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# Graded Multiplicity in Harmonic Polynomials from the Vinberg Setting

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# GRADED MULTIPLICITY IN HARMONIC POLYNOMIALS FROM THE VINBERG SETTING

by

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# ABSTRACT

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We consider a family of examples falling into the following context (first considered by Vinberg): Let G be a connected reductive algebraic group over the complex numbers. A subgroup, K, of fixed points of a finite-order automorphism acts on the Lie algebra of G. Each eigenspace of the automorphism is a representation of K. Let $\mathfrak{g}_1$ be one of the eigenspaces. We consider the harmonic polynomials on $\mathfrak{g}_1$ as a representation of K, which is graded by homogeneous degree. Given any irreducible representation of K, we will see how its multiplicity in the harmonic polynomials is distributed among the various graded components. The results are described geometrically by counting integral points on faces of a polyhedron. The multiplicity in each graded component is given by intersecting these faces with an expanding sequence of *shells*.

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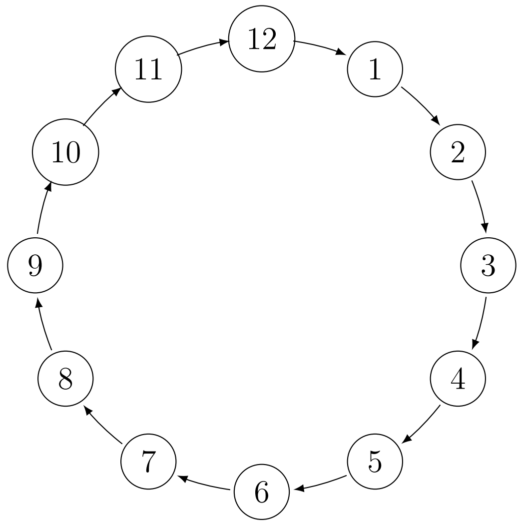
# 1 Introduction

Consider the representations of a *cyclic quiver* on $r$ nodes. If $r = 12$ we have




Recall the definition of a quiver. For each node $j$, associate a finite-dimensional vector space $V_j$. For each arrow $j \to j+1 \pmod{r}$, associate the space of linear transformations, $\mathrm{Hom}(V_j, V_{j+1})$. Set $V = V_1 \oplus \cdots \oplus V_r$, and let $K$ be the block diagonal subgroup of $G = GL(V)$ isomorphic to $GL(V_1) \times \cdots \times GL(V_r)$ acting on

$$\mathfrak{p} = \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_3) \oplus \cdots \oplus \mathrm{Hom}(V_{r-1}, V_r) \oplus \mathrm{Hom}(V_r, V_1).$$

Here we let $GL(U) \times GL(W)$ act on $\mathrm{Hom}(U, W)$ by $(g_1, g_2) \cdot T = g_2 \circ T \circ g_1^{-1}$, as usual. For $(T_1, \ldots, T_r) \in \mathfrak{p}$, we have a $K$-invariant function defined by

$$tr_p(T_1, \ldots, T_r) = \mathrm{Trace}\left[(T_1 \circ \cdots \circ T_r)^p\right],$$

for $1 \leq p \leq \min\{\dim V_j\}$. By a result of Le Bruyn and Procesi [5], the $tr_p$ generate the $K$-invariant functions on $\mathfrak{p}$. The representations considered in this paper will turn out to be equivalent to representations of a cyclic quiver, and this result about the invariant functions will be put to use in finding the decompositions.

Throughout the paper, the ground field is $\mathbb{C}$. Leaving the cyclic quiver above for a moment, we recall the the definition of $G$-harmonic polynomials. Let $G$ denote a linear algebraic group. Given a regular[1] representation, $V$, of $G$, we denote the algebra of polynomial functions on $V$ by $\mathbb{C}[V]$, which we define by identifying with $Sym(V^*)$, the algebra of symmetric tensors on the dual of $V$.

The constant coefficient differential operators on $\mathbb{C}[V]$ will be denoted by $\mathcal{D}(V)$, which we can define by identifying with $Sym(V)$. The differential operators without a constant term will be denoted $\mathcal{D}(V)_+$, and the $G$-invariant differential operators will be denoted by $\mathcal{D}(V)^G$. We define

$$\mathcal{H}(V) = \left\{ f \in \mathbb{C}[V] : \Delta f = 0 \text{ for all } \Delta \in \mathcal{D}(V)^G_+ \right\}$$

to be the *$G$-harmonic* polynomial functions. In the case of $G = SO_3(\mathbb{R})$ acting on its defining representation, $\mathcal{D}(V)^G_+$ is generated by the Laplacian $\partial_x^2 + \partial_y^2 + \partial_z^2$ and the harmonics decompose into minimal invariant subspaces, which, when restricted to the sphere, admit an orthogonal basis, namely the Laplace spherical harmonics familiar from physics [9]. In that case, decomposing the harmonic polynomials (the subject of this paper) leads to a complete set of orthogonal functions on the sphere, useful in numerous theoretical and practical applications.

In general, every polynomial function can be expressed as a sum of $G$-invariant functions multiplied by $G$-harmonic functions. That is, there is a surjection

$$\mathbb{C}[V]^G \otimes \mathcal{H}(V) \to \mathbb{C}[V] \to 0$$

obtained by linearly extending multiplication. When is this an isomorphism? That is, when is each polynomial a unique sum of products of invariants and harmonics? Equivalently, when is $\mathbb{C}[V]$ a free module over $\mathbb{C}[V]^G$, the algebra of $G$-invariant functions? For the Laplace

[1] That is, the morphism $G \to GL(V)$ is a morphism of complex linear algebraic groups.

spherical harmonics this isomorphism holds. All invariants are generated by the squared Euclidean distance function, and any polynomial can be written uniquely as a product of its radial and spherical components.

The scalar multiplication of $\mathbb{C}$ on $V$ commutes with the action of $G$. The resulting $\mathbb{C}^\times$-action gives rise to gradation on $\mathbb{C}[V]$, which is the usual notion of *degree*. The $G$-harmonic functions inherit this gradation, so we can define $\mathcal{H}_n(V)$ to be the homogeneous $G$-harmonic functions of degree $n$. We have the direct sum of $G$-representations

$$\mathcal{H}(V) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(V).$$

We now assume that $G$ is reductive in the category of linear algebraic groups. Under this assumption, every regular (hence finite-dimensional) representation of $G$ is completely reducible. Let $\{F_\mu\}_{\mu \in \widehat{\mathrm{G}}}$ denote a set of representatives of the irreducible representations of $G$.

**Problem:** For each $n$, how does $\mathcal{H}_n(V)$ decompose? That is, given $\mu \in \widehat{G}$, what is the multiplicity of the irrep $F_\mu$ inside $\mathcal{H}_n$, denoted

$$[F_\mu : \mathcal{H}_n(V)] = \ ?$$

Returning to the cyclic quiver above, the $K$-harmonic functions on $\mathfrak{p}$ form a graded representation of $K$:

$$\mathcal{H}(\mathfrak{p}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(\mathfrak{p})$$

The fact that the polynomial functions are a free module over the invariants is a consequence of the Vinberg theory of $\theta$-groups [10]. Yet, the literature on quivers does not seem to address the structure of the associated harmonic polynomials.

As a representation of $K$, the harmonics are equivalent to an induced representation. For details, see Chapter 3 of Nolan Wallach's book, entitled *Geometric invariant theory over* *the real and complex numbers* [12]. Alternatively, see *An Analogue of the Kostant-Rallis Multiplicity Theorem for $\theta$-group Harmonics* [11].

## 1.1 Background from some existing literature

The standard results concerning spherical harmonics on $\mathbb{R}^3$ were generalized by Kostant in his 1963 paper *Lie group representations on polynomial rings* [3]. This is Kostant's most often cited paper. Among many of its results, it establishes that $\mathbb{C}[\mathfrak{g}]$ is a free module over $\mathbb{C}[\mathfrak{g}]^G$ for a connected reductive group $G$.

To a combinatorialist, a natural thing to do is consider the, indeed polynomial, defined by the series

$$p_\mu(q) = \sum_{n=0}^{\infty} [F_\mu : \mathcal{H}_n(V)]\, q^n.$$

In the 1963 case addressed by Kostant, $V = \mathfrak{g}$, the adjoint representation.

These polynomials extract deep information in representation theory. For starters, they are Kazhdan-Lusztig polynomials for the affine Weyl group [2]. Outside of Kostant's setting, very little is known about them.

In the case that $\mathfrak{g}$ is of Lie type A, then $p_\lambda(q)$ was studied by Richard Stanley in [7]. Later on, connections with Hall-Littlewood polynomials were made [6]. Even combinatorial interpretations for their coefficients are known. An alternating sum formula was found by Hesselink in [1].

Then, in 1971, Kostant and Rallis obtained a generalization to the symmetric space setting in [4]. The Kostant-Rallis setting applies to each symmetric pair $(G, K)$. That is, $K$ is the fixed point set of a regular involution on a connected reductive group $G$. A natural way to generalize is to consider $K$ that are fixed by automorphisms of order larger than two. Exactly this was done by Vinberg in his 1976 theory of $\theta$-groups, published as *The Weyl group of a graded Lie algebra* [10]. Since then, an enormous amount of work has been done on $\theta$-groups, but the analog of the graded structure of harmonic polynomials still does not exist. Taking $\theta : G \to G$ to be the identity automorphism on $G = SO_3(\mathbb{R})$ we have $K = G$ acting on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^3$ and we recover the spherical harmonics example.

## 1.2 The results of this paper

In this paper, we consider an infinite family of examples. For each choice of $r \geq 2$ we will define a Vinberg pair $(G, K)$ where $K$ will be the fixed points of an order $r$ automorphism $\theta : G \to G$ for a connected reductive linear algebraic group $G$ over $\mathbb{C}$. As will be described below, we define a representation of $K$ on a space of polynomial functions $\mathcal{P}$ on an eigenspace of $d\theta$, and so also a representation of $K$ on the space of $K$-harmonic polynomials $\mathcal{H} \subset \mathcal{P}$. As a consequence of Vinberg's theory of $\theta$-groups, the polynomials will be free over the invariants and

$$\mathcal{P} = \mathcal{P}^K \otimes \mathcal{H}$$

Being interested in the structure of $\mathcal{H}$ as a $K$-representation, we will immediately notice that there is a multi-gradation on $\mathcal{H}$ such that each multi-graded component $\mathcal{H}_{\vec{n}}$ is itself a subrepresentation, for each $\vec{n} \in \mathbb{N}^r$. So we ask, what is the multiplicity of each $K$-type $F_\mu$ for $\mu \in \widehat{K}$ inside any multi-graded component $\mathcal{H}_{\vec{n}}$?

$$[F_\mu : \mathcal{H}_{\vec{n}}] = \ ?$$

There are of course many graded components $\mathcal{H}_{\vec{n}}$, one for each choice of $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$. But actually, describing the graded multiplicity of a fixed irrep $F_\mu$ will surprisingly become one dimensional, in the sense that all nonzero multiplicity will occur in $\mathcal{H}_{\vec{n}}$ for $\vec{n}$ sitting along a single ray inside $\mathbb{N}^r$ (Figure 1.1). This reduction in complexity has the additional nice property that the ray of nonzero multiplicity is completely determined by the center of $K$. It will turn out that the irreps of $K$ can be parametrized by $\vec{z} = (z_1, \ldots, z_{r-1}) \in \mathbb{Z}^{r-1}$ and $\vec{s} = (s_1, \ldots, s_{r-1}, s_r) \in \mathbb{N}^r$, where $\vec{z}$ corresponds to the center of $K$ and $\vec{s}$ corresponds to the semisimple part of $K$. The central parameter $\vec{z}$ will determine $\text{ray}(\vec{z}) \subset \mathbb{N}^r$, and all

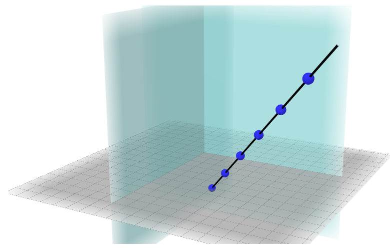


Figure 1.1: Points $\vec{n}$ in ray$(\vec{z})$ when $r = 3$

non-zero multiplicity will occur in $\mathcal{H}_{\vec{n}}$ for $\vec{n} \in$ ray$(\vec{z})$.

How is that non-zero multiplicity distributed along this ray? We give a *geometric description* of the answer in terms of counting intersection points between the faces of a polyhedron and a ray of *shells* Shell$(\Lambda_{\vec{n}})$. First, for each point $\vec{n} \in$ ray$(\vec{z})$ we will define an array of points $\Lambda_{\vec{n}} \subset \mathbb{N}^r$ and its *shell* Shell$(\Lambda_{\vec{n}}) \subset \mathbb{N}^r$, pictured in Figure 1.2. The parameter $\vec{s} \in \mathbb{N}^r$

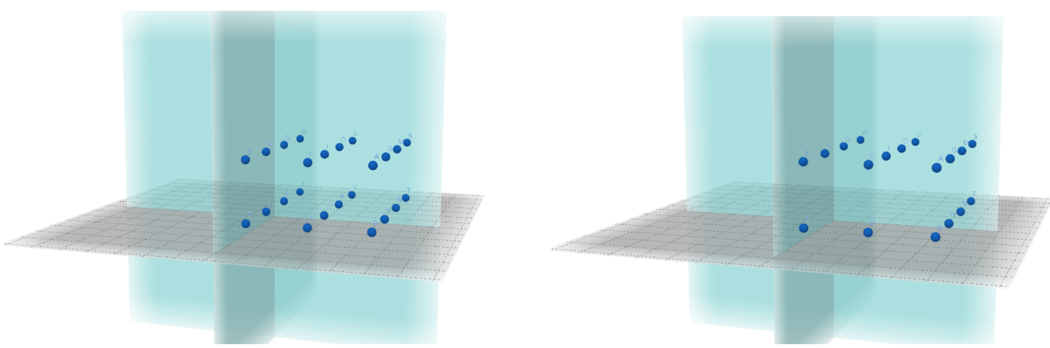


Figure 1.2: $\Lambda_{(6,5,3)}$ and Shell$(\Lambda_{(6,5,3)})$

corresponding to the semisimple part of $K$ will determine the hypersurface $\mathcal{S} \subset \mathbb{R}^r$, where $\mathcal{S}$ is simply $r$ faces of a certain polyhedron constructed from the parameter $\vec{s} \in \mathbb{N}^r$. The number of intersection points

$$\#\left(\text{Shell}(\Lambda_{\vec{n}}) \bigcap \mathcal{S}\right)$$

will count the multiplicity $[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}]$. The structure of $K$ shows up nicely in this description of graded multiplicity, because the center of $K$ determines ray$(\vec{z})$, and hence also determines the entire ray of *shells* Shell$(\Lambda_{\vec{n}})$ for $\vec{n} \in$ ray$(\vec{z})$, while the semisimple part of $K$ determines $\mathcal{S}$. For $r = 3$ we can actually draw the faces of the polyhedron inside $\mathbb{R}^3$. As an example, the irreducible representation $F_{(5,1),(7,5,4)}$ occurs with multiplicity six in $\mathcal{H}_{(6,5,3)}$, and Figure

1.3 illustrates the six intersection points between Shell($\Lambda_{(6,5,3)}$) and $\mathcal{S}$. For dimension $r > 3$,

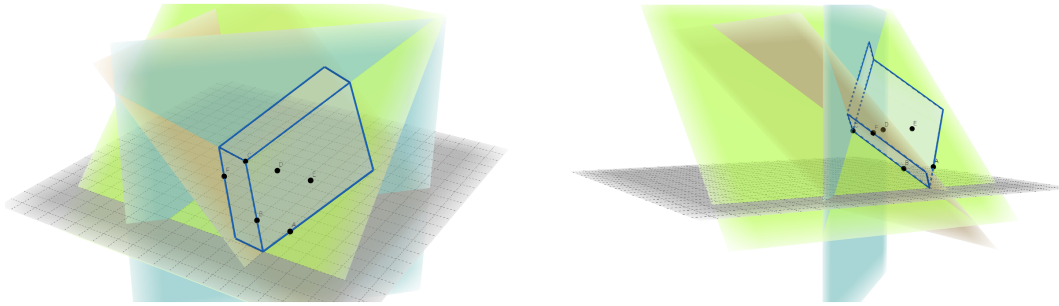


Figure 1.3: 6 Intersection points between $\mathcal{S}$ and Shell($\Lambda_{(6,5,3)}$)

despite not being able to draw the polyhedron, the same description of graded multiplicity holds, namely, the graded multiplicity is given by intersecting Shell($\Lambda_{\vec{n}}$) with $r$ faces of a polyhedron.

## 1.3 Structure of the paper

In Section 2 we describe in detail the infinite family of Vinberg pairs $(G, K)$ considered in this paper. In Section 3 we describe the irreducible representations of $K$ in terms of parameters corresponding to the center and semisimple parts of $K$. In Section 4 we see two explicit examples of our results for $r = 2$ and $r = 3$, including pictures of the hypersurface $\mathcal{S}$ and its intersections with the Shell($\Lambda_{\vec{n}}$), which count the graded multiplicity. In Section 5 we prove how a fixed graded component $\mathcal{H}_{\vec{n}}$ decomposes into irreps by using character theory. Finally in Section 6 we connect this decomposition of $\mathcal{H}_{\vec{n}}$ to integral geometry and our description of multiplicity in terms of intersections of the Shell($\Lambda_{\vec{n}}$) with the hypersurface $\mathcal{S}$, proving the assertions made for the examples in Section 4.

## 2 A specific family of Vinberg pairs

Let $G = GL(V)$ or $SL(V)$ for some finite-dimensional vector space $V$ over $\mathbb{C}$. In either case, we can construct a Vinberg pair $(G, K)$ for each choice $r \geq 2$ (since $r = 1$ corresponds to the identity automorphism). Thus, we will have two infinite families of examples. However, regardless of whether we begin with $G$ the general linear or special linear group, the results will be extremely similar. Therefore we will provide the details for only one family of examples, those where $G = SL(V)$. We will provide comments at appropriate places reminding the reader that the entire story can be worked out almost identically for the general linear group as well.

To construct a Vinberg pair $(G, K = G^\theta)$ we also need a finite-order automorphism $\theta$. What do these look like? First, simplify the question by considering inner automorphisms. If we have an inner automorphism of order $r$, it comes from conjugation by some $A \in G$, denoted $\theta_A$ sending $g \mapsto AgA^{-1}$. Requiring $\theta_A^r = 1$ forces $A^r = \lambda I$, the identity transformation. If $A^r = I$ for some $A \in G$ then this forces $A$ to be (up to conjugation) diagonal with entries from the set of $r$th roots of unity $\{1, \zeta, \zeta^2, \ldots, \zeta^{r-1}\}$. Since we are only concerned with the automorphism action $g \mapsto AgA^{-1}$, the factor of $\lambda$ will not affect the fixed points $K = G^\theta$. Thus, we can take our arbitrary finite-order inner automorphism to be simply:

conjugation by a diagonal matrix $A$ of rth roots of unity, its eigenvalues.

The simplest case is when all eigenvalues of $A$ have multiplicity 1. In this case $K = G^\theta$ is simply the diagonal subgroup isomorphic to $(\mathbb{C}^\times)^m$ for some $m$. It's irreps are all one-dimensional and the space of harmonics would decompose into one-dimensional irreps. The paper would be over too quickly.

The first difficult case is when eigenvalues have multiplicity $> 1$. In those cases $K = G^\theta$ will be a block diagonal subgroup with blocks equal in size to the multiplicities of each root of unity.

*Example.* If

$$A = \begin{bmatrix} \zeta & & & & \\ & \zeta & & & \\ & & \zeta & & \\ & & & \zeta^3 & \\ & & & & \zeta^3 \end{bmatrix}$$

then $K = G^\theta$ will be block diagonal with one 3 by 3 block and one 2 by 2 block.

In these cases the irreps of $K = G^\theta$ and the corresponding decomposition of the harmonics become much more interesting and difficult. In this paper we make a first step in this direction, by considering the inner automorphisms that fix 2 by 2 block diagonal subgroups.

## 2.1 An infinite family of examples

For each choice $r \geq 2$, we construct a Vinberg Pair $(G, K)$ where $K$ is 2 by 2 block diagonal. Let $G = SL_{2r}(\mathbb{C})$ and define the inner automorphism of order $r$

$$\theta : G \to G$$
$$g \mapsto hgh^{-1}$$

where $h$ is the diagonal $2r$ by $2r$ matrix with eigenvalues the $r$th roots of unity: $1, \zeta, \ldots, \zeta^{r-1}$, each with multiplicity 2. The fixed points are exactly the block diagonal subgroup

$$K = G^\theta = S(\underbrace{GL_2 \times \cdots \times GL_2}_{r \text{ factors}})$$

of 2 by 2 blocks with overall determinant 1. If we had taken $G = GL_{2r}$ rather than $SL_{2r}$ then $K$ would have been $GL_2 \times \cdots \times GL_2$ without the requirement that the overall determinant be 1. Thus $K$ differs only by one parameter in the center between the two cases. Both choices work out almost identically, so we provide details only for $G = SL_{2r}$ in this paper.

*Example.* For $r = 3$ and $g \in SL_6(\mathbb{C})$ we have

$$hgh^{-1} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \zeta & & & \\ & & & \zeta & & \\ & & & & \zeta^2 & \\ & & & & & \zeta^2 \end{pmatrix} g \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \zeta^2 & & & \\ & & & \zeta^2 & & \\ & & & & \zeta & \\ & & & & & \zeta \end{pmatrix}$$

The fixed points are exactly the block diagonal subgroup

$$K = G^\theta = S(GL_2 \times GL_2 \times GL_2)$$

The automorphism descends to the automorphism $d\theta$ on the Lie algebra $\mathfrak{g}$, which decomposes into eigenspaces. Following Vinberg we index the eigenspaces by $\mathbb{Z}/r\mathbb{Z}$ as $\mathfrak{g} = \mathfrak{g}_\mathfrak{o} \oplus \mathfrak{g}_\mathfrak{1} \oplus \cdots \oplus \mathfrak{g}_{\mathfrak{r}-\mathfrak{1}}$, where $\mathfrak{g}_\mathfrak{o}$ denotes the eigenspace with eigenvalue $1 = \zeta^0$, $\mathfrak{g}_\mathfrak{1}$ denotes the eigenspace with eigenvalue $\zeta^1$, and so on, where $\zeta$ is a primitive $r$th root of unity. Since $G$ acts on $\mathfrak{g}$ via the Adjoint action, then also $K$ acts on $\mathfrak{g}$ in the same way. By restriction $K$ acts on each eigenspace. Thus the polynomial functions on each eigenspace yield a representation of $K$. For the eigenspace $\mathfrak{g}_0$, this is simply $K$ acting on its Lie algebra and we are in the original case described by Kostant in 1963. For any eigenspace $\mathfrak{g}_1, \ldots, \mathfrak{g}_{r-1}$ Kostant's theory does not apply, but Vinberg's theory does. As can be quickly calculated, each $\mathfrak{g}_1, \ldots, \mathfrak{g}_{r-1}$ is isomorphic as a vector space to $r$ copies of the 2 by 2 matrices, so without loss of generality consider the $\zeta^1$-eigenspace

$$\mathfrak{g}_1 = X_1 \oplus \cdots \oplus X_r$$

where each $X_i = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$.

*Example.* Consider $r = 3$. If we take an element $X \in \mathfrak{g}$ broken into 2 by 2 blocks

$$X = \begin{pmatrix} Z_1 & X_1 & Y_1 \\ Y_2 & Z_2 & X_2 \\ X_3 & Y_3 & Z_3 \end{pmatrix}$$

we see the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where

$$\mathfrak{g}_1 \cong X_1 \oplus X_2 \oplus X_3$$

We can restrict the Adjoint action of $K$ on $\mathfrak{g}$ to the eigenspace $\mathfrak{g}_1$. If we write an arbitrary element $g \in K = S(GL_2 \times GL_2 \times GL_2)$ as $g = (g_1, g_2, g_3)$ and if $X \in \mathfrak{g}_1$ then we have an action $X \mapsto g.X$ given by

$$\begin{aligned} g.X &= \begin{pmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{pmatrix} \begin{pmatrix} & X_1 & \\ & & X_2 \\ X_3 & & \end{pmatrix} \begin{pmatrix} g_1^{-1} & & \\ & g_2^{-1} & \\ & & g_3^{-1} \end{pmatrix} \\ &= \begin{pmatrix} & g_1 X_1 g_2^{-1} & \\ & & g_2 X_2 g_3^{-1} \\ g_3 X_3 g_1^{-1} & & \end{pmatrix} \end{aligned}$$

where $X \in \mathfrak{g}_1$ is $X = X_1 + X_2 + X_3$ and each of $X_1, X_2, X_3$ is simply a 2 by 2 matrix.

**Definition 1.** For each $r \geq 2$ let $(G, K)$ be the Vinberg pair defined above where

$$G = SL_{2r}(\mathbb{C}) \text{ and } K = S\big(\underbrace{GL_2 \times \cdots \times GL_2}_{r \text{ factors}}\big)$$

Let $\mathcal{P}$ be the space of polynomials on $\mathfrak{g}_1$, the $\zeta^1$-eigenspace of $d\theta$. Define an action

$$K \times \mathcal{P} \longrightarrow \mathcal{P}$$
$$(g, f) \longmapsto g.f$$

where

$$(g.f)(x) = f(g^{-1}x)$$

This gives a representation of $K$ on $\mathcal{P}$, which we can restrict to the $K$-harmonic functions $\mathcal{H} \subset \mathcal{P}$.

We consider the $K$-harmonic polynomials $\mathcal{H}$ inside $\mathcal{P}$, and because this example falls into the Vinberg setting [10], we know that every polynomial can be written uniquely as a sum of invariants times harmonics, i.e. that

$$\mathcal{P} = \mathcal{P}^K \otimes \mathcal{H}$$

See, for example, the paper by Wallach [11]. The subject of this paper is to ask about the structure of $\mathcal{H}$ as a representation of $K$, specifically addressing the multi-graded structure of $\mathcal{H}$, as we will describe presently.

Consider $\mathfrak{g}_1 = X_1 \oplus \cdots \oplus X_r$. Since each $X_i$ is a 2 by 2 matrix, isomorphic to $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$, we consider homogeneous polynomials of degree $n_i$ on each $X_i$, which we can think of as polynomials of degree $n_1$ in the four variables coming from $X_1$, degree $n_2$ in the four variables coming from $X_2$, and so on. Thus we have a multi-gradation on $\mathcal{P}$ and hence also on $\mathcal{H}$. We denote by $\mathcal{H}_{\vec{n}}$ the multi-graded component of harmonic polynomials of homogeneous degree $n_i$ on $X_i$, where the subscript labeling the graded component $\mathcal{H}_{\vec{n}}$ is $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$.

Being interested in the structure of $\mathcal{H}$ as a $K$-representation we immediately observe that each multi-graded component is invariant, and so we can ask, for any component $\mathcal{H}_{\vec{n}}$, how does it decompose as a $K$-representation? We also ask, given an irrep $F_\mu$ of $K$, what is the multiplicity

$$[F_\mu : \mathcal{H}_{\vec{n}}] = \ ?$$

for all $\mu \in \widehat{K}$ and for all $\vec{n} \in \mathbb{N}^r$.

**Remark:** Recall the previous discussion of quivers. Consider the cyclic quiver on $r \geq 2$ nodes, where each $V_1, \ldots, V_r$ is simply $\mathbb{C}^2$, so that $V \cong \mathbb{C}^{2r}$ and consider $G = SL(V) \cong SL_{2r}(\mathbb{C})$, so that $K$ is the block diagonal subgroup isomorphic to a product of $r$ copies of $GL_2$ with overall determinant 1, denoted

$$K = S\big(\underbrace{GL_2 \times \cdots \times GL_2}_{r \text{ factors}}\big)$$

acting on

$$\begin{aligned}\mathfrak{p} &= \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \\ &= X_1 \oplus \cdots \oplus X_r\end{aligned}$$

For each $r \geq 2$ the representation of $K$ on $\mathcal{P}$ constructed above is equivalent to this representation of a cyclic quiver. Due to a result of Le Bruyn and Procesi [5], all invariant functions in $\mathcal{P}$ are generated by two. Call them

$$f = \mathrm{trace}(X_1 X_2 \cdots X_r)$$

and

$$g = \mathrm{trace}\big((X_1 X_2 \cdots X_r)^2\big)$$

We will make use of this in Section 5.2.

## 3 The irreducible representations of $K$

To describe the multiplicity of each $K$-type in the various graded components $\mathcal{H}_{\vec{n}}$, we first must describe all the irreducible representations of $K$. In Section 3.1 we describe a covering map $\varphi$ and its kernel, plus make some necessary definitions and notation. In 3.2 we see which irreducible representations of a covering group $\tilde{K}$ factor through

$$\varphi : \tilde{K} \to K$$

to become irreps of $K$ as well. In 3.3 we will see how the center $Z(K)$ acts on the multi-graded component $\mathcal{P}_{\vec{n}}$. We will also see that the multiplicity $[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = 0$ for all $\vec{n} \notin \text{ray}(\vec{z})$. In 3.4 we will see that in decomposing $\mathcal{H}_{\vec{n}}$ as a representation of $K$ it will be enough to ignore the *central characters* and decompose using the *semisimple characters* alone. All of these terms will be defined below.

This section is very detailed, but necessary. If you accept that the irreducible representations of $K$ are denoted $F_{\vec{z},\vec{s}}$ where $\vec{z} \in \mathbb{Z}^{r-1}$ and $\vec{s} \in \mathbb{N}^r$ subject to a mod 2 condition, then you may skip to Section 4: Examples, for some nice pictures and motivation.

### 3.1 The kernel of a covering map

Consider the surjective map

$$\mathbb{C}^\times \times SL_2 \to GL_2$$

given by sending

$$(w, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \longmapsto \begin{pmatrix} wa & wb \\ wc & wd \end{pmatrix}$$

Every irrep of $GL_2$ pulls back to an irrep of $\mathbb{C}^\times \times SL_2$, but $\mathbb{C}^\times \times SL_2$ has more irreps. Another way to say this is that taking the algebraic dual in the category of representations of linear algebraic groups with regular matrix coefficients, we have an injection

$$\widehat{GL_2} \hookrightarrow \widehat{\mathbb{C}^\times \times SL_2}$$

**Definition 2.** Since $K = S(GL_2^r)$ define $\tilde{K} = (\mathbb{C}^\times)^{r-1} \times SL_2^r$, then define

$$\varphi : \tilde{K} \to K$$

given by sending

$$(w_1, \ldots, w_{r-1}, g_1, \ldots, g_{r-1}, g_r) \mapsto (w_1 g_1, w_2 g_2, \ldots, w_{r-1} g_{r-1},\ w_1^{-1} w_2^{-1} \cdots w_{r-1}^{-1} g_r)$$

where $w_i \in \mathbb{C}^\times$ for $i \in \{1, \ldots, r-1\}$ and $g_i \in SL_2$ for $i \in \{1, \ldots, r\}$.

**Remark:** We need the overall determinant to be 1, so we made the arbitrary choice of multiplying the last factor $g_r \in SL_2$ by the products of inverses of all the $w_i$.

*Example.* We give a quick example of the kernel of $\varphi$, and then prove it in general in the next Proposition. For $r = 3$, the kernel of $\varphi$ is the four element set labeled by $g_{(\pm 1, \pm 1)}$ as in

$$\Big\{ g_{(-1,-1)} = (-1, -1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}),$$

$$g_{(1,-1)} = (1, -1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}),$$

$$g_{(-1,1)} = (-1, 1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}),$$

$$g_{(1,1)} = (1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \Big\}$$

**Proposition 1.** The kernel of $\varphi$ is the set of all tuples

$$(w_1, w_2, \ldots, w_{r-1}, \begin{pmatrix} w_1 & 0 \\ 0 & w_1 \end{pmatrix}, \ldots, \begin{pmatrix} w_{r-1} & 0 \\ 0 & w_{r-1} \end{pmatrix}, \begin{pmatrix} \prod w_i & 0 \\ 0 & \prod w_i \end{pmatrix})$$

where each $w_i \in \{1, -1\}$ and the product $\prod w_i$ is taken for $i \in \{1, \ldots, r-1\}$. Denote the individual elements of the kernel by

$$g_{\vec{w}} = g_{(w_1, \ldots, w_{r-1})}$$

*Proof.* Write an arbitrary element of $\tilde{K}$ (as in Definition 2) as a tuple of $w_i \in \mathbb{C}^\times$ and $g_i \in SL_2$. First we consider $i \in \{1, \ldots, r-1\}$, then we deal with $i = r$ at the end. Requiring that $w_i g_i = I_2$ is to require

$$g_i = \begin{pmatrix} w_i a_i & w_i b_i \\ w_i c_i & w_i d_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and since $w_i \in \mathbb{C}^\times$ clearly each $b_i = c_i = 0$ and $w_i a_i = 1$ and $w_i d_i = 1$. Multiplying these last two expressions together we see that $w_i^2 = 1$ since $a_i d_i - b_i c_i = 1$ becomes $a_i d_i = 1$. Thus $w_i \in \{1, -1\}$. But then $w_i a_i = 1$ and $w_i d_i = 1$ imply that $w_i = a_i = d_i$, thus for $i \in \{1, \ldots, r-1\}$

$$g_i \in \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\}$$

and $w_i = -1$ if and only if $g_i = -I_2$, as required. For $g_r$ we have that

$$w_1^{-1} w_2^{-1} \cdots w_{r-1}^{-1} g_r = I_2$$

Since each $w_i$ equals 1 or $-1$, $w_i^{-1} = w_i$, and a similar argument shows that $g_r$ can be written using $\prod w_i$ as claimed. Because there are exactly as many elements in the kernel as choices of $\{1,-1\}$ for each $w_i$, a convenient notation for the elements of the kernel is $g_{\vec{w}}$ where $\vec{w}$ has each $w_i \in \{1,-1\}$. $\square$

We consider irreducible representations of $\tilde{K}$ with matrix coefficients that are regular functions on the affine variety $\tilde{K}$. These representations are all finite-dimensional. Recall for the group $\mathbb{C}^\times$ all irreps are one dimensional and the irreducible characters are given for $z_i \in \mathbb{Z}$ by

$$w_i \longmapsto w_i^{z_i}$$

Also recall that if $g_i = \mathrm{diag}(u_i, u_i^{-1})$ is a parametrization of the maximal torus of $SL_2$ then the irreducible characters are given by

$$\begin{pmatrix} u_i & 0 \\ 0 & u_i^{-1} \end{pmatrix} \longmapsto u_i^s + u_i^{s-2} + \cdots + u_i^{-s}$$

for all $s \in \mathbb{N}$, where, as we will see, $s$ was chosen for *semisimple*. Because our representations are regular, knowing the character on the torus determines the character everywhere. For this reason when calculating characters we will restrict them to elements of the torus. We denote these irreducible characters by $\chi_s$ or $\chi_s(u_i)$ when we include the choice of toral variable. This is in contrast to our notational usage of $\chi$ to denote the character of a graded component of the harmonics or of the polynomials, as in $\chi(\mathcal{P}_{\vec{n}})$.

**Proposition 2** (Clebsch-Gordan)**.** When we have two irreducible characters of $SL_2$ parametrized by the same variable, we can expand their product into a sum as follows:

$$\chi_a(u)\chi_b(u) = \chi_{a+b}(u) + \chi_{a+b-2}(u) + \cdots + \chi_{|a-b|}(u)$$

*Proof.* This identity is well-known, but briefly, first notice that we can write $\chi_a(u)$ as

$$u^a + u^{a-2} + \cdots + u^{-(a-2)} + u^{-a} = \frac{u^{a+1} - u^{-a-1}}{u - u^{-1}}$$

Then we have the product

$$\chi_a(u)\chi_b(u) = \big(\frac{u^{a+1} - u^{-a-1}}{u - u^{-1}}\big)\big(u^b + u^{b-2} + \cdots + u^{-b+2} + u^{-b}\big)$$

Distributing and collecting terms in the correct order gives our result. □

Recall $\tilde{K} = (\mathbb{C}^\times)^{r-1} \times SL_2^r$. It's irreducible representations are standard: they are tensor products of irreps of $\mathbb{C}^\times$ and $SL_2$.

**Definition 3.** We parametrize the irreducible representations of $\tilde{K}$ by $\vec{z} \in \mathbb{Z}^{r-1}$ and $\vec{s} \in \mathbb{N}^r$ and denote them by

$$\tilde{F}_{\vec{z},\vec{s}}$$

where $\vec{z} = (z_1, \ldots, z_{r-1})$ and $\vec{s} = (s_1, \ldots, s_r)$. The $\sim$ on $\tilde{F}_{\vec{z},\vec{s}}$ simply reminds us we are talking about irreps of the covering group $\tilde{K}$.

Consider the irreducible characters of these representations. Again, when we evaluate a character at a group element, we will always restrict to the torus in each $SL_2$ factor. We will usually choose the variable $u_i$ to parametrize the torus of each $SL_2$.

**Definition 4.** We denote the character of $\tilde{F}_{\vec{z},\vec{s}}$ evaluated at a group element $g \in \tilde{K}$ by

$$\chi_{\vec{z},\vec{s}}(g) = w_1^{z_1} \cdots w_{r-1}^{z_{r-1}} \chi_{s_1}(u_1) \cdots \chi_{s_{r-1}}(u_{r-1})\chi_{s_r}(u_r)$$

where each $SL_2$ factor has been restricted to its torus as in

$$g = (w_1, \ldots, w_{r-1}, \begin{pmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} u_r & 0 \\ 0 & u_r^{-1} \end{pmatrix})$$

## 3.2 Parametrizing the irreducible representations of $K$

The identity element of $K$ must act via the identity transformation in any representation. So, if an irrep of $\tilde{K}$ is going to factor through $\varphi$, becoming an irrep of $K$, we must require that $g_{\vec{w}} \in \ker\varphi$ act by the identity transformation as well. The character of a group element acting by identity is equal to the dimension. Thus, the irreps $\tilde{F}_{\vec{z},\vec{s}}$ that factor through $\varphi$ are exactly the irreps such that evaluating the character at any $g_{\vec{w}} \in \ker\varphi$ gives the dimension.

**Proposition 3.** The irreducible representations of $K$, denoted $F_{\vec{z},\vec{s}}$, are parametrized by $\vec{z} \in \mathbb{Z}^{r-1}$ and $\vec{s} \in \mathbb{N}^r$ such that

$$\sum_{i=1}^{r-1} z_i + \sum_{i=1}^{r} s_i = 0 \pmod 2$$

These are exactly the irreps $\tilde{F}_{\vec{z},\vec{s}}$ of $\tilde{K}$ that factor through $\varphi$.

Before giving the proof we state a lemma. Recall that $\chi_s(w)$ for $w \in \mathbb{C}^\times$ denotes the irreducible character of an $(s+1)$-dimensional $SL_2$ irrep evaluated at the element of the torus parametrized by $w$ as in

$$\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \longmapsto w^s + w^{s-2} + \cdots + w^{-s+2} + w^{-s}$$

**Lemma 1.** If $s$ is odd

$$\chi_s(-1) = -(s+1)$$

$$\chi_s(1) = s+1$$

and if $s$ is even

$$\chi_s(-1) = s + 1$$
$$\chi_s(1) = s + 1$$

*Proof.* Direct calculation. □

*Proof of Proposition 3.* Consider the character of an irrep $\tilde{F}_{\vec{z},\vec{s}}$ of $\tilde{K}$ evaluated at an element of the kernel $g_{\vec{w}}$. We have that

$$\chi(g_{\vec{w}}) = w_1^{z_1} w_2^{z_2} \cdots w_{r-1}^{z_{r-1}} \chi_{s_1}(w_1)\chi_{s_2}(w_2)\cdots\chi_{s_{r-1}}(w_{r-1})\chi_{s_r}(w_1 w_2 \cdots w_{r-1})$$

Recall each $w_i \in \{1, -1\}$ since $g_{\vec{w}} \in \ker\varphi$. Each factor $w_i^{z_i}$ will equal $\pm 1$, and each factor $\chi_{s_i}(w_i)$ will equal $\pm(s_i + 1)$, and so this character evaluated on elements of the kernel will give $\pm$ the dimension. Thus, if the irrep $\tilde{F}_{\vec{z},\vec{s}}$ factors through $\varphi$ then we need to check that the character is positive, simultaneously for all elements of the kernel. For which irreps does this happen?

Consider $\vec{z} = \vec{0}$ and $\vec{s} = \vec{0}$, meaning, consider the irrep $\tilde{F}_{\vec{0},\vec{0}}$, where $\vec{0} = (0, 0, \ldots, 0)$ is shorthand for the vector full of zeros. It's easy to see that $\chi(g_{\vec{w}}) > 0$ for all $g_{\vec{w}} \in \ker\varphi$ and also that

$$\sum_{i=1}^{r-1} z_i + \sum_{i=1}^{r} s_i = 0 \pmod 2 \tag{3.1}$$

holds. Now consider, for some fixed $i \in \{1, \ldots, r-1\}$, changing $z_i$ by $\pm 1$. For any choice of $g_{\vec{w}}$, every factor keeps the same sign, except the factor $w_i^{z_i \pm 1}$ which *changes* sign for any $g_{\vec{w}}$ with $w_i = -1$. This changes the sign of $\chi(g_{\vec{w}})$, meaning the irrep no longer factors through $\varphi$. But also Equation 3.1 now fails to hold. If now we alter yet another $z_j$ by $\pm 1$, for any $j \in \{1, \ldots, r-1\}$ including the same factor $j = i$ from before, exactly the reverse happens: we go from an irrep that fails to factor, to an irrep that succeeds. At the same time Equation 3.1 changes from false to true.

Similarly, by Lemma 1, altering any $s_i$ for any $i \in \{1, \ldots, r\}$ by adding 1 introduces elements of the kernel which make the character $\chi(g_{\vec{w}})$ negative, and at the same time causes Equation 3.1 to change truth value. This works for $s_r$ as well. In this way, we can move from the irrep $\tilde{F}_{\vec{0},\vec{0}}$ to any irrep of $\tilde{K}$, and so we have shown that Equation 3.1 exactly records the ability of an irrep to factor through $\varphi$, proving the Proposition. □

## 3.3 The action of the center on $\mathcal{P}_{\vec{n}}$

In this section we will see how the center $Z(K)$ acts on the multi-graded component $\mathcal{P}_{\vec{n}}$. We will also see that the multiplicity $[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = 0$ for all $\vec{n} \notin \text{ ray}(\vec{z})$, which we will define below.

**Definition 5.** Recall $K = S(GL_2 \times \cdots \times GL_2)$. We choose a parametrization of the center $Z(K)$ by $w_1, \ldots, w_{r-1} \in \mathbb{C}^\times$ where an element $g \in Z(K)$ is given by

$$g = (\begin{pmatrix} w_1 & 0 \\ 0 & w_1 \end{pmatrix}, \ldots, \begin{pmatrix} w_{r-1} & 0 \\ 0 & w_{r-1} \end{pmatrix}, \begin{pmatrix} \prod w_i^{-1} & 0 \\ 0 & \prod w_i^{-1} \end{pmatrix})$$

where the product $\prod w_i^{-1}$ is taken over $i \in \{1, \ldots, r-1\}$. Thus $Z(K)$ is exactly the image of $\varphi$ restricted to the subset $C \subset \tilde{K}$

$$C = \{(w_1, \ldots, w_{r-1}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \subset \tilde{K}$$

The center $Z(K)$ is isomorphic to $(\mathbb{C}^\times)^{r-1}$ so we can label an irreducible representation of $Z(K)$ by an $(r-1)$-tuple of integers $\vec{z} = (z_1, z_2, \ldots, z_{r-1})$ which, because of our choice of parametrization matches our definition of $\varphi$, corresponds to the $\vec{z}$ from $F_{\vec{z},\vec{s}}$. These are simply choices of an irrep of $\mathbb{C}^\times$ in each factor. We refer to the irreducible characters of

$Z(K)$ as *central characters* and they are written

$$\prod_{i=1}^{r-1} w_i^{z_i}$$

Briefly consider a related group (it would be our $K$ had we chosen $G = GL_{2r}$)

$$H = GL_2 \times \cdots \times GL_2$$

Then its center $Z(H) = \{(w_1 I_2, \ldots, w_r I_2)\}$ where $w_1, \ldots, w_r \in \mathbb{C}^\times$. In a moment, we will set $w_r = w_1^{-1} \cdots w_{r-1}^{-1}$ to recover $Z(K)$. Throughout the paper we work with indices mod $r$, but with the representatives of the equivalence classes chosen as $\{1, 2, \ldots, r\}$ rather than $\{0, 1, 2, \ldots, r-1\}$. For example, $n_i - n_{i-1}$ is simply $n_1 - n_r$ when $i = 1$.

We have described the action of $K$ on $\mathfrak{g}_1 = X_1 \oplus \cdots \oplus X_r$ and so also on $\mathcal{P}_{\vec{n}}$. Allow $H$ to act similarly.

**Proposition 4.** *For each $\vec{n} \in \mathbb{N}^r$, $Z(H)$ acts on $f \in \mathcal{P}_{\vec{n}}$ by multiplication by a scalar depending on $\vec{n}$ as in:*

$$f \longmapsto \big(\prod_{i=1}^{r} w_i^{n_i - n_{i-1}}\big) f$$

*Example.* If $r = 3$ then $Z(H)$ acts on $\mathfrak{g}_1$ by conjugation

$$\begin{pmatrix} w_1^{-1} &&&&& \\ & w_1^{-1} &&&& \\ && w_2^{-1} &&& \\ &&& w_2^{-1} && \\ &&&& w_3^{-1} & \\ &&&&& w_3^{-1} \end{pmatrix} \begin{pmatrix} & X_1 & \\ && X_2 \\ X_3 && \end{pmatrix} \begin{pmatrix} w_1 &&&&& \\ & w_1 &&&& \\ && w_2 &&& \\ &&& w_2 && \\ &&&& w_3 & \\ &&&&& w_3 \end{pmatrix}$$

As we will see in the proof below, if $f \in \mathcal{P}_{(3,2,4)}$ then

$$\begin{aligned} f \longmapsto &(w_1/w_2)^3(w_2/w_3)^2(w_3/w_1)^4 f \\ &= w_1^{-1}w_2^{-1}w_3^2 f \end{aligned}$$

*Proof.* It's clear that for general $r$ each entry of of the matrices $X_i$ gets multiplied by $w_i^{-1}w_{i+1}$. Transferring the action to $X_i^*$ the inverse switches spots, so for each basis element $\epsilon_k$ of $X_i^*$, $\epsilon_k \mapsto w_i w_{i+1}^{-1}\epsilon_k$. Thus the homogeneous polynomials of degree $n_i$ in the $\epsilon_k$ get multiplied by $(w_i w_{i+1}^{-1})^{n_i}$. Now $f \in \mathcal{P}_{\vec{n}}$ has homogeneous degree $n_i$ in each of the variables $\epsilon_k$ from $X_i^*$ in every term, so all scalars pull out with powers and

$$f \longmapsto \left(\prod_{i=1}^{r}(w_i w_{i+1}^{-1})^{n_i}\right) f$$

Distributing the exponent and re-writing the product yields the proposition. □

**Proposition 5.** For each $\vec{n} \in \mathbb{N}^r$, the center $Z(K)$ acts on $f \in \mathcal{P}_{\vec{n}}$ by multiplication depending on $\vec{n}$:

$$f \longmapsto \left(\prod_{i=1}^{r-1} w_i^{n_i - n_{i-1} - n_r + n_{r-1}}\right) f$$

where we have parametrized $Z(K)$ as in Definition 5.

*Proof.* Since $K$ differs from $H$ above only by having an overall determinant $= 1$, the only change from the center $Z(K)$ compared to $Z(H)$ is to require determinant 1. So whereas $Z(H) \simeq (\mathbb{C}^\times)^r$, instead $Z(K) \simeq (\mathbb{C}^\times)^{r-1}$. So that $Z(K) = \varphi(C)$ as in Definition 5, let $w_r = w_1^{-1}w_2^{-1}\cdots w_{r-1}^{-1}$. Replace $w_r^{n_r - n_{r-1}}$ in Proposition 4 by $(w_1^{-1}\cdots w_{r-1}^{-1})^{n_r - n_{r-1}}$. Each $w_i$ gets an extra $w_i^{n_{r-1}-n_r}$, yielding the claimed action. □

**Definition 6.** Choose $\vec{z} \in \mathbb{Z}^{r-1}$ such that $z_1 + \cdots + z_{r-1} = 0$ mod r. Then $\vec{z}$ determines a *base point* $\vec{b} \in \mathbb{N}^r$ by setting $b_r = 0$ and $b_k$ for $k \in \{1, \ldots, r-1\}$ given by

$$b_k = \sum_{i=1}^{k} z_i - \frac{k}{r} \sum_{i=1}^{r-1} z_i$$

Define $\mathrm{ray}(\vec{z})$ to be all points $\vec{n} \in \mathbb{N}^r$ extending upwards along the vector $\vec{1} = (1, 1, \ldots, 1)$ from the base point $\vec{b}$ as in

$$\mathrm{ray}(\vec{z}) = \{\vec{b}, \vec{b} + \vec{1}, \vec{b} + \vec{2}, \ldots\}$$

or written more succinctly

$$\mathrm{ray}(\vec{z}) = \vec{b} + \mathbb{N}\underbrace{(1, \ldots, 1)}_{r \text{ ones}}$$

*Example.* Consider $\vec{z} = (5, 1)$ in the case $r = 3$. Then $\vec{b} = (3, 2, 0)$ which yields the following list for $\mathrm{ray}(\vec{z})$:

$$\mathrm{ray}(\vec{z}) = \left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}, \cdots \right\}$$

**Remark:** For some choices of $\vec{z}$ it will turn out that $\vec{b}$ has some negative components. But moving upwards from $\vec{b}$ in the direction $\vec{1}$, we eventually reach $\mathbb{N}^r$. Because $\mathrm{ray}(\vec{z})$ is defined as points inside $\mathbb{N}^r$ we only count the points $\vec{n}$ where every $n_i \geq 0$. The polynomials are graded by homogeneous degree, which is non-negative, which explains this choice.

**Proposition 6.** For all $\vec{z} \in \mathbb{Z}^{r-1}$ and $\vec{s} \in \mathbb{N}^r$ satisfying Equation 3.1, the irreducible representation $F_{\vec{z},\vec{s}}$ of $K$ occurs with multiplicity zero in every $\mathcal{H}_{\vec{n}}$ with $\vec{n} \notin \mathrm{ray}(\vec{z})$.

$$[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = 0$$

where we have chosen our parametrization of $Z(K)$ as in Definition 5, so that $Z(K) = \varphi(C)$.

*Proof.* Recall the image of $\varphi$ restricted to the subset $C \subset \tilde{K}$

$$C = \{(w_1, \ldots, w_{r-1}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \subset \tilde{K}$$

is exactly $Z(K)$, so any $F_{\vec{z},\vec{s}}$ corresponds to an action of $Z(K)$ on $\mathcal{P}_{\vec{n}}$ as in Proposition 5. From Proposition 5 we obtain an *underdetermined* linear system

$$\begin{pmatrix} 1 & & & & & 1 & -2 \\ -1 & 1 & & & & 1 & -1 \\ & -1 & 1 & & & 1 & -1 \\ & & & \cdots & & & \\ & & & -1 & 1 & 1 & -1 \\ & & & & -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ n_r \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_{r-1} \end{pmatrix}$$

The variable $n_r$ is free, and row reducing yields exactly $\vec{n} \in \mathrm{ray}(\vec{z})$ as the only non-negative integer solutions. □

**Corollary:** $F_{\vec{z},\vec{s}}$ has multiplicity zero in $\mathcal{H}$ if

$$\sum_{i=1}^{r-1} z_i \neq 0 \text{ (mod r)}$$

*Proof.* Without this requirement $\mathrm{ray}(\vec{z})$ is empty. □

## 3.4 The action of the semisimple part of $K$ on $\mathcal{P}_{\vec{n}}$

In this section we will see that in decomposing $\mathcal{H}_{\vec{n}}$ as a representation of $K$ it will be enough to ignore the *central characters* and decompose using the *semisimple characters* alone.

Consider the character of $\mathcal{P}_{\vec{n}}$, denoted $\chi(\mathcal{P}_{\vec{n}})$. We can attempt to write it as a sum of irreducible characters of $K$, thus obtaining the decomposition of $\mathcal{P}_{\vec{n}}$ into irreducible representations of $K$. Inside $\mathcal{P}_{\vec{n}}$ is the subspace of harmonics $\mathcal{H}_{\vec{n}}$, and we will try to find its character $\chi(\mathcal{H}_{\vec{n}})$ inside of $\chi(\mathcal{P}_{\vec{n}})$, thus obtaining the multi-graded decomposition of the harmonics as a $K$ representation, including the multiplicity of any given irreducible representation of $K$ within $\mathcal{H}_{\vec{n}}$ for all $\vec{n}$. So what will these characters look like?

Recall the irreducible representations $F_{\vec{z},\vec{s}}$ of $K$ are parametrized by $\vec{z} \in \mathbb{Z}^{r-1}$ and $\vec{s} \in \mathbb{N}^r$ such that Equation 3.1 is satisfied. Every such irrep comes from an irrep of $\tilde{K}$, and so by restricting $\varphi$ to the subset $\{(1, \ldots, 1)\} \times SL_2^r$ we can evaluate $\chi_{\vec{z},\vec{s}}$ at elements

$$(1, \ldots, 1, \begin{pmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} u_r & 0 \\ 0 & u_r^{-1} \end{pmatrix})$$

where the components coming from each $SL_2$ factor have been restricted to the torus, we obtain the *semisimple characters* for that irrep. From Definition 4 we have

$$\begin{aligned} \chi_{\vec{z},\vec{s}}(g) &= w_1^{z_1} \cdots w_{r-1}^{z_{r-1}} \chi_{s_1}(u_1) \cdots \chi_{s_{r-1}}(u_{r-1}) \chi_{s_r}(u_r) \\ &= 1^{z_1} \cdots 1^{z_{r-1}} \chi_{s_1}(u_1) \cdots \chi_{s_{r-1}}(u_{r-1}) \chi_{s_r}(u_r) \\ &= \chi_{s_1}(u_1) \cdots \chi_{s_{r-1}}(u_{r-1}) \chi_{s_r}(u_r) \end{aligned}$$

which we can simply denote $\chi_{\vec{s}}$ when we are only concerned with the semisimple part of $K$.

*Example.* If $r = 4$, we have $(1, 1, 1) \times SL_2^4$, and the semisimple character

$$\begin{aligned} \chi_{(1,3,0,2)} &= \chi_1(u_1) \cdot \chi_3(u_2) \cdot \chi_0(u_3) \cdot \chi_2(u_4) \\ &= \left(u_1 + u_1^{-1}\right) \cdot \left(u_2^3 + u_2 + u_2^{-1} + u_2^{-3}\right) \cdot \left(1\right) \cdot \left(u_4^2 + 1 + u_4^{-2}\right) \end{aligned}$$

corresponds to an irreducible representation of dimension $24 = 2 \cdot 4 \cdot 1 \cdot 3$.

Thus, looking at the character $\chi(\mathcal{P})$ we will find a sum of semisimple characters from $SL_2^r$, each multiplied by a different central character of $(\mathbb{C}^\times)^{r-1} \cong Z(K)$. Recall from Proposition 5 that the center's action on $f \in \mathcal{P}_{\vec{n}}$ will be by scalar multiplication by some powers that depend on $\vec{n}$:

$$f \longmapsto \left( \prod_{i=1}^{r-1} w_i^{n_i - n_{i-1} - n_r + n_{r-1}} \right) f$$

Consider $\chi(\mathcal{P}_{\vec{n}})$ for a fixed multi-graded component $\vec{n}$. In principle, if we ignore the characters coming from the center $Z(K)$ and we find a term $\chi_{(3,0,2)}$ inside $\chi(\mathcal{P}_{\vec{n}})$, we won't know if it comes as part of an irrep $F_{(-5,-1),(3,0,2)}$ or as part of $F_{(4,5),(3,0,2)}$, or some other $F_{\vec{z},(3,0,2)}$. However, by Proposition 5, we know that $Z(K)$ acts in exactly one way on all polynomials inside $\mathcal{P}_{\vec{n}}$.

Thus, fixing $\vec{n}$ determines exactly which central characters of $Z(K)$ we will see in $\chi(\mathcal{P}_{\vec{n}})$, namely from $\vec{z}$ where each $z_i = n_i - n_{i-1} - n_r + n_{r-1}$. When we write $\chi(\mathcal{P}_{\vec{n}})$ as a sum of irreducible characters, any irreducible character $\chi_{\vec{s}}$ will appear with the same multiplier, $\prod w_i^{z_i}$. In this case, fixing the multi-graded component $\mathcal{P}_{\vec{n}}$ allows us to ignore the center $Z(K)$ and decompose into a sum of irreducible characters $\chi_{\vec{s}}$ of the semisimple part of $K$ alone. Later, we proceed in this direction, and once we obtain the decomposition of $\mathcal{H}_{\vec{n}}$ for all $\vec{n}$, we can turn the question around and ask: Fixing an irrep $F_{\vec{z},\vec{s}}$, what is the multiplicity in the various $\mathcal{H}_{\vec{n}}$? But first, let us see some examples of the answer worked out in simple cases.

## 4 Examples

In this section we work out two specific examples for the cases $r = 2$ and $r = 3$. We will see that in each case the graded multiplicity

$$[F_{\vec{z},\vec{s}} \colon \mathcal{H}_{\vec{n}}]$$

is given by counting intersection points between $\mathrm{Shell}(\Lambda_{\vec{n}})$ and a hypersurface $\mathcal{S} \subset \mathbb{R}^r$. $\mathcal{S}$ will simply be $r$ faces of a certain polyhedron. At this point we make several definitions, but follow with explicit examples.

**Definition 7.** For $\vec{n} \in \mathbb{N}^r$ define a Cartesian product of sets $\Lambda_{\vec{n}}$ and the subset $\mathrm{Shell}(\Lambda_{\vec{n}})$ as follows:

$$\Lambda_{\vec{n}} = \prod_{i=1}^{r} \{n_i, n_i - 2, n_i - 4, \ldots, [n_i]_2\}$$
$$\mathrm{Shell}(\Lambda_{\vec{n}}) = \{\vec{m} \in \Lambda_{\vec{n}} \text{ such that } \vec{m} + \vec{2} \notin \Lambda_{\vec{n}}\}$$

where $[n_i]_2$ means reduce $n_i$ mod 2 and indices are taken mod $r$ from the set $\{1, \ldots, r\}$.

We follow Vinberg's *A Course in Algebra*, Chapter 7, for some of the (standard) language and results about affine geometry. For example, a *polyhedron* is an intersection of finitely many *half-spaces* (thus not necessarily bounded). Every *face* is an intersection of the polyhedron with some of its *supporting hyperplanes* [8].

**Definition 8.** If $f$ is an affine-linear function on $\mathbb{R}^r$ define

$$H_f = \{\vec{m} \in \mathbb{R}^r \colon f(\vec{m}) = 0\}$$
$$H_f^+ = \{\vec{m} \in \mathbb{R}^r \colon f(\vec{m}) \geq 0\}$$
$$H_f^- = \{\vec{m} \in \mathbb{R}^r \colon f(\vec{m}) \leq 0\} = H_{-f}^+$$

The set $H_f$ is a hyperplane. $H_f^+$ and $H_f^-$ are the half-spaces bounded by $H_f$.

**Definition 9.** For each $\overrightarrow{s} \in \mathbb{N}^r$, let $\mathcal{M} \subset \mathbb{R}^r$ be the polyhedron defined as the intersection of half-spaces

$$\mathcal{M} = \bigcap_{f \in J} H_f^+$$

where $J$ is the set of affine-linear functions $\mathbb{R}^r \to \mathbb{R}$

$$J = \{f_1, \ldots, f_r, g_1, \ldots, g_r, h_1, \ldots, h_r\}$$

and each of the $f_1, \ldots, f_r, g_1, \ldots, g_r, h_1, \ldots, h_r$ depends on the parameter $\overrightarrow{s} \in \mathbb{N}^r$ as follows, where indices are taken mod $r$ from the set $\{1, \ldots, r\}$:

$$f_1, \ldots, f_r \text{ by } f_i(\overrightarrow{x}) = x_{i-1} + x_i - s_i$$

$$g_1, \ldots, g_r \text{ by } g_i(\overrightarrow{x}) = x_{i-1} - x_i + s_i$$

$$h_1, \ldots, h_r \text{ by } h_i(\overrightarrow{x}) = -x_{i-1} + x_i + s_i$$

Each of the $f_1, \ldots, f_r$ also defines a hyperplane $H_{f_i}$, and if we intersect $\mathcal{M}$ with one of the $H_{f_i}$ we obtain a face of $\mathcal{M}$. Finally we define our hypersurface $\mathcal{S}$ to be the union of these $r$ faces formed in this way. Later, we count certain integral points on $\mathcal{S}$ to determine the graded multiplicity.

**Definition 10.** Let $\mathcal{S} \subset \mathbb{R}^r$ be the union of the $r$ faces of the polyhedron $\mathcal{M}$ defined by the affine-linear functions $f_1, \ldots, f_r$ as in:

$$\mathcal{S} = \underbrace{\left(\mathcal{M} \cap H_{f_1}\right) \cup \cdots \cup \left(\mathcal{M} \cap H_{f_r}\right)}_{r \text{ faces}}$$

## 4.1 An example with $r = 2$

Say we fix a $K$-type denoted $F_{\vec{z},\vec{s}}$ with $\vec{z} = (6)$ and $\vec{s} = (7,5)$. We ask, what is the multiplicity in $\mathcal{H}_{\vec{n}}$ for all possible $\vec{n} \in \mathbb{N}^r$? The answer is

| Graded component | $\mathcal{H}_{(5,2)}$ | $\mathcal{H}_{(6,3)}$ | $\mathcal{H}_{(7,4)}$ | $\mathcal{H}_{(8,5)}$ | $\mathcal{H}_{(9,6)}$ | other $\mathcal{H}_{\vec{n}}$ |
|---|---|---|---|---|---|---|
| Multiplicity | 1 | 2 | 1 | 1 | 1 | 0 |

This answer can be obtained by counting the intersection points between a hypersurface $\mathcal{S}$ and Shell$(\Lambda_{\vec{n}})$ for each $\vec{n}$. For example, Shell$(\Lambda_{(6,3)})$ intersects $\mathcal{S}$ twice, giving multiplicity two. So what are $\mathcal{S}$ and Shell$(\Lambda_{\vec{n}})$?

We will see how this happens in detail later, but for now recall the Clebsch-Gordan formula for the character of a tensor product of two $SL_2$ irreps:

$$\chi_{m_1}(u)\chi_{m_2}(u) = \chi_{m_1+m_2}(u) + \chi_{m_1+m_2-2}(u) + \chi_{m_1+m_2-4}(u) + \cdots + \chi_{|m_1-m_2|}(u)$$

Briefly, requiring $m_{i-1}+m_i \geq s_i$ gives us a half-space that will correspond to the affine-linear function $f_i$ from Definition 9. On the other side, requiring $s_i \geq |m_{i-1} - m_i|$ will produce a pair of half-spaces corresponding to the $g_i$ and the $h_i$ from Definition 9. For now though, we are simply considering an example from $r = 2$. We continue.

Since $\vec{s} = (7,5)$ we can build the polyhedron $\mathcal{M}$ and the hypersurface $\mathcal{S}$ defined as the union of two of its faces, both of which depend on this semisimple parameter $\vec{s}$.

$$\mathcal{M} = H^+_{f_1} \cap H^+_{f_2} \cap H^+_{g_1} \cap H^+_{g_2} \cap H^+_{h_1} \cap H^+_{h_2}$$

where

$$f_1(x_1,x_2) = x_2 + x_1 - 7$$

$$f_2(x_1,x_2) = x_1 + x_2 - 5$$

and where

$$g_1(x_1, x_2) = x_2 - x_1 + 7$$

$$g_2(x_1, x_2) = x_1 - x_2 + 5$$

$$h_1(x_1, x_2) = -x_2 + x_1 + 7$$

$$h_2(x_1, x_2) = -x_1 + x_2 + 5$$

Figure 4.1 shows the half-spaces corresponding to these affine-linear functions. It shows $H^+_{f_1} \cap H^+_{f_2}$ in the first picture followed by all half-spaces in the second picture. In this example $H^+_{g_1} \cap H^+_{g_2} \cap H^+_{h_1} \cap H^+_{h_2}$ corresponds to requiring that $|x_2 - x_1| \leq 7$ and that $|x_1 - x_2| \leq 5$, as pictured in the figure.

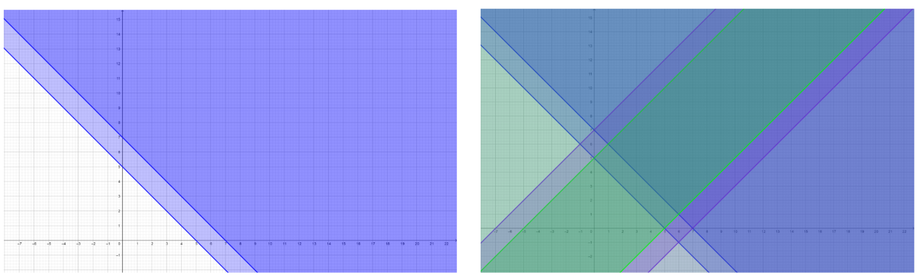


Figure 4.1: Intersecting $H^+_{f_1} \cap H^+_{f_2}$ and then all of the half-spaces giving $\mathcal{M}$

The polyhedron $\mathcal{M}$ is unbounded, but the hypersurface $\mathcal{S}$, which is simply some of its faces, will always be bounded. Intersecting $\mathcal{M}$ with each of $H_{f_1}$ and $H_{f_2}$ yields 2 faces of $\mathcal{M}$, where in this case one of the faces is the empty set. For $r > 2$ every face will be nonempty, but the construction remains identical in every dimension $r$. The final hypersurface $\mathcal{S}$ is pictured in Figure 4.2.

To find the graded multiplicity, the multiplicity in $\mathcal{H}_{\vec{n}}$, we must recall our parameter $\vec{z} = (6)$. We ask, for which $\mathcal{H}_{\vec{n}}$ could $Z(K)$ act in this way? Recall the action of the center,

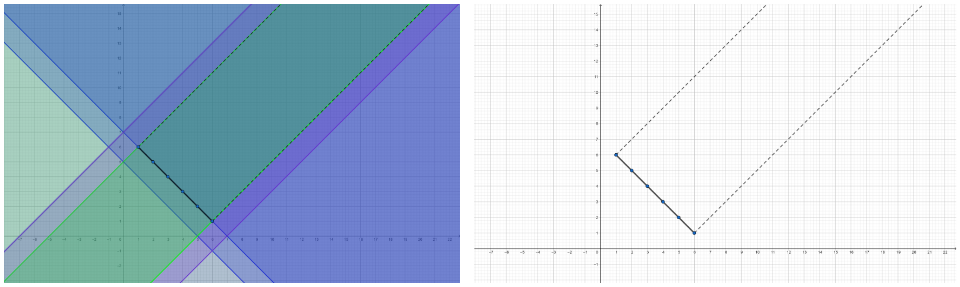


Figure 4.2: The polyhedron $\mathcal{M}$ and the hypersurface $\mathcal{S}$

where an element $g \in Z(K)$ acting by $g^{-1}Xg$, where $g$ can be parametrized by $w \in \mathbb{C}^{\times}$.

$$\begin{pmatrix} w^{-1} & & & \\ & w^{-1} & & \\ & & w & \\ & & & w \end{pmatrix} \begin{pmatrix} & X_1 \\ X_2 & \end{pmatrix} \begin{pmatrix} w & & & \\ & w & & \\ & & w^{-1} & \\ & & & w^{-1} \end{pmatrix}$$

After some brief calculation, or by utilizing Proposition 5, we see that $Z(K)$ acts on $f \in \mathcal{P}_{(n_1,n_2)}$ by

$$f \longmapsto \left(w^{2n_1-2n_2}\right) f$$

So to obtain $\vec{z} = 6$ we require $2n_1 - 2n_2 = 6$ where $n_i \in \mathbb{N}$. Thus $Z(K)$ acts by $\vec{z} = (6)$ exactly on the graded components

$$\mathcal{H}_{(3,0)}, \mathcal{H}_{(4,1)}, \mathcal{H}_{(5,2)}, \mathcal{H}_{(6,3)}, \ldots$$

Here $\vec{b} = (3,0)$ from Definition 6. Recalling the definition of $\text{Shell}(\Lambda_{\vec{n}})$ we can consider all Shells for $\vec{n} \in \text{ray}(\vec{z}) = \{(3,0), (4,1), (5,2), (6,3), \ldots\}$ and how they intersect our hypersurface $\mathcal{S}$. Figure 4.3 shows these points and their Shells, color-coded, and finally the points of intersection with $\mathcal{S}$, where the two green-colored points correspond to the fact that $F_{\vec{z},\vec{s}}$ has multiplicity 2 in $\mathcal{H}_{(6,3)}$.

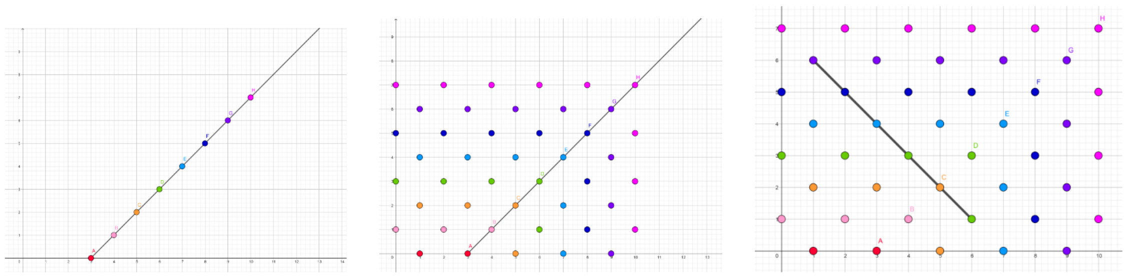


Figure 4.3: Intersecting $\mathcal{S}$ with Shell$(\Lambda_{\vec{n}})$: Graded Multiplicity

## 4.2 An example with $r = 3$

Consider the case $r = 3$ and the $K$-type denoted $F_{\vec{z},\vec{s}}$ with $\vec{z} = (5, 1)$ and $\vec{s} = (7, 5, 4)$. We ask, what is the multiplicity in each graded component of the harmonics $\mathcal{H}_{\vec{n}}$? The answer is given in Table 4.1, where all $\vec{n}$ not included correspond to multiplicity zero. Again the only

| $\vec{b} + \mathbb{N}\vec{1}$ | $\vec{n}$ | Multiplicity |
|---|---|---|
| $(3, 2, 0) + \vec{0}$ | $(3, 2, 0)$ | 0 |
| $(3, 2, 0) + \vec{1}$ | $(4, 3, 1)$ | 0 |
| $(3, 2, 0) + \vec{2}$ | $(5, 4, 2)$ | 2 |
| $(3, 2, 0) + \vec{3}$ | $(6, 5, 3)$ | 6 |
| $(3, 2, 0) + \vec{4}$ | $(7, 6, 4)$ | 7 |
| $(3, 2, 0) + \vec{5}$ | $(8, 7, 5)$ | 6 |
| $(3, 2, 0) + \vec{6}$ | $(9, 8, 6)$ | 4 |
| $(3, 2, 0) + \vec{7}$ | $(10, 9, 7)$ | 2 |
| $(3, 2, 0) + \vec{8}$ | $(11, 10, 8)$ | 1 |
| $(3, 2, 0) + \vec{9}$ | $(12, 11, 9)$ | 0 |

Table 4.1: Graded Multiplicity

non-zero multiplicities appear in a ray extending from $\vec{b} = (3, 2, 0)$ by multiples of $(1, 1, 1)$. This is due to the action of the center $Z(K)$ as in Proposition 6.

For now consider building the hypersurface $\mathcal{S} \subset \mathbb{R}^3$ out of faces of the polyhedron $\mathcal{M} \subset \mathbb{R}^3$. Since we chose $\vec{s} = (7, 5, 4)$ as our example, the affine-linear functions on $\mathbb{R}^3$ are given by

$$f_1(x_1, x_2, x_3) = x_3 + x_1 - 7$$
$$f_2(x_1, x_2, x_3) = x_1 + x_2 - 5$$
$$f_3(x_1, x_2, x_3) = x_2 + x_3 - 4$$

and also

$$g_1(x_1, x_2, x_3) = x_3 - x_1 + 7$$
$$g_2(x_1, x_2, x_3) = x_1 - x_2 + 5$$
$$g_3(x_1, x_2, x_3) = x_2 - x_3 + 4$$
$$h_1(x_1, x_2, x_3) = -x_3 + x_1 + 7$$
$$h_2(x_1, x_2, x_3) = -x_1 + x_2 + 5$$
$$h_3(x_1, x_2, x_3) = -x_2 + x_3 + 4$$

First consider the hyperplanes defined by $H_{f_1}, H_{f_2}, H_{f_3}$ pictured in Figure 4.4, where the $x_1 = 0, x_2 = 0, x_3 = 0$ planes are included for reference only. Of course, they also bound 3 of the 9 half-spaces whose intersection gives the convex polyhedron $\mathcal{M} \subset \mathbb{R}^3$. In $\mathbb{R}^3$ it's harder to picture half-spaces in our figures, and so we now illustrate all of the 9 hyperplanes that bound the half-spaces in Figure 4.5. Each pair $H^+_{g_i}$ and $H^+_{h_i}$ will be parallel half-spaces opening in opposite directions, each towards the other, because they come from requiring $|x_{i-1} - x_i| \leq s_i$.

In this case we build the polyhedron $\mathcal{M}$ and the hypersurface $\mathcal{S}$ as follows:

$$\mathcal{M} = \bigcap_{f \in J} H^+_f \text{ where } J = \{f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3\}$$

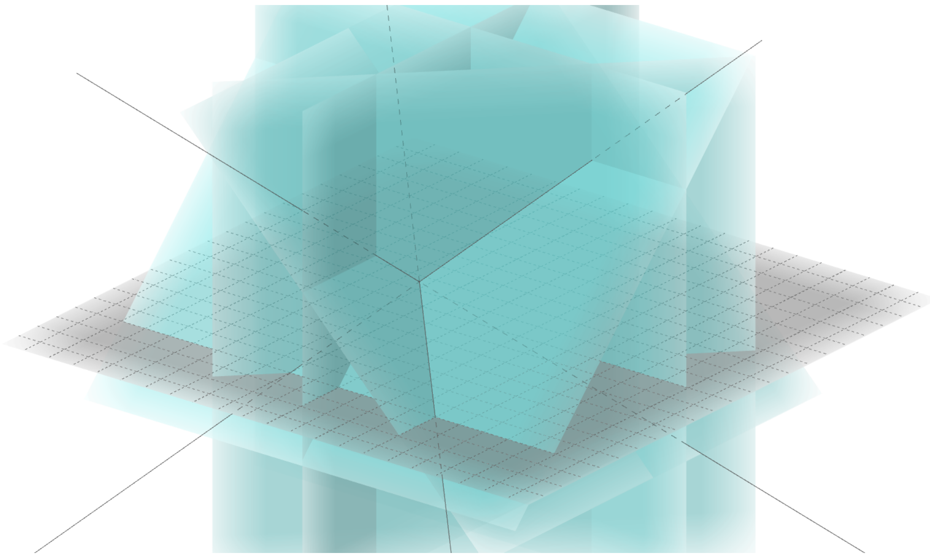


Figure 4.4: The hyperplanes $H_{f_1}, H_{f_2}, H_{f_3}$ to become the faces of the hypersurface $\mathcal{S}$

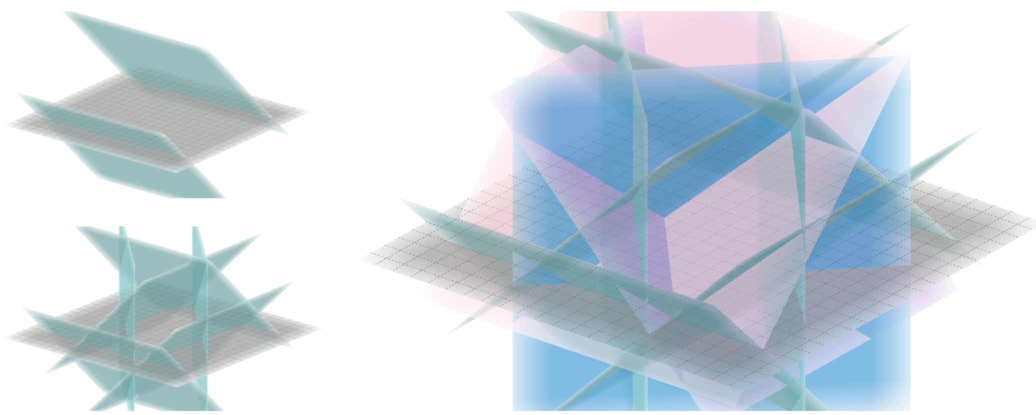


Figure 4.5: Hyperplanes used in constructing $\mathcal{M}$ and $\mathcal{S}$

and

$$\mathcal{S} = \left(\mathcal{M} \cap H_{f_1}\right) \cup \left(\mathcal{M} \cap H_{f_2}\right) \cup \left(\mathcal{M} \cap H_{f_3}\right)$$

Recall, the intersection points of $\mathcal{S}$ with each Shell$(\Lambda_{\vec{n}})$ will give us the graded multiplicity. Always $\mathcal{S}$ depends on $\vec{s}$, the parameter from our irrep $F_{\vec{z},\vec{s}}$, and in this specific example $\mathcal{S}$ came from $\vec{s} = (7, 5, 4)$ and is pictured in Figure 4.6. When we intersect with Shell$(\Lambda_{(6,5,3)})$ we find six intersection points, as explained below.

Intersecting $\mathcal{S}$ with all Shell$(\Lambda_{\vec{n}})$ for $\vec{n} \in$ ray$(\vec{z})$ we obtain exactly 28 integral points on $\mathcal{S}$, which gives the total multiplicity in $\mathcal{H}$. For the graded multiplicity we must intersect

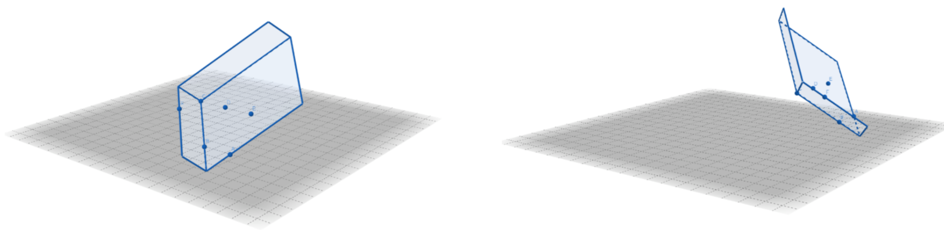


Figure 4.6: Finished hypersurface $\mathcal{S}$ and six intersection points

$\mathcal{S}$ with each particular $\text{Shell}(\Lambda_{\vec{n}})$. These Shells are harder to picture now, but they expand upward from $(3,2,0)$ and eventually hit $\mathcal{S}$. The Shells for $\vec{n} = (3,2,0), (4,3,1), (5,4,2)$ and $(6,5,3)$ are color-coded in Figure 4.7.

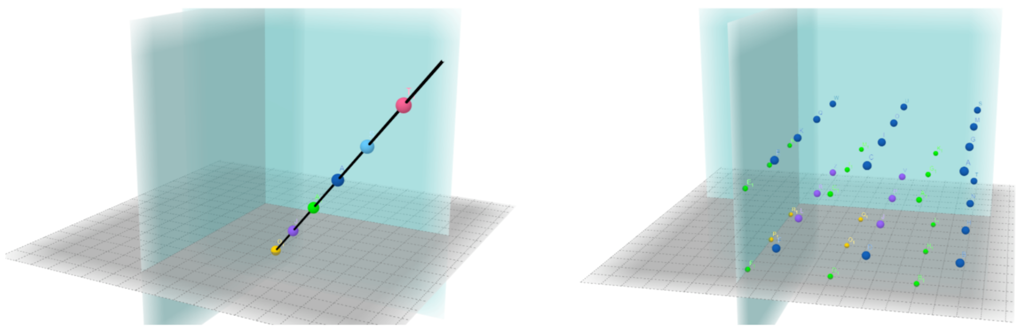


Figure 4.7: $\text{ray}(\vec{z})$ and the first four of its Shells

The last, blue Shell pictured in Figure 4.7 is $\text{Shell}(\Lambda_{(6,5,3)})$. Our irrep $F_{\vec{z},\vec{s}} = F_{(5,1),(7,5,4)}$ appears with multiplicity six inside $\mathcal{H}_{(6,5,3)}$. This fact is recorded by the six intersection points of that Shell with $\mathcal{S}$ and is pictured in Figure 4.8.

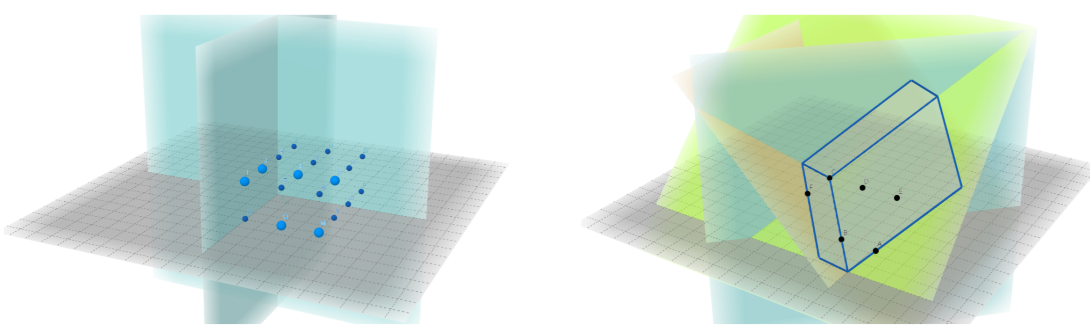


Figure 4.8: Graded Multiplicity $\#(\text{Shell}(\Lambda_{\vec{n}}) \cap \mathcal{S})$

# 5 Decomposing the Harmonics $\mathcal{H}_{\vec{n}}$

In this section we discover that the character $\chi(\mathcal{H}_{\vec{n}})$ can be written

$$\chi(\mathcal{H}_{\vec{n}}) = \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)$$

Then in Section 6 we use this result to see the geometric description of the graded multiplicity as counting intersection points between $\mathcal{S}$ and the Shell$(\Lambda_{\vec{n}})$.

## 5.1 Decomposing $\Lambda_{\vec{n}}$ and $\lambda_{\vec{m}}$ into shells

Consider a point $\vec{n} = (n_1, n_2, \ldots, n_r)$ inside $\mathbb{N}^r$. Denote the minimum of all components $n_i$ as $n_{min}$. For example if $\vec{n} = (5, 3, 6, 2, 4)$ then $n_{min} = 2$. Define two kinds of lattice polytopes, or arrays, as follows.

**Definition 11.** For each $\vec{n} \in \mathbb{N}^r$ define

$$\Lambda_{\vec{n}} = \prod_{i=1}^{r} \{n_i, n_i - 2, n_i - 4, \ldots, [n_i]_2\}$$

$$\lambda_{\vec{n}} = \prod_{i=1}^{r} \{n_{i-1} + n_i, n_{i-1} + n_i - 2, n_{i-1} + n_i - 4, \ldots, |n_{i-1} - n_i|\}$$

where $[n_i]_2$ means reduce $n_i$ mod 2 and indices are taken mod $r$ from $\{1, \ldots, r\}$.

*Example.* $\Lambda_{(3,2,3)}$ and $\lambda_{(3,2,1)}$ would be

$$\begin{aligned}\Lambda_{(3,2,3)} &= \{3,1\} \times \{2,0\} \times \{3,1\} \\ &= \{(3,2,3),(3,2,1),(3,0,3),(3,0,1),(1,2,3),(1,2,1),(1,0,3),(1,0,1)\} \\ \lambda_{(3,2,1)} &= \{4,2\} \times \{5,3,1\} \times \{3,1\} \\ &= \big\{(4,5,3),(4,5,1),(4,3,3),(4,3,1),(4,1,3),(4,1,1), \\ &\quad (2,5,3),(2,5,1),(2,3,3),(2,3,1),(2,1,3),(2,1,1)\big\}\end{aligned}$$

**Definition 12.** Define $\text{Shell}(\Lambda_{\vec{n}})$ be all $\vec{m} \in \Lambda_{\vec{n}}$ such that $\vec{m}+\vec{2} \notin \Lambda_{\vec{n}}$, and define $\text{Shell}(\lambda_{\vec{m}})$ similarly.

*Example.* $\text{Shell}(\Lambda_{(3,2,3)})$ would be everything except $(1,0,1)$ because that is the only point such that adding 2 to each $n_i$ yields another element of $\Lambda_{(3,2,3)}$.

**Proposition 7.** For every $\vec{n} \in \mathbb{N}^r$ such that $n_{min} \geq 2$ we have

$$\Lambda_{\vec{n}} = \text{Shell}(\Lambda_{\vec{n}}) \cup \Lambda_{\vec{n}-\vec{2}}$$

*Proof.* This follows from the definitions above, and also since the parity of $n_i$ does not change if you subtract 2. □

**Proposition 8.** For every $\vec{m} \in \mathbb{N}^r$ we have

$$\lambda_{\vec{m}} = \text{Shell}(\lambda_{\vec{m}}) \cup \text{Shell}(\lambda_{\vec{m}-\vec{1}}) \cup \cdots \cup \text{Shell}(\lambda_{\vec{m}-\overrightarrow{m_{min}}})$$

*Proof.* Replacing $\vec{m}$ by $\vec{m}-\vec{1}$ we go from $\lambda_{\vec{m}}$ to $\lambda_{\vec{m}-\vec{1}}$, but since $(m_{i-1}-1)+(m_i-1) = m_{i-1}+m_i-2$ and $|(m_{i-1}-1)-(m_i-1)| = |m_{i-1}-m_i|$ we have that $\lambda_{\vec{m}}$ loses the first entry in each factor becoming

$$\lambda_{\vec{m}-\vec{1}} = \prod_{i=1}^{r} \{m_{i-1}+m_i-2, \ldots, |m_{i-1}-m_i|\}$$

Thus

$$\lambda_{\vec{m}} = \text{Shell}(\lambda_{\vec{m}}) \cup \lambda_{\vec{m}-\vec{1}}$$

Applying this recursively we obtain the decomposition above. □

## 5.2 Invariants

Recall, inside the space of polynomial functions of multi-graded degree $\mathcal{P}_{\vec{n}}$ will be the subspace of harmonic functions $\mathcal{H}_{\vec{n}}$, a subrepresentation of $K$. Therefore we can consider the character of each of these representations, denoted $\chi(\mathcal{P}_{\vec{n}})$ and $\chi(\mathcal{H}_{\vec{n}})$ respectively.

**Proposition 9.** For every $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{P}_{\vec{n}}) = \sum_{j=0,1,\ldots,n_{min}} \Big(\lfloor j/2 \rfloor + 1\Big)\chi(\mathcal{H}_{\vec{n}-\vec{j}})$$

where $\lfloor j/2 \rfloor$ is the floor function. If $n_{min}$ is large this starts out as

$$\begin{aligned}\chi(\mathcal{P}_{\vec{n}}) = \chi(\mathcal{H}_{\vec{n}}) + \chi(\mathcal{H}_{\vec{n}-\vec{1}}) + 2\chi(\mathcal{H}_{\vec{n}-\vec{2}}) + 2\chi(\mathcal{H}_{\vec{n}-\vec{3}}) \\ + 3\chi(\mathcal{H}_{\vec{n}-\vec{4}}) + 3\chi(\mathcal{H}_{\vec{n}-\vec{5}}) + 4\chi(\mathcal{H}_{\vec{n}-\vec{6}}) + 4\chi(\mathcal{H}_{\vec{n}-\vec{7}}) + \ldots\end{aligned}$$

*Proof.* Due to a result of Le Bruyn and Procesi [5], all invariant functions in $\mathcal{P}$ are generated by two. Call them

$$f = \text{trace}(X_1 X_2 \cdots X_r)$$

and

$$g = \text{trace}\big((X_1 X_2 \cdots X_r)^2\big)$$

$f$ has degree $\vec{1} = (1, 1, \ldots, 1)$ and $g$ has degree $\vec{2} = (2, 2, \ldots, 2)$, meaning degree 1 or 2 in each of the $r$ components for $f$ and $g$ respectively. To see this, denote the $(a, b)$th element of $X_i$ by $X_{a,b}^{(i)}$. Matrix multiplication gives the $(a, b)$th element of the product $X_1 X_2 \cdots X_r$

as a sum

$$\sum_{j_k} X^{(1)}_{a,j_1} X^{(2)}_{j_1,j_2} \cdots X^{(r)}_{j_{n-1},b}$$

Taking the trace puts an additional sum over $a$, but changes the last index $b$ to an $a$ as well. Thus it's clear that each term will have exactly one entry from each $X_i$, so that $f$ has degree $\vec{1}$. Similarly, $g$ has degree $\vec{2}$. Because the polynomials are free over the invariants, we can write the table below. Each line corresponds to a different power of $g$, and the terms in each line correspond to different powers of $f$. The table should continue until a column corresponding to $\vec{n} - \overrightarrow{n_{min}}$, which would have a zero in at least one component. Hence at that point there are no more invariants, since both $f$ and $g$ have non-zero degree in every component. □

$$\begin{array}{llllllll}
\mathcal{P}_{\vec{n}} & = \mathcal{H}_{\vec{n}} & + f\mathcal{H}_{\vec{n}-\vec{1}} & + f^2\mathcal{H}_{\vec{n}-\vec{2}} & + f^3\mathcal{H}_{\vec{n}-\vec{3}} & + f^4\mathcal{H}_{\vec{n}-\vec{4}} & + f^5\mathcal{H}_{\vec{n}-\vec{5}} & + \dots \\
 & & & + g\mathcal{H}_{\vec{n}-\vec{2}} & + fg\mathcal{H}_{\vec{n}-\vec{3}} & + f^2g\mathcal{H}_{\vec{n}-\vec{4}} & + f^3g\mathcal{H}_{\vec{n}-\vec{5}} & + \dots \\
 & & & & & + g^2\mathcal{H}_{\vec{n}-\vec{4}} & + fg^2\mathcal{H}_{\vec{n}-\vec{5}} & + \dots \\
 & & & & & & & + \dots \\
\hline
\mathcal{P}_{\vec{n}} & = \mathcal{H}_{\vec{n}} & + \mathcal{H}_{\vec{n}-\vec{1}} & + 2\mathcal{H}_{\vec{n}-\vec{2}} & + 2\mathcal{H}_{\vec{n}-\vec{3}} & + 3\mathcal{H}_{\vec{n}-\vec{4}} & + 3\mathcal{H}_{\vec{n}-\vec{5}} & + \dots
\end{array}$$

Table 5.1: Collecting isomorphic copies of lower Harmonics

*Example.* Since $n_{min} = 1$ for $\vec{n} = (5, 1, 4, 3)$ we have

$$\chi(\mathcal{P}_{(5,1,4,3)}) = \chi(\mathcal{H}_{(5,1,4,3)}) + \chi(\mathcal{H}_{(4,0,3,2)})$$

*Example.* For any $\vec{n}$ with $n_{min} = 0$ we have $\chi(\mathcal{P}_{\vec{n}}) = \chi(\mathcal{H}_{\vec{n}})$.

## 5.3 The character $\chi(\mathcal{P}_{\vec{n}})$

**Proposition 10.** For any $\vec{n} \in \mathbb{N}^r$ with indices mod $r$ in $\{1, \dots, r\}$ we have

$$\chi(\mathcal{P}_{\vec{n}}) = \prod_{i=1}^{r} \left[ \chi_{n_i}(u_i)\chi_{n_i}(u_{i+1}) + \chi_{n_i-2}(u_i)\chi_{n_i-2}(u_{i+1}) + \cdots + \chi_{[n_i]_2}(u_i)\chi_{[n_i]_2}(u_{i+1}) \right]$$

In order to prove Proposition 10 we will use two Lemmas. We use the notation $\delta_i(\vec{n}) = (0, \ldots, 0, n_i, 0, \ldots, 0)$ simply setting all components except the $i$th to zero.

**Lemma 2.** For any $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{P}_{\vec{n}}) = \chi(\mathcal{P}_{\delta_1(\vec{n})}) \cdots \chi(\mathcal{P}_{\delta_r(\vec{n})})$$

*Proof.* We can take any multi-graded component $\mathcal{P}_{\vec{n}}$ and write it as a tensor product of the spaces of polynomials of matching non-zero homogeneous degree in only one multi-graded component, as follows:

$$\mathcal{P}_{\vec{n}} = \mathcal{P}_{\delta_1(\vec{n})} \otimes \cdots \otimes \mathcal{P}_{\delta_r(\vec{n})}$$

This is true for the same reason that $\mathbb{C}[x_1, x_2, \ldots, x_r] = \mathbb{C}[x_1] \otimes \mathbb{C}[x_2, \ldots, x_r] = \mathbb{C}[x_1] \otimes \cdots \otimes \mathbb{C}[x_r]$ is true. The result now follows because the character of a tensor product is the product of the characters. □

*Example.*

$$\mathcal{P}_{(3,2,4)} = \mathcal{P}_{(3,0,0)} \otimes \mathcal{P}_{(0,2,0)} \otimes \mathcal{P}_{(0,0,4)}$$

**Lemma 3.** For any $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{P}_{\delta_i(\vec{n})}) = \chi_{n_i}(u_i)\chi_{n_i}(u_{i+1}) + \chi_{n_i-2}(u_i)\chi_{n_i-2}(u_{i+1}) + \cdots + \chi_{[n_i]_2}(u_i)\chi_{[n_i]_2}(u_{i+1})$$

*Proof.* $\mathcal{P}_{\delta_i(\vec{n})}$ consists only of polynomials on the $X_i$ component of $\mathfrak{g}_1 = X_1 \oplus \cdots \oplus X_r$. But we know the explicit action of $K$ on any given $X_i$ is simply

$$X_i \longmapsto g_i^{-1} X_i g_{i+1} \text{ , for } (g_1, \ldots, g_r) \in K$$

Restricting to the torus in each $g_i$ and $g_{i+1}$, and letting $X_i = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ we have

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \longmapsto \begin{pmatrix} u_i^{-1} & 0 \\ 0 & u_i \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} u_{i+1} & 0 \\ 0 & u_{i+1}^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} u_i^{-1}u_{i+1}x & u_i^{-1}u_{i+1}^{-1}y \\ u_iu_{i+1}z & u_iu_{i+1}^{-1}w \end{pmatrix}$$

Since at the character level this is simply $SL_2 \times SL_2$ acting on 2 by 2 matrices, we know that the determinant of $X_i$ will be a degree 2 invariant and can compute the character using the following identity:

$$\frac{1-q^2}{\prod_{a,b=\pm1}(1-u_i^a u_{i+1}^b q)} = \sum_{k=0}^{\infty} \chi_k(u_i)\chi_k(u_{i+1})q^k$$

This identity is proved by closing geometric series and recognizing the $SL_2$ characters in the result (recall $\frac{u_i^{k+1}-u_i^{k-1}}{u_i-u_i^{-1}}$). The determinant $det(X_i)$ corresponding to $(1-q^2)$ in the identity above is not actually invariant under $K$, so all these terms lie in the same graded component, and the lemma follows. □

*Proof of Proposition 10.* Multiplying the expressions from Lemma 3 in agreement with Lemma 2, Proposition 10 follows. □

## 5.4 A quick example

Consider the case $r = 3$ and the polynomials of graded degree $\vec{n} = (3, 2, 3)$. Reading this example will make the rest of the proof easier to follow.

$$\chi(\mathcal{P}_{(3,2,3)}) = \chi(\mathcal{P}_{(3,0,0)}) \cdot \chi(\mathcal{P}_{(0,2,0)}) \cdot \chi(\mathcal{P}_{(0,0,3)})$$

$$\chi(\mathcal{P}_{(3,2,3)}) = \Big[\chi_3(u_1)\chi_3(u_2) + \chi_1(u_1)\chi_1(u_2)\Big] \cdot \Big[\chi_2(u_2)\chi_2(u_3) + \chi_0(u_2)\chi_0(u_3)\Big] \cdot \Big[\chi_3(u_3)\chi_3(u_1) + \chi_1(u_3)\chi_1(u_1)\Big]$$

Notice how each term in the first factor has a $u_1$ and a $u_2$. In fact, each term in the $i$th factor has a $u_i$ and a $u_{i+1}$ where the indices are taken mod $r$. If we expand this product we obtain $2 * 2 * 2 = 8$ terms. They are indexed by the array $\Lambda_{\vec{n}} = \{3,1\} \times \{2,0\} \times \{3,1\}$ inside $\mathbb{N}^3$, so that for each point $\vec{m} \in \Lambda_{\vec{n}}$ we get a term. The terms are listed below, so for example, the first term corresponds to $\vec{m} = (3,2,3)$.

$$\begin{aligned}\chi(\mathcal{P}_{(3,2,3)}) = \; & \chi_3(u_1)\chi_3(u_2)\chi_2(u_2)\chi_2(u_3)\chi_3(u_3)\chi_3(u_1) \\ & + \chi_3(u_1)\chi_3(u_2)\chi_2(u_2)\chi_2(u_3)\chi_1(u_3)\chi_1(u_1) \\ & + \chi_3(u_1)\chi_3(u_2)\chi_0(u_2)\chi_0(u_3)\chi_3(u_3)\chi_3(u_1) \\ & + \chi_3(u_1)\chi_3(u_2)\chi_0(u_2)\chi_0(u_3)\chi_1(u_3)\chi_1(u_1) \\ & + \chi_1(u_1)\chi_1(u_2)\chi_2(u_2)\chi_2(u_3)\chi_3(u_3)\chi_3(u_1) \\ & + \chi_1(u_1)\chi_1(u_2)\chi_2(u_2)\chi_2(u_3)\chi_1(u_3)\chi_1(u_1) \\ & + \chi_1(u_1)\chi_1(u_2)\chi_0(u_2)\chi_0(u_3)\chi_3(u_3)\chi_3(u_1) \\ & + \chi_1(u_1)\chi_1(u_2)\chi_0(u_2)\chi_0(u_3)\chi_1(u_3)\chi_1(u_1)\end{aligned}$$

The only other thing to notice is that each term in the resulting expansion above has two factors involving $u_1$, two factors involving $u_2$, and two factors involving $u_3$. We can move the factors involving $u_1$ together, so that in general, the expansion of Proposition 10 will be a sum of terms that look like

$$\prod_{i=0}^{r} \chi_{m_{i-1}}(u_i)\chi_{m_i}(u_i)$$

because each $u_i$ occurs in the $i$th factor but also in the $(i-1)$th factor. In the next subsection, we will expand each of these pairs via the Clebsch-Gordan identity into sums of irreducible characters.

## 5.5 Rewriting $\chi(\mathcal{P}_{\vec{n}})$

**Proposition 11.** For any $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{P}_{\vec{n}}) = \sum_{\vec{m}\in\Lambda_{\vec{n}}} \left( \prod_{i=1}^{r} \chi_{m_{i-1}}(u_i)\chi_{m_i}(u_i) \right)$$

*Proof.* Examine the terms in each factor of Proposition 10, repeated here:

$$\chi(\mathcal{P}_{\vec{n}}) = \prod_{i=1}^{r} \left[ \chi_{n_i}(u_i)\chi_{n_i}(u_{i+1}) + \chi_{n_i-2}(u_i)\chi_{n_i-2}(u_{i+1}) + \cdots + \chi_{[n_i]_2}(u_i)\chi_{[n_i]_2}(u_{i+1}) \right]$$

The terms in each factor are indexed by $\{n_i, n_i - 2, \ldots, [n_i]_2\}$ and so when we expand the product into a sum of terms, the resulting terms will be indexed by points $\vec{m} \in \Lambda_{\vec{n}}$. Each of these terms is a product of one choice from each of the $r$ factors. Each choice looks like $\chi_{m_i}(u_i)\chi_{m_i}(u_{i+1})$. Thus in the resulting product there will be two factors involving each $u_i$, but coming from neighboring indices $m_{i-1}$ and $m_i$. See Section 5.4 for an example. □

**Proposition 12.** For any $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{P}_{\vec{n}}) = \sum_{\vec{m}\in\Lambda_{\vec{n}}} \left( \sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}} \right)$$

where $\chi_{\vec{p}}$ denotes the semisimple character $\chi_{p_1}(u_1)\chi_{p_2}(u_2)\cdots\chi_{p_r}(u_r)$.

*Proof.* We first consider $\prod_{i=1}^{r} \chi_{m_{i-1}}(u_i)\chi_{m_i}(u_i)$. By the Clebsch-Gordan identity, we can multiply two irreducible characters and obtain a sum of irreducible characters.

$$\prod_{i=1}^{r} \chi_{m_{i-1}}(u_i)\chi_{m_i}(u_i) = \prod_{i=1}^{r} \left[ \chi_{(m_{i-1}+m_i)}(u_i) + \chi_{(m_{i-1}+m_i-2)}(u_i) + \cdots + \chi_{|m_{i-1}-m_i|}(u_i) \right]$$

But now it's clear that when we expand *that* product, the terms in the resulting sum will be indexed by points $\vec{p} \in \lambda_{\vec{m}}$. If we let $\chi_{\vec{p}}$ denote $\chi_{p_1}(u_i)\chi_{p_2}(u_2)\cdots\chi_{p_r}(u_r)$ then we have a new way of writing $\chi(\mathcal{P}_{\vec{n}})$, as above, where the $\chi_{\vec{p}}$ are the characters of irreducible representations of the semisimple part of $K$. □

## 5.6 Finding $\chi(\mathcal{H}_{\vec{n}})$ inside $\chi(\mathcal{P}_{\vec{n}})$

**Proposition 13.** For any $\vec{n} \in \mathbb{N}^r$ such that $n_{min} \geq 2$ we have

$$\chi(\mathcal{P}_{\vec{n}}) - \chi(\mathcal{P}_{\vec{n}-\vec{2}}) = \sum_{\vec{m}\in \mathrm{Shell}(\Lambda_{\vec{n}})} \left[\Big(\sum_{\vec{p}\in \mathrm{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}}\Big) + \Big(\sum_{\vec{p}\in \mathrm{Shell}(\lambda_{\vec{m}-\vec{1}})} \chi_{\vec{p}}\Big) + \cdots + \Big(\sum_{\vec{p}\in \mathrm{Shell}(\lambda_{\vec{m}-\overrightarrow{n_{min}}})} \chi_{\vec{p}}\Big)\right]$$

with the convention that the sums over $\mathrm{Shell}(\lambda_{\vec{m}-\vec{j}})$ are empty whenever $\vec{m}-\vec{j}$ has any component $m_i - j < 0$.

*Proof.* Since $\Lambda_{\vec{n}} = \mathrm{Shell}(\Lambda_{\vec{n}}) \cup \Lambda_{\vec{n}-\vec{2}}$ whenever $n_{min} \geq 2$, we have

$$\begin{aligned}\chi(\mathcal{P}_{\vec{n}}) &= \sum_{\vec{m}\in\Lambda_{\vec{n}}} \Big(\sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}}\Big) \\ &= \sum_{\vec{m}\in \mathrm{Shell}(\Lambda_{\vec{n}})} \Big(\sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}}\Big) + \sum_{\vec{m}\in\Lambda_{\vec{n}-\vec{2}}} \Big(\sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}}\Big)\end{aligned}$$

But the right-hand summand is simply $\chi(\mathcal{P}_{\vec{n}-\vec{2}})$, which means

$$\chi(\mathcal{P}_{\vec{n}}) = \sum_{\vec{m}\in \mathrm{Shell}(\Lambda_{\vec{n}})} \Big(\sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}}\Big) + \chi(\mathcal{P}_{\vec{n}-\vec{2}})$$

Since

$$\lambda_{\vec{m}} = \mathrm{Shell}(\lambda_{\vec{m}}) \cup \mathrm{Shell}(\lambda_{\vec{m}-\vec{1}}) \cup \cdots \cup \mathrm{Shell}(\lambda_{\vec{m}-\overrightarrow{m_{min}}})$$

we have

$$\chi(\mathcal{P}_{\vec{n}}) - \chi(\mathcal{P}_{\vec{n}-\vec{2}}) = \sum_{\vec{m}\in\mathrm{Shell}(\Lambda_{\vec{n}})} \left[\left(\sum_{\vec{p}\in\mathrm{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}}\right) + \left(\sum_{\vec{p}\in\mathrm{Shell}(\lambda_{\vec{m}-\vec{1}})} \chi_{\vec{p}}\right) + \cdots + \left(\sum_{\vec{p}\in\mathrm{Shell}(\lambda_{\vec{m}-\overrightarrow{m_{min}}})} \chi_{\vec{p}}\right)\right]$$

Each term corresponding to a choice of $\vec{m}$ will be a sum as above, which will clearly have a different number of terms for different $\vec{m}$, since $m_{min}$ is different for each $\vec{m}$. Since $\vec{n}$ itself is an element of $\mathrm{Shell}(\Lambda_{\vec{n}})$, we may have $\vec{m} = \vec{n}$, which is when $m_{min}$ is maximized. So instead, we take the convention that the sums over $\vec{p} \in \mathrm{Shell}(\lambda_{\vec{m}-\vec{j}})$ are empty unless $j \le m_{min}$, and we write each sum using $n_{min}$ as in the Proposition. This introduces potentially empty sums, but simplifies the expression in a way that becomes useful during Propositions 14 and 15. □

Now consider distributing the outer sum over the inner ones, and examine just one term.

**Proposition 14.** Re-indexing the double sum: For any $\vec{n} \in \mathbb{N}^r$ and $j \in \mathbb{N}$ we have

$$\sum_{\vec{m}\in\mathrm{Shell}(\Lambda_{\vec{n}})} \left(\sum_{\vec{p}\in\mathrm{Shell}(\lambda_{\vec{m}-\vec{j}})} \chi_{\vec{p}}\right) = \sum_{\overrightarrow{m'}\in\mathrm{Shell}(\Lambda_{\vec{n}-\vec{j}})} \left(\sum_{\vec{p}\in\mathrm{Shell}(\lambda_{\overrightarrow{m'}})} \chi_{\vec{p}}\right)$$

*Proof.* This proof is actually quite straight-forward, but the notation can be confusing. Both sides are sums of $\chi_{\vec{p}}$, and so we need only check that the sets of indices $\vec{p}$ are the same, so the proposition is nothing but a re-indexing of the sum.

Consider showing $\subseteq$. Choose $\vec{p} \in \mathrm{Shell}(\lambda_{\vec{m}-\vec{j}})$ for some $\vec{m} \in \mathrm{Shell}(\Lambda_{\vec{n}})$. Since we have chosen a $\vec{p}$, we chose it from a non-empty sum, and so $j \le m_{min}$. Define $\overrightarrow{m'}$ by defining $m'_i = m_i - j$. We have that $p_i \in \{m_{i-1}-j+m_i-j, m_{i-1}-j+m_i-j-2, \ldots, |m_{i-1}-j-(m_i-j)|\}$, where there is some $k$ such that $p_k = m_{k-1} - j + m_k - j$, since we are in the Shell. But this is the same as saying that $p_i \in \{m'_{i-1} + m'_i, m'_{i-1} + m'_i - 2, \ldots, |m'_{i-1} - m'_i)|\}$, and for the same $k$, $p_k = m'_{k-1} + m'_k$. Thus $\vec{p} \in \mathrm{Shell}(\lambda_{\vec{m}'})$ as needed. Now, is $\vec{m}' \in \mathrm{Shell}(\Lambda_{\vec{n}-\vec{j}})$? Since $\vec{m} \in \mathrm{Shell}(\Lambda_{\vec{n}})$ then $m_i \in \{n_i, n_i-2, \ldots, [n_i]_2\}$, and there is some $k$ such that $m_k = n_k$. Since

$j \leq m_{min}$ then $m'_i = m_i - j$ is always $\geq 0$, so we know $m'_i \in \{n_i - j, n_i - j - 2, \ldots, [n_i - j]_2\}$, and for the same $k$, $m'_k + j = n_i$ so $m'_k = n_i - j$. This means $\vec{m}' \in \text{Shell}(\Lambda_{\vec{n}-\vec{j}})$ as required.

And now to show $\supseteq$. Pick $\vec{p} \in \text{Shell}(\lambda_{\vec{m'}})$ for some $\vec{m'} \in \text{Shell}(\Lambda_{\vec{n}-\vec{j}})$. Define $\vec{m}$ by defining $m_i = m'_i + j$. Is $\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})$? We have that $m'_i \in \{n_i - j, n_i - j - 2, \ldots, [n_i - j]_2\}$, which means that $m'_i + j \in \{n_i, n_i - 2, \ldots, [n_i - j]_2 + j\}$, which also means that $m_i \in \{n_i, n_i - 2, \ldots, [n_i]_2\}$. We also have the existence of $l$ such that $m'_l = n_l - j$, implying $m_l - j = n_l - j$, which means $m_l = n_l$. Thus $\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})$. Is $\vec{p} \in \text{Shell}(\lambda_{\vec{m}-\vec{j}})$? We have that $p_i \in \{m'_{i-1} + m'_i, m'_{i-1} + m'_i - 2, \ldots, |m'_{i-1} - m'_i|\}$ where for some $k$ we have $p_k = m'_{k-1} + m'_k$. Then since $m_i = m'_i + j$ we have $p_i \in \{m_{i-1} - j + m_i - j, m_{i-1} - j + m_i - j - 2, \ldots, |(m_{i-1} - j) + (m_i - j)|\}$ with $p_k = m_{k-1} - j + m_k - j$. This means $\vec{p} \in \text{Shell}(\lambda_{\vec{m}-\vec{j}})$. This completes the proof. □

**Proposition 15.** For any $\vec{n} \in \mathbb{N}^r$ such that $n_{min} \geq 2$ we have

$$\begin{aligned}
\chi(\mathcal{P}_{\vec{n}}) - \chi(\mathcal{P}_{\vec{n}-\vec{2}}) = &\sum_{\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p} \in \text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) \\
&+ \sum_{\vec{m} \in \text{Shell}(\Lambda_{\vec{n}-\vec{1}})} \left( \sum_{\vec{p} \in \text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) \\
&+ \cdots \\
&+ \sum_{\vec{m} \in \text{Shell}(\Lambda_{\vec{n}-\overrightarrow{n_{min}}})} \left( \sum_{\vec{p} \in \text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)
\end{aligned}$$

*Proof.* This follows easily from the previous two Propositions. First distribute the outer sum over the inner sums in Proposition 13 and then re-index each resulting double sum by using Proposition 14. □

## 5.7 Decomposing $\mathcal{H}_{\vec{n}}$ by induction

Finally we come to the decomposition of $\mathcal{H}_{\vec{n}}$. The claim is that

**Proposition 16.** For any $\vec{n} \in \mathbb{N}^r$ we have

$$\chi(\mathcal{H}_{\vec{n}}) = \sum_{\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p} \in \text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)$$

where the $\chi_{\vec{p}}$ are irreducible characters

$$\chi_{p_1}(u_1)\chi_{p_2}(u_2)\cdots\chi_{p_r}(u_r)$$

of $SL_2^r$ coming from the semisimple part of $K$.

*Proof.* The proof is by induction. Consider the graded component $\vec{n}$. Let's refer to the minimum coordinate $n_i$ as $n_{min}$. The base case is when $n_{min} = 0$. In this case, $\Lambda_{\vec{n}} = \text{Shell}(\Lambda_{\vec{n}})$, since in the factor of $n_{min}$, $\{n_i, n_i - 2, \ldots, [n_i]_2\}$ degenerates to simply $\{0\}$ and so every point is in the Shell. Then every $\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})$ has at least one component, say $k$, where $m_k = 0$. $\lambda_{\vec{m}} = \prod\{m_{i-1} + m_i, m_{i-1} + m_i - 2, \ldots, |m_{i-1} - m_i|\}$ by definition, but then the $k$th factor degenerates to simply $\{m_{k-1}\}$. This means that $\text{Shell}(\lambda_{\vec{m}}) = \lambda_{\vec{m}}$. Then

$$\chi(\mathcal{P}_{\vec{n}}) = \sum_{\vec{m} \in \Lambda_{\vec{n}}} \left( \sum_{\vec{p} \in \lambda_{\vec{m}}} \chi_{\vec{p}} \right) = \sum_{\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p} \in \text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)$$

Since there are no invariants when $n_{min} = 0$, this is also the character $\chi(\mathcal{H}_{\vec{n}})$ (see Proposition 9), which completes the base case $n_{min} = 0$.

Consider now the case $n_{min} = 1$. Then there is one invariant, $f$, of multi-graded degree $(1, 1, \ldots, 1)$. In this case

$$\chi(\mathcal{P}_{\vec{n}}) = \chi(\mathcal{H}_{\vec{n}}) + \chi(\mathcal{H}_{\vec{n}-\vec{1}})$$

At this point, we know both $\chi(\mathcal{P}_{\vec{n}})$ and $\chi(\mathcal{H}_{\vec{n}-\vec{1}})$, meaning we can solve for the unknown

$\chi(\mathcal{H}_{\vec{n}})$. Since when $n_{min} = 1$, $\text{Shell}(\Lambda_{\vec{n}}) = \Lambda_{\vec{n}}$, we have

$$\begin{aligned}
\chi(\mathcal{P}_{\vec{n}}) &= \sum_{\vec{m}\in\Lambda_{\vec{n}}} \left( \sum_{\vec{p}\in\lambda_{\vec{m}}} \chi_{\vec{p}} \right) \\
&= \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} + \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}-\vec{1}})} \chi_{\vec{p}} \right) \\
&= \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) + \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}-\vec{1}})} \chi_{\vec{p}} \right) \\
&= \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) + \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}-\vec{1}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) \\
&= \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) + \chi(\mathcal{H}_{\vec{n}-\vec{1}})
\end{aligned}$$

where we have used Proposition 14, as well as our induction base case of $n_{min} = 0$. It's now clear that $\chi(\mathcal{H}_{\vec{n}})$ is leftover, and is exactly as claimed.

Consider the graded component $\vec{n}$, where $n_{min} \geq 2$. The final induction step assumes we have shown the claim is true for all $m_{min} < n_{min}$. Since by the induction step we know the character of any $\mathcal{H}_{\vec{n}-\vec{j}}$ for $j > 0$ then Proposition 15 becomes

$$\chi(\mathcal{P}_{\vec{n}}) - \chi(\mathcal{P}_{\vec{n}-\vec{2}}) = \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right) + \chi(\mathcal{H}_{\vec{n}-\vec{1}}) + \cdots + \chi(\mathcal{H}_{\vec{n}-\overrightarrow{n_{min}}})$$

We can break up Proposition 9 into two separate lines as in the table below. But recognizing

| | | | | | | |
|---|---|---|---|---|---|---|
| $\chi(\mathcal{H}_{\vec{n}})$ | $= \chi(\mathcal{P}_{\vec{n}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{1}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{2}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{3}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{4}})$ | $-\dots$ |
| | | | $-\chi(\mathcal{H}_{\vec{n}-\vec{2}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{3}})$ | $-2\chi(\mathcal{H}_{\vec{n}-\vec{4}})$ | $-\dots$ |
| $\chi(\mathcal{H}_{\vec{n}})$ | $= \chi(\mathcal{P}_{\vec{n}})$ | $-\chi(\mathcal{H}_{\vec{n}-\vec{1}})$ | $-2\chi(\mathcal{H}_{\vec{n}-\vec{2}})$ | $-2\chi(\mathcal{H}_{\vec{n}-\vec{3}})$ | $-3\chi(\mathcal{H}_{\vec{n}-\vec{4}})$ | $-\dots$ |

Table 5.2: Subtracting characters in two ways

the second line of subtractions in the table as nothing but $\chi(\mathcal{P}_{\vec{n}-\vec{2}})$, and comparing the two equations, we see that

$$\chi(\mathcal{H}_{\vec{n}}) = \sum_{\vec{m}\in\text{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in\text{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)$$

as claimed. □

# 6 Connecting $\chi(\mathcal{H}_{\vec{n}})$ to affine geometry

In the previous section we found the decomposition of $\mathcal{H}_{\vec{n}}$ into irreducible representations of $K$. We know from Proposition 16 that

$$\chi(\mathcal{H}_{\vec{n}}) = \sum_{\vec{m}\in \mathrm{Shell}(\Lambda_{\vec{n}})} \left( \sum_{\vec{p}\in \mathrm{Shell}(\lambda_{\vec{m}})} \chi_{\vec{p}} \right)$$

But now, given a $K$-type $F_{\vec{z},\vec{s}}$, we ask for the multiplicity in all the various $\mathcal{H}_{\vec{n}}$,

$$[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = \ ?$$

**Proposition 17.** If $\vec{n} \notin \mathrm{ray}(\vec{z})$ then the multiplicity is zero. Otherwise, for $\vec{n} \in \mathrm{ray}(\vec{z})$ we have

$$[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = \#\{\vec{m} \in \mathrm{Shell}(\Lambda_{\vec{n}}) \text{ such that } \vec{s} \in \mathrm{Shell}(\lambda_{\vec{m}}))\}$$

*Proof.* By the definition of $\mathrm{Shell}(\lambda_{\vec{m}})$, any $\vec{s}$ can appear once, or not at all. Thus this proposition follows directly from Propositions 16 and 6. □

**Proposition 18.** For $\vec{s}, \vec{m} \in \mathbb{N}^r$, then $\vec{s} \in \mathrm{Shell}(\lambda_{\vec{m}})$ if and only if $\vec{m}$ satisfies

1. $|m_{i-1} - m_i| \le s_i \le m_{i-1} + m_i$ for all $i$.
2. $s_i = m_{i-1} + m_i \pmod 2$ for all $i$.
3. There exists at least one index $k$ such that $s_k = m_{k-1} + m_k$.

*Proof.* This simply turns around the definition

$$\lambda_{\vec{m}} = \prod_{i=1}^{r} \{m_{i-1} + m_i, m_{i-1} + m_i - 2, \ldots, |m_{i-1} - m_i|\}$$

combined with the definition of $\mathrm{Shell}(\cdot)$. □

We are interested in the multiplicity of our $K$-type $F_{\vec{z},\vec{s}}$ inside $\mathcal{H}_{\vec{n}}$ where $\vec{n} \in \text{ray}(\vec{z})$. We know all the points $\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})$, and we need only ask: how many of them satisfy the requirements from Proposition 18? The answer is our multiplicity.

Moving toward a geometrical description of these multiplicities, recall that a linear inequality defines a half-space in $\mathbb{R}^r$. It will be beneficial to think back to the examples in $r = 2$ and $r = 3$ given in Section 4. We intersect half-spaces to create a polyhedron $\mathcal{M} \subset \mathbb{R}^r$, and then by requiring some of the inequalities to be equalities we obtain $r$ faces of $\mathcal{M}$ which we call $\mathcal{S} \subset \mathbb{R}^r$, a hypersurface. We then intersect the $\text{Shell}(\Lambda_{\vec{n}})$ with $\mathcal{S}$ to obtain the graded multiplicity $[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}]$.

Recall from Section 4, Definition 9 defined $\mathcal{M} \subset \mathbb{R}^r$ as an intersection of half-spaces which depended on the parameter $\vec{s} \in \mathbb{N}^r$. Also Definition 10 defined the hypersurface $\mathcal{S} \subset \mathbb{R}^r$ as the union of $r$ faces of $\mathcal{M}$, hence also depending on the parameter $\vec{s} \in \mathbb{N}^r$. Note that condition (1) from Proposition 18, regarded as a linear inequality on $\vec{x} \in \mathbb{R}^r$ give exactly the half-spaces used in constructing $\mathcal{M}$. Then note that condition (3) simply means that $\vec{m} \in \mathbb{N}^r \subset \mathbb{R}^r$ lies on one of the $r$ faces of $\mathcal{M}$ which define the hypersurface $\mathcal{S}$. In summary, condition (1) defines the polyhedron and condition (3) requires $\vec{m}$ to lie on one of $r$ faces.

And now we state the main result.

**Proposition 19.** Consider the irrep $F_{\vec{z},\vec{s}}$ and inquire about its multiplicity $[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}]$. If $\vec{n} \notin \text{ray}(\vec{z})$ then the multiplicity is zero. Otherwise, for a fixed $\vec{n} \in \text{ray}(\vec{z})$, the multiplicity of $F_{\vec{z},\vec{s}}$ inside $\mathcal{H}_{\vec{n}}$ is given by the number of intersection points between $\mathcal{S}$ and $\text{Shell}(\Lambda_{\vec{n}})$. In other words,

$$[F_{\vec{z},\vec{s}} : \mathcal{H}_{\vec{n}}] = \#\Big(\text{Shell}(\Lambda_{\vec{n}}) \bigcap \mathcal{S}\Big)$$

where the parameter $\vec{s}$ determines the polyhedron $\mathcal{M}$ and also the hypersurface formed from $r$ of its faces $\mathcal{S}$.

At this point we need to invoke some results from [11] before finishing with a proof of Proposition 19. We need this simply to satisfy condition (2) of Proposition 18.

**Lemma 4.** If the multiplicity of $F_{\vec{z},\vec{s}}$ in $\mathcal{H}$ is non-zero, then we have

1. For all $k \in \{1, \ldots, r-1\}$

$$z_k + \frac{1}{r}\sum_{i=1}^{r-1} z_i = s_k \pmod 2$$

2.

$$\frac{1}{r}\sum_{i=1}^{r-1} z_i = s_r \pmod 2$$

*Proof.* The proof of this fact requires calculations very similar to those done in [11]. Let $M$ be the centralizer in $K$ of a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{g}$ consisting of semisimple elements, where $\mathfrak{g}$ is the Lie algebra of our group $G = SL_{2r}$. Then the number of $M$-fixed vectors is equal to the multiplicity of $F_{\vec{z},\vec{s}}$ in the harmonics $\mathcal{H}$ (but completely ignoring the gradation!). When you require the number of M-fixed vectors to be non-zero, the two conditions on $\vec{z}$ and $\vec{s}$ stated in this Lemma follow. Thus, these conditions are satisfied by any irrep $F_{\vec{z},\vec{s}}$ occurring in $\mathcal{H}$ with non-zero multiplicity. Rather than include such detailed calculations, we leave them as an exercise in line with [11], since although technical, they are straight-forward, and have been considered elsewhere. □

**Lemma 5.** If $\vec{m} \in \text{Shell}(\Lambda_{\vec{n}})$ for $\vec{n} \in \text{ray}(\vec{z})$ then for all $i \in \{1, \ldots, r\}$

$$m_{i-1} + m_i = s_i \pmod 2$$

*Proof.* If we can show $n_{i-1} + n_i = s_i \pmod 2$ then we are done, since $\vec{m}$ is obtained from $\vec{n}$ by simply subtracting multiples of 2 from various components.

Recall that if $\vec{n} \in \text{ray}(\vec{z})$ then for $k \in \{1, \ldots, r-1\}$

$$n_k = n_r + \sum_{i=1}^{k} z_i - \frac{k}{r}\sum_{i=1}^{r-1} z_i$$

First consider $k \in \{1, \ldots, r-1\}$. Doing all calculations mod 2 we have

$$\begin{aligned} n_{k-1} + n_k &= n_r + \sum_{i=1}^{k-1} z_i + \frac{k-1}{r} \sum_{i=1}^{r-1} z_i + n_r + \sum_{i=1}^{k} z_i + \frac{k}{r} \sum_{i=1}^{r-1} z_i \\ &= z_k + \frac{1}{r} \sum_{i=1}^{r-1} z_i \pmod 2 \\ &= s_k + s_r + s_r \pmod 2 \\ &= s_k \pmod 2 \end{aligned}$$

where we have used Lemma 4 in the second-to-last step. Last we consider $k = r$. Again doing calculations mod 2 we have

$$\begin{aligned} n_{r-1} + n_r &= \big(n_r + \sum_{i=1}^{r-1} z_i + \frac{r-1}{r} \sum_{i=1}^{r-1} z_i\big) + n_r \\ &= \frac{1}{r} \sum_{i=1}^{r-1} z_i \pmod 2 \\ &= s_r \pmod 2 \end{aligned}$$

This concludes the proof. □

*Proof of Proposition 19.* By the definition of $\mathcal{S}$, properties (1) and (3) of Proposition 18 are satisfied. By Lemma 5, property (2) is also satisfied. Thus, any $\vec{m} \in \mathrm{Shell}(\Lambda_{\vec{n}})$ that lies on the hypersurface $\mathcal{S}$ will satisfy Proposition 18, and $\vec{s}$ will be in its $\mathrm{Shell}(\lambda_{\vec{m}})$. □

At this point, we have proved everything necessary to understand why the examples in Section 4 are true. We have a geometric description of the total multiplicity in $\mathcal{H}$ given by counting the intersections between a certain hypersurface $\mathcal{S}$ and an expanding sequence of *shells*, and we have a geometric description of the graded multiplicity in $\mathcal{H}_{\vec{n}}$ given by intersecting a given *shell* with that same hypersurface.

## REFERENCES

[1] W. H. Hesselink. *Characters of the nullcone*. In: Mathematische annalen 252.3 (1980), pp. 179182.

[2] Shin-ichi Kato. *Spherical functions and a q-analogue of Kostants weight mul-tiplicity formula*. In: Inventiones mathematicae 66.3 (Aug. 1982), pp. 461468. issn: 1432-1297.

[3] Bertram Kostant. *Lie Group Representations on Polynomial Rings*. In: American Journal of Mathematics 85.3 (1963), pp. 327404.

[4] Bertram Kostant and Stephan Rallis. *Orbits and representations associated with symmetric spaces*. In: American Journal of Mathematics 93.3 (1971), pp. 753809.

[5] Lieven Le Bryun and Claudio Procesi. *Semisimple representations of quivers*. In: Transactions of the American Math Society 317.2 (1990), pp. 585598.

[6] I.G. Macdonald. Symmetric functions and Hall polynomials. Oxford Classic Texts in the Physical Sciences: Oxford Mathematical Monographs. Oxford University Press, 2015. isbn: 9780198739128.

[7] Richard P. Stanley. *The stable behavior of some characters of SL(n,C)*. In: Linear and Multilinear Algebra 16.1-4 (1984), pp. 327.

[8] Ernest B. Vinberg. A Course in Algebra. Graduate Studies in Mathematics ; Vol. 56. American Mathematical Society, 2003. isbn: 978-0-8218-3318-6.

[9] Ernest B. Vinberg. Linear Representations of Groups. Basler Lehrbucher, a series of advanced textbooks in mathematics ; Vol. 2. Birkhauser Verlag, 1989. isbn: 3-7643-2288-8.

[10] Ernest B. Vinberg. *The Weyl group of a graded Lie algebra*. In: Mathematics of the USSR-Izvestiya 10.3 (1976), p. 463.

[11] Nolan R. Wallach. *An Analog of the Kostant-Rallis Multiplicity Theorem for -Group Harmonics*. In: Progress in Mathematics 323.1 (2017), pp. 603625.

[12] Nolan R. Wallach. Geometric invariant theory. Universitext. Springer International Publishing, 2017. isbn: 978-3-319-65905-3.

# 7 Curriculum Vitae

Alexander Heaton

**Education**

**Marquette University High School** graduated 2007

**UW Madison** attended for one year, 2007-2008

**Marquette University** attended for three years. Graduated 2011 with Bachelor's degree.

**UW Milwaukee** attended for five years, 2014-2019

**Publications**

**Graded multiplicity in harmonic polynomials from the Vinberg setting**

*Submitted for publication* but also available here: eprint arXiv:1805.03178

**Branching from the general linear group to the symmetric group and the principal embedding**

*Submitted for publication* with Songpon Sriwongsa and Jeb F. Willenbring

Also available here: eprint arXiv:1812.06211

**Awards**

**Mark Lawrence Teply Award**, 2018

for graduate students showing remarkable potential in their research field

**Morris and Miriam Marden Award in Mathematics**, 2018

for a mathematical paper of high quality

**Ernst Schwandt Teaching Award**, 2017

for outstanding graduate student teaching

**GAANN Fellowship**, 2014 - 2016

Graduate assistance in areas of national need

**Gold Medal Award**, 2011

for highest graduating G.P.A. of all undergraduates at Marquette University

### Service

**UWM Math Circle**, 2017-2019

met weekly to work with young students from the community

**STEM for a day: UWM Black Cultural Center**, April 2018

gave presentations about research mathematics to area high school students

**Spotlight on Mathematicians of Color**, April 2018

organized this talk jointly with UWM multi-cultural centers for undergraduates interested in math

**Group Theory and Physics Seminar**, 2017-2018

organized and led a joint physics/math student seminar

**Graduate Student Colloquium**, 2016-2018

organized a weekly colloquium for graduate students

**Undergraduate Math Club**, March 2018

gave a talk to undergraduate math majors about the classification of closed surfaces (two-dimensional manifolds)

**Conferences and Talks**

**UW-Madison Algebraic Combinatorics Seminar** - Madison, February 2019

presented *Branching from* $GL_n$ *to* $\mathfrak{S}_n$*, Plethysm, and the Principal Embedding*

**Joint Mathematics Meetings (JMM)** - Baltimore, January 2019

**AMS Special Session on Group Representation Theory and Character Theory**

presented *Graded multiplicity in harmonic polynomials* (4:30pm Jan 19)

presented *Embedding* $\mathfrak{sl}_k$ *in* $\mathfrak{sl}_\mathfrak{n}$ *as a small subalgebra and the symmetric group* (8am Jan 19)

**MAA Contributed Paper Session on It's Circular: Conjecture, Compute, Iterate**

presented *Grids in the circle* (1pm Jan 18)

**Southeastern Lie Theory Workshop X** - Georgia, June 2018

presented *Graded multiplicity in harmonic polynomials*

**Summer School on Lie Theory** - Georgia, June 2018

**GSCC Combinatorics Conference** - Dallas, April 2018

presented *Graded multiplicity in harmonic polynomials*

**MAA-Wisconsin Sectional Meeting** - April 2017

presented *Symmetries of spacetime:* $SL_2(\mathbb{C})$ *and the Lorentz group*

**Workshop: Atlas of Lie Groups and Representations** - Utah, July 2017

**UWM Graduate Student Colloquium**

presented *Braid Groups* - Spring 2019

presented *Emmy Noether: Symmetry and Conservation Laws* - Fall 2018

presented *Fourier Analysis and Groups* - Spring 2017

presented *An intuitive introduction to generating functions* - Fall 2017

presented *Lie groups and Lie algebras* - Spring 2016

presented *Introduction to projective geometry* - Fall 2015

**Teaching**

I was the sole and primary instructor for the following courses:

Math 103 Contemporary Applications of Mathematics (Fall 2016)

Math 231 Calculus and Analytic Geometry I (Spring 2017)

Math 232 Calculus and Analytic Geometry II (Fall 2017)

Math 233 Calculus and Analytic Geometry III (Spring 2018)

Math 240 Matrices and Applications (Fall 2018)

Math 276 Algebraic structures for elementary education majors (coming Spring 2019)

**Languages**

English (native), Spanish (advanced)

Python, LaTeX, Magma, SAGE, Mathematica, Matlab