2$*

$$\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \alpha_{N-k+1} \tag{A3}$$

where

$$\alpha_{N-k+1} := \sum\sum_{1 \le i < j \le N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i,N-k+1} W_{j,N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \ .$$

Proof. Combining Lemmas A.2 and A.3 yields the following

$$(b-a)\mathbb{E}[U_n[a,b]] \le \mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N | \mathcal{F}_k^N] - \tilde{S}_k^N\right)]$$

for all $n \le N$. Moreover we have

$$\begin{aligned}
\mathbb{E}[\tilde{S}_{k+1}^N | \tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k} | \mathcal{F}_{N-k+1}] \\
&= \sum\sum_{1 \le i < j \le N-k+1} \phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i,N-k+1} W_{j,N-k+1} Q_{i,j}^{N-k+1} \ ,
\end{aligned}$$

according to Lemma 3.1. Therefore we obtain

$$\begin{aligned}
\mathbb{E}[Y_N^N] &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N | \mathcal{F}_k^N] - \tilde{S}_k^N] \\
&= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \sum\sum_{1 \le i < j \le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
&\qquad\qquad \left. \times\ W_{i,N-k+1} W_{j,N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \\
&\le \mathbb{E}[\tilde{S}_N^N] + \left| \sum_{k=1}^{N-1} \sum\sum_{1 \le i < j \le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left. \times\ W_{i,N-k+1} W_{j,N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \right| \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1 \le i < j \le N-k+1} \left| \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left. \times\ W_{i,N-k+1} W_{j,N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \right| \ .
\end{aligned}$$

Now using Jensen's inequality yields

$$\begin{aligned}
\mathbb{E}[Y_N^N] \le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} & \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
& \left.\times\, W_{i,N-k+1} W_{j,N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right] \\
\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} & \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
& \left.\times\, W_{i,N-k+1} W_{j,N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right] .
\end{aligned}$$

The latter inequality above holds because $1 - \epsilon_k \le 1$ for all $k \le N-1$. $\square$

Remark A.6. For the almost sure existence of the limit $\lim_{n\to\infty} S_n$, it remains to show that the upper bound on the right hand side of (A3) is finite.

In addition to the almost sure existence of $S(q)$, one may need that

$$S_\infty = \lim_{n\to\infty} \mathbb{E}[S_n]$$

almost surely, in order to identify S_∞. This could be established by the following Lemma (compare Neveu (1975), Lemma V-3-11).

Lemma A.7. *The following statement holds true:*

$$S_\infty = \lim_{n\to\infty} \mathbb{E}[S_n|\mathcal{F}_\infty] = \lim_{n\to\infty} \mathbb{E}[S_n]$$

almost surely, if the limits above exist.

Proof. Let $a > 0$ and let S_n converge to some limit S_∞ almost surely as $n \to \infty$. Now consider that we have

$$\lim_{n\to\infty} \min(S_n, a) = \min(S_\infty, a)$$

almost surely, because $\min(\cdot, a)$ is continuous (see van der Vaart (2000), Theorem 2.3). But $\min(S_n, a)$ is bounded by a. Hence applying the Dominated Convergence Theorem yields

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] &= \mathbb{E}[\lim_{n\to\infty} \min(S_n, a)|\mathcal{F}_\infty] \\
&= \mathbb{E}[\min(S_\infty, a)|\mathcal{F}_\infty] \ .
\end{aligned}$$

Note that S_k is measurable with respect to $\mathcal{F}_n$ whenever $k \geq n$, therefore S_∞ must be $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. Consequently S_∞ must be F_∞-measurable. Moreover, for $a \in \mathbb{R}$, $\min(\cdot, a)$ is a continuous function. Thus $\min(S_\infty, a)$ is $\mathcal{F}_\infty$-measurable as well. Hence

$$\lim_{n\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] = \min(S_\infty, a)$$

almost surely. Thus we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[S_n|\mathcal{F}_\infty] &= \lim_{n\to\infty} \lim_{a\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] \\
&= \lim_{a\to\infty} \lim_{n\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] \\
&= \lim_{a\to\infty} \min(S_\infty, a) \\
&= S_\infty \ . \qquad \text{(A4)}
\end{aligned}$$

almost surely. Moreover we obtain

$$\mathbb{E}[S_n|\mathcal{F}_\infty] = \mathbb{E}[S_n]$$

for all n, by applying Lemma 3.4. Now the latter together with (A4) implies the statement of the lemma. □

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