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Prospective Early Childhood Teachers’ Evolving Conceptions of Using a Mathematics Learning Trajectory to Guide Intentional Teaching

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PROSPECTIVE EARLY CHILDHOOD TEACHERS' EVOLVING CONCEPTIONS OF USING A MATHEMATICS LEARNING TRAJECTORY TO GUIDE INTENTIONAL TEACHING

by

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ABSTRACT

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This qualitative, phenomenological study investigated how fifteen early childhood preservice teachers' (PSTs) mathematical knowledge needed for teaching and early mathematics learning trajectory knowledge impacted the intentionality of instructional decision-making. The central research question asked: In what ways do early mathematics learning trajectories inform prospective early childhood teachers' instructional decisions in ways that are likely to advance student learning on the subitizing trajectory? The literature review revealed numerous studies focused on the usefulness of learning trajectory knowledge on prospective elementary and inservice teachers' mathematical knowledge for teaching, lesson planning, instruction, and assessment, but no studies were found regarding early childhood pre-service teachers' understanding of an early mathematics learning trajectory to guide intentional instructional decision-making.

A semi-structured interview protocol with stimulus texts was designed to elicit early childhood PSTs' understanding of subitizing, the subitizing trajectory, and the influence of each on their instructional decision-making. Five themes emerged from the

analysis of this data offering insights into the intentionality of early childhood PSTs' decision-making to advance student learning: (1) demonstrates an understanding of subitizing, (2) recognizes and validates the importance of subitizing for young children, (3) articulates learning trajectory progression through dot arrangements, (4) demonstrates an awareness of the developmental nature of children's mathematical thinking, and (5) centers instructional decisions on children's thinking.

Findings from this study suggest early childhood PSTs (a) demonstrated a keen interest in understanding children's thinking and were capable of crafting instructional opportunities that aligned with the subitizing learning trajectory, (b) developed a complex and nuanced understanding of the subitizing trajectory, and (c) engaged in a cycle of instructional decision-making highlighting an intricate relationship between subject matter knowledge, pedagogical content knowledge, and learning trajectory knowledge.

Dedication

Aubrey, Natalie, Rose, and Janine

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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CHAPTER ONE

INTRODUCTION

Children's early mathematics experiences are foundational for their future success as mathematics learners (Claessens, Duncan, & Engel, 2009). A substantial body of research highlights not only the capability of young children in learning mathematics, but the importance of mathematical development in young children (Balfanz, 1999, Baroody, Lai, & Mix, 2008; Gallistel & Gelman, 1991; Ginsburg, Lee, & Boyd, 2008; NRC, 2001; NRC, 2009; Sarama & Clements, 2009; Seo & Ginsburg, 2004). In fact, early mathematics is a significant predictor of later academic success in elementary school, and even into middle and high school (Duncan et al., 2007; Ritchie & Bates, 2013; Watts, Duncan, Siegler, & Davis-Kean, 2014). Surprisingly, early mathematics not only predicts later success in mathematics, but also predicts later reading achievement even better than early reading skills (Claessens & Engel, 2013; Duncan et al., 2007). This evidence is consistent for children regardless of income level and gender (Seo & Ginsburg, 2004) highlighting the importance of mathematics learning in preschool.

Mathematical knowledge begins during infancy and undergoes extensive development over the first five years of life (Baroody 2004; Liu, Bowman-Thomas, & Siegler, 1996; Ginsburg & Seo, 2004; NRC, 2009; Piaget, 1952). Moreover, young children have a surprising capacity to learn substantial mathematics. Indeed, "young children possess a remarkable ability to formulate, represent, and solve simple mathematics problems and to reason and explain their mathematical activities. They are positively predisposed to do so and to understand mathematics when they first encounter it" (NRC, 2001, p.6).

Unfortunately, most children in the U.S. have a discouraging lack of opportunity to engage in rich mathematical experiences (Clements, 2013). Too many young children start their formal schooling behind in mathematics, laying the foundation for persistent gaps in achievement (Clements, Baroody, & Sarama, 2013; Watts, Duncan, Siegler, & Davis-Kean, 2014). These negative effects are in one of the most important subjects in academic life and affect children's overall life course (Duncan & Magnuson, 2011; Sarama & Clements, 2009; Furtak, 2009). Given the critical role of a strong start in mathematics, the National Council of Teachers of Mathematics (NCTM) and National Association for the Education of Young Children (NAEYC) took the position that early childhood programs should "provide for children's deep and sustained interaction with key mathematical ideas" (NAEYC/NCTM, 2010, p. 6).

Extensive research in the past two decades has focused on understanding how children's thinking changes and evolves over time in specific content domains. Researchers observed that children follow typical developmental pathways in learning mathematics, leading to the articulation of detailed learning trajectories (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Confrey et al., 2012; NRC, 2009; Fosnot & Dolk, 2001a, 2001b, 2002, 2010; Sarama & Clements, 2009). Researchers hypothesized how mathematics learning trajectories might be useful to classroom teachers, though few if any have begun to explore how to situate learning trajectories within early childhood prospective teacher education. Therefore, the purpose of this study was to explore how an understanding of mathematics learning trajectories influences early childhood teachers' instructional decision-making that is likely to result in advancing young children's learning of mathematics.

Purpose of the Study and Research Questions

The Conference Board of Mathematical Sciences (2001) argued that teachers should study the mathematics they teach in depth. Ball, Thames, and Phelps (2008) conceptualized this “professionally oriented subject matter knowledge in mathematics” (p. 389) as mathematical knowledge for teaching (MKT). Learning trajectories, initially viewed as a tool to chart a course for student learning (Clements & Sarama, 2014), are valuable sites to deepen and refine teachers’ MKT (Wilson, Sztajn, Edgington, & Confrey, 2014). Despite the fact that standards for new teachers recommend that teachers develop “a deep and flexible understanding of their content areas” (CCSSO, 2011, p. 8), many beginning early childhood teachers are typically left underprepared to engage in teaching mathematics (Daro, Mosher, & Corcoran, 2011).

This research study examined the effects of an understanding of the subitizing learning trajectory (Sarama & Clements, 2014) on prospective early childhood teachers’ instructional decision-making. I conjectured that when early childhood prospective teachers come to understand young children’s developmental growth on the subitizing learning trajectory they will make instructional decisions that will intentionally advance young children’s subitizing skill and ability. Specifically, this descriptive qualitative study investigated the following research question and attendant questions:

Question: In what ways do learning trajectories inform early childhood prospective teachers’ instructional decisions in ways that are likely to advance student learning on the subitizing trajectory?

Attendant Question #1: *What understandings do early childhood prospective teachers have regarding the subitizing learning trajectory?*

Attendant Question #2: *Do early childhood prospective teachers draw upon their knowledge of* early mathematics *learning trajectories as they make instructional decisions?*

This study contributes research to two fields—mathematics education and early childhood teacher education.

Background of the Problem

Improving early childhood mathematics education has been the focus of recent national discussions (e.g., Clements et al., 2013; Early Learning STEM Symposium, 2016). Key advocacy groups for both early childhood and mathematics education—the National Association for the Education of Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM)—issued a joint position statement on the importance of early mathematics (NAEYC & NCTM, 2010). The National Mathematics Advisory Panel (NMAP, 2008) focused on mathematics learning for Pre-K to Grade Eight. To that end, the National Research Council (NRC, 2009) issued a set of recommendations for early childhood mathematics teaching and learning. This increased interest in early childhood mathematics education brings the work of early childhood teachers, and those that are responsible for preparing early childhood teachers to teach mathematics well, to the forefront of key issues in national policy agendas in the United States.

The surprising importance of early mathematics is highlighted for several reasons. First, mathematical proficiency has become as important a gatekeeper as literacy and, thus, critically important for all members of society to achieve (Sarama & Clements, 2009). Additionally, considerable evidence suggests that proficiency with early mathematics skills is the strongest predictor of later mathematics and reading achievement (Claessens & Engel, 2013; Duncan, Dowsett, Claessens, Huston, Pagani, Engel, Brooks-Gunn, Sexton, Duckworth & Japel, 2007; Duncan & Magnuson, 2013). Finally, Watts et al. (2014) found that when children are able to make substantial gains in their mathematical skills upon entering school regardless of their school-entry skills, they are able to make consistent gains in mathematics throughout school.

Other studies link various aspects of the relationship between early mathematics and later achievement. Krajewski and Schneider (2009) found that early mathematics was a stronger predictor of later mathematics achievement than even intelligence or memory abilities. According to Duncan and Magnuson (2011) children with persistent problems attaining mathematics skills are less likely to graduate from high school or go to college, and that mathematics achievement in adolescence actually predicts subsequent labor market success.

What is the status of early mathematics in the United States? International comparisons indicate that children in the United States perform worse in mathematics, and their lagging mathematics development is evident as early as preschool (NRC, 2009). Domestically, wide gaps in performance among variously advantaged and disadvantaged groups persist, and appear to be increasing (Claessens & Engel, 2013; Sarama & Clements, 2004). Specifically, low socioeconomic status and some minority groups are risk factors for low mathematics achievement, which has been attributed to lack of opportunities to learn mathematics (Clements & Sarama, 2009). Further, children who live in poverty and who are members of linguistic and ethnic minority groups demonstrate significantly lower levels of mathematics achievement than their majority, middle class peers (Clements & Sarama, 2011).

Statement of the Problem

The predictive power of early mathematics skills confirms the need for high-quality mathematics learning experiences during the early years (Anders & Rossbach, 2015; Hachey, 2013; Sarama & Clements, 2009). Intentionally planned and expertly implemented instructional experiences early in the lives of young children can help to improve mathematics achievement and help prevent or counter the development of mathematics learning difficulties (NRC, 2009). University teacher education programs are uniquely positioned to support prospective teachers in learning how to nurture and instill mathematical skill and confidence in their future students. This is an exciting yet daunting challenge.

A major focus in early mathematics education is arguably to provide high-quality mathematics education for all children, from the earliest years (Clements et al., 2011). Reform efforts suggest that effective teaching of mathematics is required for young learners and should be centered on the mathematical knowledge needed for teaching (Ball & Bass, 2000; Ball et al., 2008) and an understanding of developmental learning progressions (Daro et al., 2011).

At the heart of effective mathematics teaching, Ma (1999) places a "profound understanding of fundamental mathematics." She noted that "a teacher with profound understanding of fundamental mathematics is not only aware of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics, but is able to teach them to students" (p. xxiv). In support, Ball's (2000) construct of professionally oriented knowledge reiterates that it is not just what mathematics teachers know, but "how they know it and what they are able to mobilize mathematically in the course of teaching" (p. 243).

Recently, the concept of learning trajectories has gained momentum as a tool to help future educators learn how to examine and understand students' mathematical thinking, as such they have both theoretical and pedagogical value. Learning trajectories in mathematics education are research-based frameworks developed to document in detail the likely progressions, over long periods of time, of students' reasoning about big ideas in mathematics. Seen as an "anticipated, empirically grounded learning path established prior to instruction that affords the teacher a framework around which instructional choices and decisions can be made"

(Simon, 1995, p. 139), learning trajectories are hypotheses that describe stages of thinking, knowledge, or skills that students are likely to go through as they develop an understanding of mathematical ideas (Clements & Sarama, 2014, Daro et al., 2011). Trajectories address both the possible order and nature of the stages in the growth of students' mathematical understanding as well as how teachers can build upon this knowledge to realize more effective teaching practices.

Recent findings (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge, & Gould, 2005; Brown, 2010; Edgington, 2012; Mojica, 2010; Sarama, Clements, Wolfe, & Spitler, 2016; Wilson, 2009) suggest that knowledge of mathematics learning trajectories support practicing and prospective teachers' understanding of student thinking, deepens their mathematical knowledge for teaching, and reinforces their mathematics teaching identity. Knowledge of developmental paths enhances teachers' understanding of children's thinking, helping teachers assess children's level of understanding and intentionally offer instructional activities meant to meet each student at their unique location on the trajectory. When teachers understand the developmental progressions for each major domain or topic of mathematics, and sequence activities based on them, they build mathematics learning environments that are particularly developmentally appropriate and effective (Clements & Sarama, 2014; Confrey, Maloney, & Corley, 2014; Duschl, Maeng, & Sezen, 2011; Furtak, 2009). Though useful at the level of curriculum, assessment, and standards (Corcoran, Mosher, & Rogat, 2009), evidence is only beginning to emerge to suggest how learning trajectories can be utilized in teacher education to provide a framework for intentional, equitable, and effective teaching practices.

The development of the research problem for this study was based on the need for early childhood prospective teachers to learn how to intentionally advance young children's mathematics learning. The conceptual framework (see Figure 1.1) guiding this study suggests that prospective teachers' knowledge of mathematics learning trajectories and their developing MKT coalesce to support intentional instructional decisions that facilitate young children's mathematical growth.

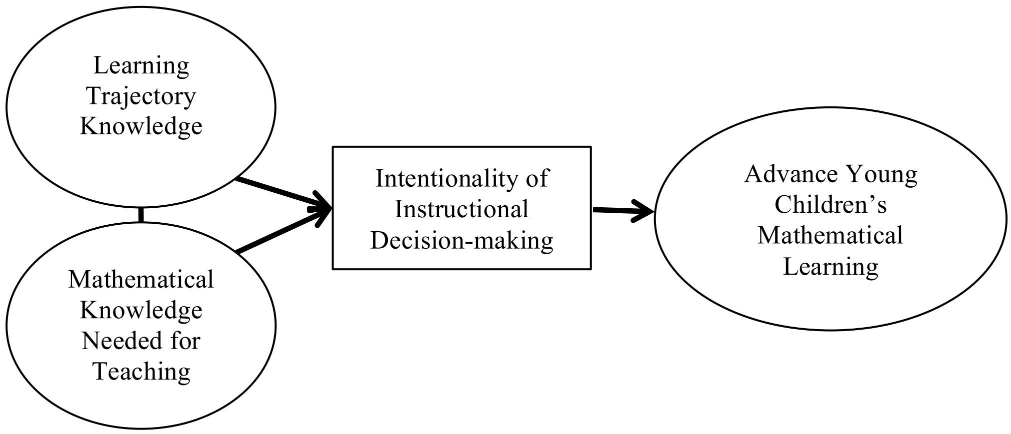


Figure 1.1. The conceptual framework for this study.

Significance of the Study

This qualitative study focused on improving the capacity of prospective early childhood teachers' of mathematics to advance student learning. One major goal of teacher education is to prepare prospective mathematics teachers to create environments where all students engage in high levels of academic performance. This is a monumental task, and mathematics educators face many challenges in supporting prospective teachers as they develop the necessary skills to create this type of mathematical environment for children.

Often, prospective teaches hold the same mathematics misconceptions as students (Graeber, Tirosh, & Glover, 1989) and enter teacher education programs with little to no experience in working with students on mathematical ideas. Many prospective teaches suffer from the negative effects of math anxiety and lack of confidence in their own mathematical ability and ability to teach mathematics. These negative beliefs lead to undervaluing the teaching of mathematics or prevent effective teaching (Bursal & Paznoska, 2006; Gresham, 2007; Harper & Daane, 1998).

As prospective teachers make sense of students' mathematical understanding, they often use their own reasoning as a lens, unable to distinguish children's thinking from their own (Bursal & Paznoska, 2006). This would suggest that an important goal for teacher educators is to support prospective teachers' shift from using their own thinking as a primary lens to process student reasoning, to having tools to help evaluate student thinking as they monitor learning goals and adjust instruction and tasks as necessary.

Teacher educators have an opportunity to provide early childhood prospective teachers with appropriate tools to ensure they are effective novice teachers. What aspects of mathematics are important, which less so? How do we diagnose what a child knows? How do we build on that knowledge—in what directions and in what ways? One tool that has the potential to answer questions and help early childhood prospective teachers become more effective teachers of mathematics is mathematics learning trajectories. Thus, this study seeks to provide insight into whether prospective teachers employ an understanding of a learning trajectory as they make instructional decisions intended to advance children's mathematical thinking.

Definition of Terms

The following definitions are provided to ensure uniformity and understanding of key terms used throughout the study. The terms included are: counting principles, developmentally appropriate practice, intentionality, intentional teaching, early childhood education, mathematical knowledge for teaching, mathematics learning trajectories, and subitizing.

Counting Principles

Counting includes three principles: the stable order principle, the one-to-one correspondence principle, and the cardinal principle. The stable order principle captures the fact that the count words are applied in a consistent order. One-to-one correspondence means that every individual item in a collection of objects is tagged with one and only one count word and each count word is applied to one and only one individual item. Finally, the cardinal principle entails that the last count word stated represents the number of individual items enumerated during the count (Gellman & Gallistel, 1986).

Developmentally Appropriate Practice

Teaching practices that respond to and promote individual children's optimal learning and development (NAEYC, 2013).

Early Childhood Education

The care and education of children in the earliest stages of childhood. According to the National Association of Young Children (NAEYC), it spans the human life from birth to age 8 (Retrieved from: https://nces.ed.gov/ipeds/cipcode/cipdetail.aspx?y=55&cipid=87297)

Intentionality

Intentionality means to act purposefully, with a goal in mind and a plan for accomplishing the goal (Espstein, 2014).

Intentional Teaching

Teaching that is grounded in defined learning objectives for children, selects instructional strategies likely to help children achieve the objectives, uses assessments that identify learning progress, and adjusts instructional strategies based on evidence of student thinking. When enacting intentional teaching, teachers systematically introduce content using developmentally based methods while respecting children's individual approaches to learning (Epstein, 2014).

Mathematical Knowledge for Teaching (MKT)

Ball et al., (2008) defined mathematical knowledge for teaching as “the mathematical knowledge needed to carry out the work of teaching mathematics” (p. 395).

Mathematics Learning Trajectories

Learning trajectories are research-based descriptions of how students’ thinking evolves over time from informal ideas to increasingly complex understandings and formal ideas, recognizing that each student’s path can be unique. Learning trajectories address both the possible order and nature of points in the growth of students’ understanding (Sarama & Clements, 2009). A complete learning trajectory includes three aspects: the goals of learning, the thinking and learning processes of children at various levels, and the sequence of learning activities aligned to the levels. See Appendix A for the Subitizing Learning Trajectory (Sarama and Clements, 2009), the trajectory featured in this study.

Subitizing

Subitizing is defined as the automatic recognition of quantity without counting and viewed as a hallmark of a young child’s developing sense of number and quantity (Clements, 1999). Subitizing includes two types, perceptual and conceptual. Perceptual subitizing is perceiving the whole quantity of a set of objects. Conceptual subitizing is seeing smaller quantities inside the larger and combining those smaller quantities to get the total. For example, a young child may “just know” or perceive that pattern A is six. Another child may recognize one set of four and one set of two and quickly combine them or conceptually compose them to make six. (See Figure 1.2.)

Perceptual subitizing Conceptual subitizing

"I see six!" "I see six because I see four and two. When I combine them I know it is six!"

Figure 1.2. The difference between perceptual and conceptual subitizing.

Organization of the Dissertation

Chapter 1 encompasses the statement of the problem, research questions, significance of the study, and definition of terms. Chapter 2 contains the review of literature and research related to this study. The methodology and procedures used to gather data for the study are presented in Chapter 3. The results of analyses and discussion of themes from the study are contained in Chapter 4. Chapter 5 provides a summary of the study and themes, conclusions drawn from the findings, a discussion how these themes relate to the field, and recommendations for further study.

CHAPTER TWO

LITERATURE REVIEW

This chapter presents a synthesis of the literature that frames the purpose and rationale for this study. The methods used to locate pertinent literature are first summarized. The literature review then begins with a synthesis of the research highlighting the critical need for impactful mathematics teaching and learning at the early childhood level. After briefly discussing this study's definition of a learning trajectory I review research on the knowledge needed for teaching and specifically the knowledge needed to teach mathematics well. Next, mathematics learning trajectories are discussed. Included is a synthesis of the varying perspectives on learning trajectories, commonalities among the perspectives, historical context of learning trajectories, learning trajectory based instruction, and a critical analysis of the learning trajectory construct. Then I review studies on researchers' initiatives to translate learning trajectories into useable tools for teacher. Finally, subitizing and the subitizing learning trajectory are examined as subitizing provides the content focus for the study and is used as an exemplar of mathematical knowledge needed for teaching.

Methods Used to Locate and Select Pertinent Literature

The search for pertinent literature began with a search using Google Scholar and all University of Wisconsin-Milwaukee Library databases, for 'pearls' using the following authors' names: Shulman; Ball and Bass; Clements and Sarama. I then searched Google Scholar and all University of Wisconsin-Milwaukee Library databases for relevant papers that addressed prospective teacher knowledge. Search terms included the following: pedagogical content knowledge, content knowledge for teaching, mathematical knowledge for teaching, mathematics education, practice-based teacher education, instructional practices, core teaching practices, and high-leverage teaching practices. A broad search was also conducted for learning trajectories. Search terms included the following: learning progressions, learning trajectories, learning trajectories and teacher education, learning trajectories and preschool teachers and math, developmental learning progressions.

Articles and resources on learning trajectories judged to have significant influence and impact on the developing research base were selected. To provide empirical validation for the development and use of learning trajectories I included publications from the field of science education. The literature that investigates mathematical knowledge needed for teaching and effective teaching practices is vast and explores a variety of avenues. I selected articles and publications judged to have significant influence and impact on learning trajectories, mathematical knowledge needed for teaching, and effective mathematics teaching practices.

Due to a lack of direct evidence supporting the focus of this study I will make a warrant-based claim. The literature suggests that evidence exists for the potential use of learning trajectories in curriculum development, assessment, and instruction. The relationship, however, between prospective teachers' knowledge of learning trajectories and how that knowledge may or may not inform instructional decisions that advance young children's mathematical growth has yet to be explored in the research base. Therefore, this study investigated the extent and ways in which early childhood prospective teachers applied their understanding of subitizing and the subitizing learning trajectory to intentionally advancing children's subitizing ability.

Terminology regarding early childhood education is often used inconsistently (Kagan, Kauerz & Tarrant, 2008). In this paper, the following terms are used:

- *early childhood education* when discussing the care and education of children from birth to age eight.
- *early mathematics* when discussing mathematics programming for children ages birth

through age eight.

- *preschool mathematics* when discussing mathematics programming for children age three to age five.
- *elementary school* when describing the education for children Grade 1 to Grade 5.
- *early childhood education (ECE) teachers* includes all personnel whose primary role is to provide direct instructional services for young children. Included in this category are lead teachers, assistant teachers, aides, and family childcare providers.
- *early childhood preservice or prospective teachers* (PSTs*)* include university students pursuing a bachelor's degree or post baccalaureate certification in early childhood education.

Early Mathematics

Position statements by national associations and research from both mathematics and early childhood educators have articulated the need to provide a solid foundation in mathematics education for young children. In 2000, NCTM updated *Principles and Standards for School Mathematics* to include a section on prekindergarten. Shortly thereafter, the National Association for the Education of Young Children (NAEYC) and NCTM (2002) issued a joint position affirming the important foundation high-quality, challenging, and accessible mathematics education provides for children ages three through six. Following the release of the National Research Council's (2009) report on early childhood mathematics, NAEYC and NCTM (2010) issued a revision of their join position statement that argued, "children should experience effective, research-based curriculum and teaching practices" in mathematics (p. 1).

Although virtually all young children have the capability to learn and become competent in mathematics, for most, the potential to learn mathematics in the early years of school is not currently realized (NRC, 2001; NRC, 2009). Historically, little attention has been paid to teaching mathematics to young children before they enter elementary school. This stems, at least in part, from generally negative attitudes about mathematics on the part of the American public as well as to beliefs that early childhood education should primarily consist of a nurturing environment that promotes social-emotional development, with academic content primarily focusing on language and literacy development.

Comparative studies demonstrate the poor mathematical achievement of American children to children from other industrialized countries, particularly for children of color and those living in poverty. Decades of evidence make it clear that many children in the United States are not meeting international standards (Geary et al., 1996; Ginsburg, 2009). Many contend that American children may be among the most poorly educated mathematics students in the industrialized world and that they are falling more and more behind their Asian and European counterparts (NRC, 2001; NRC, 2009).

A Historical Perspective

Prior to the onset of the twenty-first century, mathematics education in early childhood in the United States was not an emphasis (Geary et al., 1996) with the focus placed on the development of social skills and literacy skills (Epstein, 2014). Mathematics as an instructional subject had traditionally been considered above the preschool and kindergarten levels (Balfanz, 1999). Therefore, the teaching of mathematics in early childhood has often been viewed as developmentally inappropriate (Ginsburg, 2009). Because of this, mathematics instruction was delayed until elementary school (Balfanz, 1999), with little mathematics being studied prior to first grade beyond the counting of small quantities and the recognition of basic geometric shapes.

Young children were historically considered not cognitively capable of engaging in the thinking needed to understand mathematics (Piaget, Inhelder, & Szeminska, 1960). Learning theorists at the beginning of the twentieth century viewed young children as incapable of learning mathematics. Thorndike (1922), for example, concluded that young children were so mathematically inept that “little is gained by [doing] arithmetic before grade 2, though there are many arithmetic facts that they can [memorize by rote] in grade 1” (p. 198). In line with this perspective, a review of mathematics education in the United States showed that virtually no mathematics was offered from kindergarten through second grade in the early 1900s (Balfanz, 1999). Beginning with the progressive movement in the1920s, mathematics as a subject was gradually introduced into the elementary grades, becoming established at the early elementary level by the 1960s (Blair, Gamson, Thorne, & Baker, 2005). During this period, it was argued that the formal development of mathematical knowledge of children should be delayed until elementary school (Brownwell, 1941; Starkey, Klein, & Wakeley, 2004).

Piaget (1952) explored children’s developing knowledge before elementary school, presenting young children as mathematically curious and as actively constructing mathematical knowledge as they interacted with their physical and social world. Young children were deemed incapable of abstract and logical thinking until the concrete-operational stage, around age 7. Theoretically viewed as unable to construct a true concept of number and or an understanding of arithmetic. Latter interpretations (Ginsburg & Golbeck, 2004) of Piaget’s research focused on children’s deficiencies reinforcing the contention that young children could not benefit from early instruction in mathematics.

Shifting Perspectives

By the end of the twentieth century developmental psychologists transferred focus from what young children could not do to what they could do, initiating a new dominant trend in research (Gelman & Gallistel, 1978). This paradigm shift produced convincing evidence that young children—from infancy—are much more powerful mathematicians than previously known. Developmental studies found overwhelmingly that young children engage in diverse types of mathematical thinking in their everyday interactions with the social and physical world (Seo & Ginsburg, 2004). Clements & Sarama (2004a) in particular strongly argued, “prekindergarten children have the interest and ability to engage in significant mathematical thinking and learning” (p. 11).

How important is early math? A landmark set of studies found that preschool math concepts were the most powerful predictor of later learning (Claessens & Engel, 2013; Duncan, Claessens, & Engel, 2004; Claessens, Duncan, & Engel, 2009; Duncan et al., 2007). The finding was consistent for both boys and girls from high and low socioeconomic backgrounds.

Duncan and colleagues (2007), using six large-scale longitudinal studies involving up to 36,000 children, assessed the association between skills and behaviors that emerge during the preschool years and later academic achievement. While controlling for variables known to influence children’s academic performance such as socioeconomic status, mother’s education, family structure, and child heath they isolated the actual predictive powers of early math, reading, attention, and socio-emotional skills on academic achievement. Study results demonstrate that, among the aforementioned variables, early math skills were the strongest predictor of later academic performance. Furthermore, researchers found that “early mathematics skills predicted reading, math, and science achievement as well as grade retention from kindergarten through eighth grade” and that the “importance of these math skills for subsequent achievement increases or is maintained over time” (Claessens & Engel, 2013, p. 2).

Watts and colleagues (2014) found that early-grade (e.g., preschool, kindergarten) gains in mathematical skill were significant predictors of mathematics achievement at Grades 1, 3, 5 and age 15. Study results revealed that students (n=1,364) who make substantial gains in their mathematical skill, regardless of their school-entry skills at 54 months (4 ½ -years of age), made gains in mathematics throughout their schooling. Reading and working memory, by comparison, were found to be less predictive of later achievement. The authors found that this pattern held even as students transitioned from elementary to high school, where mathematics becomes considerably more complex. These results demonstrated the importance of prekindergarten mathematics knowledge and early math learning for later achievement.

Having established the strong predictive relation between early mathematics achievement and a broad range of later academic abilities, what effects might early mathematics ability have beyond the classroom? Using a large (n=18,558), nationally representative (England, Scotland, Wales), longitudinal sample spanning 1958 to 2009, Ritchie and Bates (2013) investigated the significance of mathematics skills in early childhood to socioeconomic success (SES) at mid-life. Results suggested that mathematics ability at age seven was substantially and positively associated with future socioeconomic success attained at age 42, regardless of gender.

A key takeaway from these findings confirms that children's mathematics learning in the first six years of life has profound, long-lasting outcomes for students in their later years and into adulthood. What children know early affects them for many years after (NMP, 2008) and ensuring strong math knowledge for early learners can help to provide more equitable opportunities for academic success and future economic success.

What mathematics deserves priority? It is evident high-quality early mathematics instruction matters and children's success as mathematical learners is more important than previously understood. Mathematics education research recommends PreK-Grade 2 mathematics center on number and operations, and geometry and measurement (Clements & Sarama, 2004b; NRC, 2009). These ideas, which are important preparation for school and for life, are also genuinely mathematical, with importance from a mathematician's perspective. Moreover, they are interesting to children, who enjoy engaging with these ideas and exploring them. Of aforementioned areas number and operations is arguably the most important. Number and operations includes early counting and cardinality, early operation sense, subitizing, comparing and ordering, and composing. Additional research offers critical insight into which of these key mathematical understandings deserve more time in early childhood mathematics programming.

In a four-year longitudinal study, Krajewski and Schneider (2009) identified specific quantity-number competencies (QNC) (e.g., knowledge of number-word sequence, quantity to number-word linkage) as more predictive of mathematical achievement in fourth grade than non-specific precursors (e.g., number naming speed, nonverbal intelligence, socio-economic status). Results revealed specific quantity-number competencies constituted an important prerequisite for the comprehension of school mathematics.

Using longitudinal data from a primarily low-income and minority sample of children (n=1,375) Nguyen and colleagues (2016) identified advanced number competencies as most predictive of mathematics achievement in fifth grade, more so than basic numeracy, geometry, patterning, and measurement skills. Basic number competencies included rote counting, one-to-one correspondence, number recognition, and perceptual subitizing (e.g., instant recognition of quantity). Advanced number competences included counting objects with cardinality, counting forward or backward from a given number, and conceptual subitizing (e.g., composing small groups to name a quantity).

This collection of studies points to the profound importance of early mathematics learning to prepare preschool children for school and life success. The strong predictive relation between early mathematics achievement and a broad range of later academic abilities establishes the urgent need to increase young children’s intentional engagement with mathematics in preschool (Moss, Bruce, & Bobis, 2016). High-quality mathematics learning opportunities, prior to formal schooling, and in the first years of school, are crucial. If preschoolers lag in early number-quantity competencies or advanced number competencies, and are offered appropriate interventions, gaps can be closed, but it must be done early in a child's educational experience (Nguyen et al., 2016).

Pedagogical Implications. A recent review of the National Association for the Education of Young Children (NAEYC) Guidelines over the last few decades reveals a shift toward productively integrating academic instruction with playful learning and efforts to develop social skills (NAEYC, 2009; Epstein, 2014). Given opportunities to learn, children develop an informal knowledge of mathematics that is surprisingly broad, complex, and sophisticated (Ginsburg, 2008; NRC, 2009; Sarama & Clements, 2004). Young children are interested in and enjoy learning mathematics and engage in a significant level of mathematical activity during free play. They explore patterns, compare sizes, and count objects. This is true for children regardless of income level or gender (Seo & Ginsburg, 2004).

As we consider how to best engage young children in mathematics learning experiences, the question of effective teaching-learning practices arise. Fuson, Clements, and Sarama (2015) suggest that learning mathematics with understanding is a primary goal of early mathematics instruction. They continue, "Unfortunately, most of us learned mathematics without much understanding. Our experience can limit our vision to rote learning, such as telling or showing, with little thinking by children" (p. 64). An alternative approach widely proposed in early childhood settings in that that a child discovers mathematical concepts and understandings themself through interacting with objects or in play. Fuson and colleagues (2015) caution that these "oversimplified … dichotomies," (p. 64) play versus academics, adult-directed versus child-directed, and child-centered versus teacher-centered/directed, disregard the complexities and interactive nature of learning and are potentially damaging to children's mathematical growth.

Can children learn mathematics solely through playing? The National Research Council (2009) asserts that the intuitive foundational mathematics skills young children naturally develop during play are not enough. While play offers extensive opportunities to develop dispositions and habits of mind valued in mathematics education (e.g., curiosity, creativity, persistence) it "does not guarantee mathematical development" (NAEYC/NCTM, 2002, p. 6) and has the potential to negatively impact the continuity and coherence of children's learning opportunities and experiences (Day-Hess & Clements, 2017). "Children do learn from play, but it appears they learn so much more with artful guidance and challenging activities provided by their teachers" (Seo & Ginsburg, 2004, p. 103). Epstein (2009) refers to this artful guidance as intentional teaching.

Intentional teaching involves teachers "adapting teaching to the content, type of learning experience, and individual child with a clear learning target as a goal" (NRC, 2009, p. 226) and does not imply didactic learning approaches such as worksheets, rote memorization, or seat work. It is through playful and intentional teaching that children advance beyond their intuitive mathematics thinking (Ginsburg, 2009; Hachey, 2013). Evidence suggests that child-centered, playful learning programs promote sustained academic performance compared to more traditional, academically focused programs (e.g., Diamond, Barnett, Thomas, & Munro, 2007; Lillard & Else-Quest, 2006; Marcon, 2002). Recognizing the importance of play in preschool mathematics programming, considerable discrepancies exist concerning how play-based pedagogies are conceptualized and implemented (Chein, Howes, Burchinal, Pianta, Ritchie, Bryant al. 2010).

Fisher, Hirsh-Pasek, Newcombe, and Golinkoff (2013) identified guided play or playful instruction as a promising middle ground to free play and direct instruction. Researchers implemented three instructional models to teach the geometric properties of four shapes to seventy four-to five-year olds: free play, guided play, and direct instruction. Results revealed that children taught shapes in the guided play condition showed improved shape knowledge compared to the other groups, an effect that was still evident one week after the intervention. Findings suggest that scaffolding techniques that heighten engagement, direct exploration, and facilitate "sense-making," such as guided play, undergird shape learning.

Impactful early mathematics experiences for children hinge on intentional teaching. Intentional teaching requires complex pedagogical skills, deep understanding of priority big ideas, and insight into how children acquire that knowledge. This is a "heavy lift" in mathematics for prospective and in-service teachers for several reasons. Many teachers of young children report a negative attitude toward mathematics and low confidence in their own mathematics abilities often related to past experiences learning mathematics (Maloney & Beilock, 2012; Copley, 2004; Harper & Daane, 1998; Lee & Ginsburg, 2007). Moreover, they admit to not seeing themselves as teachers of mathematics and do not place a high value on teaching mathematics (Ginsburg & Ertel, 2008; Maloney & Beilock, 2012).

Evidence supports a strong desire for teachers to view mathematics through a positive lens and to teach mathematics that ensures their students are capable and confident mathematics learners (Anders & Rossbach, 2014; Bursal & Paznoska, 2006; Hembree, 1990; Tobias, 1987). These studies suggest teacher education programs should attend to the emotional and motivational aspects of teaching as well deepening teachers' knowledge of mathematics and varied, yet intentional, pedagogical approaches.

Effective mathematics teaching is mediated by a teacher's capacity to develop young children's mathematical understanding and skill in key mathematical domains (Ball et al., 2008; Bobis et al., 2005; Daro et al., 2011; NRC, 2009; Sarama & Clements, 2009; Shulman, 1986; von Glaserfled, 1987). To that end, several empirically-based frameworks (e.g., Clements & Sarama, 2014) delineate children's mathematics growth from birth to age eight. Translating those frameworks into usable tools for early childhood teachers is a relatively new phenomenon. This study contributes to this nascent, yet burgeoning body of research.

This Study's Definition of a Learning Trajectory

In this study, a learning trajectory is defined as a learning path, or a known learning sequence, which delineates predictable development of young children's mathematical thinking form birth to age eight in specific content domains. Each learning trajectory identifies overarching big ideas and concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning. Children's progress along these pathways does not occur with maturation, but is the result of appropriate learning experiences.

In regard to this study, a complete learning trajectory consists of three components—(1) an overarching mathematical goal, (2) a developmental progression of children's reasoning, understanding, and abilities, and (3) aligned learning activities. The trajectories answer four key questions, listed in Figure 2.1 along with a summary of each component.

Key Questions	Learning Trajectory Component
Where am I trying to go with children's mathematics learning?	The **goal** of the trajectory names an overarching big idea of key mathematical ideas (e.g., subitizing, counting, comparing) identified as generative of children's future success.
Where are children now in their mathematical thinking? What is the next important mathematical idea to target?	The **developmental progression** offers a narrative description of mathematics learning for the specified goal, identified by successive levels of children's reasoning, understanding, and abilities that move from informal to more sophisticated mathematical thinking.
How can I foster children's mathematics learning along the continuum?	The **learning activities** are intentionally selected and carefully designed tasks, matched to each level, which promote growth and advancement on the trajectory.

Figure 2.1. Learning trajectories help answer these questions.

Knowledge Needed For Teaching

In order to successfully navigate students' ideas during instruction, teachers need to develop not only their subject-specific knowledge, but also knowledge about how students learn the subject (Furtak, 2009). Learning trajectories have potential to be invaluable teacher preparation and professional development tools since they contain information regarding knowledge of student ideas and student learning, as well as suggestions for strategies or actions to help students learn.

Pedagogical and Content Knowledge

In 1986, Lee Shulman introduced pedagogical content knowledge (PCK) to the landscape of research on teaching and teacher education. At the time, the term called attention to a new and special kind of teacher knowledge that links content and pedagogy, a "particular form of content knowledge that embodies aspects of content most germane to its teachability" (p. 9). In addition to general pedagogical knowledge and knowledge of the content, Shulman (1986) suggests that teachers need to know what topics children find interesting or difficult and which representations most useful for teaching a specific content area.

Shulman (1987) articulated seven general dimensions of teacher knowledge. Figure 2,2 identifies the seven ideas that define “a sophisticated, professional knowledge that goes beyond simple rules such as how long to wait for students to respond” (Ball et al., 2008, p. 391). Each dimension works in concert with the others to articulate the important role of content knowledge and to situate content knowledge in the larger landscape of professional knowledge for teaching.

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter.
- Knowledge of learners and their characteristics.
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures.
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.
- Content knowledge. Common knowledge of the discipline.
- Curriculum knowledge, with particular grasp of the materials and programs that serve as “tools of the trade” for teachers.
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special from or professional understanding.

(Shulman, 1987, p. 8)

Figure 2.2. Shulman’s major categories of teacher knowledge. Seven dimensions that articulate the professional knowledge needed for teaching.

The current study centers on two of the Shulman’s components: content knowledge and pedagogical content knowledge. Literature in mathematics education (Ball, 2000; Ball et al., 2008; NRC, 2001, 2010) and professional consensus agree that mathematics teachers, regardless of the level or age of the students they teach, rely on a combination of mathematical content knowledge and pedagogical content knowledge.

Content knowledge includes knowledge of the subject, its key structures, and its big ideas, thus pushing knowledge beyond simple facts and concepts (Shulman, 1986, 1987). Additionally, content knowledge is not merely the content students will learn. When viewed through the lens of teaching it encompasses what teachers know about their subject and what knowledge they are able to apply in the course of teaching (Ball, 2000).

Shulman (1987) suggested that pedagogical content knowledge is of special interest because it identifies distinctive bodies of knowledge for teaching. Shulman (1986) defined pedagogical content knowledge as comprising:

> The most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others…pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics. (p. 9)

It is this interconnectedness between teaching and content that defines pedagogical content knowledge as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987 p. 8).

The introduction of pedagogical content knowledge surfaced questions about the content and nature or teachers' specialized subject matter understanding. Shulman (1986) invited consideration of the following: What are the sources of teacher knowledge? What does a teacher know and when did they come to know it? How does the teacher prepare to teach something never previously learned? How does learning for teaching occur?

Mathematical Knowledge for Teaching

Claiming that the concept of PCK proposed by Shulman (1986, 1987) was underdeveloped Ball et al. (2008) built on Shulman's work to conceptualize mathematical knowledge for teaching (MKT). By focusing their work on the careful study of the mathematical knowledge needed for teaching Ball and colleagues' work resulted in "refinements of the popular concept of PCK and the broader concept of content knowledge for teaching" (p. 390). Subsequently they argue several particular types of knowledge are unique to teaching.

Scaffolding from Shulman's thinking regarding the role and importance of PCK, Ball (2000) suggested, "Knowing subject matter knowledge and being able to use it is at the heart of teaching all students" (p. 243). To that end, Ball et al. (2008) defined mathematical knowledge for teaching as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). What is noteworthy regarding this definition is that it begins with teaching, not teachers (Ball, 2000; Ball & Bass, 2000; Ball et al., 2008). It is concerned with the tasks involved in teaching and the mathematical demands of these tasks. With an intentional focus on the work of teaching the work is now framed as seeking to "unearth the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching" (Ball et al., 2008, p. 395).

The diagram in Figure 2.3 articulates Ball et al.'s (2008) framework for mathematical knowledge for teaching. It is organized around two large domains: pedagogical content knowledge (PCK) and subject matter knowledge (SMK). The domain of pedagogical content knowledge is most related to knowledge that emerges from a focus on the learner's cognitive development and is based on teachers' understandings of the learner's thinking. The subject matter knowledge domain represents aspects of teacher knowledge that are centered on the logic of the discipline. Each of these domains is further divided into three categories of teacher knowledge.

Within pedagogical content knowledge, knowledge of content and students is defined as the “knowledge that combines knowing about students and knowing about mathematics (Ball et al., 2008, p. 401) so that teachers may anticipate what students are likely to think as well as what they find confusing, interesting, or motivating. Knowledge of content and teaching refers to knowledge about the design of instruction in ways that brings together mathematical understandings and understandings of the pedagogical choices that effect learning. This includes selecting examples, sequencing tasks, and evaluating advantages and disadvantages of various representations. In addition, pedagogical content knowledge includes knowledge of content and curriculum; an understanding of the ways a particular concept is developed with curricular materials.

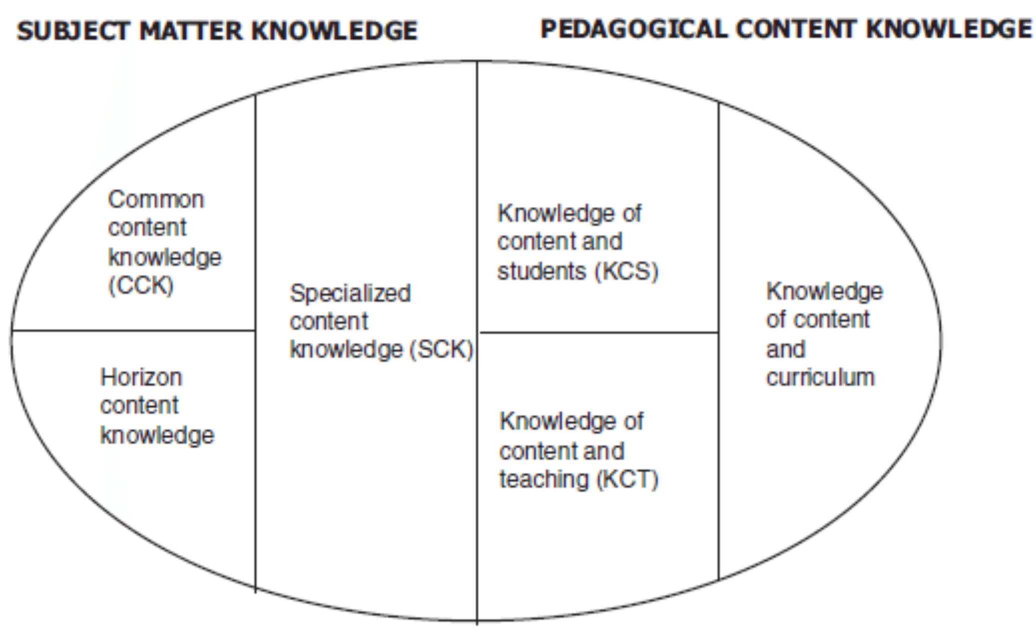


Figure 2.3. Domains of mathematical knowledge for teaching (Ball et al., 2008, p. 403). A framework for the mathematical knowledge needed for teaching.

Within the broad category of subject matter knowledge, Ball et al. (2008) explained that common content knowledge is the mathematical knowledge and skill used in settings other than teaching. Horizon content knowledge represents “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (p. 403). Finally, specialized content knowledge is the mathematical knowledge and skill that is unique to teaching.

In summary, Ball et al.’s (2008) framework suggests teachers draw upon a broad array of mathematical and pedagogical knowledge as they teach, affirming the depth of knowledge essential for effective teaching. This is particularly true for early childhood teachers when teaching mathematics. Effective mathematics teachers of young children possess an intimate understanding of the mathematics they teach which allows them to focus on their students’ mathematical thinking and subsequently make instructional decisions to advance their children’s learning.

The usefulness of refining Shulman’s eight categories into a conceptual map of the mathematical content knowledge for teaching (Ball et al., 2008) brings to the fore three critical ideas that may help hone teacher education and professional development efforts. Considerations include (1) developing teachers’ specialized content knowledge if it proves to be a greater predictor of student achievement than advanced content knowledge, (2) identifying varying aspects of teacher preparation and development which are shown to influence teachers’ PCK and SMK more than others, and (3) creating materials for teachers as well as teacher education and professional development. Incorporating work on learning trajectories as sites for teacher education and professional development may prove to be an essential next step in these efforts.

Mathematics Learning Trajectories

The meaningful development of mathematical knowledge stems from constructing a well-interconnected web of mathematical concepts and skills (NMAP, 2008). By connecting new information to previously learned knowledge, children are able to develop deep and flexible mathematical understanding (Hatano, 2003; Piaget, 1952). This often entails learning mathematical concepts and skills in an empirically delineated sequence. Such a sequence of the development of mathematical concepts and skills is called a learning trajectory (Battista, 2006; Clements & Sarama, 2004b, 2009; Confrey, 2012; Simon, 1995).

Understanding how students' mathematical ideas develop and how to apply such understandings to every aspect of teaching centers the work of teaching on student thinking. Such understanding is particularly important at the early childhood level because children often interpret mathematical situations, even those that seem obvious to adults, quite differently from adults (NRC, 2009). The younger the child the more important teachers' use of children's thinking and learning as starting points (Clements & Sarama, 2014).

The first use of the term "learning trajectory" as applied to mathematics education is credited to Martin Simon (1995) while reporting on his own work with prospective teachers. Simon proposed the notion of a hypothetical learning trajectory as a model of how students' learning might progress over a period of time, with particular attention on students' mathematical experiences necessary to prompt that learning. He framed a learning trajectory as an "anticipated, empirically grounded learning path established prior to instruction that affords the teacher a framework around which instructional choices and decisions can be made" (Simon, 1995, p. 139).

Learning trajectories articulate developmental progressions of children's mathematical thinking. These progressions play a special role in children's cognition and learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development. The National Research Council (2009) described learning trajectories as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 213). Simon (1995) placed a premium on the developmental nature of student thinking as he viewed the trajectory as a tool to hypothesize "how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In support, Clements and Sarama (2004) suggested that learning trajectories entail:

> Descriptions of children's thinking and learning in a specific domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking. (p. 83)

Battista (2006) conceptualized developmental progressions using the concept of levels of sophistication through which a student progresses from pre-instructional reasoning to different cognitive plateaus ending in formal mathematical concepts. Confrey, Maloney, Nguyen, Mojica, and Myers (2009) specified a learning trajectory was:

> A researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (p. 347)

Weber and Lockwood (2014) defined learning trajectories as "predictive or descriptive representations of the development of students' mathematical knowledge over time" (p. 46).

Though perspectives differ regarding to what a learning trajectory is, they contain common elements. First, all learning trajectories synthesize research on student thinking to describe predictable pathways of learning overtime. Levels of understanding that increase in sophistication typically delineate this pathway. Second, learning trajectories do not function independent of instruction and are influenced by interactions between instruction and students' prior knowledge. This implies that advancement on the learning trajectory is not a consequence of maturation, but hinges on appropriate instruction. Third, learning trajectories are not descriptions of a rigid pathway of learning; rather, they are approximations of the variety of partial understandings, critical conceptual markers, and likely steps along the way. Thus, learning trajectories differ from the sequence of topics typically used in instruction, which are most often based on disciplinary logic.

The word "trajectory" gives the impression of a specific linear pathway, but not all researchers or theorists view learning trajectories strictly in this manner. While a trajectory presents a progression of learning for a particular mathematical concept, each trajectory can take on a variety of representational forms, such as webs and networks (Simon, 1995), pathways (Sarama & Clements, 2009), connected hexagons (Confrey et al., 2012), or a landscape (Fosnot & Dolk, 2001a, 2001b, 2002, 2010).

Common Components of a Learning Trajectory

Learning trajectories may vary in span, grain size, use of misconceptions, and level of detail, but each focuses on one or more specific mathematical understanding(s), proposes the mathematical knowledge students need to have to form a coherent view of that idea, and describes a sequence of activities and instruction to engage students in learning the idea in the way the researcher proposed (Weber & Lockwood, 2014).

The mathematics learning goal. The first aspect of learning trajectories is the establishment of a mathematical goal. The goal, typically referred to as *big ideas of mathematics*, represents clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking, and important to future learning (Clements & Sarama, 2014). These goals identify a clear picture of the big ideas of mathematics children should learn. An example of a big idea for young children is subitizing or the ability to quickly recognize cardinality of sets of objects.

The developmental progression. The second part of a learning trajectory consists of developmental progressions most commonly delineated as stages of thinking, each more sophisticated than the last, through which children progress on their way to achieving the mathematical goal. Developmental progressions underlie learning trajectories (Clements & Sarama, 2004b; Furtak, 2009; Sarama & Clements, 2009; Simon, 1995; Weber & Lockwood, 2014; Wilson, Sztajn, Edgington, & Confrey, 2014). Most stages are levels of thinking—a "distinct period of time of qualitatively distinct ways, or patterns, of thinking" (Clements & Sarama, 2014, p. 5) that apply only within a specific big idea.

In essence, developmental progressions emphasize learning models that reflect natural developmental progressions identified in theoretically and empirically grounded models of children's thinking, learning, and development (Clements & Sarama, 2004; Confrey et al., 2014; Sarama & Clements, 2009). The models describe the processes involved in the construction of the mathematics goal across several distinct structural levels of increasing sophistication, complexity, abstraction, and generality.

Instructional activities. The third aspect of learning trajectories is an instructional sequence. These are composed of key tasks designed to promote learning at a particular conceptual level or benchmark in the developmental progression. Sarama and Clements (2004b, 2009) described their process for developing a coherent instructional sequence for a learning trajectory. First, the specific mental constructions and patterns of cognition that constitute children's thinking at each level are hypothesized. Second, tasks are designed that require children to apply the actions of the goal level of thinking. Third, the tasks are sequenced corresponding to the order of the developmental progressions to complete the hypothesized learning trajectory.

Simon (1995) proposed the learning trajectory as a framework to help teachers think about how students' learning may evolve and he did not include suggestions for teaching. Rather, its purpose was to emphasize "the importance of having a goal and a rationale for teaching decisions and the hypothetical nature of such thinking" (Simon, 1995, p. 136). According to researchers (Clements & Sarama 2004b; Confrey et al., 2014; Daro et al., 2011; Duschl et al., Furtak, 2008; Sarama & Clements, 2009) no proposed task sequence is the only, or the best, path for learning and teaching, only that it is hypothesized to show promise in furthering children's mathematical thinking and skill.

In summary, learning trajectories are hypotheses of mathematical growth and development that are rooted in empirical study of the ways in which students' thinking grows in response to relatively well-specified instructional experiences (Clements & Sarama, 2004b; Simon, 1995; Steffe, 2004). A complete hypothetical learning trajectory includes all three aspects: the goals of learning, the thinking and learning processes of children at various levels, and the sequence of learning activities in which they might engage. In essence, the instruction and tasks support students in developing the ways of understanding in the trajectory and the specific elements of the tasks provide insight into how the students' ways of understanding develop (Weber & Lockwood, 2014).

Instructional Frameworks Guided by Learning Trajectory Knowledge

A well-known example of a program that concentrated on students' cognitive development is Cognitively Guided Instruction (CGI) (Carpenter et al., 1989). CGI offered elementary teachers a framework presenting levels of sophistication in the strategies children used for solving various addition and subtraction word problems. In a subsequent study, Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996) found that teachers who had a "research-based model of children's thinking" (p. 496) offered more opportunities for children to solve problems and were more likely to elicit and base their instruction on children's current thinking and understanding.

More recently, Sztajn, Confrey, Wilson, and Edgington (2012) proposed a theoretical connection between research on learning and research on teaching called *Learning Trajectory Based Instruction* (LTBI), defined as teaching that "uses learning trajectories grounded in student thinking as the basis for instructional decisions" (p. 147). They put forth the LTBI framework as one avenue to describe the ways in which teachers' knowledge of learning trajectories guides their instructional decisions.

To better articulate the affordances of LTBI Sztajn and colleagues (2012) placed learning trajectories at the center of four highly used frameworks for examining mathematics teaching, namely mathematical knowledge for teaching (Ball et al., 2008), task analysis (Stein, Grover, & Heningson, 1996), discourse facilitation practices (Stein, Engle, Smith, & Hughes, 2008), and formative assessment (Heritage, 2008). Sztajn and colleagues (2012) argued that conceptualizing each of these teaching categories around learning trajectories served as a unifying element for instruction and advanced a theory of teaching purposefully centered around research on learning.

As a teacher's mathematical knowledge is a central focus of this study, it is necessary to discuss how mathematical knowledge for teaching (MKT) as defined by Ball and colleagues (2008) is reinterpreted when centered on LTBI (Sztajn et al., 2012). As a reminder, Ball and colleagues (2008) defined six subcategories of teacher knowledge under subject matter knowledge (SMK) or pedagogical content knowledge (PCK). Under PCK, *knowledge of content and students* was defined as the knowledge that combines knowing about students and knowing about mathematics. *Knowledge of content and teaching* was knowledge about the design of instruction for a particular content. *Knowledge of content and curriculum* encompassed knowledge how mathematical content is presented in instructional resources. Under SMK, *common content knowledge* was defined as knowledge of mathematics not specific to teaching whereas *specialized content knowledge* was the kind of mathematical knowledge that is specific to the work of teaching. Specialized content knowledge was exemplified as the knowledge teachers need to explain patterns in student errors or decide whether a nonstandard approach would work in general. The *horizon content knowledge* category represented the coherence of mathematical topics over the span of mathematics included in the curriculum.

Sztajn and colleagues (2012) refine the six original components of MKT (Ball et al., 2008) as they defined learning trajectory based instruction as a framework for teaching. Each refined component references learning trajectories and highlights the "importance of the logic of the learner and of the learning trajectories' ordered expected levels of sophistication in defining LTBI" (p. 149). Each of the six components when viewed through the lens of LTBI is discussed below. See Table 2.1 for a side by side comparison of the six categories of MKT (Ball et al., 2008) and the related LTBI (Sztajn et al., 2012) refinement.

Table 2.1. Reinterpretation of MKT defined through LTBI (Sztajn et al., 2012, p. 154)

Components of Learning Trajectory (LT) Based Instruction		
Category	Ball et al., 2008 MKT Original Definition	Sztajn et al., 2012 LT-based Interpretation
Pedagogical Content Knowledge (PCK)		
Knowledge of Content and Students	Knowledge that combines knowing mathematics and knowing students.	Knowledge of the various levels of the trajectory through which learners progress from less to more sophisticated ways of thinking. Knowledge of the cognitive steps that support development of the ways learners approach certain tasks.
Knowledge of Content and Teaching	Knowledge of how to design instruction for a particular piece of content.	Knowledge of ways to support learners' cognitive development through progressively more sophisticated levels of the trajectory to help student voices develop into mathematical perspectives. Knowledge of how to select and target tasks to promote individual movement along the trajectory and content-rich classroom discourse.
Knowledge of Content and Curriculum	Knowledge of the ways a particular concept is developed with curricular materials.	Knowledge of how to utilize student voice to choose and adapt curricula that is in line with mathematics disciplinary perspectives.
Subject Matter Knowledge (SMK)		
Common Content Knowledge	Knowledge of mathematical knowledge and skill used in settings other than teaching.	Knowledge of concepts and procedures represented at each level of the trajectory needed to perform the tasks associated with each level, all the way to the end (top) of the trajectory.
Specialized Content Knowledge	Knowledge of the mathematical knowledge and skill that is unique to teaching.	Knowledge of how to use one's mathematical perspective to test the appropriateness of various solutions and representations learners propose in their own voice; unpacking each level of the trajectory, explaining the mathematical issues behind the levels.
Horizon Content Knowledge	Knowledge of how "mathematical topics are related over the span of mathematics included in the curriculum" (p. 403).	Knowledge of the most sophisticated understanding that is found at the end (top) of a particular trajectory, representing the ultimate mathematical goal of a trajectory.

Sztajn and colleagues (2012) consider PCK through the voice and actions of the student, knowledge of content and students is defined as knowledge of the various levels of the trajectories through which learners progress from less to more sophisticated ways of thinking. This includes an understanding of student thinking and ways in which learners at varying levels of the trajectory approach mathematical tasks. Knowledge of content and teaching encompasses knowledge, selection, and implementation of tasks appropriate for students at different levels on the trajectory. Knowledge of content and curriculum includes knowing how use student voice to select and adapt mathematics instructional materials as provided by a school district or educational agency. More broadly, pedagogical content knowledge in the context of learning trajectory based instruction might help teachers answer such questions as:

- Can I provide an example of student thinking or student voice for each level of the trajectory?
- Can I articulate how student thinking grows and develops over time on this trajectory?
- When I select and implement tasks with my students, do I know how to adapt the tasks without compromising the mathematics or opportunities for student growth?
- Do I understand how to support student math talk in a way that advances individual student understanding and when applicable the class as a whole?

According to Sztajn and colleagues (2012) SMK as conceptualized through learning trajectory based instruction places a focus on the mathematics of the learning trajectory. Common content knowledge in relation to learning trajectories is viewed as knowledge of concepts and procedures represented at each level of the trajectory needed to perform tasks for that level. Specialized content knowledge refers to possessing the necessary mathematical knowledge necessary to make sense of student generated solutions and representations. This requires an understanding of the mathematics behind the each of the levels of a learning trajectory. Finally, horizon content knowledge assumes an understanding of the mathematical big idea developed in the learning trajectory and where it is situated in the broader landscape of their students' mathematical work. Generally speaking, subject matter knowledge in the context of learning trajectory based instruction might help teachers answer such questions as:

- Do I understand the mathematics (concepts and procedures) inherent to each level of the trajectory?
- When my students share their mathematical thinking and representations can I identify what is mathematically salient and build from it during instruction, staying true to their current level of understanding?
- Do I understand the importance of the mathematical big idea of the learning trajectory for my students' current and future learning?

Conceptualizing teaching on learning trajectories has the potential to benefit the teaching and learning of mathematics including more learner-centered classrooms rich with mathematics conversations (Clements & Sarama, 2008; Clements, Sarama, Wolfe, & Spitler, 2013), instructional decisions based on student thinking (Mojica, 2010; Wickstrom, 2014; Wilson, Sztajn, Edgington, & Meyers, 2015), improved understanding of student thinking (Wickstrom, 2014; Wilson, 2009), the selection of developmentally appropriate activities (Brown, Sarama, & Clements, 2007), and anticipation of students' thinking (Edgington, 2012).

Critical Analysis of the Learning Trajectory Construct

Critics of learning trajectories encourage researchers and teachers alike to carefully consider the widespread use and application of this research for guiding teaching (Sikorski & Hammer, 2010). Empson (2011) reminds the reader that Simon (1995) considered a learning trajectory to be a teaching construct. Thus, teachers hypothesize learning trajectories, or use hypothesized learning trajectories, to support planning tasks with the goal of bridging students' current thinking with possible future thinking. In essence, it is the hypothesized learning trajectory that helps teachers grapple with critical instructional decisions. In essence, Empson viewed the learning trajectory as one of the many means of instructional decision-making, not an end.

Empson (2011) proposed three key points to keep in mind as the field continues to move toward organizing the teaching of mathematics on learning trajectories. First, learning trajectories are not really new to mathematics education. As an example, Empson cites Gibb's (1956) study on children's thinking about subtraction word problems as one catalyst for Carpenter, Fennema, Peterson Chiang, and Loef (1989) to study how teachers use this information about children's thinking to guide their teaching resulting in the Cognitively Guided Instruction framework. Second, learning trajectories focus on specific domains of conceptual development, which may limit other valued aspects of the mathematics curriculum. Third, teachers and teaching matter, as such, simply following the instructional sequence suggested by a learning trajectory is not a guarantee of student knowledge acquisition.

Anderson and colleagues (2012) reported that researchers and other leaders in science and mathematics have raised a number of concerns about trajectories. They suggested the theoretical framings found in learning trajectories inadequately account for the ways culture, race, and context shape learning. They challenged developers to expand the methodologies used for development and validation to ensure diverse student populations are represented in the trajectories. In support, Lesh and Yoon (2004) express concerns with issues of equity and diversity inherent in the conceptualization, development, and implementation of learning trajectories. They argued, though mathematical learning is multidimensional and occurs through connections across multiple domains, trajectories have the potential to reduce learning to a hierarchical, linear path devoid of cultural contexts seemingly ignoring the many identities students bring to the learning environment. They urge, though progress along a trajectory is critical, it should not come at the expense of students' identities.

Myer et al. (2015) posit that use of learning trajectories to guide instruction has the potential to foster equitable access to mathematics. They suggest it is not the LTBI model itself that is equitable or inequitable, but the use of the model. For example, while trajectories support teachers' view of student learning along a continuum, they may also allow for solidifying deficit views that justify pre-conceived ideas about "high" and "low" children, or ideas about students who do not follow the typical path as mathematical deviants.

Pertinent to this discussion, Wilson, Sztajn, Edgington, Webb and Meyers (2017) examined twenty-two elementary teachers' discourse in a yearlong professional development setting to understand the ways in which learning a mathematics learning trajectory impacted aspects of teachers' discourse about students as learners. Results indicated that over time, some discursive patterns for explaining students' academic performance changed to incorporate the trajectory, while others remained unaffected. For example, when teachers analyzed student thinking in relation to age or grade level, the developmental nature of the trajectory was central to their discussions and they credited student growth as an outcome of increased opportunity and experience. On the other hand, learning trajectory knowledge did not shift teachers' beliefs that innate ability determines success in mathematics. Indeed, teachers' use of descriptors such as "low" and "high" to characterize student mathematical activity continued throughout the professional development experience. Researchers suggest more research is needed to examine the potential of learning trajectories to change the ways teachers conceptualize students and learning.

The critics of learning trajectory based instruction offer much food for thought as we consider the wider application of learning trajectory research for use by teachers. Proponents offer a myriad of benefits for using learning trajectories to benefit improved teaching and learning. High profile, and extremely compelling examples include increased MKT, careful attention to children's thinking, and selection of tasks that move children to more sophisticated levels of thought and rigor. Critics of learning trajectories offer equally compelling concerns. A narrow view of mathematics as a discipline and progressions that have the potential to function devoid of student identity are not to be taken lightly. Due to this juxtaposition, more discussion and research are needed to understand teachers' use of LTBI in creating equitable classrooms and challenging potential inequitable assumptions about what students can or cannot do. To that end, researchers have begun to investigate how to translate learning trajectories into useable tools for teachers. In the next section, I summarize research regarding the use of learning trajectories in professional development settings and with prospective teachers in a university mathematics teacher education course.

Learning Trajectories As Tools for Teachers

Validation studies of learning trajectories addressing the accuracy of the developmental progression of skill and knowledge are well underway (Confrey, 2012; Confrey et al., 2014; Sarama & Clements, 2009; Weber & Lockwood, 2013). Considerable effort has gone into designing curricula and assessments based on learning trajectories and validating their effectiveness with learners (Battista, 2004; Clements & Sarama, 2008; Clements, Sarama, Spitler, Lange & Wolfe, 2011; Confrey, 2012; Duschl, Maeng, & Sezen, 2011). Recently, learning trajectory research has expanded to include a focus on instruction by examining the ways learning trajectories might be useful in preservice teacher education (Ivers, Fernandez, Linares & Choy, 2018; Mojica, 2010) and professional development settings (Bobis et al., 2005; Clements, Sarama, Wolfe & Spitler, 2016; Edgington, 2014; Edgington, Wilson, Sztajn, & Webb, 2016; McCool, 2009; Wickstrom, 2014; Wilson, 2009; Wilson, Sztajn & Edgington, 2013; Wilson et al., 2014).

Bardsley (2006) conducted a case study of 14 pre-kindergarten teachers on their use of Building Blocks (Clements & Sarama, 2007), a curriculum based on empirically supported learning trajectories on early-childhood mathematical concepts. She concluded that teachers' motivation for participating in the professional development influenced how they used the curriculum materials. Teachers who wanted classroom activities were more likely to focus on moving students through the levels. However, teachers who participated to learn better mathematics used the curriculum as a structure for making instructional decisions.

Bobis and colleagues (2005) examined three professional development projects in Australia and New Zealand that drew upon research in young children's mathematical learning and in particular early mathematics learning trajectories (Sarama & Clements, 2009). Teachers learned to utilize the student diagnostic assessments from Math Recovery (Wright, Martland, & Stafford, 2006) and interview protocols to better understand students' mathematical thinking. Results revealed that as teachers increased their understanding of mathematics developmental pathways they increased their use of hand-on activities, emphasis on thinking strategies, efforts to challenge and extend children's understanding, effective use of materials, and formative assessment practices. As a result data reflected a significant growth in student achievement and teachers' MKT. Additional findings revealed knowledge of developmental learning trajectories supported an increased confidence in teaching mathematics, enjoyment of mathematics, and commitment to making mathematics learning engaging for their young learners.

Wilson (2009) investigated teachers' uses of a learning trajectory for rational number reasoning, referred to as the equipartitioning learning trajectory (Confrey, 2012), in instruction. Rational number includes the topics of multiplication, division, fraction, ratio, rate, decimals, percentages, similarity, and scaling. Thirty-three Kindergarten to Grade 2 teachers participated in twenty hours of professional development. They studied the learning trajectory for equipartitioning and key instructional practices, including clinical interviewing, task selection and adaptation, analysis of student work, and classroom interactions. Findings from the study indicated that the introduction of the learning trajectory assisted teachers to varying degrees in identifying specifically what students needed to learn next, deepening their own understandings of equipartitioning, and facilitating coherent instruction.

Wickstrom (2014) investigated teachers perceived advantages and disadvantages of using two learning trajectories, length and area measurement (Sarama & Clements, 2009), to improve classroom instruction and student learning. Study participants included three fourth-grade teachers all teaching in a diverse, high-needs school. Each of the teachers participated in professional development on using learning trajectories as a tool to formatively assess individual student's thinking as a means to inform classroom instruction. Findings indicated that though teachers regularly noticed their students' thinking after professional development on learning trajectories, they did not necessarily alter their instruction in response. Teachers in the study described the learning trajectory document and its language as a barrier to their learning suggesting the importance of attending to both the ways in which teachers are introduced to learning trajectories and the ways learning trajectories are represented for teachers in professional development.

Wilson, Sztajn, Edgington, and Confrey (2014) conducted a retrospective analysis of three purposefully selected teachers who were part of a larger design experiment in a school-based professional development setting, specifically a thirty-hour summer institute spread over six consecutive days. The summer institute offered teachers opportunities to learn about the first twelve levels of the equipartitioning trajectory (Confrey, 2012) and develop appreciation for the importance of equipartitioning in students' mathematical development. Findings indicated that professional learning tasks focusing on pedagogical content knowledge present in learning trajectories supported teachers' learning of subject matter knowledge and that teachers' learning of a learning trajectory is mediated by their MKT.

In a large scale study involving sixty-four preschool (four-year-old) teachers, Sarama and colleagues (2016) evaluated the effects of a research-based model for scaling up educational interventions on teachers' practices in early mathematics in the short and long term.. The intervention, a professional development program based on young children's mathematics learning trajectories, had a substantial positive effect on teachers' instructional practices, some of which mediated student outcomes. Teachers also demonstrated sustained levels of fidelity as long as six years after the end of the intervention. Notable is these teachers' encouragement and support for discussions of mathematics and their use of formative assessment. Finally, teachers taught the curriculum with increasing fidelity over the following six years without support from the project.

Edgington and colleagues (2016) investigated how elementary teachers learned about and used four learning trajectories for number and operations (Sarama & Clements, 2009) in their teaching. A researcher developed learning trajectory profile table was created in an attempt to make the multiple learning trajectories simultaneously accessible and to facilitate teachers' use of learning trajectories to talk about students' mathematical thinking in more detailed ways. Results revealed that teachers found the profiles useful for recognizing and labeling the details of students' thinking and in connecting content across multiple learning trajectories to consider a broader image of students as mathematics learners. Though teachers drew upon the learning trajectory profiles when analyzing individual student thinking they did not translate the details of the trajectory profiles to whole class instruction.

Whereas the previous studies indicated learning trajectories supported changes in teacher knowledge of students' mathematical thinking, Mojica (2010) reported that prospective teachers' learning of a learning trajectory resulted in changes in mathematical content knowledge. She conducted a design study with fifty-six prospective elementary teachers to investigate learning of the equipartitioning learning trajectory (Confrey, 2012) over an eight-week period, within a semester long elementary mathematics methods course. Additional results suggested that elementary prospective teachers' knowledge of a learning trajectory enhanced their ability to leverage student thinking to advance learning and guided their instructional decisions.

Learning trajectories can support both practicing and prospective teachers' refinement of learning and teaching models by providing a conceptual framework (Corcoran et at., 2009) for understanding differences across students' thinking and organizing these understandings as they change overtime. The studies reviewed provide evidence of the utility of learning trajectories in deepening inservice and prospective teachers' understanding of the content they will teach and improving their ability to recognize and attend to their children's thinking during instruction. Mathematics learning trajectories have the potential to be a valuable tool for teachers as they consider instructional decisions that will be of the most benefit to their students' current and later growth.

Subitizing

Mathematics education researchers and practitioners agree that a central objective of early mathematics education is developing children's number sense (Baroody et al., 2006). Even before they learn to count, young children come to an informal understanding of quantity by subitizing, or recognizing the cardinality of small sets of objects without counting (Sarama & Clements, 2009). This important aspect of early number development serves as the content focus for this study

Early Studies in Subitizing

Subitizing utilizes visualization in recognizing an amount rather than counting it. Kaufman, Lord, Reese, and Volman (1949) first coined the word subitizing as the fast, highly accurate method of quantifying collections of six items or less without having to count. Initially referred to as a "judgment of numerousness" (p. 498), the word subitize originates from the classic Latin adjective subitus, meaning sudden, and the medieval Latin verb subitare, meaning to arrive suddenly. Early subitizing studies (e.g., Kaufman et al., 1949; Saltzman & Garner, 1948; Taves, 1941) featured a stimulus such as circles, dots, and squares of varying orientations and sizes quickly shown to participants who were asked to state how many objects they saw. Researchers recorded reaction times, accuracy of guess, and answer confidence. Three early studies informed the present study as they provided insight into the recognition of subitizing as a way to discriminate numerousness as well as to explore variables that impact one's subitizing abilities.

Taves (1941) investigated the methods used by participants (N=133 adults) to report the number of dots briefly shown and the degree of confidence in the correctness of their reporting. The number of dots ranged from two to one hundred eighty with the arrangements randomly shown one at a time for 0.20 seconds each. Confidence was self-reported and was estimated on a six-point scale ranging from zero to five. Zero meant no confidence, a sheer guess, and five meant complete certainty. Taves' results suggest that participants were confident in reporting up to six dots, at eight however, participants' confidence in reporting fell rapidly and was variable from that point forward. Taves claims that the sharp shift noted between six and eight dots indicated two ways subjects reported numerousness. First, a small number of items, from one to seven, were named by simply stating "how many" without counting and second, quantities greater than eight were named by counting.

Saltzman and Garner (1948) studied the effect of a large number of variables on the discrimination patterns hypothesized by Taves (1941). They wished to find out whether the discrimination of quantity was affected by such things as: (1) a participant's knowledge of the stimulus-range; (2) practice; (3) regularity of the spacing of the stimulus-objects, (4) participants' distance from the stimulus-objects; (5) brightness of the background on which the stimulus-objects appeared; and (6) size of the stimulus-objects. In the majority of their experiments the stimulus-objects varied in number from two to ten. Two different methods were used to cue participants: (1) the dot patterns were exposed for 0.5 seconds and the accuracy of the participant's discrimination was recorded or (2) the dot patterns were exposed until the participants responded. The participant's reaction time—the interval between the exposure of the dot patterns and the beginning of the participant's verbal response—was measured.

Study results revealed that no more than three circles were correctly identified 100% of the time. With repetition the reports became more accurate. The authors believed that this was due to increased familiarity with the stimulus-materials though it was also discovered that repetition had relatively little effect on the accuracy of judgments for six circles and above. This is interesting as Taves (1941) noted a similar phenomenon beginning with six circles, as well. In regard to the six variables studied, Saltzman and Garner (1948) found that all of the variables—knowledge of the stimulus range, practice, regularity in spacing, brightness, and size—affected both accuracy and reaction-time. The most important finding, that influences the present study is that the accuracy of naming “how many” breaks down and becomes less reliable after six.

Kaufman et al. (1949) suggested that a judgment of numerousness, or how many objects a group contains, is made in two ways: (a) it may be comparative—more numerous or less numerous than a specific number or (b) it may be absolute. The focus of their study was the direct reporting of a number method, a special form of the absolute judgment method, where, after a brief look so that counting is impossible, a number is assigned to represent how many things are in any given collection of objects.

Nine adult participants (eight female and one male) in the study were shown an irregular dot pattern of one to two hundred dots for 0.2 seconds and asked to state how many dots they saw. The instructions to one group of four participants emphasized speed, and directions to the other group of five emphasized accuracy. Response-time was measured and quantity perceived was recorded, as was the participant’s degree of confidence of the accuracy in their response. Results suggested that when participants saw one to six dots the correct number was usually reported and participants were more confident in their answers. When participants were shown more than six dots answers were less accurate and participants’ confidence fell rapidly and was variable from that point forward.

In summary, findings in the above studies suggested that in adults subitizing mediates accurate judgments of numerosity for quantities up to five or six. Quantities six or greater were named using counting or estimating. In addition, a variety of factors influence subitizing abilities including the amount of time participants viewed the dot pattern, number of dots in the arrangement, regularity in spacing between dots in the arrangement, size of the dots, and previous experiences subitizing quantities.

Subitizing Abilities in Young Children

Developmental counting theories conflict with regard to the origin and importance of subitizing in the evolution of children's counting skills and number competency. Though most researchers agreed children eventually develop the ability to subitize, some posit that subitizing is nothing more that fast counting (Brownwell, 1928; Douglass, 1925; Gelman & Gillistel, 1978). These researchers acknowledged that preschool children can subitize small quantities of one, two, or three but assert that this ability appears to develop after children were able to quantify a set by counting. They argued that children’s ability to abstract number is related to their ability to count and that young children count first even when estimating (naming) small numbers.

Other theorists (Carper, 1942; Douglass, 1925; Hannula & Lehtinen, 2014; Le Corre, Van de Walle, Brannon, & Carey, 2006) suggested that subitizing small numbers precedes the development of counting and plays a pivotal role in guiding the development of counting skills and numeric reasoning abilities. Further, a study by Fitzhugh (1978) suggested that three- and four-year-old children (N=62) successfully subitized sets of one or two but appear unable to quantify sets of two objects by counting. Her results suggested that subitizing is the earliest quantifier used by young children and that the ability to subitize sets of at least two objects is a necessary precursor to the child’s discovery that the counting procedure can be used to quantify sets of objects.

A precursor to subitizing, “spontaneous focus on numerosity (SFON),” (Hannula & Lehtinen, 2005, p. 235) highlights children’s tendency to focus on the numerical aspects of their environment, and is seen as a distinct, mathematically significant process. Results of a series of studies (Batchelor, Inglis, & Gilmore, 2015; Hannula & Lehtinen, 2005;) suggest that SFON builds children’s subitizing ability, which in turn supports the development of cardinality, counting, and arithmetic skills. SFON at three years of age predicts development of cardinality knowledge a year later and in four-year-olds is related to verbal counting ability a year later. To the extent that this is true, subitizing forms a foundation for all learning of number (Sarama & Clements, 2009).

Types of Subitizing

The purpose of subitizing is to state the size of a set of objects quickly and without counting. This important mathematical ability is foundational to children’s learning of number (Gallistel and Gelman, 1991). There are two types of subitizing—perceptual subitizing and conceptual subitizing (Clements, 1999). Perceptual subitizing is perceiving the entire quantity of a set of objects and naming those quantities without needing to count them. Conceptual subitizing is perceiving subgroups inside a larger arrangement and then being able to name the total amount without needing to count.

Perceptual subitizing is usually limited to collections containing four or fewer items. Perceptual subitizing is closest to the original definition of subitizing, defined as recognizing a quantity without consciously using other mental or mathematical processes and then naming it. For example, one might see three dots on a die as illustrated in Figure 2.4 and quickly say “three,” by perceiving the three dots intuitively and simultaneously.

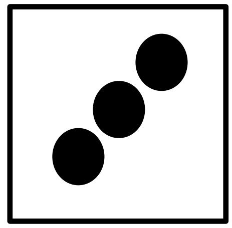


Figure. 2.4. A dot pattern perceptually subitized as "three."

The second type of subitizing, conceptual subitizing, involves seeing a collection of objects as composed of smaller groups and then quickly combining these groups to name the cardinality of the entire collection. The total number of dots is perceived subgroups inside a larger arrangement and then being able to name the total amount without needing to count. In the case of the six-dot domino as seen in Figure 2.5 one might see each side of the domino as three and quickly combine those groups to name the cardinality of the entire set. Conceptual subitizers are capable of viewing number and number patterns as units of units.

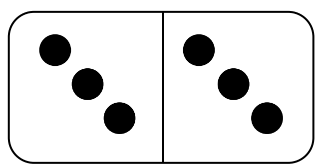


Figure 2.5. A six-dot domino that can be conceptually subitized as "three" and "three" for a total of "six."

The Subitizing Learning Trajectory

The goal of the subitizing learning trajectory (see Appendix A for a full list of the ten levels of the trajectory) is sophisticated conceptual subitizing ending with unitizing quantities that support place value understanding and multiplicative thinking (Clements & Sarama, 2014). Beginning at about age 2, children begin to name groups of one, two, and sometimes three objects without counting. By age 5, with instruction and experience, children can conceptually subitize groups with five items. By age 8, most children can enumerate larger sets of items by identifying groups of items, using place value knowledge, and drawing on multiplication ideas provided they engage with appropriate learning opportunities and experiences.

As evidenced by earlier studies, subitizing skills are acquired in a gradual, step-by-step manner. For example, Wynn (1992) found that children initially differentiate “one” from “more than one” at about thirty-three months of age. Between thirty-five and thirty-seven months, they differentiate between one and two, but not larger numbers. A few months later, at thirty-eight to forty months, they can identify all quantities that they can count, four and higher, at about the same time. However, research in natural, child-initiated settings shows that the development of these abilities can occur much earlier, with children working on one and two around twenty-four months of age. Further, some children begin with “two” rather and “one.” Study results suggest that number knowledge develops in levels, over time (LeCorre et al., 2006).

It is important to recognize that although young children are sensitive to quantity, intentional interactions with others in recognizing and naming quantity is essential to learning to subitize as it does not develop on its own (Baroody et al., 2006). Naming small groups with numbers, before counting, helps children understand number words and their cardinal meaning. In addition, mathematizing everyday experiences with small quantities, for example, asking for “three blocks” as opposed to “some more blocks,” helps young children develop early number recognition and can help lay the foundation for subitizing.

Important factors in determining the difficulty of subitizing tasks include the size of the collection, the spatial arrangement of objects (Kaufman et al., 1949; Saltzman & Garner, 1948; Sarama & Clements, 2009; Taves, 1941), and deliberate practice with subitizing (Hannula & Lehtinen, 2005). In regard to quantities, collections of four and below prompt perceptual subitizing and quantities five and above prompt conceptual subitizing. Children usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966). Figure 2.6 illustrates such a progression.

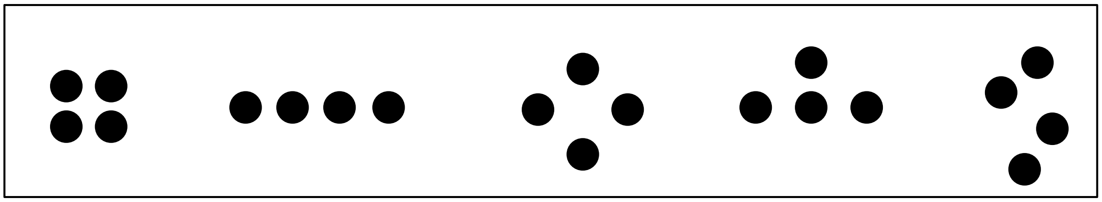


Figure 2.6. Spatial patterns of four that move from easiest to more difficult (Clements & Sarama, 2014).

In conclusion, subtizing small numbers appears to precede counting, supports the development of counting ability (LeCorre et al., 2006), and plays an important role in the development of early number knowledge and number reasoning (Gallistel & Gelman, 1991). As a result, it appears to be a foundation for all learning of number as children use subitizing to discover critical properties of number, such as conservation, part-whole relationships, and compensation. As subitizing skills grow and develop over time and with experience, unitizing as well as arithmetic capabilities benefit. Thus, "subitizing is a critical competence in number" (Sarama & Clements, 2009, p. 51).

Conclusion

The purpose of this chapter was to review and synthesize literature to situate the current study. Though useful at the level of curriculum, assessment, and standards development, it remains to be shown how learning trajectories can be incorporated into teachers' practice and become a tool to understand students' thinking, for planning instructional activities, for interacting with students during instruction, and for assessing students' understandings. Thus, bringing learning trajectories into the classroom through teacher education is one critical area of knowledge that needs to be investigated.

Daro et al. (2011) suggested that learning trajectories can lead to improved instruction and student achievement by providing teachers with a conceptual structure that informs and supports their ability to respond appropriately to evidence of their students' differing stages of progress. Informed and effective pedagogical decisions within the context of learning trajectory based instruction hinges on teachers' pedagogical and subject matter knowledge and frames the mathematical knowledge needed for teaching (Brown et al., 2007; Sztajn et al., 2012). This is true for both in-service and prospective teachers.

In order for early childhood mathematics instruction to be effective it must be done intentionally attending to the rigors of the discipline of mathematics in ways young children find engaging and interesting (Brown et al., 2007; Clements & Sarama, 2004b, 2007, 2014; Hachey, 2013). Opportunities to engage in significant mathematics activity are especially important for low-income children. These children, on average, demonstrate lower levels of competence with mathematics prior to school entry, and the gaps persist or even widen over the course of schooling (citation needed). Providing young children with extensive, high-quality early mathematics instruction can serve as a sound foundation for later learning in mathematics and contribute to addressing long-term systematic inequities in educational outcomes.

Teaching matters (Ball & Forzani, 2011; Daro et al., 2011; Empson, 2011; NCTM, 2014) and effective teaching is intentional (Espstein, 2014). Learning trajectory research may prove useful as long as learning trajectories are used to empower and support teachers to incorporate their children's thinking into instructional decision making. When teacher's understand the developmental progressions for each major domain or topic of mathematics, and sequence activities based on them, they build mathematics learning environments that are particularly developmentally appropriate and effective (Clements & Sarama, 2014; Confrey et al., 2014; Duschl et al., 2011; Furtak, 2009). One framework for teaching, learning trajectory based instruction (LTBI) (Sztajn et al., 2012), offers an opportunity to intentionally center teaching decisons on students' thinking. For LTBI to be successful teachers must understand the mathematics in the trajectory and be able to articulate how student thinking advances as children advance on the trajectory. As research on learning trajectories increases and is brought to bear on some of the most vexing problems in teaching and learning mathematics it is worth considering its role in early childhood prospective teacher education.

CHAPTER THREE

METHODOLOGY

The chapter begins with a restatement of the research questions and continues with a discussion of the researcher's theoretical framework. The chapter continues by detailing the researcher's study design, which includes the research context, data collection methods, and a presentation of the pilot study used to further inform this dissertation. This is followed by a discussion of preliminary data analysis procedures and data quality checks. The chapter finishes with a presentation of the researcher's statements on trustworthiness and dependability, and researcher reflexivity.

Problem Statement and Research Questions Restated

Researchers are calling for early childhood mathematics instruction to become more intentional and adaptive in moving students toward meeting learning goals through the use of learning trajectory based instruction (Anders & Rossbach, 2015; Brown, Sarama, & Clements, 2007; Daro et al., 2011; Hachey, 2013; Jung & Conderman, 2013). Indeed, Corcoran et al. (2009) and Daro et al. (2011) advocated for using learning trajectories when designing both pre- and in-service teacher education in an effort to help teachers better analyze students' understanding and misconceptions. Sztajn et al. (2012) posited, "teachers' understanding of how the logic of the learner progresses over time, combined with contextual factors, can serve as justification for their decisions" (p. 152). While an articulation of developmental growth has contributed greatly to the knowledge base of how students learn mathematics, much still needs to be learned about how to translate learning trajectories into usable tools for both pre- and in-service teacher education (Corcoran et al., 2009; Daro et al., 2011). The current study was guided by the following research questions:

Question: *In what ways do learning trajectories inform prospective early childhood teachers' instructional decisions in ways that are likely to advance student learning on the subitizing trajectory?*

Attendant Question #1: *What understandings do prospective early childhood teachers have regarding the subitizing learning trajectory?*

Attendant Question #2: *Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?*

Theoretical Framework

A theoretical framework underlies the philosophical assumptions of a study and makes explicit "a basic set of beliefs that guide actions" (Guba, 1990, p. 17). The theoretical framework makes clear the researcher's assumptions about the nature of reality and knowledge. At the core of the theoretical framework is the researcher's inquiry paradigm. Inquiry paradigms ensure that research is theory-driven, nestled in belief systems that offer different purposes for doing research and different ways of making meaning (Glesne, 2011). An inquiry paradigm is "a loosely bonded groupings of assumptions, philosophies, and theories" (p. 6) that shape every aspect of the research process. This study was situated in the interpretivist paradigm because multiple interpretations exist of how learning trajectories can inform early childhood prospective teachers' instructional decisions.

Interpretivist Paradigm

Interpretive research argues that reality is socially constructed, thus not one single, observed reality exists (Merriam, 2009). Researchers functioning from the interpretivist paradigm argue that knowledge is fluid and contextually bound, meaning knowledge is not constructed in a vacuum, thus what is known is always negotiated within cultures, social settings, and relationships with other people. Interpretivist researchers believe that knowledge is constructed, not found (Denzin & Lincoln, 1994). They view their role as understanding the complex world of lived experiences from the point of view of those who live it (Schwandt, 1994). Thus, in the interpretivist paradigm, the responsibility of the researcher is to watch, listen, act, record, and examine throughout the study. The researcher's values are inherent in all aspects of the research process allowing "the researcher in the interpretive approach [to be] the instrument through which the topic is revealed" (Angen, 2000, p. 391).

The interpretivist framework served as a guide in describing, understanding, and interpreting early childhood prospective teachers' efforts to understand student thinking and how they use that information to guide instructional decisions. Assuming that the nature of reality and truth is "socially constructed, complex, and ever-changing" (Glesne 2011, p. 8), it was important to investigate how study participants described and interpreted their decision making in the context of their prospective preparation. As inquiry paradigms are contextually bound, the primary vehicle for informing instructional decisions made by early childhood prospective teachers originated in the mathematics method class. Thus, significant insight into the methods class is central to this study.

The Case for Qualitative Research

Qualitative methods were selected to address the study's needs. Qualitative research by definition is exploratory and is best used when the goal of the research is to grasp meanings, motives, reasons, and patterns that are usually unnoticed with quantitative approaches (Patton, 2015). Qualitative research methods helped the researcher explore and surface a "complex detailed understanding" (Creswell, 2013, p. 48) of prospective teachers' rationale for their instructional decisions. The exploratory nature of qualitative inquiry allowed the researcher to listen with openness and curiosity to the motives of prospective teachers' decision making.

Curry (2015) explained qualitative research as a "strategy for systematic collection, organization, and interpretation of contextual information" that is at once thoughtful, deliberate, and strategic. Data are physically obtained through the researcher, allowing a more complete view of the context to be considered, including the complexities inherent in human behaviors and interactions (Merriam, 2009). While traditional scientific approaches to research seek to test hypotheses or find causal relationships, the goal of qualitative research is to describe and make sense of a phenomenon in its natural setting from the view of the participants (Creswell, 2013).

"Real time" instructional decisions are influenced by multiple factors and are therefore difficult to quantify. For example, it has been shown that a teacher's competence and confidence with the mathematics they are expected to teach is grounded in their mathematical knowledge for teaching (Ball et al., 2008). A qualitative approach provides a pathway that allowed the researcher to unravel the complexities of instructional decision-making.

Finally, Miles, Huberman, and Saldaña (2014) advocated for qualitative methods when exploring an under-researched area and developing hypotheses. Though learning trajectories have been used to develop standards, curriculum, and assessments, this phenomenon and its usefulness as teacher development tools have yet to be fully explored. Emerging research on teachers' use of learning trajectories show that as teachers make sense of trajectories, these trajectories can support selection of instructional activities, interactions with students in classroom contexts, and use of students' responses that further learning (Sztajn et al., 2012). Thus, phenomenology is a suitable approach in this qualitative study.

A Phenomenological Research Strategy

Qualitative inquiry focuses on "finding meaning in context" (Merriam, 2009, p. 2) and phenomenology describes the common meaning for several individuals of their lived experiences of a concept or a phenomenon (Creswell, 2013). Phenomenologists focus on describing what all participants have in common as they experience a phenomenon. The basic intent of phenomenology is to reduce individual experiences with a phenomenon to a description of "what people experience and how it is that they experience what they experience" (p. 117).

Phenomenological research strategies detail an in-depth and contextually framed exploration of a single phenomenon (Creswell, 2013) guiding the researcher to "develop a composite description of the essence of the experience for all individuals" (Cresswell, 2013, p. 76). This requires "methodologically, carefully, and thoroughly, capturing and describing how people experience some phenomenon" (Patton, 2015, p. 115). Through discovering patterns that emerge after close observation, careful documentation, and thoughtful analysis of the research topic, the qualitative researcher "uncovers the meaning of a phenomenon for those involved" (Merriam, 2009, p. 5).

The phenomenon that is the focus of this inquiry is prospective teachers' understanding of the subitizing learning trajectory and what influence if any that understanding has on the instructional decision making of prospective teachers intended to advance young children's mathematical development. This approach allowed me to not just describe the phenomenon but to make an interpretation of the meaning of a shared experience.

The shared experience in this study is that each of the participants successfully completed the same mathematics methods course planned and taught by the researcher. The researcher maintained a "strong relation to the topic of inquiry" (Creswell, 2013, p. 80) throughout the extent of the study and therefore needed to "*bracket* [emphasis in original] himself or herself out of the study by discussing personal experiences with the phenomenon" (p. 78). This allowed the researcher to partly set aside personal experiences with the phenomenon so that the focus could be placed on the experiences of the participants in the study.

Intentionality of instructional decision-making is hard to assess though it is viewed as an essential component of adaptive, effective teaching. To unravel the complexities of intentionality in decision-making, the research context must first be observed and described. Through sharing the research context, important insights into the early childhood mathematics methods class frame the shared, lived experience of each participant.

Research Context

The early childhood education (ECE) mathematics methods course at Lakewood University, a pseudonym, is the context for this study. Lakewood University (LU) is a large, Midwestern, urban university. At LU, the School of Education offers undergraduate and post-baccalaureate certification programs for those pursuing Early Childhood Education (ECE) through its Department of Curriculum and Instruction and the Department of Exceptional Education, respectively. Thus, students who enrolled in the ECE mathematics methods class come from a variety of certification programs.

Department of Curriculum and Instruction ECE Certification Programs

The Department of Curriculum and Instruction offered two pathways to teacher certification for prospective early childhood teachers, a traditional undergraduate program and a post-baccalaureate certification program.[1] Specific to their mathematical preparation for

[1] ECE undergraduates and students in the ECE Post Baccalaureate Certification Program could pursue certification add-ons for English as a Second Language and Bilingual Education.

teaching, undergraduate students completed two mathematics teaching content courses with a C or better, followed by an early childhood mathematics methods class.

The mathematics content courses, offered through the Department of Mathematical Sciences, followed the recommendations of the Conference Board of Mathematical Sciences (CBMS, 2001; 2010) for elementary teachers, defined as teachers of Kindergarten through Grade Five. Both content courses studied the mathematics that the prospective teachers teach from the perspective of a teacher. The first course included a focus in the theory of arithmetic of whole numbers, fractions, and decimals with an introduction to algebra, estimation, and problem-solving strategies. The second course was a continuation of the first with a focus in geometry, statistics, and probability.

The two mathematics content courses were typically taken during a student's freshman year, prior to their admittance into the professional program sequence. The mathematics content courses were not required for those pursuing ECE certification as a post-baccalaureate student. Those courses were considered optional and it is the ECE faculty advisor who made the final recommendation regarding the need for the mathematics content courses.

Department of Exceptional Education EC Special Education Certification Programs

The ECE mathematics methods class was a required component for students pursuing an ECE Special Education undergraduate degree and an ECE Special Education certification with a master's degree option in the Department of Exceptional Education.[2] Like their regular education counterparts, ECE Special Education undergraduate students completed the same two mathematics teaching content courses offered through the Department of Mathematical Sciences with a C or better to be eligible to enroll in the mathematics methods course. Students seeking

[2] EC Special Education undergraduates and students in the EC Special Education Certification Program may pursue certification add-ons for Autism Spectrum Disorder and Transitions for Students with Disabilities.

ECE Special Education Post-Baccalaureate Certification are not required to take the two mathematics teaching content courses offered through the Department of Mathematical Sciences, as they already possess a bachelor's degree.

Students enrolled in the ECE Special Education Certification with a master's degree option were all post-baccalaureates students. Study participants seeking this certification entered the program with little to no early childhood teaching experience and identified as "career changers." The certification program is a two-year course of study that included two summer sessions, two fall semesters, and two spring semesters with student teaching taking place during the final spring semester. In addition, students in this program included an optional pre-intern or intern position where they served as either a paraprofessional or the teacher of record, respectively. All classrooms where the pre-interns and interns served were in the neighboring large, urban school district.

Clinical Experiences

Each ECE certification program – ECE General Education, ECE General Post-Baccalaureate, ECE Special Education, and ECE Special Education Post-Baccalaureate – offered clinical experiences each semester of the professional sequence with two exceptions. One exception included both undergraduate programs, ECE General Education and ECE Special Education. These programs offered a one-credit field experience that was taken prior to the start of the professional sequence. This course placed prospective ECE majors in an early childhood educational setting (e.g., early care center or early childhood classroom) for one-half day per week. The second exception is the ECE Special Education Post-Baccalaureate Certification Program, which had no clinical experience during the fall semester of Year II. Table 3.1 displays a more detailed summary of the clinical experiences for the ECE program and the EC Special Education program.

Early Childhood Special Education clinical experiences. ECE Special Education undergraduate students and ECE Special Education Post-Baccalaureate students engaged in field placements that span the years for which they seek certification. Students completed their field experiences in early care centers, community agencies, and early childhood classrooms. The placements included urban and suburban settings. Clinical experiences were paired with a literacy-focused teaching methods course.

Early Childhood General Education clinical experiences. Similar to their Special Education counterparts, undergraduate and post-baccalaureate students pursuing certification as early childhood teachers participated in supervised clinical experiences each of the four semesters prior to student teaching. Each clinical experience offered opportunity for university students to broaden their firsthand experiences with young children of varying ages. Similar to the ECE Special Education program, field placements are in urban and suburban settings and each field experience was paired with a literacy-focused methods course.

Table 3.1. Summary of clinical experiences pre- and post- mathematics methods class.

Program	Pre-mathematics methods class clinical experience*	Supervised Clinical Experiences Concurrent with Mathematics Methods Class	Post-mathematics methods class clinical experiences*	Student Teaching Experiences*
ECE General undergraduate	**	None	***	5 full days per week
ECE General Post-Baccalaureate	2 half days per week	None	2 half days per week AND 2 full days per week	5 full days per week

EC Special Education	2 half days per week	None	2 half days per week	5 full days per week
EC Special Education Post-Baccalaureate	2 half days per week	None	2 half days per week	5 full days per week

*A clinical experience lasts for one semester.
**Clinical experiences for ECE students pre-mathematics method class consists of either 1 or 2 half days per week.
***Clinical experiences for ECE students post-mathematics methods class consists of either 1 or 2 half days per week or full-time student teaching.

Placement of the Early Childhood Mathematics Methods Course

The placement of the EC mathematics methods course varies from program to program as displayed in Table 3.2. The ECE undergraduate program classifies the mathematics methods course as non-sequenced. Non-sequenced classes may be taken any semester after prerequisites are met. Therefore, undergraduate students in the ECE program enrolled in the mathematics methods course one to three semesters prior to student teaching. Students seeking a bachelor's degree in early childhood special education were scheduled to take the mathematics methods class during the fifth semester of an eight-semester program. Students pursuing an ECE Special Education Post–Baccalaureate certification took the mathematics methods class during the fall semester of Year II of a two year program.

Table 3.2. Placement of the early childhood mathematics methods course.

Program	Typical placement of EC mathematics methods in the program timeline.
EC Special Education	Fifth semester of an eight-semester program.
EC Special Education Post-Baccalaureate	Third semester (Fall) of a two-year program.
ECE General	Non-sequenced course. Taken any semester after meeting prerequisites.
ECE General Post-Baccalaureate	Non-sequenced course. Taken either Fall or Spring semester.

Clinical Experiences in Mathematics Education

The Association of Mathematics Teacher Educators (AMTE) in their *Standards for Preparing Teachers of Mathematics* noted that effective mathematics teacher education programs provide prospective teachers opportunities to learn in clinical settings. Thoughtfully designed clinical experiences support the "development of beginning teachers who can skillfully do the work of mathematics teaching" (AMTE, 2017, p. 40) and point to the importance of learning through engagement in teaching (Ball & Forzani, 2011). Well-developed clinical experiences provide prospective teachers with opportunities to develop skill with teaching practices, and insight into mathematics content and into students as learners of that content. However, as noted Table 3.2, each of the ECE certification programs lacked a supervised clinical experience specifically for mathematics education.

The Early Childhood Mathematics Methods Course

The ECE mathematics methods course was offered each fall and spring semester and occasionally during the summer session. The semester long courses included fourteen sessions, one each week of the semester, each lasting two hours and forty minutes. The syllabus for the course can be found in Appendix B. In an attempt to engage prospective ECE teachers with actual children and their mathematical thinking and development, two child interviews were required course assignments. Many students, particularly those that are serving as full-time teachers, regularly implemented class activities with their students. These experiences were voluntary, non-structured, and unsupervised. To nurture curiosity in children's mathematical thinking and to study best practices, the class watched and reflected on approximately twenty short videos during class or as part of homework assignments throughout the semester.

The research regarding mathematical knowledge needed for teaching (Ball et al., 2008) underpinned all aspects of the development and implementation of the methods course. The course approached mathematics teaching and learning from a developmental perspective and included the mathematical development of children from birth to age eight. Cornerstones of the course included learning trajectory research, using and connecting mathematical representations, selecting appropriate mathematical tools that support the development of mathematical thinking in young children, and strategies for eliciting and listening to student mathematical thinking to scaffold learning experiences. Productive and unproductive beliefs regarding the teaching and learning of mathematics (NCTM, 2014) were incorporated into class sessions and were grounded in mindset research (Boaler, 2013; Dweck, 2006).

The course featured four learning trajectories developed by Sarama and Clements (2014). The trajectories, along with the class session in which it was introduced, included:

1. Counting: Class Session 2
2. Recognizing Number and Subitizing: Class Session 4
3. Composing and Decomposing: Class Session 8
4. Adding and Subtracting: Class Session 10

At numerous times throughout the semester, early childhood prospective teachers placed young children's mathematical thinking demonstrated on a video or piece of student work on the appropriate learning trajectory and then planned learning opportunities designed to meet and advance student growth along the learning trajectory. In addition, the students carefully analyzed the results of the two child interviews, placed the children on appropriate levels on the learning trajectory, provided rationale for their placement, and suggested learning opportunities meant to advance the child on the trajectory. At no point during the semester did the participants engage in an activity that replicated the content of the interview protocol developed for this study.

As each learning trajectory is unique to the mathematical idea it is developing, each was introduced to the students in its own unique way. What is common is that a learning trajectory was introduced to the students after they have had considerable opportunities to develop an understanding of the mathematical big idea profiled by each trajectory. This understanding was developed through multiple paths and with multiple tools. For example, in preparation for the *Recognizing Number and Subitizing Trajectory* students read an article (Huinker, 2011) that discussed the meaning and importance of subitizing in young children's mathematical development, watched numerous videos of children subitizing quantities, engaged in various class activities intended to support subitizing including mathematical games, and learned how to recognize subitizing opportunities children encounter as they play. Students learned to distinguish between perceptual and conceptual subitizing. Approximately two, two-hour and forty minutes class sessions, were dedicated to subitizing and development of subitizing in young children. The goal was to view subitizing as foundational to a young child's developing sense of number and quantity and to ensure ECE students understood its importance in children's mathematical experiences.

Many tools common to early childhood classrooms were utilized during the methods course to show how teachers could intentionally support subitizing in their classrooms. Tools included Unifix cubes, dot arrangements, five frames, ten frames, and rekenreks (math racks). Before receiving a copy of the subitizing trajectory students worked with a partner to place a "cut apart" version of the levels of the trajectory in order. This was done to help ECE prospective teachers recognize the developmental nature inherent to both the learning trajectories and young children's mathematics thinking. Once the students were satisfied with the placement of the levels they received a copy of the trajectory and used it check their sequence. After receiving the subitizing learning trajectory they watched three brief videos of children subitizing, placed each on the subitizing trajectory, and provided a rationale for their placement consistent with content learned during the methods class.

Participants

Potential participants for this study were enrolled in one of two sections of the early childhood mathematics methods course taught at Lakeshore University during the Fall 2016 semester. Twenty-one participants were enrolled in one section of the methods course, and twenty-three in the other section. To be eligible for the study, participants stated a commitment to complete an undergraduate major in either Early Childhood Education or Early Childhood Special Education and secure state teacher certification. Eligibility for participation in the study was extended to students in the Early Childhood Education Post Baccalaureate teacher certification program and the Special Education Post Baccalaureate teacher certification program.

Creswell (2013) suggested that phenomenological studies can vary in size from three to four individuals to ten to fifteen therefore a recruitment goal of fifteen participants was established for the study. Participation in the study was voluntary and participants were invited to participate through an email that was sent after semester grades were finalized from the researcher's university email to the participants' university email. Follow up emails were sent to eligible participants that did respond to the first invitation to participate. Fifteen students responded to the inquiry email and all participated in the study.

All study participants were female. Seven were non-traditional students and eight were traditionally aged. Ten participants were White American, one participant was African American, one participant was Asian American, and three participants were bilingual (English-Spanish) and identified as Latina. Each Spanish speaking participant was an immigrant to the United States and identified with their country of birth, which included Mexico, Puerto Rico, and El Salvador. Table 3.3 displays the participant's program, ECE General Education or EC Special Education, and teaching status, traditional literacy-focused field experience or full-time teaching, at the time of the study. In-depth participant profiles are provided at the beginning of Chapter Four.

Table 3.3. Study participants according to program and teaching experience at the time of the study.

	Traditional (Literacy Field Experience)	Teaching Full-time	Total
ECE General	9	3	12
ECE Special Education	1	2	3
Total	10	5	15

Data Generation Tools

Two tools were used to collect data. The first was the mathematics methods class course syllabus and the second was a semi-structured interview. Documents such as the syllabus constitute a particularly rich source of information that can deepen fieldwork and qualitative analysis (Merriam, 2009; Patton, 2015). Interviews allow the researcher to gather participants' insights on the experiences, feelings, opinions, and knowledge directly related to the research question (Merriam, 2009).

Document: Course Syllabus

The guiding document for the early childhood mathematics course was the course syllabus, found in Appendix B. The course syllabus was a "program implementation documentation" (Patton, 2015, p. 377), and it provided information regarding the study that could not be gleaned through interviews. To be considered relevant, documents must evolve from the topic of inquiry, its authenticity must be assessed, and its purpose validated (Merriam, 2009). One of the greatest advantages to using documents is stability (Merriam, 2009). Documentary data are considered objective and nonreactive measures in that they "exist independent of a research agenda" (p. 156). This was true in regard to the course syllabus.

The syllabus was instructor developed therefore questions relating to the authenticity, the purpose for which it was produced, and specifications regarding its use can be addressed. The course syllabus provided insights into the pre-determined sequence of topics of study, learning opportunities, and expected course outcomes. The syllabus clearly stated course goals and guidelines outlining what students would need to do to reach those goals including required readings and assignments, and the grading scheme. Not evident in the course syllabus were the weekly homework tasks intended to provide continuity of learning from one week to the next, specifics regarding the more comprehensive assignments, and detailed teaching plans for each session. This lack of detail is a limitation inherent in the use of documents in qualitative research (Patton, 2015).

Interview: Scenario-based Protocol

In addition to the course syllabus, data was collected through a sixty-minute, semi-structured, face-to-face interview. I conducted interviews with willing participants at a location of their choosing. It was important to me to conduct each interview in a comfortable, private, and easy to access setting. I accomplished this by allowing participants to select the time and location of their interview. Interviews took place at local coffee shops, the student union, at my campus office, and when applicable in participants' classrooms after school hours. Each interview was completed between three to eight weeks after the conclusion of the methods course. Finally, during each interview, I recorded the information gathered using an electronic recording device, which was transcribed verbatim, along with paying attention to each participant's tone of voice and degree of engagement. Photographs were taken to capture participants' use of and interactions with the materials I brought to each interview.

The following section presents the rationale for using a qualitative interview as a data collection tool for this study. Background defining a qualitative interview is shared, the interview protocol is discussed in-depth, and the pilot studies that aided in the development of the interview protocol are detailed.

Background. Interviews are one of the most widely used techniques for conducting qualitative research (Glesne, 2011; Merriam, 2009). Social science researchers utilize interviews to find out what is in and on people's minds, in order to explore and learn why they do what they do (Glesne, 2011). Qualitative interviews stem from the belief that "the perspective of others is meaningful and knowable and can be make explicit" (Patton, 2015, p. 426).

In its most simplified form the qualitative interview affords the researcher and the participant an opportunity to engage in a conversation focused on questions related to a research study. A more careful examination encourages the researcher to utilize a qualitative interview when they seek to uncover what cannot be directly observed and to understand what has been observed (DiCicco-Bloom & Crabtree, 2006; Merriam, 2009) According to Patton (2015), the purpose of the qualitative interview is to capture how those being interviewed "view their world, to learn *their* terminology and judgments, and capture the complexities of *their* individual perceptions and experiences" (emphasis in original) (p. 442). Fontana and Frey (1998) suggest an open-ended and semi-structured interview guide be used in order to gather information in the distinct areas that the researcher examined.

This research study employed a semi-structured interview (Merriam, 2009) with stimulus texts (DiCicco-Bloom & Crabtree, 2006; May, 1991; Torronen, 2002). A semi-structured interview ensures minimal variation to the questions and that the same basic lines of inquiry are pursued with each person interviewed (Patton, 2015).

Torronen (2002) suggested that stimulus texts may be used as "clues, microcosms, or provokers" (p. 343) during the semi-structured interview process. Examples of stimulus texts include films, photos, sketches, scenarios or news items that are used for encouraging interviewees to speak about the research topic. Used as microcosms, the stimulus text prompts interviewees to compare their world against that of the stimulus text. When stimulus texts are used as provokers, the researcher chooses cultural products that challenge, with the aid of probing questions, the interviewees to work with established meanings, conventions, and practices of the phenomenon under investigation. For this study, stimulus texts included dot arrangements, counters, five and ten frames, rekenreks, and a whiteboard and dry-erase markers.

The Protocol. The semi-structured interview featured the *Recognizing Number, and Subitizing Trajectory* included in Appendix A (Clements & Sarama, 2014). The trajectory describes developmental growth of children's understanding and skill with subitizing from ages birth to age eight. The classroom scenarios, dot arrangements, and follow-up questions focused on *Level 4 Perceptual Subitizer to Four* through *Level 7 Conceptual Subitizer to Ten.* Table 3.4 displays the section of the subitizing trajectory upon which this study centered.

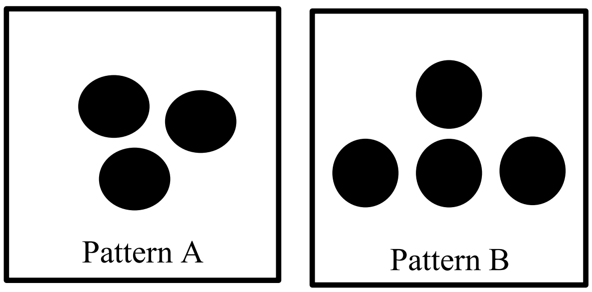
The interview protocol featured a classroom scenario and related questions. (See Appendix C for the full protocol.) The scenario engaged participants in preparing and facilitating a mathematics lesson for a five-year old kindergarten class as suggested in the fictitious instructional resources provided by a fictitious school district. The interview protocol was designed to surface participants' knowledge of subtizing, ideas of intentional teaching (Epstein, 2014), and developmentally appropriate instruction (NAEYC, 2009) surfaced as naturally as possible.

Table 3.4. Levels four through seven of the subitizing learning trajectory. Adapted from Clements and Sarama (2014).

Level	Level Name	Age	Description
4	Perceptual Subitizer to 4	4	Progress is made when a child instantly recognizes collections up to four when briefly shown and verbally names the number of items. For example, when shown four objects briefly, says "four.
5	Perceptual Subitizer to 5	5	The child instantly recognizes briefly shown collections up to five and verbally names the number of items. For example, when shown five objects briefly, says "five."
6	Conceptual Subitizer to 5	5	The child can verbally label all arrangements to five shown only briefly. For example, a child at this level would say, "I saw 2 and 2 and so I saw 4."
7	Conceptual Subitizer to 10	5	The child can verbally label most briefly shown arrangements to six, then up to ten, using groups. For example, a child at this level might say, "In my mind, I made two groups of 3 and one more, so 7.

The interview protocol consisted of four main stimulus texts (Torronen, 2002). The first stimulus texts were the Set 1 dot patterns shown in Figure 3.1. Participants were asked if they would use those three patterns with the fictitious group of five-year olds. The second stimulus text included the six dot patterns in Set 2 (shown in Figure 3.2). The primary task was to order those patterns for classroom use and to justify that order. A more focused discussion of Pattern F was the third stimulus text and the final stimulus text involved participants adding in a pattern of their design and choosing. I discuss each of the four stimulus texts and related interview questions and tasks below. (See Appendix D for an uninterrupted version of the interview questions and supporting research.)

Set 1 Dot Patterns



Set 2 Dot Patterns

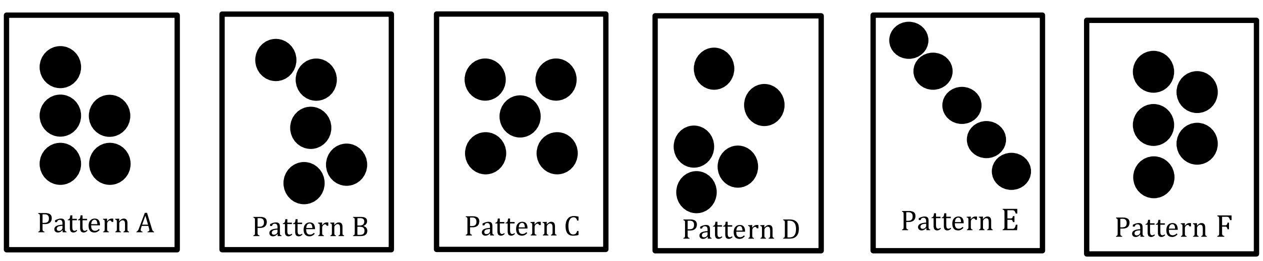


Figure 3.1. Set 1 and Set 2 dot patterns used during this study's interview to elicit subitizing.

The Set 1 dots arrangements opened the fictitious lesson and participants were asked if they would utilize these dot patterns with their students and if so, why and how. Each Set 1 dot pattern was placed on its own card so participants could re-orientate the images and alter the order in which they were placed. Table 3.5 displays the interview questions used with the Set 1 dot patterns, the purpose of the questions in the context of the interview, and the literature supporting the questions.

Table 3.5. Questions and supporting rationale used with Set 1 dot patterns.

Interview Question	Purpose	Research Base
Would you use these dot patterns with your kindergarten students? Tell me why?	Do participants: • recognize this as an activity that prompts subitizing? • name subitizing? • identify subitizing as important to children's early number sense?	Ball et al., 2008 Baroody, 1986 Clements, 1999 Douglas, 1925 Epstein, 2014 Sztajn et al., 2012
How might you use these with your kindergarten students?	Do participants identify instructional strategies that prompt subitizing?	Clements, 1999 NAEYC, 2009 Huinker, 2011 Markovits & Hershkowitz, 1997 Risden, 1978 Sztajn et al., 2012
What responses do you expect from the children with these dot patterns?	Do participants anticipate a variety of student responses and acknowledge the vary levels of sophistication in each response? Do those responses provide insight into the developmental progression of subitizing or the subitizing trajectory?	Ball et al., 2008 Baroody et al., 2006 Clements, 1999 Carper, 1942 Sarama & Clements, 2009 Sztajn et al., 2012
What do those responses suggest?	Do participants verbalize the difference between seeing quantity and counting by ones? Do those responses suggest knowledge of the subitizing learning trajectory?	Ball et al., 2008 Clements, 1999 Fitzhugh, 1978 Huinker, 2011 Risden, 1978 Sztajn et al., 2012

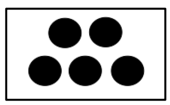
The second stimulus text featured the Set 2 dot arrangements. Participants were asked to share how they might use those dot patterns with their kindergartners and in what order they would use them. As with Set 1, each Set 2 dot arrangement was placed on its own card. This allowed participants the freedom to move the patterns around, re-orient them if desired, and arrange them in their desired order. All dot patterns in Set 2 represented the quantity of five to better distinguish difficulty level. In addition, participants addressed student thinking and misconceptions multiple times throughout this scenario. Table 3.6 displays the interview questions associated with Set 2 dot patterns, the purpose of the questions in the context of interview, and the literature supporting the questions.

Table 3.6. Questions and supporting rationale used with Set 2 dot patterns.

Interview Question	Purpose	Research Base
Can you place them in order as to how you might use them with your kindergartners? Explain for me why you placed them in that order?	Do participants order the patterns from easier to more challenging patterns and provide justification for her decisions? Will they place the patterns in an order that reflects the subitizing trajectory?	Ball et al., 2008 Beckwith& Restle, 1966 Brownwell, 1928 NAEYC, 2009 Epstein, 2014 Sarama & Clements, 2009 Sztajn et al., 2012
What would you hope to hear from students that tell you that they are ready to move to the next pattern?	Do participants mention both conceptual and perceptual subitizing either formally or informally? Does the rationale provided indicate application of mathematical knowledge needed for teaching?	Ball et al., 2008 Brownwell, 1928 Clements 1999 Epstein, 2014 Sarama & Clements, 2009 Sztajn et al., 2012

The third stimulus text featured Pattern F. Participants were asked how they might respond to a child's wrong answer to the question of "How many dots do you see?" The purpose of this scenario was to investigate the participants' interest in student thinking, flexibility with instructional decisions and understanding of the subitizing trajectory. Table 3.7 displays the interview questions associated with Pattern F, the purpose of the questions in the context of interview, and the literature supporting the questions.

Table 3.7. Questions and supporting rationale used with Pattern F.

Interview Question	Purpose	Research Base
Imagine after you showed Pattern F the students gave a non-sensible response. What might a non-sensible response tell you about their understanding? How would you follow-up?  Pattern F	What instructional strategies do participants suggest? Do the instructional strategies support subitizing and an understanding of quantity?	Ball et al., 2008 NAEYC, 2009 Epstein, 2014 Sarama & Clements, 2009 Sztajn et al., 2012
You want the children to continue to engage with “dot pattern flash” how could you adjust the activity to meet them where they are?	Do participants draw upon their understanding of learning trajectories and developmentally appropriate instruction as they explore children’s misconception? Do they suggest a tool to help elicit thinking and understanding from the child? What rationale do they provide for their instructional decisions?	Ball et al., 2008 NAEYC, 2009 Epstein, 2014 Sarama & Clements, 2009 Sztajn et al., 2012

The fourth and final stimulus text asked participants to develop a dot pattern to include in Set 2 and identify the placement of this new dot pattern into the order they had previously established. Discussing the development of the added-in pattern as well as its placement in the Set 2 sequence affords valuable insight into each participant’s understanding of subitizing, the subitizing trajectory, and participant’s ability to center instruction on children’s thinking. Table 3.8 displays the interview questions associated with this final task, the purpose of the questions in the context of interview, and the literature supporting the questions.

Table 3.8. Questions and supporting rationale used to prompt participants create a pattern to add in to Set 2.

Interview Questions	Purpose	Research Base
• If you were to suggest a pattern to include in this collection, what would it be, where would you place it, and why? • What different responses might you anticipate getting from your students? • How would those responses help you decide if it is an appropriate next step?	What understandings inform the participant's thinking as she recommends next steps? Is the recommended pattern appropriate for the progression of the order of the Set 2 cards and the subitizing learning trajectory?	Ball et al, 2008 NAEYC, 2009 Epstein, 2014 Sarama & Clements, 2009 Sztajn et al., 2012

While conducting the interviews, I intentionally set aside, or bracketed (Moustakas, 1990), any judgment or preconceived ideas I held regarding effective early childhood mathematics teach and learning due to my extensive experience as a former teacher of mathematics of young children, a K-8 district mathematics coach and teaching specialist, professional development provider for PreK-Grade 2 teachers, and many years developing and teaching mathematics education courses at Lakewood University. Bracketing ensures the researcher sets aside all preconceived experiences or notions to more wholly engage with the experiences of participants in the study. As Moustakas (1990) states:

> The data generated is dependent upon accurate, empathic listening; being up to oneself and to the participants; being flexible and free to vary procedure to respond to what is required in the flow of dialogue; and being skillful in creating a climate that encourages the participants to respond comfortably, accurately, comprehensively, and honestly in elucidating the phenomenon (p. 48).

Bracketing continued after each interview as I took detailed notes to record my own thoughts, emotions, and all other considerations I encountered during the research process. This post-interview reflection allowed me to honor the thoughts, feelings, and experiences of study participants.

The Pilot Study

The current study was informed by a small qualitative, pilot study conducted during Fall 2015. The pilot study included four white, female, traditional students enrolled in the ECE program. All had successfully completed the same mathematics methods course six weeks prior to the beginning of the pilot study. At the time of the pilot study three of the participants were two semesters away from their student teaching experience and one was student teaching. Each aspect of the current study was informed by the pilot study though its greatest influence can be seen in the interview protocol. The implications of the pilot study for the current study impacted the conceptual framework for this study, prompted a more narrow content focus, and revealed the phenomenon of intentional teaching.

The intent of the pilot study was to investigate whether a purposeful study of two specific learning trajectories, the counting learning trajectory and the subitizing learning trajectory, would foster early childhood prospective teachers' mathematical knowledge for teaching and their ability to plan instruction that meets and advances young children's mathematical knowledge. (See Appendix E for the Pilot Study Coding Manual.) Data from the pilot study suggested that prospective early childhood teachers do engage in decision making that accessed both their developing mathematical knowledge needed for teaching and their knowledge of the developmental stages children progress through as they learn mathematics.

The pilot study resulted in four findings. First, prospective early childhood teachers were able to overcome early negative experiences with mathematics to establish themselves as beginning teachers of mathematics of young children. Second, the development of a mathematics teaching identity surfaced as key to prospective early childhood teachers' ability to effectively facilitate learning in young children. Third, prospective early childhood teachers engaged in decision making informed by their nascent knowledge of children's developmental stages of learning mathematics. Fourth, the mathematics methods course played a significant role in contributing to study participants' mathematical knowledge needed for teaching.

Specific to subitizing, pilot study participants recognized and relied on their knowledge of the subitizing learning trajectory as they responded to prompts regarding appropriate next steps in instruction and addressing student misconceptions. By placing the learning trajectory at the center of decision-making prospective early childhood teachers could target the mathematics, elicit student thinking, and suggest further learning experiences that would encourage more sophisticated subitizing.

The pilot study provided several implications for further study. First, it was clear that all participants were able to justify the use of varied dot arrangements as critical to young children's mathematical development, and in particular, subitizing. Each participant identified a sequence of patterns that scaffold mathematical understanding aligned with the subitizing learning trajectory. What was not as clear was the rationale or reasoning behind the participants' decision making. This is why the study focused on one trajectory and furthered probed participant's instructional decisions around that one trajectory. This detail was brought to my attention during my final interview of the pilot study. Justice (a pseudonym) quickly sequenced the dot arrangements as the other three participants had done but it was what happened "in between" the dot patterns that captured my interest. After analyzing the transcript it was evident that Justice displayed a complex understanding of young children's development of subitizing as she sequenced the dot patterns. Not only did Justice clarify how levels of reasoning became more sophisticated from dot pattern to dot pattern, she was intentional in her justification for a child's readiness to advance through the sequence of dot patterns. It is this intentionality in decision-making that I wished to further explore in the current study.

Methods Used to Organize and Analyze Data for the Current Study

In this section I review the analysis procedures employed during the study. Figure 3.2 displays the data analysis processes I employed and at what point of the study each was implemented. What follows is a description of the data analysis strategies utilized and how they were used during this study.

Transforming the Data into a Readable Text

All interviews were digitally audio-recorded. To make interview data more accessible for analysis each interview was transcribed. Transcribed interviews included exact language used by both the researcher and the participant and insightful actions or physical responses of the interviewee noted during the actual interview. Interviews were transcribed within one to three weeks of completion. Handwritten observations noted by the researcher during the interview were included with the interview transcript along with any general analytic comments. As an example, on each transcript I noted the participants' disposition, level of engagement, and confidence throughout the interview. These comments served as the beginning for analytic memos written after each interview.

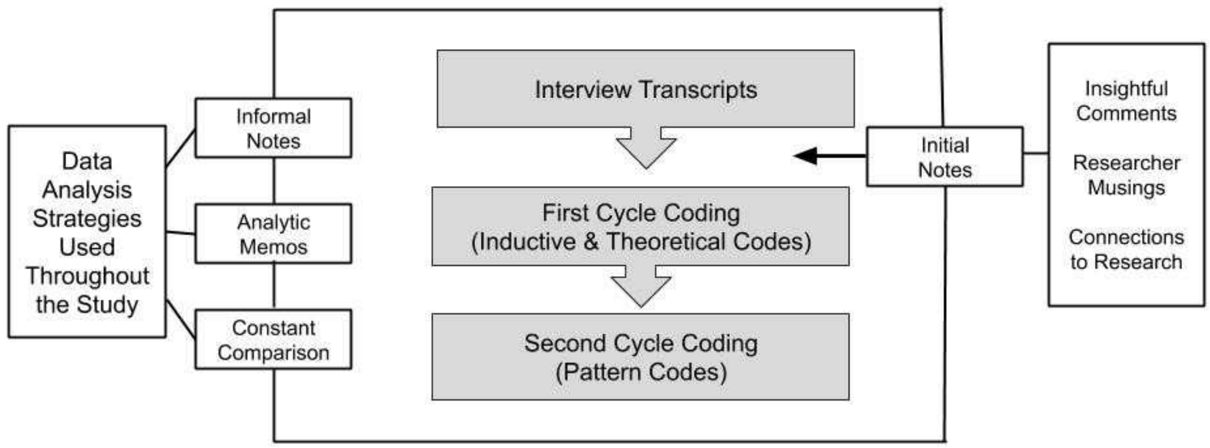


Figure 3.2. Data analysis procedures employed for this study. After interviews were transcribed initial notes were used prior to more formal procedures such as constant comparison, analytic memos, and informal notes.

Initial Notes and First Cycle Coding

Data analysis began as a careful read of each transcript. As I read I made initial comments and notes on hard copies of the interview transcripts. My initial notes included inductive codes, cross-references to material in another part of the data or another interview, particularly insightful comments, or my personal musings and internal commentary on what I was reading. As I read the initial notes provided me with ideas for analytic consideration (Saldaña, 2016) and lead to the creation of first cycle codes. As an example, my initial jottings surfaced the idea that study participants were intensely interested in children's mathematical thinking and extensively explored many ways to elicit their thinking.

By employing inductive coding at the earliest stages of data analysis I established codes and themes from the raw data by identifying words and phrases that were similar in order to group the data into related categories (Cresswell, 2013). These first impressions, referred to as "clusters of meaning" (p. 82), provided a transitional link between the raw data and eventual codes. For example, as I read the transcribed interviews I noted broad themes such as questions participants would ask the children to surface their thinking, rationale for ordering the dot patterns, and clear evidence of participants' understanding of subitizing and why it is important for young children to develop. These broad ideas supported the development of inductive first cycle codes.

According to Glesne (2011), coding is a "progressive process of sorting and defining and defining and sorting" (p. 194) and is used to "discern themes, patterns, processes, and to make comparisons and build theoretical explanations." In the case of interview data, first cycle coding categories "summarize in a word or short phrase - most often as a noun - the basic topic of a passage of qualitative data" (Saldana, 2009, p. 70). As an example, study participants' keen interest in student thinking mentioned above was assigned the first cycle code, "Organized proposed instruction around student readiness." See Appendix F for this study's coding manual.

Related inductive first cycle codes were grouped together under theoretical categories. Maxwell (2013) defined theoretical categories as "primarily *descriptive* [italics in the original], in a broad sense that includes descriptions of participants' concepts and beliefs; and they stay close to the data categorized, and don't inherently imply a more abstract theory" (p. 108). This process of grouping and naming related first cycle codes under descriptive categories offered me the opportunity to gain insight on the interviewee's language, perspectives, and views (Saldaña, 2009). Table 3.9 displays inductive first cycle codes and related theoretical categories. The theoretical categories allowed me to "work with loosely held chunks of meaning" (Miles et al., 2014, p. 87), and to remain open to reconfiguring and renaming categories as the data took shape as I "mercilessly cross-checked" (p. 87) to identify the most compelling themes.

Table 3.9. Descriptive codes identified during first cycle coding

Inductive First Cycle Codes	Theoretical Category
• Defined subitizing. • Stated the difference between conceptual and perceptual subitizing. • Identified why subitizing is important. • Focused on quantity.	Understood the big idea of subitizing
• Articulated how subitizing skills grow over time. • Provided rationale for order of dot patterns that mirrored developmental growth. • Awareness of developmental growth in math.	Viewed subitizing growth as developmental
• Introduced new representation to support understanding of quantity. • Organized proposed instruction around student readiness • Introduced new representation to support understanding of quantity.	Considered student's current level of understanding
• Articulated strategies to elicit student thinking. • Used student reasoning as a starting point for instruction.	Started with students' thinking
• Addressed Pattern E in a way that revealed why subitizing is important. • Kept a focus on understanding quantity.	The Case of Pattern E
• Adjusts the pattern to a smaller quantity. • Asks the child to count the dots. • Prompted child to "show me why you think there are ten" • Told them they are wrong. • Translated the pattern to another representation. • Applied one-to-one correspondence to verify "how many."	Managing a Misconception: The Case of Pattern F
• Justified pattern based on developmental nature of subitizing trajectory. • Based new pattern on children's potential subitizing skill. • Discussed development of the pattern in light of children's thinking.	Introduced a new pattern
• Subitizing is not as easy as it looks. This develops over time. • Children need to know the number words before they can say "how many." • Don't expect your students to see everything	Personal thoughts about teaching mathematics to young children

the way you see it. • I hated math. I don't want that to happen to my students.	

Constant Comparison

Constant comparison involves "the continuous sorting and contrasting of the elements of the dataset" (Trochim & Donnelly, 2008, p. 285). In the context of this study, constant comparison involved comparing one segment of data with another segment of data. The data included interview transcripts, photographs taken during the interview that captured the order of the dot arrangements developed by the participants, and manipulatives used by the participants during the interview. This continuous back and forth helped me notice similarities and differences between and among participant comments, unifying big ideas, and tentative theoretical propositions. According to Strauss and Corbin (1993) constant comparison is employed to help "protect the researcher from accepting any of those voices on their own terms, and to some extent forces the researcher's own voice to be questioning, questioned, and provisional" (p. 280).

I engaged constant comparison throughout the breadth of the study and specifically at the early stages of inductive first cycle coding. For example, as I read each interview transcript it became clear that each participant has some working knowledge of subitizing. Some talked about what it was, others offered various examples of subitizing, and others shared why subitizing was important to young children's mathematics development. This regular back and forth resulted in the emergence of the following inductive First Cycle codes (a) defined subitizing, (b) stated the difference between conceptual and perceptual subitizing, and (c) identified why subitizing is important, and (d) focused on quantity. I used the theoretical category "understood the big idea of subitizing" to group those codes.

Analytic Memo Writing

Analytic memos were completed throughout the study and were used to capture my "private and personal musings before, during, and about the entire enterprise" (Saldaña, 2009, p. 34). As an example, an analytic memo was immediately written after each interview was completed and before the interview was transcribed and included thoughts about the participants' reactions to the interview process, the interview protocol, or my general inquiry processes.

One analytic memo that became particularly important to the study included the pattern each participant choose to add into the sequence they created with the Set 2 cards. This memo helped me identify the intentionality of each participant's decision and whether or not it was inline with the developmental progression as suggested by the subitizing learning trajectory. This memo is found in Appendix G.

Moving from Theoretical Codes to Second Cycle Pattern Codes

To support systematic data analysis all interview transcripts were loaded into NVivo 11, a data analysis software program used to analyze qualitative data. Each interview was reviewed in NVivo by applying the inductive first cycle codes and the theoretical categories listed in Table 3.4 as preliminary themes.

The theoretical categories were further modified as I continued to immerse myself in the data. My goal was to establish more stable Pattern Codes. Therefore each theoretical category was changed in NVivo to match its related pattern code. Table 3.10 displays the theoretical categories and the resulting Pattern Codes. The pattern codes served as my coding manual.

Table 3.10 Theoretical categories and the related pattern codes.

Theoretical Category	Pattern Code
Understood the big idea of subitizing	Knowledge of subitizing
Viewed subitizing growth as developmental	Learning trajectory knowledge and specifically subitizing learning trajectory
The Case of Pattern E	Dot pattern order mirrors learning trajectory progression
The Case of Pattern F	Intentional decision meant to advance learning based on understanding of subitizing learning trajectory.
Considered student current level of understanding	Intentional decision meant to advance learning based on understanding of subitizing learning trajectory.
Started with students' thinking	Actions that honor student current capabilities
Introduced a new pattern	Dot pattern order mirrors learning trajectory progression
Rationale for dot pattern order	Dot pattern order mirrors learning trajectory progression
Personal thoughts about teaching mathematics to young children	Personal beliefs about children and teaching that influence decision making

Dependability and Credibility

Credibility refers to the extent to which research findings can be replicated (Lincoln & Guba, 1985; Merriam, 2009) and considers whether the results are an accurate interpretation of the participants' meaning. To ensure credibility, the findings in a qualitative inquiry must make sense (Maxwell, 2013) and represent a compelling whole that allows the researcher and reader to feel confident about the observations, interpretations, and conclusions (Creswell, 2013). Strategies used to enhance the dependability of a study allow for stronger congruence between the participants' construction of reality and the researcher's interpretation of this reality (Merriam, 2009). Cresswell (2013, pp. 250-253) described eight strategies often used in qualitative research to contribute to its credibility. Figure 3.3 displays the eight strategies. Cresswell recommended that qualitative researchers engage in at least two of the eight procedures in any given study. For the current study, I employed the following procedures: (1)

triangulation of data; (2) use of detailed, thick descriptions; and, (3) disclosure of researcher positionality and bias.

- *Prolonged engagement and persistent observation* in the field includes close, long-term contact with study participants
- *Triangulation of data* entails use of multiple and different sources, methods, and theories to provide corroborating evidence
- *Peer review or debriefing* engages a reviewer who "keeps the researcher honest; asks hard questions about methods, meanings, and interpretations" (p. 251).
- *Negative case analysis* refines working hypotheses as the inquiry advances in light of negative or disconfirming evidence.
- *Clarifying researcher bias* identifies the researcher's positionality and any biases or assumptions that impact the inquiry.
- *Member checking* involves engaging participants' views in the credibility of the findings and interpretations.
- *Rich, thick descriptions* illustrate in detail the participants or setting under study
- *External audits* engages an outside consultant to review both the process and the product of the study

Figure 3.3 Creswell's eight strategies used to validate qualitative research.

Triangulation entails the use of multiple methods, multiple sources of data, multiple investigators, or multiple theories to confirm emerging findings (Merriam, 2009). The fifteen interview transcripts produced during the data collection phase of the study served as multiple sources of data. Employing triangulation with interview data meant comparing interviews from participants with diverse perspectives and life stories. As an example, in-service teachers accounted for one-third of study participants. Teachers seeking bilingual certification, born outside of the United States, and for whom English was an additional language, accounted for one-fifth of participants. Slightly less than half of study participants self-identified as non-traditional students completing degrees as post-baccalaureate students. The diversity of participants allowed me to run queries using Nvivo 11 software to consider how the groups responded to the same question or task in the interview. This helped me explore the phenomenon of intentionality of decision-making.

Rich, thick descriptions throughout engage the reader in the research context as they are "a highly descriptive, detailed presentation of the setting and in particular, the findings, of a study" (Merriam, 2009, p. 227). Rich, thick descriptions are presented in the form of quotes from participants, field notes, and documents and help the reader determine if the overall findings ring true. As a result, I present carefully described, detailed vignettes and narratives supported by direct quotes from the interviews. I used my field notes to ensure I remained true to their thoughts and feelings. My goal was to ensure consistency of findings in my study.

Positionality supports a narrative placement of researcher objectivity and subjectivity (Lave & Wenger, 1991). I come to this qualitative research project having served as the instructor of record for the methods class and taught this same class for many prior semesters. I developed each aspect and attended to each detail of the course including content, text selection, homework assignments, and projects. My experience in K-12 teaching includes fifteen years as an elementary classroom teacher in bilingual (Spanish-English) settings, and as a mathematics specialist and coach for two different school districts for a total of seven years. I hold a master's degree in elementary and middle school mathematics. My intense interest in early childhood mathematics in general, and learning trajectories specifically, lead me to develop and facilitate numerous professional development opportunities for elementary and early childhood teachers at the local, state, and national level. To minimize any bias toward my participants, I set aside any preconceived ideas or judgments during the data analysis phase of the study. This approach helped me improve efficiency and gives credibility to the dissertation study (Maxwell, 2013).

Dependability establishes the study's finding as consistent and repeatable (Creswell, 2013). To ensure dependability, I documented coding schemes and themes as well as crosschecked all of the data sources to identify commonality of themes (Maxwell, 2013). I used a research journal to document each step of the inquiry including revisions to the interview questions, participants' affect during the interviews, and responses I wished to remember. I utilized analytic memos to help me think about what I was seeing in the data, and what I might be learning as the study unfolded. Using these tools and techniques assisted in the enhancement of trustworthiness of the research study findings (Creswell, 2013).

Limitations of the Current Study

On the one hand phenomenology strives to understand a common experience and bring meaning to it and may contribute to the development of new theories and understandings. On the other hand, phenomenological research does not produce generalizable data (Trochim & Donnelly, 2008; Patton, 2015). Phenomenology requires researcher interpretation, making phenomenological reduction an important component to reduce biases, assumptions, and pre-conceived ideas about the experience or phenomenon.

As the study participants were students in a class I developed and taught, it was essential I be mindful of potential pitfalls and challenges. Glesne (2011) stated, "Backyard research can be extremely valuable, but it needs to be entered with heightened consciousness of potential difficulties" (p. 43). The fact that I have taught this course numerous times at the same institution meant that I needed to be conscious of the biases I held as a researcher-participant regarding the students enrolled in the course, previous experiences I had teaching the coarse, and preconceived notions of early childhood prospective teachers' strengths and struggles with mathematics as a discipline.

Second, I served as the instructor for the methods course and as the researcher for this study. It is quite possible that study participants said what they thought I wanted to hear as their former instructor. In addition, as the instructor it was possible my own biases and pre-conceived outcomes could impact the results of the study. In response to that possibility I engaged what Moustakas (1994) referred to as epoche, or bracketing, as a way by which researchers "set aside our prejudgments, biases, and preconceived ideas" (p. 85). Engaging the concept of epoche allowed me to approach the studied phenomenon with a totally new perspective. The principles of bracketing helped reduce biases I have as influenced by my position as instructor for the course, a former classroom teacher, a former district mathematics specialist, and designer and facilitator of professional development for early childhood teachers.

Third, the participants in the study were voluntary and likely agreed to participate in the study due to positive experiences with the mathematics methods course and with me. It is worth considering how the results of the study could be impacted by engaging participants that did not find the course as helpful to their future work as a teacher of early childhood mathematics.

Fourth, the research participants must be able to articulate their thoughts and feelings about the experience being studied (Moustakas, 1994). It may be difficult for them to express themselves due to language barriers, age, cognition, embarrassment, and other factors. Two of the interviews were conducted in Spanish. As Spanish is my second language the interviews may have preceded quite differently if a native speaker conducted the interviews.

Finally, it cannot be dismissed that a researcher with no prior teaching experience nor knowledge of young children's mathematics learning trajectories may not have found the same level of cooperation I experienced. As an example, each of the participants remained actively engaged in the scenarios and the conversation for over one hour. This may be reflective of the positive relationship each participant and I shared.

My hope is that this study will inform future researchers and educators who prepare prospective early childhood teachers to improve the teaching and learning of mathematics for children ages three through six. By placing children's thinking at the center of teaching, researchers may be able to ensure equitable access to high quality mathematics instruction and make real gains on closing the persistent achievement gap that negatively impacts the current and future opportunities for children of color as well as those living in low resource communities.

CHAPTER FOUR

FINDINGS

The goal of this phenomenological study was to qualitatively explore how an understanding of mathematics learning trajectories supports early childhood prospective teachers to become effective teachers. The following research questions steered the course of this study:

Overarching Research Question: In what ways do learning trajectories support early childhood prospective teachers' preparation to become effective teachers?

Attendant Question 1: What understandings do early childhood prospective teachers have regarding the subitizing trajectory?

Attendant Question 2: Do early childhood prospective teachers draw upon their knowledge of learning trajectories as they make instructional decisions?

Examining the influence of learning trajectory knowledge on prospective ECE teachers' instructional decision-making provides further insight into the skillset of a well-prepared beginning teacher of mathematics (AMTE, 2017).

Within this chapter, the reader is provided with a brief introduction to each participant, a discussion of the five major themes and related sub-themes. Data analysis and reporting in phenomenology requires the researcher to provide a contextual description of "how" the phenomenon was experienced by the group of participants (Creswell, 1998). This description is provided during the discussion of the major themes and sub-themes through the voice of study participants.

Participant Profiles

The findings from this study represent responses from fifteen female participants that were students in an early childhood mathematics methods course taught by the researcher. Five study participants taught young children on a full-time basis as either a teacher of record, a special education intern serving as a full-time teacher, or paraprofessional. The remaining ten participants were traditional undergraduates whose experience with young children ranged from serving as a babysitter or nanny to field experiences completed as partial requirements for other classes in their early childhood education program. All participants were seeking state teacher certification. At the beginning of the interview I reviewed each participant's demographic data that was voluntarily collected on all students enrolled in the early childhood mathematics methods class. The demographic information for the study participants is found in Table 4.1.

Table 4.1 Study participants' demographic information.

#	Pseudonym	Student Status	Teaching Status	Bilingual or Monolingual	Program	Location of Interview
1.	Karina	Non-Traditional	Yes	Monolingual	Special Education	Researcher's on-campus office
2.	Jaeden	Traditional	No	Monolingual	Special Education	Researcher's on-campus office
3.	Karaleen	Non-Traditional	Yes	Monolingual	General Education	Participant's 5K classroom
4.	Mandisa	Non-traditional	Yes	Monolingual	Special Education	Local coffee shop
5.	Marisol	Non-traditional	Yes	Bilingual	General Education	Participant's 5K classroom
6.	Sasha	Non-traditional	No	Monolingual	Special Education	Participant's 3K classroom
7.	Amalie	Non-traditional	Yes	Monolingual	Regular Education	Local coffee shop
8.	Cecilia	Traditional	No	Monolingual	General Education	Researcher's on-campus office
9.	Flora	Traditional	No	Bilingual	General Education	Researcher's on-campus office
10.	Crystal	Non-traditional	No	Monolingual	General Education	Researcher's on-campus office
11.	Karolyn	Traditional	No	Monolingual	Special Education	Researcher's on-campus

						office
12.	Kayla	Non-traditional	No	Monolingual	General Education	Researcher's on-campus office
13.	Amber	Traditional	No	Bilingual	General Education	Researcher's on-campus office
14.	Justine	Traditional	No	Monolingual	General Education	On-campus coffee shop
15.	Marie	Traditional	No	Monolingual	General Education	Researcher's on-campus office

As we were well acquainted from spending the previous semester together I began each interview by asking each participant to respond to a question that would provide insight into their personal motivations for choosing teaching as a career and why, in particular, a focus on early childhood education. Listening to their motivations for becoming a teacher provided a non-threatening beginning to the interview and offered me a window into their developing beliefs about young children's learning and their role as an early childhood teacher. Information from the first question is used to introduce the reader to each participant. All personal identifying information has been changed to ensure the privacy of all participants except for their status as a traditional or non-traditional student, whether they were seeking bilingual certification, and if they were pursuing certification as a general education or special education early childhood teacher.

Karina, Participant 1—is a white female, non-traditional student pursuing a Master's degree in Early Childhood Special Education along with an add-on certification for Autism Spectrum Disorder. At the time of the mathematics methods course she taught 3K-5K students in a large urban mid-western school district as an Intern. Prior to her work as a teacher, she served as an army reservist and was employed as an occupational therapist. Her interest in teaching was sparked due to the lack of opportunity to practice as an occupational therapist for children. She heard of the post baccalaureate early childhood special education program through the husband of a friend. Her decision to become a teacher has brought her nothing but joy and shared:

> This is the first job I've ever had that when I wake up in the morning I can be completely exhausted, and I usually am exhausted, but I'm excited. I am always happy and I wake up and I'm like, 'Yes! I get to go to work today!'

As to why she selected to pursue a degree in special education, Karina stated:

> I love special education because it's kind of like a puzzle. I'm very mathematical and I'm very mathematically minded. I like to figure things out and each kid is like a puzzle to me. Not that they're not individuals. But each kid is their own little puzzle, and I love figuring them out. I think that's why I like special education so much but it's awesome when they get it and you see when they get it.

Karina will complete her Master's degree in Spring 2020. She plans to stay at her current assignment once she finishes her degree. Karina was animated and engaged throughout the interview. She regularly used her work with her students as a touchstone or launching point for her rationale or as examples throughout the interview.

Jaeden, Participant 2—is a white, traditional, full-time student pursuing a degree in early childhood education. Jaeden initially began her university studies as an Occupational Therapy (OT) major. Her interest in OT was sparked by the regular participation of special education students in her high school gym class. After developing a bond with a special education student her special education teacher invited her to shadow him for a few days. Teaching was an immediate call for her after that experience, though she gravitated toward OT after spending a few days shadowing the occupational therapist at her high school. After taking a few OT courses she realized, “I really wasn't doing something that I loved. I thought at that time that I wanted to be an occupational therapist because they made more money than teachers, but I also realized that I really loved teaching more.” Jaeden plans to student teach in Fall 2020. She expressed no overt fear of mathematics throughout the duration of the course or during the interview.

Karaleen, Participant 3—is an African-American, non-traditional, full-time student working as a paraprofessional in a 5K classroom in a religious charter school in the same large Midwestern city as the university. Karaleen worked in banking for twenty years prior to making the decision to pursue a degree in teaching. Her interest in teaching began after serving as a Sunday School teacher for many years for the church where her mother served as Sunday School Director. Karaleen shared:

> I found teaching Sunday School really interesting, I really enjoyed doing it, I liked read through the lesson plans given to me, but I liked being able to do my own thing. It was then that I decided that I liked it enough that it was something that I wanted to go to school for. I actually left a career in banking and have now begun a career in teaching. I'm so happy that I made the switch. I love it! It's a lot of work, and it's really rewarding. I love to see the kids starting at one level in the early part of the year and then see how they develop towards the end of the year.

After completing her degree in summer 2018 Karaleen hopes to secure a full-time teaching position at the school where she currently serves as a paraprofessional. The classroom teacher with whom she worked was so pleased with the knowledge Karaleen shared from her mathematics methods class that at the time of the interview they regularly co-planned and co-taught mathematics to their five-year olds. Karaleen held a very strong fear of mathematics, which was clearly evident throughout the mathematics methods class. During the interview she was animated, engaged, and excited to share with me how she was implementing knowledge gained in the class with her kindergarten students.

Mandisa, Participant 4—a white, nontraditional, full-time student pursuing a degree in Early Childhood Special Education. She serves as a full-time 3K teacher for a community-based daycare center in the same large urban Midwestern city as the university. Mandisa will graduate in Spring 2018 and plans to leave the state to pursue her teaching career, faulting what she views as an “unfriendly political climate towards teachers and teaching” and believes that she will “make more [money] and have a better quality of life when it comes to the job of teaching” in another part of the country.

Before pursuing a career in teaching Mandisa, “wasn't doing anything for years and years” and ultimately began pursuing an associate’s degree as an Administrative Assistant (AA) at a local community college. She found herself enjoying her Humanities courses much more than her AA classes and about the same time “decided that I wanted to be doing something that was going to be more helpful. I wanted to feel more useful. So, I decided to be a teacher.” Mandisa expressed she wished she had learned mathematics in the way that she learned to teach it through the methods class. “Math was not a very happy place when I was in elementary school” and cited an emphasis on rote memorization as a significant contributor to her poor memories of her early mathematics experiences. One of her big takeaways from the methods class was “Math is not a contest or a race. As a teacher I really want my students to understand the subject and I believe *they* really want to understand it, too.” Mandisa remained interested and confident throughout the interview.

Marisol, Participant 5—is a Puerto Rican, Spanish dominant, nontraditional student. At the time of the interview Marisol served as a 5K teacher for a language immersion school in a large school district in the same city as the university. She is pursuing a bilingual education add-on certification. Due to my ability to speak Spanish I gave Marisol the choice of whether we would conduct the interview in Spanish or English. She selected Spanish. Marisol moved to the mainland to pursue better opportunities for her daughter and son and has lived in the same urban city as the university for the last two years. Her interest in teaching was apparent even as a young child. According to her mother, Marisol did play with dolls, but she was always teaching them something—how to read, write, or play with toys. Throughout her elementary years she served as the “teacher’s helper.” Marisol holds a bachelor’s degree in early childhood education from Puerto Rico and is proudly the “the only teacher in her whole family.” Prior to her current position as a 5K teacher she taught four-year olds in a daycare setting for about five months and then at a private charter school that serves Spanish speaking families in her community.

Marisol regularly referenced her work with her students to support her rationale or frame examples throughout the interview. In addition, her nine-year old daughter’s struggle with mathematics was a front-and-center concern for Marisol throughout the class and she referenced her a few times during the interview. In one notable comment Marisol shared, “The quantities that she [her daughter] is working with are getting bigger and if they can't see the smaller quantities, the bigger quantities really don't mean anything. I watch my daughter when she does problems like ‘three times five’ and she will make three groups of five, usually using tally marks, and then she will go back and count them by ones. She's in third grade, and she's not seeing groups.” For this reason, Marisol places a concerted effort into carefully scaffolding her students’ ability to see, name, and understand quantity. Marisol was animated, confident, and engaged throughout the interview.

Sasha, Participant 6—is a white, non-traditional student pursuing a Master's degree in Early Childhood Special Education. At the time of the interview Sasha had just moved from being a Special Education Pre-Intern Paraprofessional to a Special Education Intern for 3K-5K students in a large urban Midwestern school district located in the same city as the university. After graduating high school Sasha pursued a twelve-year career in retail sales and management. After losing her last job retail job due to store closings Sasha made the choice to return to school and completed a Pre-Law and Political Science degree. She was always interested in teaching and after learning of the Early Childhood Special Education post baccalaureate program she enrolled and was accepted into the program. To be sure of her decision she worked as a substitute teacher "just to make sure that it [teaching] was something that I really wanted to do. I was never a really great student and I wanted to give back and help people. That is why I decided to do early childhood special education."

Sasha remained very nervous throughout the course of the interview. That could be attributed to the fact that she took pride in being an exceptionally thorough and precise student in class. She possessed a solid working knowledge of course content and with the responsibility of full-time teaching looming in her near future stated:

> I definitely feel that after our course I have a much better understanding of mathematics. I feel much more comfortable and much more confident teaching it. I do think there should be other courses. One class is not enough. I like the way that we built from one idea to the next. And I feel like with reading, I loved all my literacy teachers, but if it was taught as intensely as this class I feel we would all be really, really prepared to teach. But I don't feel really, really prepared to teach literacy. Well I actually think that for those of us in particular in special education, we really need more. Because we have so many

> children at so many different levels, and we need to understand those levels to meet their needs. And one class just isn't enough. We really need to work more on how do we teach math, in particular now that we know how important it is to children's future.

Amalie, Participant 7—is a white, non-traditional student pursuing certification as an early childhood teacher. She graduated in 2012 with a bachelor's degree in Human Development. After working as a bank teller for two years Amalie served as a counselor and part-time educational therapist for adolescents in a residential care setting. Her responsibilities as an educational therapist included all aspects of teaching all content areas as well as writing individual education plans. Though Amalie enjoyed the work of the educational therapist she was not able to move into that position on a full-time basis, as she did not have state teacher certification. She returned to her university studies as a post baccalaureate to secure state teacher certification as an early childhood teacher. To that end, she is currently working as a 3K teacher for a private, for-profit day care center. After finishing her certification in Spring 2018, Amalie plans to return to the residential care center as the educational therapist. Amalie was comfortable and confident throughout the interview.

Cecilia, Participant 8—is a white, traditional student pursuing an early childhood education degree. Cecilia, the middle child of five sisters, became interested in working with young children when her family became a foster family for three young siblings. Cecilia found herself helping care for her foster siblings and "loved every minute of it." As a teenager she was a highly requested babysitter and even developed and taught a few educational classes (e.g., cooking and drawing) for young children in her church and neighborhood. Overtime, she established longstanding relationships with six families for whom she has worked for eight years. Cecilia states,

> I've always worked with kids, and I really loved every minute of it. So when I got to college I either wanted to be a high school English teacher or an early childhood teaching. So the last minute when I really had to make a decision I decided that my passion is really with the little ones. And I mean the babies.

Cecilia will graduate in Fall 2019 and hopes to secure a position as a Kindergarten teacher for three-, four-, or five-year olds. Cecilia was comfortable and relaxed throughout the course of the interview.

Flora, Participant 9—is a traditional, Spanish dominant, student pursuing an early childhood education degree with an additional certification in bilingual education. Flora immigrated to the United States with her family when she was a preschooler. Due to my ability to speak Spanish I gave Flora the choice of whether we would conduct the interview in Spanish or English. She selected Spanish.

As a young girl Flora frequently entertained younger siblings and cousins by playing school and she was always the teacher. Pursuing a degree in education was an easy choice as she identified herself as able to develop caring relationships with young children with ease. She would like to teach kindergarten in the bilingual program and is excited that Spanish is the primary language of instruction. Flora is committed to the local Spanish-speaking community as she was raised in a family of Mexican descent in the same large Midwestern urban city as the university. Being educated in this same community, she is uniquely positioned to empathize with and address the challenges Spanish-speaking families and children face as they enter into the American educational system. For example, Flora stated:

> In the third grade, English was introduced as a language of instruction in my classroom. When we first started with English I was like "Whoa, wait a minute like what's going

> on?" I guess I never knew that there really was a language barrier until that year. I had some troubles growing up in school because of the language barrier in third, fourth, and fifth grade. I went on to middle school and to my shock *everything* was in English. It was so difficult because I was mostly used to speaking Spanish. When we shifted into all English I was like "Whoa!" That was even more of a change than before.

Beyond her role as a teacher, Flora sees herself as an advocate for her families and young students. Her own experiences fuel this desire and she shared:

> That's one reason that keeps pushing me forward [in my education]. I know there are a lot of parents out there that don't have a lot of education and they really need me. I also know that sometimes the parents may let the children figure it out on their own because that's what happened to me. I'm not saying that was bad I just am saying that I wish someone had been there to help me. Parents play such an important role and I want to make sure that I can do what I can to make sure that they're a part of their child's education.

Finally, Flora altruistically wishes to help her families maneuver the American educational system to the benefit of their children. Flora stated:

> So I really wanted to do something to make a difference for the kids when they start really little so that when they grow up they will have a better experience than I did. My parents tried helping but they didn't go to school beyond second grade in Mexico so they didn't really have much education and did not know how things worked here. My sister and I we basically had to figure things out by ourselves and my parents left it up to us to figure it out. As my sister was older I decided I would follow what she did. I was following her because I thought she knew more than I did. When I look back at my

education I wish I had done things a little differently. I guess I would have liked to attend a better school. My elementary and middle schools had good teachers but the materials and the curriculum were very old and the instruction was all over the place. In middle school the teachers were really just really focused on teaching science, math, and social studies and they never really thought about what we might need as second language learners to actually learn.

Flora was comfortable and relaxed throughout the interview. Flora plans to student teaching in Fall 2017.

Crystal, Participant 10—is a white, nontraditional student pursuing a degree in early childhood education. Crystal actively participated in all aspects of the interview though remained quiet and reserved throughout the whole experience. This mirrored her approach to class. She rarely participated in whole group discussions during the semester though she was earnest in completing her assignments and connecting the various assignments back to her own children and stepchildren.

She selected early childhood education as a career choice after an unsuccessful attempt in a nursing program at a for-profit university. Crystal shared, “I always knew I wanted to do something that would make a difference in people's lives. I always knew that I really enjoy working with kids. I have my own, and I thought that this was kind of my second best choice that I would enjoy.” Crystal student teaches in Fall 2017 and hopes to teach five-year old kindergarten after she completes her degree. Kindergarten is appealing to her as “they are just so young and I really believe that kindergarten is the foundation. I think to serve as an early influence in their lives is something that I really, really want to do.”

Karolyn, Participant 11—is an Asian American, traditional student pursuing a degree in Early Childhood Special Education. Karloyn was student teaching at the time of our interview. Karolyn remembers always wanting to be a teacher and was a sought after babysitter in her community. Her interest in working with young children with special needs was spurred when, as a junior in high school, she was able to spend time with her mother who worked as an Occupational Therapist. Karolyn developed a bond with a young boy with autism and a diagnosed behavioral disorder. After getting to know him she was convinced that working with young children with special needs was what she wanted to do. Karolyn was a recent transfer to the university that is the site of this study. She was animated, confident, and relaxed throughout the course of the interview.

Kayla, Participant 12—is a white, traditional student pursuing a degree in Early Childhood Education. Initially pursuing a degree in journalism and broadcast, Kayla quickly tired of that work and took a semester off to think through possible next steps. Her mom mentioned that she always enjoyed working with children, thought she was good at it, and encouraged her to pursue a degree in education. Kayla returned to the university as an Early Childhood Education (ECE) major and knew immediately that she had found her "academic home." In Kayla's own words, "I really fell in love with the program here." Two semesters into her ECE program, Kayla moved to California and taught preschool for a year and a half. She enjoyed that work but returned to the university to complete her degree. Kayla completed her degree in Fall 2017. Kayla was relaxed, confident, and engaged throughout the entire interview.

Amber, Participant 13—is a bilingual (Spanish-English), traditional student pursuing a degree in early childhood education with an additional certification as a bilingual teacher. We conducted the interview in English per Amber's choice. Amber came to America as a five-year old from El Salvador and entered American schooling in five-year old kindergarten. Amber shared:

> Kindergarten was a blur. I don't remember anything. We moved here and I did not know any English at all. We lived with my Grandpa when we first moved here and I was in a bilingual classroom for Kindergarten and Grade One. We moved to a community when I started second grade with no bilingual services *AND* there was only one ESL teacher for the whole district and I saw her only once a week. I did not know how to read totally in English. There was not much diversity in my old school. I was new to them and they did not really know how to help me. It was crazy! That is why I want to be a bilingual teacher.

Amber has been teaching since she was a little girl. Her stuffed animals were her first students and she kept them very busy. Though she considered being a judge or a lawyer, it was the experience of teaching Sunday School that helped her decide she really wanted to teach. Amber stated, “While teaching Sunday School, I saw how one person could really mold little people and how other people can really discredit young kids. They are smart!” Amber was actively engaged in all aspects of the interview and regularly connected her comments back to her experiences as an early elementary student. Amber’s commitment to her future students’ surfaced throughout the interview and the following is a solid example. She explained:

> We can look at any of this work in math or any other content area and say that the kids learn differently but, when you're adding a whole different language and culture, it's a lot different and I want to be sure that I understand, or work to understand, how a child might see something. I felt like my teachers did the surface of what they could have done and I want to be one of those teachers who dig deeper to help my students. I’ve been in

> that situation where there were not the resources for me. There wasn't a need for it at the time because there was just me but that should never be an excuse to not start something or do something to help.

Amber student teaches Fall 2017.

Justine, Participant 14—is a white, nontraditional student pursuing a degree in early childhood education. She began her university studies as a Russian language major with the goal of becoming a Russian translator. Not pleased with the limited amount of opportunity to speak Russian, Justine declared herself a History major. Though she loved the learning, she was not seeing a direct career path as a history major. As a mother of three she felt that teaching would give her "a nice second income and I could be home with my kids all summer." After recently experiencing a divorce she was worried about the limited financial opportunities of a teaching position but will finish her ECE degree, regardless. She was completing her student teaching in a Grade One classroom at the time of the interview. She was actively engaged in the interview though she was very self-critical of what she could or could not remember from the mathematics methods class.

Marie, Participant 15—is a white, traditional student pursuing a degree in early childhood education. Marie has always seen herself as a teacher. When she was in elementary school she would collect extra worksheets from her teachers and take them home to "play school." She has served as a nanny for many families and at the time of the interview was a nanny for twin three-year old boys whom she frequently referenced. In addition to her work as a nanny, Marie organized and ran a summer camp for twenty-five elementary-aged children in her home neighborhood for many years. Eventually she sees herself running her own daycare center as her interest lies with "the really 'little littles.'"

Marie was very engaged though uncharacteristically nervous during the interview. In contrast, during class Marie was consistently relaxed and outgoing. Marie student teaches in Spring 2018.

Overall the interviews were conducted in a relaxed yet professional manner. The responses were insightful and the participants were willing and eager and to share their knowledge with me. I rarely see my students after the conclusion of each semester though as they move into student teaching they reach out via email or phone as they plan their first lessons. This study gave me the opportunity to dig a bit deeper into each participants' motivation for teaching and their ability and willingness to share their developing understandings and beliefs on effective mathematics teaching and learning for young children. I thoroughly enjoyed the opportunity to reconnect with each one. I was encouraged by their stories and inspired by their commitment to their future students. As we had a previously established relationship as teacher and student, I was wondering how our relationship might change as I approached the interview as a peer-to-peer interaction. I do believe that our relationship resulted in my participants opening up and sharing detailed responses to my structured interview questions.

Overview of Themes

Through analysis of the interview data five themes emerged. The first three themes and related sub-themes are organized under subject matter knowledge (Ball et al., 2008) and address the understandings prospective teachers had regarding the developmental nature of children's mathematics abilities and subitizing. The major themes for subject matter knowledge include (1) Demonstrates an understanding of subitizing; (2) Recognizes and validates the importance of subitizing for young children; and, (3) Articulates learning trajectory progression through dot arrangements.

The final two themes and related sub-themes, organized under pedagogical content knowledge (Ball et al., 2008), illuminate participants' instructional decisions as they engaged their understanding of the learning trajectory to advance children's subitizing skill in relation to the subitizing trajectory. The major themes for pedagogical content knowledge include: (1) Understands the developmental nature of children's subitizing skill and ability and, (2) Centers instructional decisions on children's thinking. Despite differences among participants, common perspectives emerged regarding subitizing and the subitizing learning trajectory, young children's ability to subitize, the role subitizing plays in children's understanding of number, and the teacher's role in developing that understanding.

Subject Matter Knowledge Findings

Professionally oriented subject matter knowledge (SMK) in mathematics is at the heart of effective teaching. Ball et al. (2008) suggested, "Teaching may require a specialized form of pure subject matter knowledge—'pure' because it is not mixed with knowledge of students or pedagogy" (p. 396). Though SMK functions independent of knowledge of students and knowledge of teaching, it is knowledge needed specifically for the work of teaching. Instruction guided by learning trajectories includes teachers' knowledge of the logic of the discipline, specifically one's understanding of the mathematical goal and developmental stages of the learning trajectory. In this study, participants demonstrated SMK as they defined the big idea of subitizing, articulated why subitizing is important to intentionally develop in young children, and demonstrated understanding of the progression of the subitizing learning trajectory. The three themes for subject matter knowledge inform the first attendant research question: What understandings do early childhood prospective teachers have regarding the subitizing trajectory?

Theme 1: Demonstrates an Understanding of Subitizing.

> *I would be happy if one my students, who are three and four, said "three" if they saw Pattern A (see Figure 4.1) because there are three dots. I would also be happy because the response would be instantaneous. That helps me believe that they are not counting, that they are able to subitize. I think that's really important especially with a lower number like three. It's also a simple dot pattern, which should help them subitize. I guess if they are still taking some time and counting by ones that would be okay, but eventually I would want them to recognize that there are three.*
>
> *Sasha, Special Education Pre-Intern*

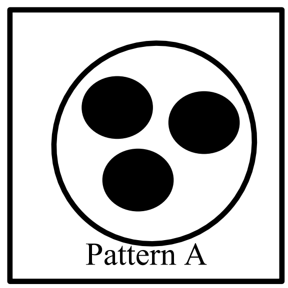


Figure 4.1. Sasha refers to Pattern A from Set 1 to demonstrate how a child subitizes the pattern and states "I see three" without counting.

Study participants demonstrated their knowledge of early subitizing as an automatic perceptual process (Kaufman et al., 1949) used to identify the numerosity of small collections of objects up to and around four and that conceptual subitizing is needed for larger quantities or for irregular or nonstandard arrangements of dots or objects (Clements, 1999).

For this study, the mathematical goal is subitizing. Study participants successfully identified the learning goal when asked to whether not they would use the Set 1 (See Figure 4.2) with their hypothetical class of five-year old kindergarten students.

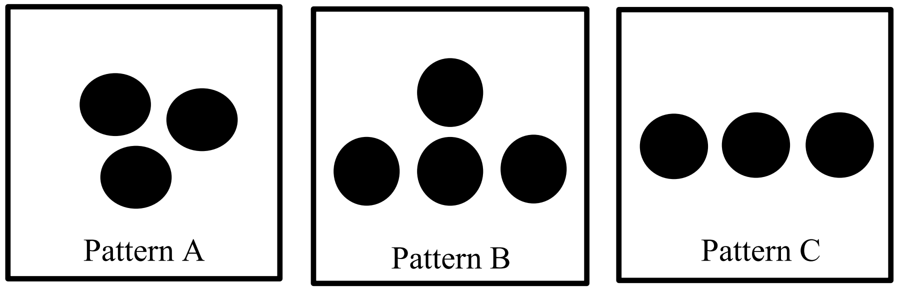


Figure 4.2. The three dot patterns in Set 1.

Crystal clearly articulated the goal of using dot patterns (see Figure 4.3) with young children and juxtaposed it with counting by ones when she shared:

> From what I remember, if I remember correctly, this is subitizing and subitizing begins with the idea of twoness and threeness and it's the idea of seeing numbers instantly. I could use these patterns to see if the children could subitize small numbers. Just name "how many" without counting. I think that they might count by ones first because I think that would be the level that they're at. Counting, like one-to-one correspondence. They might subitize after they have counted first or they might just subitize and say "three."

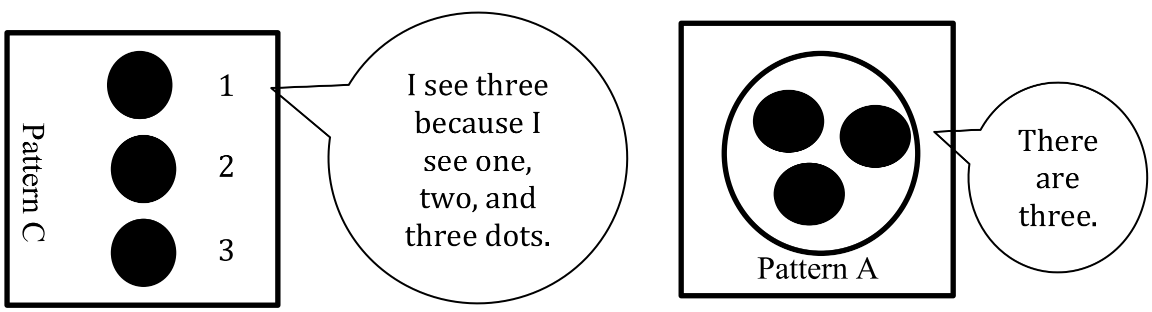


Figure 4.3. Crystal demonstrates the difference between counting and subitizing.

In step with Crystal, Sasha's response highlights the instantaneous recognition of "how many" as central to subitizing abilities. We see similar knowledge reiterated by Cecilia:

> I think that this is where we talk about perceptual and conceptual subitizing, correct? Oh yeah, there is that one where you just kind of see it instantly, and the other kind where you would see the two parts and then you take the parts and put them together again to

get the total. And I think that if you see it right away it's going to be a smaller amount, correct? And for the other one, like this pattern of four (Set 1, Pattern B), they would have to see it in groups. You would have to see them in groups and then put it together again. This other one with the three you would just see it in a glance. I don't remember which is which, sorry.

Each of the fifteen participants identified the mathematical goal of using dot patterns with young children as *quickly seeing how many* though eight struggled to remember the term subitizing. An example would be Kayla. After talking through how she would hope children would name the number of dots for the patterns in Set 1, I asked what she would learn about her young learners if they could name those quantities quickly without counting. Kayla got very nervous and stammered a bit as she replied, "Well it tells me that…ummm…just that…I don't know the word. I just forgot that. I think that it would tell me that they can see numbers in different forms or in different ways." Though she clearly identified perceptual subitizing when she shared, "I think it's good to focus on the whole number first and then help them kind of see the groups but I would focus on the whole quantity first." She clearly could not recall the mathematical term *subitize.* In a related example, Marie shared:

> "With these dot patterns I guess we are after one-to-one correspondence, or…wait…If they just see the three or the four without counting, then they are subsi, subsidizing, no wait...subitizing?? Is that the correct word? Is this conceptual or perceptual subitizing? Whichever one is the one that you see it automatically."

Karaleen talked about subitizing as the relationship between subitizing and cardinality as she articulated her expectations for her five-year old students as they worked with dot patterns. She shared:

I want them [the children] to look at it like a group, and name the total like cardinality but not count…to be able to just look at it and say, "That's three, or, that's four." Kind of like memory but not really.

In a similar vein, Sasha offered a clear example of conceptual subitizing by naming the many ways she saw Pattern B of Set 1, "I see it as a three and one. But you could see a one and a three this way, or a two and a one and a one. When you put each on back together, it is four." Figure 4.4 illustrates Sasha's understanding of conceptual subitizing. Sasha is unable to recall the type of subitizing she employs to find the total dots on the card though her example clearly represents conceptual subitizing.

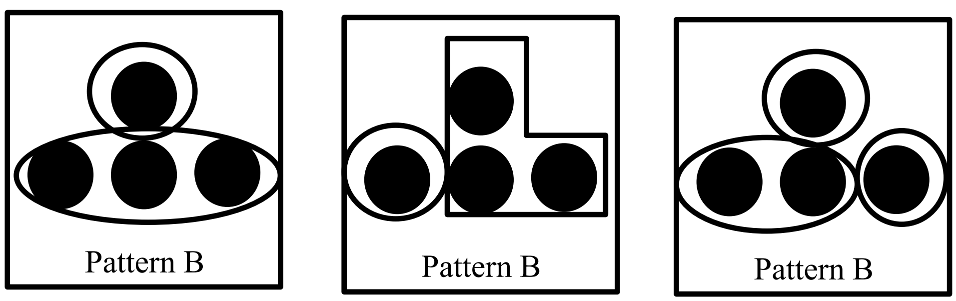


Figure 4.4. Sasha shares different ways she conceptually subitizes Pattern B.

Mandisa responded in similar fashion when asked how she thought children might respond to Pattern A of Set 1 (see Figure 4.5). Mandisa explained:

So I'm hoping that they would get to the point where they would be able to tell me very quickly "I see two over here and I see one more and I know that it's a total of three." I would probably at this stage in the game accept a little bit of 'One…Two…Three.' Like counting the dots individually."

When asked what mathematical skill the children demonstrate if they tell her "it's a total of three," Mandisa carefully stated, "…seeing the two quantities and putting them back together again…” It is clear that she understands the goal is subitizing though she is unable to name either type.

In a related example, when asked how children might respond to Pattern C (Set 1) Amalie states, “I would hope that they be familiar with that dot pattern. I hope that they would just be able to spit the number out without counting.” This is a clear example of perceptual subitizing.

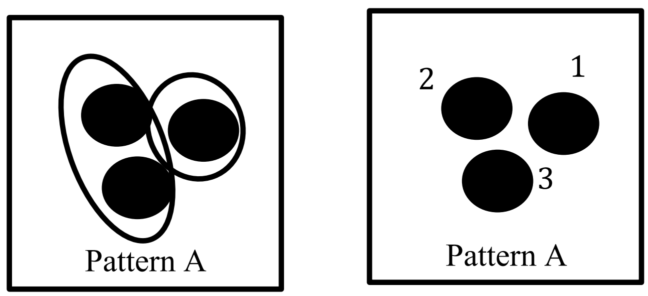


Figure 4.5. Mandisa’s compares counting by ones to conceptual subitizing.

In a related example, when asked how children might respond to Pattern C (Set 1) Amalie states, “I would hope that they be familiar with that dot pattern. I hope that they would just be able to spit the number out without counting.” This is a clear example of perceptual subitizing.

Standard dot patterns, like those on the face of the die were quickly singled out as cues to perceptual subitizing. Flora carefully and independently sequenced the three dot patterns in Set 1 and then shared how children would likely be able to name how many right way in the first pattern of three as shown below in Figure 4.6.

> I guess I would pick this one first. I think this would be a fast one for them to see. The three dots are in a straight line and I think they would just recognized them as three. They look like the three on the dice. If they would be more spread out I think that they would want to count them but right now they're very close together and they're in a line. They would just see three. I think that in a line it's easier to see and I can just see the three as

well. I think that when they are in a bigger group and maybe more spread out the children may need to count them by ones until they are ready to find groups and combine them.

Figure 4.6. Flora orders Set 1 cards and signals out "three" as a quantity children easily subitize.

Flora's straightforward comment was significant for two reasons. First, she carefully attended to the idea that if one is to subitize a quantity the regularity of the spacing matters (Saltzman & Garner, 1948). "If they would be more spread out I think that they [the children] would want to count them." Second, she identified the importance of the size of the set or numerosity (Taves, 1941), stating that "when they are in a bigger group… the children may need to count them by ones until they are ready to find groups and combine them or with smaller quantity they can just name the total." Taves suggested that smaller quantities from one to six are named by stating "how many" without counting and larger quantities are named by counting.

Each participant suggested that children would quickly name "how many" and acknowledged that the spatial arrangement of the dots influences how difficult the patterns were to subitize. Study participants commonly expressed this idea when they related the size of a set with a shape or common image. For example, Amber suggested she would start with Pattern C (Set 1). She explains:

> I would start here because it looks like a stop and go sign. They [the children] might see that and that would help them (Pauses) Yeah…because you know kids will know green, yellow, red? (She points to the three dots from top to bottom on Pattern C as she says the

colors.) (See Figure 4.7.) They might just see it as three because there are three, like the stoplight.

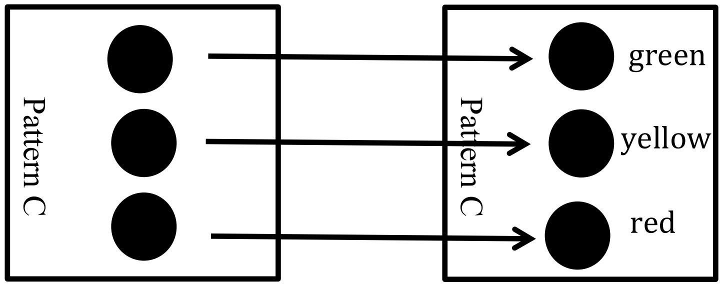


Figure 4.7. Pattern C Set 1 shown as a stoplight as described by Amber.

Kayla suggested that children might very quickly see three in Pattern C (Set 1) (see Figure 4.8) after a quick look by noticing that it forms a straight line.

> We could also do the flashing of the dot patterns and I could ask them [the children] “How many dots did you see? How did you see them?” So they would tell me if I show them the one with the three in a line (she traces the three dots from top to bottom with her fingers) they could tell me “I see three ... Also, after I flash the dot pattern or let them look at it I can ask them “How do you know that it's three?” They might say “Well, I see it in a line or it looks like on a die. One…two…three. I see that's going down and I see that that's three.” So they would just see it.

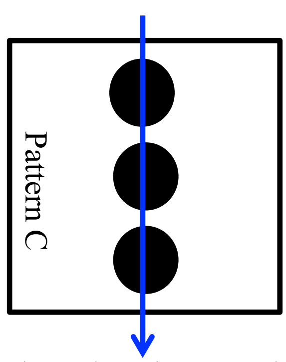


Figure 4.8. Kayla traces the three dots downward in a line as she state “three.”

Finally, Amber and Kayla thought that the children might recognize Pattern A (Set 1) (see Figure 4.9) as three because the pattern might spark the children to see a triangle. As Kayla noticed (Touching Pattern A, Set 1), she explained:

> They [the children] could also tell me 'I saw the triangle and I know that that makes three.' As they talk about three, the dots in Pattern A, I would probably want to hear that they say 3…because it looks like a triangle.

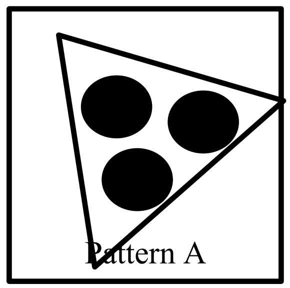


Figure 4.9. Kayla shows how children might see Pattern A as a triangle.

To center one's teaching on student thinking as described by learning trajectory research requires understanding of the mathematical goal of the learning trajectory. Study participants were thoughtful in sharing their developing understanding of subitizing. Articulating why subitizing is critical to young children's mathematical development is an essential aspect of teaching with learning trajectories and is the next theme.

Theme 2: Recognizes and Validates the Importance of Subitizing for Young Children

> *And when we first started to use the dot patterns in class I kept thinking to myself, "Why is this important? Seriously, they're just dots!" Now I think, "Wait a minute this is REALLY important and these dot patterns lay the foundation for so much!" I mean the kids need to SEE that three is three and that three can be expressed in so many different ways. I see now that being able to subitize is going to help them learn what numbers mean and then to add and subtract with understanding.*
>
> *Jaeden, Early Childhood Special Education Major*

The second theme to emerge addressed the importance of subitizing. Jaeden recognizes the important space subitizing occupies in the landscape of young children's mathematical lives.

The connecting thread of this theme is the belief by participants that subitizing was a mathematically significant process and can be utilized as a launch for young children's understanding of number and quantity. Three sub-themes appeared in the data analysis relating to this theme: (1) subitizing helps young children understand quantity; (2) perceptual subitizing supports cardinality; and (3) conceptual subitizing lays the foundation for early addition and subtraction.

Sub-theme 1: Subitizing helps young children understand number as quantity. Several study participants identified subitizing as key to helping young children understand number as quantity. The ability to hold a mental image of a small quantity in one's minds' eye and quantify the total was viewed as mathematically significant for young children. To that end, the idea of seeing quantity was a common thread for study participants. For example, Karina, a special education intern working with three-, four-, and five-year olds with special needs, shared:

> When I am working with dot patterns with my students I know my focus is quantity. I want them to see a dot pattern of three in their mind, like on dice, because when they hear the word 'three' I think they will hold onto that what three means.

Karaleen highlighted the importance of quantity developed through subitizing dot patterns as she reflected on her class of five-year old kindergarten students. She commented,

> All these patterns…help those kids see quantities, not just numbers. I don't know...for some reason, when my kids see these (dot patterns) it's easier for them to visualize the quantity versus looking at just the symbol. That is something that I will go back to regularly.

Similarly, Mandisa, teaching three-year olds in a Head Start program stated, "Working with dot patterns is about what quantity is. That is very important." In agreement, Marisol, reflecting on the importance of subitizing for her kindergarten students shared, “They need to be able to subitize small quantities, and right now they are not seeing quantity. I understand now that they cannot move forward with addition until they have a much better understanding of small quantity.”

Non-teaching study participants also identified subitizing as a support to understanding quantity. Amber reflected, “I guess this [using dot patterns] is about naming how many, so quantity.” When asked to justify the use of dot patterns with small children Jaeden emphasized, “…that's exactly what I'm talking about. I'm talking about quantity. And subitizing as well.” Recognizing the difference between naming a numeral and naming a quantity Flora added, “They don't necessarily need to know the number but it's important that they see the quantity. I also want them to know that the quantity represents the number and that the number represents the quantity.”

Immediately recognizing or labeling a dot pattern (e.g., equating a rectangular array of four as seen on a die as “four”) is known as verbal number recognition (Baroody et al., 2006) and is considered key to a conceptual understanding of number. Prompting verbal number recognition through Quick Images was common for all participants. This is not a surprise as Quick Images was a central activity in the mathematics methods class. During Quick Images children are given a three to five second look at an arrangement of dots. This amount of time is enough to allow the children see perceive a small quantity of dots but not long enough to permit counting. After a second brief look to verify their perception children are asked, “How many dots did you see?” and “How did you see them?” Jaeden used this activity to address how subitizing builds number sense and connections between numbers. She explained:

> As the children respond to these dot patterns, I am listening for whether or not they are able to subitize. It also lets me know what they understand about quantity. It also gives me a nice picture into what they understand about number sense. I think that that's a lot of it, that idea of number sense. It's the relationship between the different numbers that's important. One thing I would watch for is when the children see the pattern do they raise up their finger and just go one, two, three really fast? Then I know they're counting by ones. I know they're not subitizing and they need a little bit more work.

Crystal suggested, "…if they say four after I flash the pattern then I know that they understand the quantity. The total. And I would also want them to understand that there are different ways to make four."

In addition to prompting verbal recognition of number, Flora, who is pursuing bilingual certification, offered Quick Images as a way to surface quantity and then connect that quantity to its symbolic representation. Her comments lend support to the idea that counting competencies are interdependent. Her intentional focus on helping children develop a conceptual understanding of number was clear as she ensured children made critical connections between representations. She shared:

> I think what's nice about the dot patterns is that you can use them in a lot of ways to help the children recognize quantity. For example if I show them a pattern with the three dots and I asked them "Can you tell me how many you see?" Their answer will tell me if they understand quantity. They don't necessarily need to know the number (symbol) but it's important that they see the quantity and name it.

In support of Flora's intentional move to include symbols in Quick Images, Baroody et al. (2006) suggest that seeing different examples of a quantity labeled with a numeral offers children the needed experience to recognize numerals. Amber, also pursuing bilingual certification, suggested that the ability to work within three different representations would have made a significant difference for her as a K5 English Language Learner (ELL). She adds:

> As an ELL we never see enough pictures. I think dot patterns help us, and help us remember numbers better than words. So as a language learner you could write the symbol "2", but I would have no idea how to read that. So if I see the symbol "2" and I see a dot pattern with two dots and I hear the word "two" there are different representations for that same idea so then I get it.

Kayla suggested that by linking number words to dots patterns young children will not simply memorize how numbers look and how to say them they will understand that numbers have meaning. The count words will carry meaning and each time they say a count word, they are stating a quantity. She expanded on that idea below:

> I guess I want the kids to understand that numbers mean something and they're not just a symbol or a word. Yes I would want them to see that the number three written three actually stands for three objects. Without these dot patterns, I think that kids will just memorize the words for counting like "one, two, three." They might not understand that those words actually mean something. Like they stand for an amount, like the number one stands for the quantity of one. I don't want them to repeat what they hear. I want them to understand it. So the focus of this work is helping the kids understand the idea of quantity.

Quantity surfaced as a key idea for each of the fifteen participants. In fact, throughout the course of each interview participants regularly articulated the desire and fortitude to support young children's conceptual development of quantity through subitizing.

Sub-theme 2: Subitizing engages children's reasoning about cardinality. Karaleen recognized the relationship between subitizing and cardinality and the usefulness of that relationship for her students when she shared,

> I want them [the children] to look at it [a dot pattern of three] like a group, and name the total. Like cardinality but not count…to be able to just look at it and say 'That's three, or, that's four.' Kind of like memory but not really."

A common theme among the participants seeking bilingual certification was the need to ensure that the children know the number names in the target language in order to connect perceptual subitizing to cardinality. For example, Marisol, who was teaching 5K at the time of the study, emphasized:

> I started using the dot patterns because I think the majority of the group is very low. They are really struggling with counting and are unable to tell me "how many" even with these smaller quantities. I found out they did not know the number names.

Flora added that, "I would actually start with "How many dots do you see? I want them comfortable with those number names." Amber mentioned she would use the dot patterns to "see if they have…if they can have cardinality."

Starting with children's strengths resonated with all participants. Mandisa was one of many participants that suggested that some children may need to count a dot pattern first, before perceptually subitizing small quantities. Mandisa provided the following insights:

> The patterns help my students see a visual of the number. They can count the dots and if they could not just recognize the amount they can use their fingers to count each dot. Like, they can look at the dots and put a finger for each dot they see. That is what some

> of my students are doing now. Some count by ones to tell me how many dots and some subitize. Magically we usually come up with same number!

Sasha, an Early Childhood Special Education Pre-Intern noted that after showing her students a dot pattern she would hope they would subitize small quantities. She explained:

> I would be happy if they responded with "three" because the response would be instantaneous. That helps me believe that they are not counting, that they are subitizing. I think that is really important, especially with a lower number like three. It's also a simple dot pattern. I guess if they are still taking some time and counting by ones I would still be OK with that as well.

Amber added support for letting children count before an intentional push on subitizing. She suggests, "Like maybe they count them. They need to know the number names so they have something to say when we subitize." Mandisa, Sasha, and Amber all agree that having number names accessible as a needed prerequisite to the concurrent development of subitizing and cardinality.

Acknowledging the hard work that goes into connecting subitizing and cardinality Marisol offered a strategy she was currently implement with her bilingual five-year old kindergartners. Worried that her children were counting by ones at the midpoint of their 5K experience her goal was to support "seeing groups" as opposed to counting by ones.

> I give each child a small white board and a marker. I show them a pattern, like this one of three, and they draw the pattern the way they see it. After they draw the pattern they could tell me "I see two and one more, and then I would want them to make a circle around the two dots to show me where those are and then one leftover dot. If they say, "I see three" then they circle the full dot pattern. After that I want them to **tell** me how many

they see. The drawing helps them see the dot pattern as a group. Right now, some kids are doing that but when it is time to combine the smaller quantities some go back and count by ones to find the total. (She tilts her head, shrugs her shoulders, and offers a worried smile.)

Lastly, many study participants viewed perceptual subitizing as a way to support cardinality and conservation as they discussed the many ways children might see one particular dot pattern and recognize that the cardinality of the set has not changed. Cyrstal suggests this as important, "Because if they say four then I know that they understand the quantity. The total. And I would also want them to understand that there are different ways to make four." As example Jaeden offered the following discussion linked to two different patterns of three. (See Figure 4.10.) Jaeden explained:

> Like in Pattern C they would just see three and name it "three" and in Pattern A they might see three as a triangle and name it "three" or they might see it as two and one and put it together to get three. As they see "three" in many ways they also get to make a relationship with the number three…and the dots are not always in the same little pattern. I think that's really important for them, that they see lots of different ways to see these numbers.

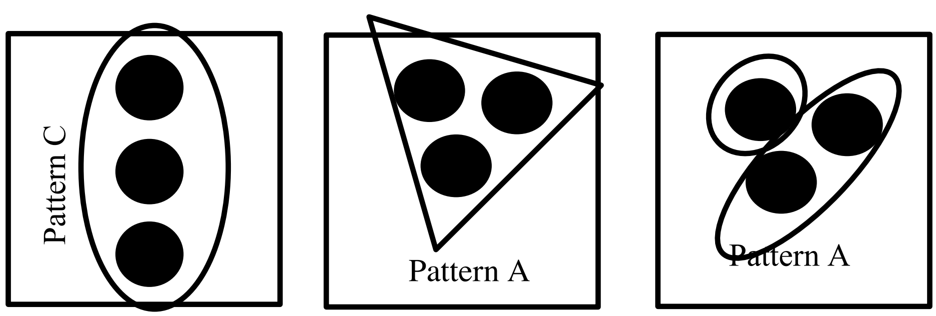


Figure 4.10. Jaeden offers an example of three different ways children may subitize three.

Mandisa highlighted a benefit of subitizing as the flexible perspective of quantity she saw in her young learners. She shares:

> What I'm looking for is that they recognize the same quantity in many ways for example on dot patterns, on dice, on number cards, on the five frame. No matter how we look at it five is five. Over time, they would not need to count. They would have so much experience that they would know that all these different ways are just five.

Study participants regularly cited the opportunity subitizing activities offered as a way to advance young children's ability to apply the cardinal principle. Moving children from counting by ones to seeing quantities in groups was acknowledged as a key understanding of this important goal.

Sub-theme 3: Subitizing lays the foundation for early addition and subtraction. Thirteen out of fifteen study participant's identified children's abilities to perceptually subitize as beneficial to what they viewed as more formal instruction on addition and subtraction. Nine of the thirteen participants provided examples for their reasoning. The remaining four simply shared that subitizing would help them "learn how to add and subtract later on" (Justine).

A closer examination of the interview data reveals the depth of participants' knowledge concerning the role subitizing plays in students' understanding of addition and subtraction. As example, consider how Karina frames the role of subitizing as she considers the work her three- and four-year old special education students will face once they are a little older. She explains:

> We aren't adding and subtracting yet but it [being able to subitize] would just give them a leg up because they won't have to sit there and count. And usually when they count they forget what they've counted because they can't write it down, so even if they count

> for one, two, three, four, they can't write four, so then they forget. I think that using these patterns will help us visually into addition when they're ready for it.

Karina sees the foundation subitizing lays for her young learners developing understanding of number and quantity and serving as a visual entry point into addition concepts. In accordance, Kayla links the idea of quantity to addition as she highlights the various number relationships children develop as they conceptually subitize. According to Kayla:

> …subitizing will help them when they move into addition and subtraction with meaning. Like being able to see a number and know what it stands for, like the number four means four dots. So when they see the number four I want them to be able to picture four dots in their head and then if they're going to add three to the four then I want them to be able to see three dots in their head and then add those four dots and three dots together to get seven. Or even see three dots and three dots to get six and the one more to get seven.

Kayla's many examples highlight how dot patterns, when used with intention, provide young children a meaningful entry point into beginning addition. According to Bowman, Donovan and Burns (2000) learning and development will most likely occur when new experiences are built on what a child already knows and is able to do. Kayla articulates this point quite succinctly.

In keeping with a focus on conceptual understanding for addition concepts, Mandisa suggests seeing groups and combining groups should precede the introduction of symbols. Mandisa believes that,

> Seeing the dots and combining the dots is not as abstract as just seeing the symbol for three and the symbol for one and adding them. So they should just be able to see

the quantities really quickly and know how much they mean and combine them. So visually seeing addition by seeing the dot patterns. (See Figure 4.11.)

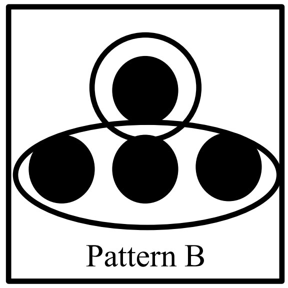


Figure 4.11. Seeing four as three and one. Mandisa's reference to Pattern B, Set 1 where the quantities three and one combine to make four.

Three participants addressed the novelty of conceptually subitizing dot patterns and how it supported their own understanding of addition. They shared that it was helpful for them to *see* the operation of addition. This point became very important to Amber, who immigrated to America from El Salvador as a five-year old and was taught in an exclusively English environment with extremely limited ESL support. She commented,

> I have never seen dot patterns before and now that I know, and I put myself in the shoes of the kids, and how they would see them, ummmm....just like seeing things in groups, because I feel like we just move so fast like right to addition... 2+2! Being able to see it and then breaking it down has been really helpful for me.

Marie, equally intrigued by the use of dot patterns to support her developing flexibility with quantity, provided the following thoughtful comment about the role conceptual subitizing played in her ability to make equivalent expressions for addition basic facts. She goes on to say,

> ...eventually it will really help with addition and subtraction but like even moving beyond to like ten. Ten is a big number for kids. I know it was a big number for me...so I like when we did things like 9 + 5 is the same as 10 + 4. Putting numbers together like that and...I mean...that is how I see this going. That is excelled very much from what we are

doing now...but moving up to ten. Then seeing like "Oh, this! (pointing at Pattern C from Set 2 shown in Figure 4.12). If this is doubled then five and five is ten. That is so cool to me."

Kayla, Karolyn, and Sasha each articulated the connectedness of children's learning and how part-whole understanding provides a natural bridge for thoughtfully moving from counting to addition and subtraction. They each addressed how they see part-whole understanding as nested between early counting and addition and subtraction and how important it is to intentionally develop.

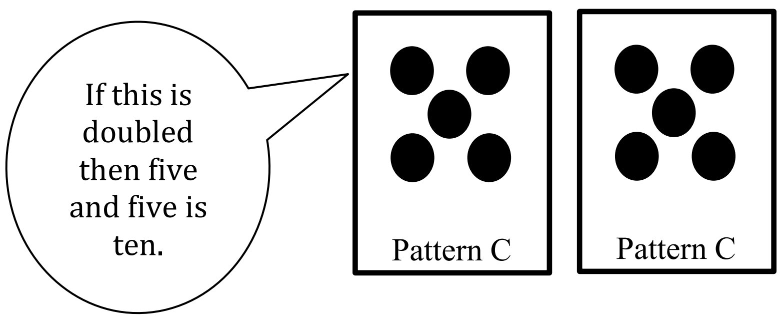


Figure 4.12. Two sets of five. Marie uses Pattern C to demonstrate that two five patterns show ten.

Sasha addressed part-whole knowledge as laying the foundation for informal addition and subtraction. She shared, "I guess [when children are subitizing] we really wouldn't even be adding at this point. We would just be breaking numbers apart. But I guess in a way when we put it back together again we are adding." Karolyn added:

> In the beginning there I introduced one, two, three, four, five, six to my kindergartners...counting. It's also important that they start to think about how they can take a number and decompose that number into two parts or find the groups that make up that number. It's all these different factors that go into their

> stages of development. We think that we should go right to adding and subtracting but from your class we learned to…that we don't.

Kayla frames the progression suggested by Karolyn and Sasha. She states, "I guess I think about the progression that we talked about in class and how we want to move children from counting to part-whole to addition and subtraction."

Finally, one particularly interesting comment that illustrates how participants articulated the relationship between addition and subtraction and conceptual subitizing came from Marisol. Marisol's interview occurred just one month into her position as the 5K teacher at a language immersion program. Addition and subtraction was the suggested unit for her kindergarten students. After learning about the bridging powers of conceptual subitizing from early counting to addition she tried some dot patterns with her class the next day. To her surprise, the children were unable to perceptually nor conceptually subitize, preventing them from managing quantities with meaning. After sharing this experience with me, she commented,

> I see the subitizing work as the key to future growth. When I first learned about this in class, I came back and I did this with my whole group of students. They have never had this experience before and I knew that they needed it. I understand now that they cannot move forward with addition until they have a much better understanding of small quantity. I have stopped doing addition and subtraction until they are able to see the groups and combine them without counting.

Theme 3: Articulates Learning Trajectory Progression through Dot Arrangements

> *I would probably start with this one, Pattern C (see Figure 4.13), because I think the children would look at it and say, "I think that I know this one, I know that it's five, because it looks like the pattern that you see on the dice." Then I would probably select Pattern A. The kids may see the four as a square and then the one on the top, or they may see Pattern A as a five because it looks like Pattern C but the dot is not in*

the middle, it is on top. I see them [the two patterns] very connected because they build from each other. The idea that patterns build from each other is important.

--Kayla, ECE Regular Education Major

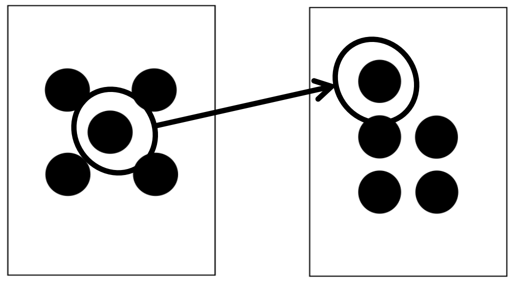


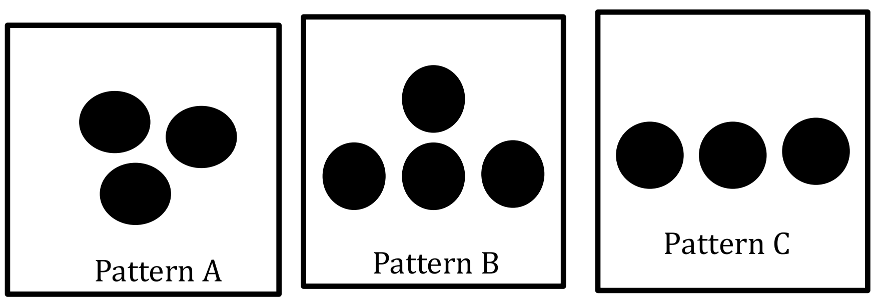
Figure 4.13. Seeing five differently. This figure displays Kayla's connection between Pattern A and Pattern C.

In the opening quote, Kayla hypothesizes that Pattern C in Set 2 is an appropriate pattern to begin with as it is one that young children may recognize from a die. Her suggestion is to move from Pattern C to Pattern A as the children might see five but notice that the inside pip is now on top of an arrangement of four, no longer in the middle. This seemingly simple observation is significant in that Kayla acknowledges that moving one dot shifts how one might perceive the quantity of five. Kayla is thoughtful in ordering the dot pattern cards and does so in a way that supports children's transition from perceptual to conceptual subitizing. She anchors her decisions on children's ability to reason about quantity. Kayla's developing subject matter knowledge ensures her decisions are both intentional and developmentally appropriate.

Two central components of this study are the sets of dot arrangement cards intended to prompt subitizing in young children. (See Figure 4.14.) Set 1patterns are composed of three and four dots and are meant to prompt perceptual subitizing. Each Set 2 card contains five dots in different arrangements. Some, such as Pattern C, can be subitized perceptually and the rest are subitized conceptually. As part of the interview protocol participants were first asked to order Set 2 patterns as they would use them with five-year olds, and second, to explain why they created that particular order. The task of organizing six different dot arrangements of five intentionally investigated study participants' understanding of the levels of the subitizing trajectory.

A teacher's skill at increasing their children's ability to subitize is closely aligned to their understanding of the detail and nuances inherent to the subitizing trajectory. This knowledge, when coupled with their responsiveness to their children's thinking, creates learning environments that are mathematically powerful and productive. Recognizing when children are ready for a more sophisticated or nonstandard dot arrangement, or a larger quantity of dots, is a critical pedagogical decision that may, at first glance, seem straightforward.

Set 1 Dot Patterns



Set 2 Dot Patterns

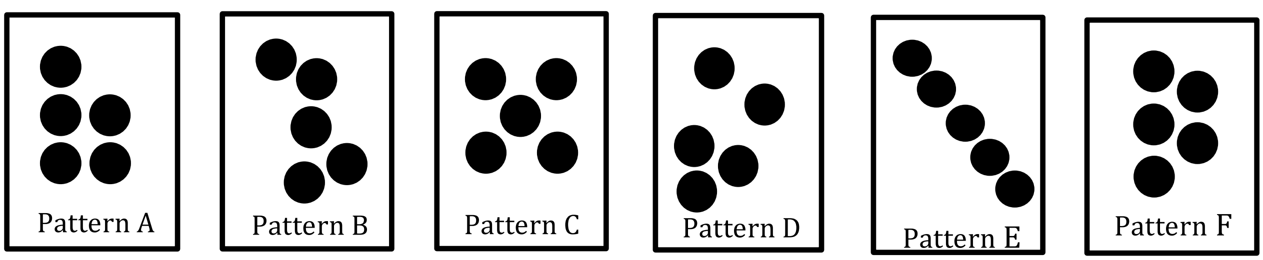


Figure 4.14. Set 2 dot patterns used during this study's interview to elicit subitizing.

Three sub-themes provide evidence of study participants' subject matter knowledge as it relates to an understanding of subitizing and the subitizing learning trajectory. They include participants' rationale for (a) ordering the Set 2 dot patterns, (b) managing Pattern E, and (c) creating and placing their own dot pattern in the sequence.

Sub-theme 1: Order Matters. This sub-theme examines both the order established by study participants and the explanations provided for the order. Both are discussed in relation to the subitizing learning trajectory and what this reveals about participants' subject matter knowledge.

What order did participants select? What does the order reveal about their SMK? I begin this section with a discussion of a typical participant developed sequence for Set 2 dot patterns. Figure 4.15 displays the order Marie established for the six dot arrangements. Her order represents a fairly typical order established by study participants. The order is: Pattern C, Pattern A, Pattern F, Pattern D, and Pattern B. Pattern E was intentionally separated from the group as displayed. According to Clements and Sarama (2014) Pattern C is typically perceptually subitized due its familiar arrangement. Pattern A can be perceived as Pattern C with the center dot shifted, and conceptually subitized as four and one more, or even six with one missing. Pattern F, Pattern D, and Pattern B prompt conceptual subitizing as the arrangements increase difficulty. It was common for participants to view Pattern F, Pattern D, and Pattern B as smaller groups that would need to be combined to find the total. Pattern E rounded out Marie's sequence. Pattern E was the focus of much discussion by study participants and is taken up in the next own sub-theme.

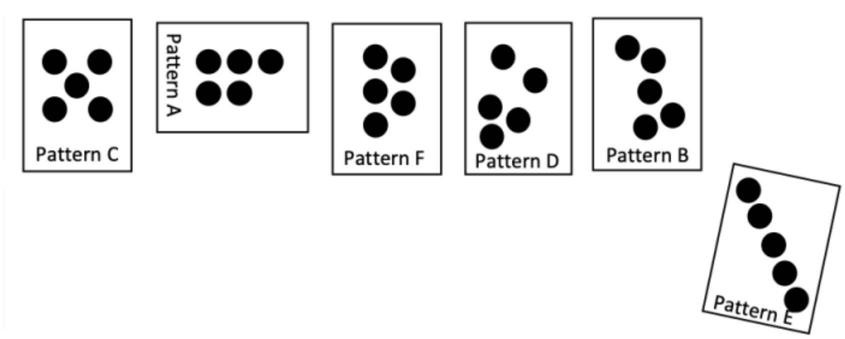


Figure 4.15. This sequence, established by Marie, is typical of how study participants ordered the six dot pattern cards.

Marie's recognizes the shift in rigor as she talks through the sequence. She explains:

> I started with Pattern C. I know that is five because of the dice pattern. I think they [the children] would see that as five. Then to move from Pattern C to Pattern A I would see the 4 [on Pattern A] and then one more. There is more thinking and such need for Pattern A because there are two groups and you need to add them to get the five. The same thinking fits for Pattern F. There are two groups, three and two. It is not how you might see it on dice so a bit more challenging. Then you get over to Pattern B were the dots are random and Pattern D too but not as much. I guess B and D could go in either order. So you are moving from one group (Pattern C), finding two groups (Pattern A and Pattern F), to maybe finding three groups (Pattern B and Pattern D). Pattern E, I'm not so sure. I think you are back at one-to-one correspondence for that one. Having a pattern like that all in a row might jump start the one-to-one counting but be very helpful in seeing groups.

Marie's order matches the level of difficulty as determined by Clements and Sarama (2014) with the exception of Pattern E. They categorized the six cards as follows: (a) Easy Patterns—Pattern A and Pattern C, (b) Middle Difficulty Patterns—Pattern E and Pattern F, (c) Difficult Patterns—Pattern B and Pattern D. (See Figure 4.16.).

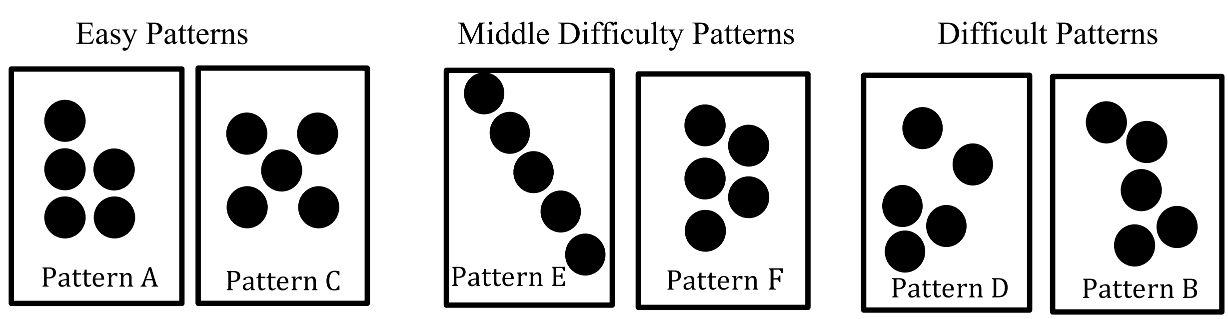


Figure 4.16. The dot pattern cards from Set 2 organized according to difficulty.

Linking the difficulty categories to the subitizing learning trajectory makes explicit the increase of expectations for student thinking . The easy arrangements are appropriate for *Level 6—Perceptual subitizer to five.* The middle difficulty arrangements are appropriate for *Level 7—Conceptual subitizer to five.* The challenging arrangements are appropriate for *Level 8—Conceptual subitizer to ten*. Therefore placing Pattern A or Pattern C in the first or second position of the sequence follows the progression of the subitizing learning trajectory. The same would be true for placing Pattern E or Pattern F in the third or fouth position and for placing Pattern B or Pattern D in the fifth or sixth position.

Table 4.2 displays the participants' accuracy in ordering the dot cards according to difficulty level. Nine participants (Group 1) correctly ordered the six cards according to difficulty categories and also ordered the patterns from easiest to difficult. This order is in line with the progression outlined by levels 6-8 of the subitizing learning trajectory. Three of these nine participants, of which Marie was one, ultimately excluded Pattern E from their sequence. As all the other cards were correctly placed, I counted these three as having the correct order. Rationale for this choice is outlined in the next sub-theme, Pattern E—Honoring Subitizing and Working Within the Progression. Three participants (Group 2) placed four of the six cards in appropriate difficulty categories. Two participants (Group 3) placed three of the six cards in appropriate difficulty cateogories. One participant (Group 4) placed two of the six cards in appropriate difficulty categories.

Table 4.2 might lead us to conclude that Group 1 participants understood the progression of the trajectory as demonstrated through the order of the dot patterns. It might be suggested that the remaining groups did not demonstrate an understanding that is in alignment with the developmental progression of the learning trajectory. The correct order of the cards is important as is participants' reasoning for the order they developed. Participants' rationale for card placement is addressed in the next section where I take up participants' reasoning for their established order.

Table 4.2. Accuracy of overall order of dot pattern cards according to difficulty category.

Group	Accuracy of Order of Set 2 Cards	Number of Participants
1	All cards in correct categories	9
2	Four cards in correct categories	3
3	Three cards in correct categories	2
4	Two cards in correct categories	1
	Total	15

A deeper look into the placement of the cards affords the opportunity to attend to the subtle shifts inherent in levels six though eight of the subitizing trajectory. Table 4.3 displays the number of participants who accurately organized the six cards according to the three discrete difficulty categories. The table helps us see the number of participants that placed both, either, or neither of the cards in the correct difficulty category.

Reviewing the Easy Patterns column, ten participants placed Pattern A and Pattern C in either the first or second position of the sequence. Four participants place either Pattern A or Pattern C in the first or second position, but not both. One participant placed neither Pattern A nor Pattern C in the first or second postion. Repeating that same reasoning for Middle Difficulty Patterns, nine participants placed Pattern E and Pattern F in the either the third or fourth position. Five placed either Pattern E or Pattern F in the third or fourth position, but not both. One participant placed neither Pattern E nor Pattern F in third or fourth position. Reviewing the Difficult Pattern column we see eleven placed Pattern B or Pattern D in the fifth or sixth position and four placed either Pattern B or Pattern D in fifth or sixth position, but not both. (Refer to Appendix H for a breakdown of how individual participants sequenced the six dot patterns.)

Table 4.3. Number of participants correctly placing cards according to difficulty category.

Patterns Categorize Correctly	Easy Patterns (A & C)	Middle Difficulty Patterns (E & F)	Difficult Patterns (B & D)
Both Patterns	10	9	11
One Pattern	4	5	4
Neither Pattern	1	1	0
Total Participants	15	15	15

Which dot patterns did participants misplace and at what frequency? Participants misplaced four patterns, Pattern A, Pattern D, Pattern E and Pattern F. Pattern A, an easy arrangement, was misplaced five times, each to middle difficulty category. Pattern D, a difficult arrangement, was misplaced four times. It was placed once in the easy category and three times in the middle difficulty category. Interestingly, the middle difficulty patterns, Pattern E and Pattern F, were misplaced a total of seven times. Pattern E and Pattern F were scattered throughout the sequence, misplaced into the easy category three times and the difficult category four times, suggesting participants had a difficult time determining where and when to use them with small children.

Successful implementation of instruction guided by early mathematics learning trajectories begins with a teacher's understanding of the mathematics as outlined in the learning trajectory. This knowledge is identified as a central component of subject matter knowledge (Sztajn et al., 2012). Participants' demonstrated their understanding of subitizing knowledge as discussed in level 5 and level 6 of the subitizing trajectory through the order of the six dot arrangements. The data in Table 4.2 suggests that nine participants (Group 1) possessed more SMK as their order closely aligned to the developmental progression of the trajectory. Table 4.2 further suggests three participants (Group 2) somewhat attended to the developmental progression of the trajectory and three participants (Group 3 and Group 4) were unsuccessful in attending to the trajectory as they sequenced the cards.

Table 4.3 displays the number of participants correctly placing cards according to the three difficulty levels—easy, middle difficulty, and difficult—as established by Clements and Sarama (2014). In what ways might the data displayed in Table 4.3 support the conclusion that some participants possessed more SMK than others*?* Table 4.3 suggests that participants recognized differences between hard patterns and easy patterns with greater success than middle difficulty patterns as the easy and difficult patterns were accurately categorized more frequently than the middle difficulty patterns. Ten participants correctly categorized easy arrangements and eleven correctly categorized difficult arrangements. Pattern E and Pattern F, the medium difficulty cards, challenged participants' ability to recognize and articulate the shift from perceptual to conceptual subitizing. This is significant as *level 7—conceptual subitizer to five* signals a change in cognition, from quantifying five as a whole, to quantifying and combining two or more groups to name five.

What does participants' justification reveal about learning trajectory understanding? Sztajn et al. (2012) suggested that subject matter knowledge in relation to learning trajectories includes knowledge of concepts and procedures represented at each level of the trajectory and applying one's mathematical understanding to interpret student thinking at each level of the trajectory. To provide further insight into SMK, participants were asked to justify the sequence of the dot arrangement cards.

The most popular strategy for ordering the patterns was to move from what participants referred to as more organized patterns to less organized patterns or from more familiar to less familiar patterns. Figure 4.17 displays the order established by Mandisa. Her order mirrors the level of difficulty established by Clements and Sarama (2014). Eight additional participants placed either an easy dot pattern in the first or second position, either medium difficulty pattern in the third or fourth position, or either challenging pattern in the fifth or sixth position.

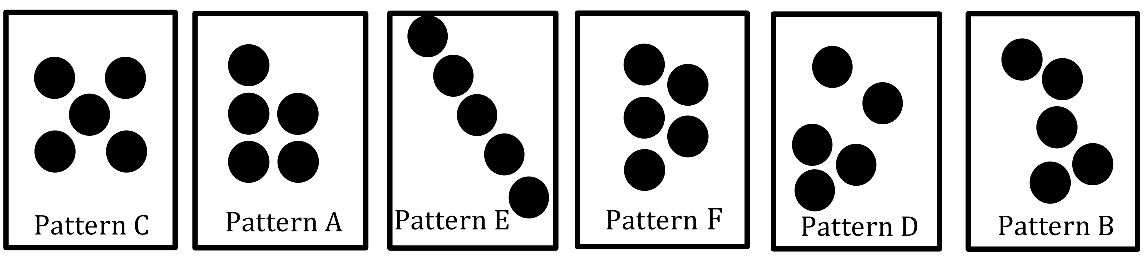


Figure 4.17. Mandisa's dot pattern order for Set 2 moves from easy to medium difficulty to challenging patterns.

Though the order of the cards provides insight into participants' understanding of the progression of the subitizing trajectory, their reasoning adds important detail. Mandisa's reasoning makes clear her thinking and her ability to notice the level of difficulty inherent to each card. She explains:

> I placed them in an order that would begin with the easiest to recognize to more difficult, in my opinion. This one [Pattern C] is one that I would hope that they would have seen on dice numerous times and just call it five. And in the next pattern [Pattern A] they should easily be able to see that it's four and one. And then combine them to see that it's five. Diagonally, Pattern E, I think that would be pretty easy for them to see five. Pattern

> F would be a little bit more difficult, because I think they would have to see the three and the two. The next pattern [Pattern D] has that separation. They would have to know to see the two groups and then put them back together again to get five. So they would be forced to see the three and be forced to see the two and then combine them. The last pattern [Pattern B] would be a little bit more advanced, I think, because it's not quite as organized as the other patterns. I can see a rhyme or reason on the other cards, for Pattern B, it's a little more challenging. It seems more random.

Thirteen of the fifteen study participants began the sequence with Pattern C, what many participants generally referred to as the “standard dice pattern” for five. Crystal’s response is representative of the group when asked to explain why Pattern C was first. She stated, “I put Pattern C first because I think the children would be most familiar with that pattern. Just because I think they would have seen it on the dice.” Participants chose to lead with a pattern that would prompt perceptual subitizing. In support, Jaeden shared, “So I'd start with something really easy and a pattern that I think they would recognize quickly.” Other reasons for leading with Pattern C included, “they [the children] might just know it from playing games,” “it is well-known,” “familiar,” and “the most common of all the patterns shown.”

Justine and Sasha also began with Pattern C as they viewed it as a stepping stone to success. Sasha shared, “Children will likely recognize that arrangement…it would set them up for more success later on down the line.” In agreement Justine stated, “I want the kids to feel successful at first, I guess. So I’ll start with something like this (Pattern C) that I think they would be familiar with.” They anticipated the children would experience success with this pattern; therefore, it was positioned first in the sequence.

It is important to note that not all participants organized the dot cards according to the trajectory. Amalie and Flora began with Pattern A and Pattern E respectively and though they selected a pattern other than then standard dot arrangement for five (Pattern C) their reasoning though their reasoning regarding the desire to start children off with an organized, familiar pattern held.

Amalie selected Pattern A, an easy pattern, and based her decision on the last pattern she used from Pattern B from Set 1 (see Figure 4.18). She shared:

> So basically when I look at the patterns, I go with familiarity first. So, which would be some of the most common patterns? So, when I looked at Pattern A, I noticed the four pattern on the bottom. The other thing I want to say is that I'm basing my decisions on quantity. So we ended the last sets, the pink cards with four. So I want to start with four, which is the hope, and then they would add on one more to get to five. So it would sound like four, and one more is five."

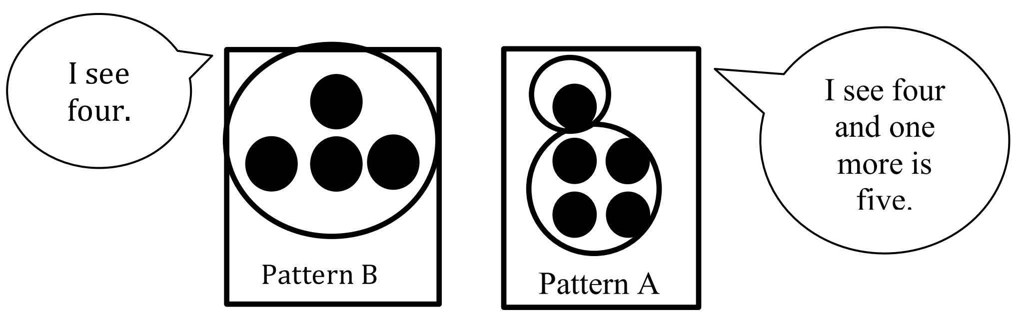


Figure 4.18. Amalie uses children's familiarity with the quantity of four from Set 1 (on the left) to guide her decision for her first card from the Set 2 cards (on the right).

Flora began her sequence with Pattern E. She found the linear arrangement very organized and supportive of children's ability to subitize. She stated, "I think that it's easier for children to start with patterns that are more organized. For example more straightforward patterns like Pattern E. Some patterns are just easier that others. " We see Flora begin with

Pattern E (see Figure 4.19) and then continue her sequence with Pattern C, Pattern A, Pattern F, Pattern D, and Pattern B. Talking through the sequence we hear Flora describe how she would hope children would manage the changes in arrangements.

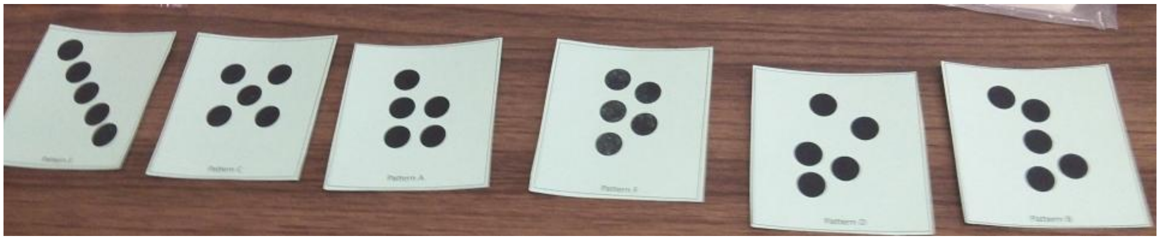


Figure 4.19. Flora’s order for Set 2 cards progresses from, in her opinion, more organized to less organized.

She explains:

> I want them to see the original pattern and know that they can make the next pattern by rearranging some of the dots. I want them to be flexible like that. Like moving from Pattern E to Pattern C is really just moving two dots (see Figure 4.20) and then moving from Pattern C to Pattern A is moving one dot. I want them to know that it's five because if I take the one from the middle [from Pattern C] and I move it out, I see that it is 4 and 1. And the four and the one, is now Pattern A. I hope they would just say, “I know that it's five because I just move the one from the middle and I put it up on top.” So again, I want them to see the parts but I also want them to see that there are five.

Flora makes it clear as she moves from pattern to pattern her intention is children’s flexibility with quantity. Her hope is they mentally map from one arrangement to the next, seeing the quantity of change but still represent five. She does initially focus on seeing groups as others did and includes an ability to mentally arrange and rearrange patterns of five.

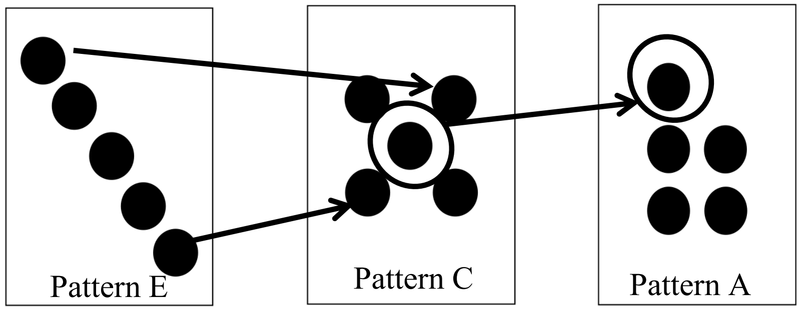


Figure 4.20. This displays the connection Flora hopes children see as they move from subitizing Pattern E to subitizing Pattern C to subitizing Pattern A.

What insight can we gain regarding participants' SMK as they discussed their order for the remaining patterns: Pattern B, Pattern D, Pattern E, and Pattern F? When discussing the order for these patterns participants tended to refer to "seeing twos and threes" or "finding groups." Meaning participants transitioned from patterns that could be perceptually subitized to patterns that could be conceptually subitized. Each participant signaled this transition with nine suggesting seeing two and three, and the remaining six emphasizing seeing groups to find the whole.

Eight participants began their sequence with Pattern C and then followed with Pattern A and Pattern F (see Figure 4.21). Participants suggested that Pattern A and Pattern F provided children opportunities to conceptually subitize by seeing two and seeing three and combining them to reach five. As Karina stated, " I want them to see the three and the two." Marie, when asked to talk a bit more about what it meant to see two and three, elaborated, "After Pattern C, the first pattern in my sequence that they just see five, I think they are ready to move from one dot pattern of five to the next when they can see the different groups. It would be important to me that they see three and two."

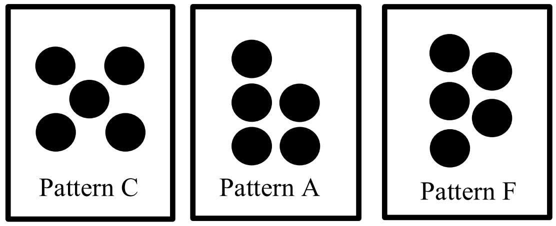


Figure 4.21. Over half of study participants began with Pattern C, followed by Pattern A, and then Pattern F. They viewed this sequence as supporting conceptual subitizing.

Jaeden also began her sequence with Pattern C, Pattern A, and Pattern F. She suggested that before children can quickly see two parts and compose them to get five they need to first understand what five means, and second, be able to subitize smaller quantities. She explains,

> I guess I didn't mention this before but they have to *know* five to even get into these harder patterns. I think they need to see five in different ways. It won't matter if it's a two, two, and a one or if it's a three and a two. It's still 5. If they weren't ready they either wouldn't say anything when I show it to them, or they would have to literally count every dot. And I would know that they're doing that because their fingers would be up in the air and they would be pointing and saying the number words out loud.

In line with Jaeden, Karolyn explains, “If they [the dot arrangements] are harder to see as a whole…If they can see the whole and then see the parts, Pattern A and Pattern F I think are good next steps for that [conceptual subitizing].”

Cecilia was one of six participants that focused on the more general idea of finding groups in lieu of specifically finding groups of two and three. She shared:

> I feel like it's about the ability of the children to see the groups and put them back together again. When they are able to clearly see different ways that is important. For example questioning them after they have talked about how they saw it [Pattern F] in one way asking them if they can see that total of five in another way. If they can see the total

multiple ways and get the correct total I think they're ready to keep moving on to more challenging patterns.

Children's ability to manage quantity weighed heavily on the participants' rationale for the overall order of the dot patterns. Pattern B and Pattern D prompted their push on quantity. Identified by Clements and Sarama (2014) as difficult patterns, all fifteen participants had either Pattern B or Pattern D in the fifth or sixth position and eleven of the participants placed both pattern B and Pattern D in the final two positions of the sequence. Marisol summarized quite succinctly her placement of Pattern B and Pattern D at the end of the sequence. She stated:

> I think maybe I put these last (see Figure 4.22) because in Pattern D I see two quantities. I think that makes it harder. The other one [Pattern B] I see the whole and I don't see the parts right away. I really need to think about how I would break it [Pattern B] apart and put it back together. Yeah, I don't see the parts right way with Pattern B. With this one [Pattern D] it is the opposite. I see the parts but I don't see one whole until I put it together.

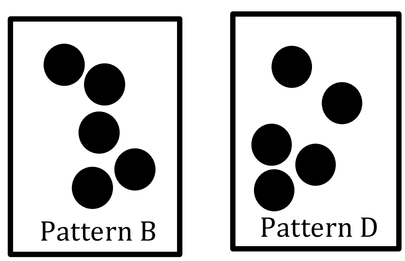


Figure 4.22. Marisol discusses why Pattern D and Pattern B are challenging. In Pattern D she quickly see the parts not the whole. In Pattern B she sees the whole but does not see quickly see the parts.

Pattern D and Pattern B were overwhelmingly identified as challenging due to the irregularity of the organization of the dots. Many participants commented that the children would likely find Pattern B the most challenging due to, as Sasha stated, "its snakelike appearance." Karina added, "These patterns would be very abstract. No clear groups surfaced" and children would need to work hard to find those groups. Marie felt that after Pattern C and Pattern A the order of Pattern F, Pattern B, and Pattern D (she omitted Pattern E) did not really matter as children would likely need to conceptually subitize each of them. She explained, “So you are moving from [seeing] one group (Pattern C), two groups (Pattern A and Pattern F), to three groups (Pattern B and Pattern D).”

Pattern E, the straight line of five dots, showed by far the most variability in position. It appeared once in the first position, once in the second position, three times in the third position, four in the fourth position, once in the fifth position, and five in the sixth position. In fact three of the participants who placed it in the fifth position eventually omitted Pattern E completely from the sequence. Amalie’s comment collectively describes participants’ angst for Pattern E. She shares:

> It's all linear. I felt like if I was a kindergarten student and I saw that pattern I would probably just shout out a random number. I don't think I'd be able to hold that pattern in my head to be able to subitize the quantity to get to five.

More discussion regarding the anomaly of Pattern E is addressed in the subsequent sub-theme.

Did participants consciously or knowingly attend to the progression outlined by the subitizing trajectory? Four participants, Karolyn, Karina, Flora, and Karaleen included the idea of attending to a learning progression as they ordered the six Set 2 dot pattern cards. Karolyn and Karina, both early childhood special education majors, used the phrase “learning progression” and addressed its influence on their order of the dot patterns. Karolyn shared, “So when I figure out what to do I have that particular progression emblazoned in my mind. Move from easier patterns to more complicated patterns.” Karina added, “It's all these different factors that go into their stages of development… the trajectories are helpful in helping me figure out kind of what to do when and what to expect."

During the interview, two participants, Flora and Karaleen, tangentially referred to learning progressions or trajectories. For Flora, knowing "where everybody is at and then move forward" was particularly key. She referenced the idea of knowing where her students are in mathematics and moving forward three different times throughout the interview. She shared a particularly insightful comment when asked if there were other factors that contributed to how she ordered the Set 2 cards. She stated,

> Like, I think carefully about what kids already know. Based on what you know about what they know, you can pretty much move forward. I don't think that a teacher can really start somewhere with a child and move them forward if you really don't know where they are. I see the progressions helping with that a lot.

Karaleen echoed a similar sentiment when asked about other factors that contributed to the order of the dot patterns. She stated, "I would also go back to how children grow developmentally and think about where children are and where they need go. The order of the cards helped me think about that."

Intentional decision-making was on display as participants shared their rationale for ordering the Set 2 dot patterns. They attended to the subtle changes in difficulty as they moved from one pattern to the next and offered insight into the importance of a careful scaffold from easier to more challenging patterns. Some participants referred directly back to the idea of the progression or trajectories as they rationalized their preferred order.

Sub-theme 2: Pattern E—Honoring subitizing and working within the progression. Particularly insightful conversations centered on the decision to include or remove Pattern E (see

Figure 4.23) from the Set 2 dot patterns. Identified as a medium difficulty arrangement (Clements & Sarama, 2014) Pattern E pushed on participants' perspectives of subitizing and the value of subitizing for young children's mathematics learning. The majority of the participants selected patterns for the third and fourth positions that would prompt children to conceptually subitize five, so this pattern created some interesting disequilibrium.

I begin by sharing a selection of opening comments about Pattern E and what they might reveal regarding participants' SMK. Then, for the participants that viewed Pattern E as viable, I provide evidence of intentional actions they would take to ensure children subitized the quantity as opposed to counting the dots one by one to reach five.

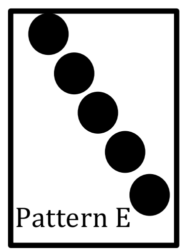


Figure 4.23. Pattern E. This pattern posed significant conversation with most study participants as they felt it did not promote subitizing.

As participants sifted their way through the six dot patterns, Pattern E was commonly put off to the side or tentatively placed in the sequence. More often then not it was moved or removed at various times. Below are participant comments regarding the presence of Pattern E in the collection of dot arrangements. I found them particularly helpful when framing the quandary of Pattern E and what it revealed about participants' understanding of subitizing, their agency as decision makers, and their commitment to children's success as early subitizers. I use comments from Karolyn, Kayla, Jaeden, Marie, and Cecilia to open.

Karolyn: It [Pattern E] is such a weird pattern. I know that it only has five on it, I know that now but imagine if there were two more dots added to it and there were seven total. The patterns would look very similar. It doesn't give children the opportunity to do the group thing. They just see a line. At least that's the way I see that pattern right now.

Kayla: (Laughing.) I don't know how to explain Pattern E. It counters what we want them to do which is finding groups. I think... yep, Pattern E is kind of interesting. I don't know that I would use it. Yeah it just doesn't connect to what we were doing with them [the kindergartners].

Jaeden: And E is just something else! I wouldn't want to show that pattern to the class. I think they would be like, "Can I see that again?" Honestly, I think I would have to see it again to know how many dots are going diagonally. My fear with Pattern E is that they would have to count. And I don't want them to have to count. They won't see any organization to that pattern. They would probably just see dots going down in a line.

Marie: But with this one (Pattern E) they might go back to counting just 'cuz it is the line. I would not use it as is.

Cecilia: Pattern E, it is pretty clear but I still think, it is like...(sighs heavily)...there's no group in this one they're going to have to somehow count it in their head. (She moves Pattern E out of the sequence. See Figure 4.24.) Okay, well let me think. Well, if I don't abandon ship totally with that particular pattern…Gosh I don't really know (speaks tentatively). Yes, it's the counting idea, that's a big part of why I'm struggling.

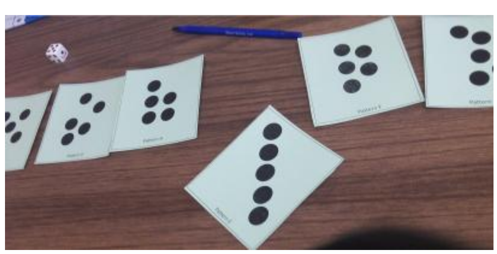


Figure 4.24. Cecilia pulls Pattern E out of the sequence she has designed, unsure of its purpose.

These five comments reinforced for me that study participants understand what subitizing is and want to be sure that the dot patterns and the order are as encouraging as possible for their fictitious five-year olds. They viewed the linear arrangement of Pattern E as too many to perceptually subitize and not organized in such a way to promote conceptual subitizing.

Attending to the developmental appropriateness of the patterns was important, but what surfaced as most important was encouraging subitizing as a way to understand quantity. If the children would not be able to subitize Pattern E, the pattern was questioned. For example, Karina shared a concerned surfaced by three other participants. She began:

> I think that it's just one that they would memorize. But I think it's hard, because when you look at it all you see is a line, and you start to question well are there four in that line or are there five in that line? The patterns that come before it are so much easier to see the groups.

Similarly, Marisol stated, "This pattern does not lend itself to the idea of subitizing because I really don't see the group." Along that same line, Crystal shared, "If I think that finding groups will be hard and they will need to count by ones than I don't want it in the middle of this sequence." Kayla pondered, "I think you would have to give them more time to see it." In line with the previous comments, Jaeden laughingly picks up Pattern E (see Figure 4.25) and states:

> The one [pattern] where I don't see any groups, is pattern E that's why I put it at the end. I think that this is just really hard. If they're just learning their fives this would be even more challenging. I think they have to have a good grip on the number five.

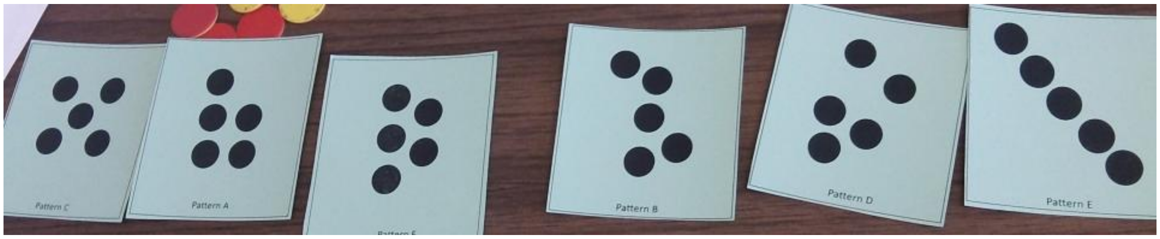


Figure 4.25. Jaeden's sequence for the dot arrangements shows Pattern E at the end.

Eventually Jaeden omitted Pattern E from her sequence, as did Amber, Kayla, Karolyn and Marie. Removing Pattern E was done with thought and intention and connected back to the goal of subitizing. Kayla suggested that "yeah it just doesn't connect to what we were doing with them." Amber furthered:

> I never ever see numbers in a line [like in Pattern E]. I guess I feel like that with dot patterns you want to see them in groups and this one is kinda like all together...it just seems weird. I guess I want the kids to see the groups and I feel like in the line you don't really see groups. You kinda see one group together but not in a way that you can like split them up so you're not counting them. I guess it IS organized like Pattern C but not in a way that the mind sees it in groups. (See Figure 4.26.) Cuz it kinda looks like there are more than five in a way. I would not use it.

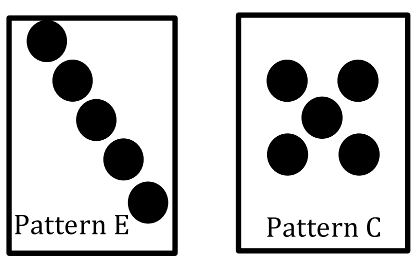


Figure 4.26. Amber compares Pattern E to Pattern C as she weighs whether or not to include Pattern E in her sequence.

When asked if she would consider including Pattern E at some point in the future Marie responded, "Honestly no. I don't like that one. Yeah...I mean I see it as a train. I kinda want to count by ones." Indeed twelve of fifteen participants commented about the feelings of discord they felt toward Pattern E. On the other hand, three did not question Pattern E. They included Karaleen, Mandisa, and Sasha. Karaleen stated Pattern E would be one "the kids would need to memorize. Mandisa shared, "Kids that have experience with dot cards would just know this is five. I think they would just have to see it as five." Sasha followed, "I feel like it's pretty easy. It is just in a straight line. It's not as confusing [as Pattern B or Pattern D.]"

What do participants' responses reveal about their SMK? Of the twelve participants who expressed concern with Pattern E, ten were so intrigued they took it upon themselves to describe how to engage children to subitize Pattern E. As an initial instructional strategy each suggested placing one counter on one dot to help children see five and four went one step further and paired Pattern E with a five frame or ten frame. I defer to Karolyn, Marisol, and Crystal to clarify these strategies. What unites each approach is the participants' willingness to the nurture children's subitizing abilities.

Bothered by the fact that there was "no way to group" Pattern E, Karolyn reached for a five frame and some small counting bears. First, she placed one bear on each dot and then transferred the bears one by one to the five frame. (See Figure 4.27.) Keeping the focus on subitizing and staying within the developmental trajectory for learning, Karolyn commented:

> So once I get the bears onto the dot pattern then we can see how the five bears on the dot pattern are the same quantity as the five bears that we would put into the five frame. One bear goes into one empty space. So now we see that five is five.

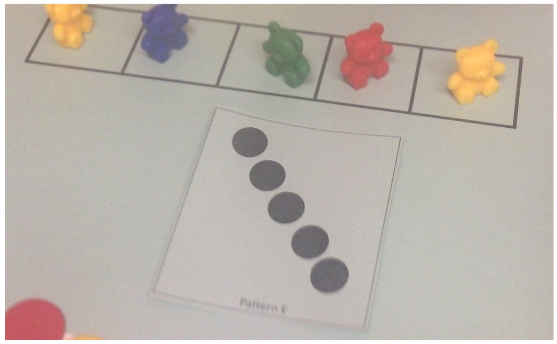


Figure 4.27. Karolyn uses the five frame in a effort to help subitize Pattern E.

Marisol opted for a similar way to see quantity that would prompt subitizing, though she reached for the ten frame as opposed to the five frame. She rotated the card so the dots were horizontal as opposed to vertical as displayed in Figure 4.28. She continued:

> Like I said earlier I needed to count them one, two, three, four, five, to make sure that there were five. So I wish it was just this way. (She rotates the pattern so the dots are horizontal.) Well I think if they are working with a ten frame they might just know that it's five. There, I think that would help. Even if I put it next to the ten framed I'm still not convinced to that I would even use this in the sequence of patterns. I don’t like it that they have to count them.

Figure 4.28. Marisol rotates Pattern E and places it next to the top row of the ten frame to emphasize five.

For a final example of participants’ willingness and intentionality to help children subitize the dots arrangement displayed in Pattern E, Crystal suggests putting two white bears and three red bears on top of the bears to suggest groups:

In Pattern E they are all grouped together. I suppose the kids could count them all. Maybe I could use two different colors of teddy bear counters. (See Figure 4.29.) You could actually do this with the other dot patterns as well if they were having a hard time seeing the groups. I think that this could help them stay away from counting by ones. For the children that need more experience, this might make the groups more clear.

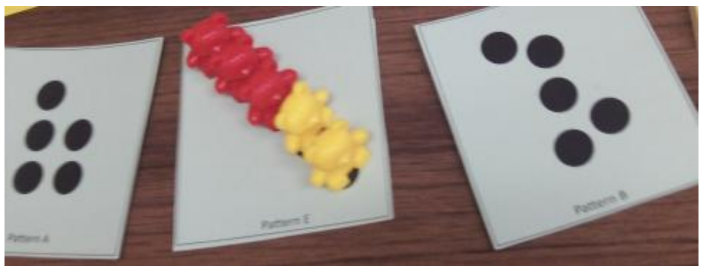


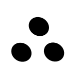
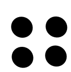
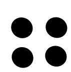
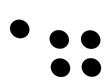
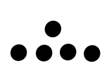
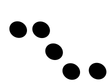
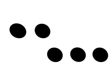
Figure 4.29. Crystal places three orange bears and two yellow bears on the dots of Pattern E in an attempt to help children "see" groups.

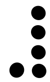
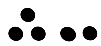
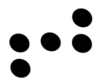
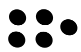
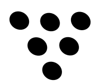
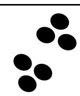
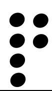
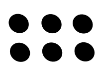
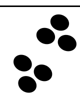
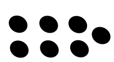
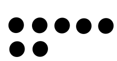
The conversations sparked by Pattern E were unexpected and every single participant reflected on the affordances and drawbacks of its use. With some sort of modification many participants begrudgingly kept it in while others quickly removed it from the sequence. Critical to each conversation was the idea that children were asked to subitize, not count, and multiple efforts were made to prompt subitizing. The reasons to completely omit or to modify Pattern E were well developed, thoughtfully articulated, and stayed true to the big idea of the trajectory, that being subitizing.

Sub-theme 3: Rationale for "Added in" Pattern. Perhaps one of the most interesting ways study participants displayed their subject matter knowledge was through the pattern they developed to include in their established Set 2 sequence. This task, developed to provide insight into subject matter knowledge, provided a wide open platform for participants to apply their learning trajectory knowledge and demonstrate their understanding of subitizing. The participants eagerly embraced this opportunity. The following information was gathered for each participant (a) the pattern, (b) explanation of why that pattern (c) placement of the pattern in relation to their Set 2 sequence and, (d) explanation for why that location. This information highlighted the intentionality of the decisions made and whether or not those decisions were inline the progression outlined by the subitizing trajectory (Clements & Sarama, 2014).

The placement of the dot pattern became important as it demonstrated an understanding of the progression of the subitizing trajectory. Four participants designed a pattern to place at the beginning of their Set 2 sequence. Six participants designed a pattern that they would place at the end of their Set 2 sequence. Four participants developed patterns they would place somewhere in the middle of their Set 2 sequence. Two participants develop two patterns, one to place before and an additional pattern to place after their Set 2 sequence. Table 4.3 displays each pattern developed its location in the sequence, and the participant's reason for that pattern.

Table 4.4. Dot patterns and placement as developed by each participant.

Participant	Size	Pattern	Placement	Reason for pattern and placement
Karaleen	Three		Before	"If they were struggling I would go lower. It would help me focus on grouping."
Amber	Four		Before	"They can learn the four pattern and that for Pattern C they are just adding one more dot in the middle."
Karolyn	Four		Before	"It is a dice pattern and a smaller number and it is pretty similar to the five dot pattern."
Justine	Five		Before	"I would start with three and then I would start with the four dot pattern and then one more off to the side is five."
Sasha	Five		Before	"If they knew the four in Set 1 I would do this one. The pattern is four and one more."
Amalie	Five		Middle	"This is similar to what they've seen before and where they are going. So when they get to Pattern E it would be such a shock."
Karolyn	Five		Middle	"This is more challenging because it is spaced out. So your eyes would need to see one group and then see the other group and then put them

				together to get the total." Scaffold between Pattern B and Pattern D.
Kayla	Five		Middle	"Doing this first might make Pattern E not be so challenging. There is a line of four and one on the side, so another way to see five."
Mandisa,	Five		Middle	"This would help them see the two groups. So you have a triangle and two more in straight line." Scaffold between Pattern F and Pattern D.
Marie	Five		Middle	"I think this makes the groups more clear." Scaffold between Pattern D and Pattern B.
Marisol	Five		Middle	"The children can see the four and one in my pattern and the four and one in Pattern A." Scaffold from Pattern C to Pattern A.
Cecilia	Six		Undecided	"It is pretty easy to see these different groups. I can see where you would get lots of different answers." Initially seen as a scaffold between Pattern A and Pattern F. Ultimately decided she did not know where to place it.
Crystal	Six		After	"There is one three and another three. They use patterns they know to get to six."
Flora	Six		After	"I think they would see the four and then could add the one and one more for six."
Jaeden	Six		After	"This is organized. The kids could see three and three. If they can find Pattern A in here, they just add one more on."
Karina	Six		After	"It pushes the concept of three which I like because we are seeing two quantities."
Jaeden	Seven		After	"If they are comfortable with six, and they knew six right away, I would add one more. That way they could see it as six and one more."
Karaleen	Seven		After	"We see the five and two. I like to incorporate this work with the ten frame."

The patterns developed were diverse and aligned with the established progression of the subitizing learning trajectory. Three different quantities were selected for patterns placed before the first card of the Set 2 sequence, three, four, and five. Each participant that placed patterns at the start of the Set 2 sequence suggested they would use the patterns to help children ease into the six patterns of Set 2. For example, Sasha used her pattern of four (see Figure 4.30) to scaffold from the final pattern in Set 1 to the first pattern in her Set 2 sequence, that being Pattern C. She stated:

> So if they understood four from before I just do this one next. I would put this at the beginning and they could see the four and one more. I think it would be nice to do this pattern right after the four card from Set 1 and then right before the recognizable dot pattern of five.

Sasha's suggestion of a pattern of four and one more is in line with *Level 6: Perceptual Subitizer to 5* and *Level 7: Conceptual Subitizer to 5* of the subitizing learning trajectory. These two levels engage children in quickly recognizing quantities up to five as whole amounts as well as seeing and combining two small quantities to make a whole.

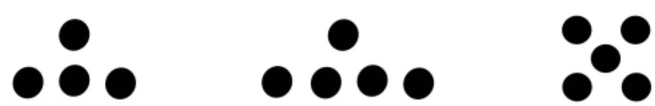


Figure 4.30. Sasha's new dot pattern of "four in a line and one more" placed between a quantity of four and the more standard pattern of five.

Karaleen, who developed the pattern of three, (See Table 4.4) stated this pattern would allow her to "focus on grouping. We could find smaller numbers inside of three." The pattern of three would be used "if they [students] were struggling with five," meaning the quantity was too big for them to successfully subitize. She intentionally scaffolded back to *Level 4: Perceptual Subitizer to Four.* This surfaced as important to her as she identified the final patterns at the end of her sequence (Pattern D and Pattern B) as patterns the children would need to conceptually subitize. Karaleen knew these patterns would be a challenge so she would start from the beginning to "set them up for success." Justine echoed the similar reasoning when she suggested that she would not start with a pattern of five and would instead step back to patterns of two, three, and four. This backtracking demonstrated her understanding of the trajectory levels proceeding *Level 6: Perceptual Subitizer to Five.*

Two different quantities were developed for placement within the Set 2 sequence, five, and six. Each of the participants who developed patterns for use in the middle of the sequence envisioned each pattern being used as either a bridge between two challenging patterns or to help the children more successfully subitize Pattern E. As example, Amalie developed her "Z" pattern to building an understanding for working with Pattern E. She explained:

> It gives the center diagonal of three, but this it also has the dot at the top and the dot at the bottom. So then you move into a linear pattern [like Pattern E] they can shift the dots into a pattern they've seen before and subitize the quantity.

Along a similar line, Marisol would use her pattern to scaffold from Pattern C to Pattern A and suggested it would "help the children recognize the similarities between the patterns. That would help them see many ways to see five." Marie would place her pattern of five between Pattern D and Pattern B. In her own words, "Pattern B just throws me off" and the structure of the pattern would held the children prepare for the more unstructured pattern. (See Figure 4.31.) She stated:

> Make the groups more clear. They [the children] can show me that they see two, and one, and two. The can see the one, three, one. Or, they can move this guy [dot] over to make a four and one and it is the same as Pattern A.

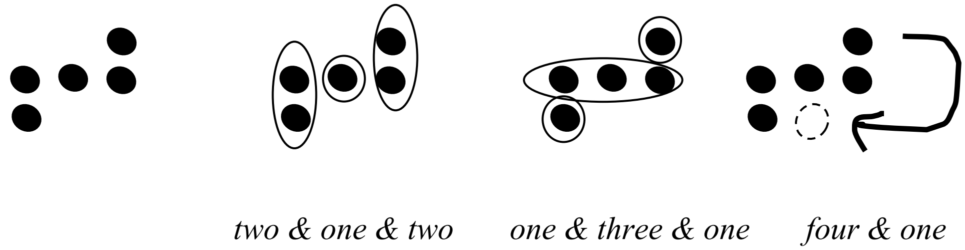


Figure 4.31. Marie shows the different ways children could conceptually subitize the pattern of five she developed.

The patterns developed for use in the middle of the Set 2 sequence are in line with the expectations of the subitizing learning trajectory *Level 6: Conceptual Subitizer to Five.* The only pattern that would not fit this level would be Marisol's pattern of six. Her pattern fits nicely with *Level 7: Conceptual Subitizer to Ten*. Marisol had a hard time justifying the location of her pattern and eventually defaulted to not knowing where she would place it. She stated, "I guess I'm not sure yet where it would go. I never really thought that one dot would make that big of a difference, but now I am thinking it is a big step."

Two different quantities were selected for patterns placed at the end of the sequence, six and seven. Each placed an emphasis on conceptual subitizing and moved directly into working within *Level 7: Conceptual Subitizer to Ten*. Each offered an appropriate scaffold from the last pattern in their Set 2 sequence. When matched to the subitizing learning trajectory each pattern could be used to successfully prompt subitizing in young children working at *Level 7: Conceptual Subitizer to Ten*, and begin to move into informal addition. For example, Jaeden first developed a pattern of six as you would see on the face of a die. She then added one more on saying "If they were comfortable with six, and they knew this was six right away, I would add on more on…That way they could see it as six and one more." Karaleen, who clearly saw her work with her own class of five-year olds in this exercise, created a pattern of seven that looked it was on ten frame, meaning five dots in a row and two in the second row. Figure 4.32 displays how Karaleen would move the dots onto the ten frame to help the children see five and two as seven. Karaleen continued:

> I have a total of seven, but what I like about this is we can talk about groups. So I can see the five and two. If I wanted to bring in a number sentence I could (and she writes 5 + 2 = 7). I like to incorporate this work with the ten frame. So I

would use the ten frame and the dots together. So my work around this would eventually lead the kids to number sentences.

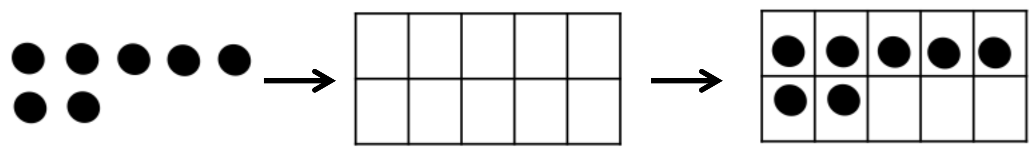


Figure 4.32. Karaleen transfers her pattern of seven to the ten frame to emphasize seven as a quantity of five and two.

Participants demonstrated their subject matter knowledge, specifically their specialized content knowledge, as they engaged in ordering dot patterns, rejecting or modifying Pattern E, and developing a pattern to add in to the Set 2 sequence. As evidenced in the data, each participant thoughtfully applied this understanding to meet and advance children's learning. Intentional instructional decisions start with an understanding of how the big idea of a learning trajectory progresses. In this next section I explore the pedagogical content knowledge of learning trajectory based instruction based instruction and the role it plays in instructional decision-making.

Pedagogical Content Knowledge Findings

Pedagogical content knowledge (PCK) is knowledge that emerges from a focus on the learner's cognitive development. While teaching guided by learning trajectories, pedagogical content knowledge is demonstrated by one's ability to engage and apply learning trajectory understanding to be responsive to and capitalize on children's thinking with the intent to advance children's learning. For this study, PCK entailed believing that mathematical thinking in young children grows developmentally and centering instruction on children's thinking. Each theme unpacks the characteristics of intentional teaching inherent to learning trajectory based instruction and informs the second attendant research question: Do early childhood prospective teachers draw upon their knowledge of learning trajectories as they make instructional decisions? The two themes for pedagogical content knowledge illuminate participants' instructional decisions as they engaged their understanding of the subitizing learning trajectory to advance children's subitizing abilities. Those themes include: (1) Understands the developmental nature of children's subitizing skill and ability; and, (2) Centers instructional decisions on children's thinking.

Theme 1: Understands the Developmental Nature of Children's Subitizing Skill and Ability

> *Developmentally I think about how children grow. How we start with quantities zero to five and then going to ten, and then working within ten for a long time. Because if a child is not, if they haven't, mastered up to five they may not be ready to move past that to work with quantities like six, seven, eight, nine, and ten.*
>
> --Karaleen, ECE Regular Education Major

Teachers' awareness of the developmental nature of children's mathematical growth allows them to carefully plan and structure learning opportunities unique to each child (Clements & Sarama, 2014; Daro et al., 2011). To provide evidence for such knowledge I looked for instances where study participants explicitly discussed children's mathematical growth as developmental, acknowledged the role of quantity to support growth, and sequenced Set 2 patterns to prompt growth. As study participants shared their understanding of subitizing, they used that information many times to adjust or offer tasks to scaffold engagement. They aimed at advancing children on the Subitizing Learning Trajectory, further demonstrating their specialized content knowledge and showing the strong link between pedagogical content knowledge and subject matter knowledge.

Sub-theme 1: Acknowledgement of developmental growth in mathematics. The idea that children are born with innate mathematical abilities is surprising to many prospective early childhood educators. The application of learning trajectory research supports both this belief.

Karina, perhaps stating the obvious, recognized that children's growth follows predictable benchmarks. She shared,

> So I know that literacy has an order in which you teach things. No matter how slow or how fast the kid learns they are going to learn in this particular progression. And I did not realize that math had some of those same progressions. I'm a memorizer, math has always been easy for me, so as a kid I don't remember progressing through these different levels. Like these different mathematical stages. I honestly did not even know that they existed.

The idea of a progression for knowledge acquisition and growth in early mathematics was a novel idea for seven of the fifteen prospective teachers who explicitly acknowledged the developmental nature of children's acquisition of mathematics skill and ability. As example, when considering how to best meet the needs of her future students Karaleen offered, "I would also go back to how children grow developmentally and think about where the children are and where they need to go."

When asked to articulate how she might know when a child is ready to move on to a more sophisticated dot pattern as well as what knowledge she was tapping in to as she made that decision, Flora, a non-teaching prospective teacher seeking bilingual certification, shared:

> I think carefully about what young kids already know. Based on what you know about what they know, you can pretty much move forward. I don't think that a teacher can really start somewhere with a child and move them forward if you really don't know where they are. That's pretty much how I like to base my decisions so far. No matter which content area it is, literacy, math, English, I try to first start with

> what they know. And then I move forward from where you need help with to get to the next level and to keep improving.

Effective teaching in the early childhood mathematics classroom is grounded in an understanding of developmental growth that allows teachers to select and target leaning opportunities that encourage student learning (NAEYC, 2009). Karolyn, who was student teaching in a 5K classroom at the time of the interview, centered on the idea that children progress as mathematical thinkers following developmental stages. She shared:

> Developmentally I think about how children grow. I guess the, the, trajectories could be helpful in helping me figure out kind of what to do when and what to expect. I do have those levels in mind as I think through how children might respond and what I might do next with them.

Indeed, early mathematics trajectories do lend support to teacher's ability to assess and monitor children's growth as doers of mathematics (Daro et al., 2011). Trajectories provide a progression of children's thinking and provide teachers with a tangible tool for recognizing and honoring children's thinking. In addition, learning trajectories give teachers permission to linger on an important concept and not push too hard or too fast on children's developing understandings. Amalie, a fulltime 3K teacher pursuing her teaching certification, recognized this. She articulated:

> If they don't get it [subitize correctly] right away I would know that we're going to get there, we're going to get to seeing four and one is five, but if they don't see it right away that's okay. I can check with the learning trajectory and that helps me figure out where to go next and if I should be worried about where they are now.

Effective early childhood teachers recognize that young children will not think like adults nor mirror the thinking of their teacher (Clements & Sarama, 2014). Jaeden, a traditional prospective teacher pursuing an early childhood special education degree, explored this idea as she shared:

> I think you have to base it [instruction] on the kids. I also think you have to be careful to not expect your kids to see everything the way you see it, you know? So even when maybe there's a three and a two [in a dot pattern], even though I would not have seen it that way, maybe kids will and I need to be open to that.

The helpfulness of using a learning trajectory to identify where children's skill set lies was articulated by Karina, a teaching intern for three- to five-year olds with special needs. She explained:

> I did little pieces of assessments to be able to track their IEP (Individual Education Plan) goals. So to track their growth toward their IEP goals, I had to figure out where each of them were. The learning trajectory helped me do that. So, for a couple of my older students, their goals are addition and subtraction by the end of the year. So to get them adding and subtracting by the end of the year, I had to figure out where they were on the addition and subtraction trajectory to be able to scaffold to where they needed to be.

As Karina demonstrates, learning trajectory knowledge supports a teacher's responsiveness to children and use of emerging in-the-moment opportunities to capitalize on student thinking.

Finally, Marisol, a full time 5K teacher, acknowledged the importance of opportunity and experience as she reflected on her students' struggles to subitize. At the time of the interview Marisol had been with her class for four weeks. After learning about subitizing and the subitizing trajectory she attempted subitizing tasks with her students and discovered "that the class is very behind. I look at the subitizing trajectory and many of them are below where they should be given their age." This caused her great concern and she concluded:

> I guess I would base instruction off of how much experience the children have with these different patterns. If they work with ten frames, if they work with dice, if they work with different types of patterns…I guess it makes sense that the more experience they have with the [dot] patterns the easier it will be for them to see how many dots there are, and to explain how they see them.

The above quotes demonstrate that prospective early childhood teachers believe that children grow developmentally in their mathematical abilities and they wish to honor this belief as they engage mathematically with young children. In addition, study participants view math skills as fluid and "grow-able" and that to properly target instruction a teacher must anchor their instructional decisions on identified developmental benchmarks and their children's thinking and not their own. This understanding is foundational to a focus on learner's cognitive development, the heart of pedagogical content knowledge, and lays the groundwork for a teacher's ability to engage in effective instruction.

Sub-theme 2: Amount and arrangement of dots impact growth in subitizing. Study participants employed their understanding of subitizing and the subitizing trajectory as they talked about the impact of the number of dots and the arrangement of dots on children's ability to subitize. Cecilia articulated this thought as she considered whether or not it was acceptable that her students count by ones to name "how many" in a dot pattern. She shared:

> I think at the very start I probably would accept that [counting by ones] but I know that the goal of using these dot patterns is to get the total without counting. It's important that

they know what the total is and it's also important that they tell me how they got to the total.

It was Sasha who recognized the importance of seeing a pattern, decomposing it into smaller parts, and then recomposing to state the total as she described her rationale for subitizing work with her students. She explained, “I want the kids to decompose those dot patterns without counting by ones and then I want them to put it back together again without counting by ones to see the total.”

Though each study participant offered numerous examples of how the quantity and arrangement of dots impact children’s ability to visualize quantity, it was Amalie who clearly stated:

> The different structures and size of patterns support the children’s thinking and how they subitize. For example, the more structured patterns that are five or less allow them to use information they may have gained by playing games with their family. So it might be things that they already know. By starting with those I hope they would build some confidence and feel really excited about it.

Many participants agreed that starting with smaller quantities was a good way to judge student ability and readiness. Flora suggested, “I think that is easier to start with patterns that are more organized. For example, more straightforward arrangements.” Justine offered,

> I feel like you had us (in class) start with smaller patterns like these (touches the “three” and “four” from Set 1) because these would be recognizable like a dice pattern, which they might have at home, and then build up to ten.

In fact, the idea of initially working with smaller, more easily recognizable patterns was expressed by each of the fifteen study participants.

Expanding on Justine's thought, Karolyn explained the benefit for children's thinking as she extended an easier arrangement to another, in this case the pattern of three displayed in a vertical line to a pattern of four which could be seen as "one more" as shown in Figure 4.33. Karolyn explained:

> I think I would do this one first. (She pointed to the pattern with the dots in a vertical line.) Like I like the idea that this pattern has three and this other pattern has four. The quantity of three is very early in the trajectory. Like if they could see the three then they could look at the pattern with the four as "three in a line and then one more."

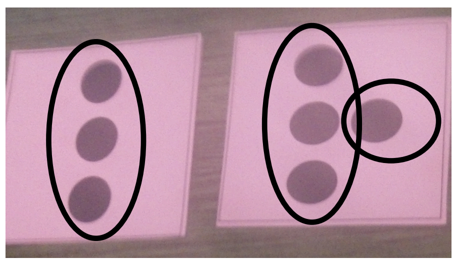


Figure 4.33. Karolyn places the first two patterns in her order and explains the relationship between the first pattern (on the left) and second pattern (on the right).

When asked to organize Set 2 cards (see Figure 4.34) in the order they would use them with the fictitious classroom of five-year olds, thirteen of the fifteen study participants placed Pattern C first. Participants commented they wanted to "begin with the easiest to recognize" as they conjectured that young children may recognize it from dice and "just call it five." A common thought for the majority of the participants, Sasha clarified:

> I began with Pattern C because it is on a dice. And I believe that children will likely recognize that arrangement. I would start with that one first because it would set them up for more success later on down the line. I hope that first pattern would be a refresher and something that they would feel confident about.

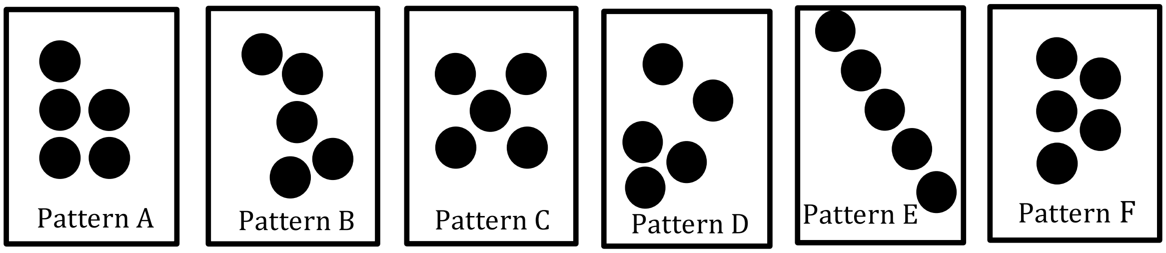


Figure. 4.34. The six dot arrangements of five that participants were asked to organize in their suggested order of use.

Karaleen, an experienced preschool teacher, shared a similar sentiment regarding Pattern C, and included the idea that this pattern was included early in in the five-year old kindergarten curriculum.

> I think they would have seen that pattern before. You're talking about early in the year, and I know that on our SmartBoards we have dice that we can "shake." So my students have seen this pattern before, and they've seen them for a while now. I think that they would look at Pattern C, and would just kind of say, "I see it as four on the outside and one on the inside so I know that it's five" or they might just know it is five." (See Figure 4.35.)

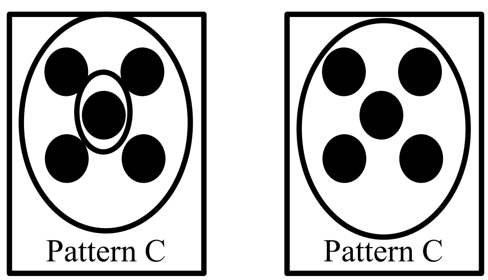


Figure 4.35. Karaleen states that her five-year old kindergarten students would see Pattern C as four and one or they might just see it as five.

Karalee couches her discussion of what she would do and why in a classroom scenario. She actively moves between representations and relies on her knowledge of content and students as she rationalizes how children might subitize Pattern C.

A key component to a learning trajectory concerns the increasing sophistication of the mathematics as children progress from the beginning to the end of the trajectory. In the case of the subitizing trajectory the mathematics increases in sophistication as children move from perceptual to conceptual subitizing. A teacher might increase the quantity of items in the arrangements or keep the quantity of items the same and shift the arrangement to push for deeper understanding of quantity and more sophisticated reasoning.

Pattern B and Pattern D were not very popular with the participants. (See Figure 4.36.) None would omit either pattern as some did with Pattern E, but many questioned their own ability to immediately see how many as well as young children's ability to see how many. Karina, a 3K-5K Special Education Intern, shared:

> Well I'm thinking that they [my students] might see chunks inside of those patterns, but this one (pointing to Pattern B) is so abstract. There is no organization to it, I think that would make it hard for my students particularly. I also think it would be hard for regular education students at first, as well. There is no pattern, and we're used to seeing patterns to things. The organization and the grouping make it easier.

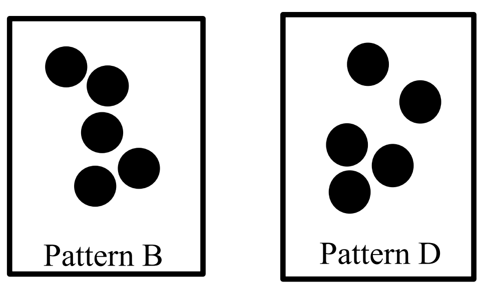


Figure 4.36. Karina believed that Pattern B and Pattern D would be challenging for regular education students and most certainly for her young learners with special needs.

Every participant commented on the challenge of both patterns though Pattern B was specifically called out as, according to Cecilia, "just hard." It ended at the end of the Set 2 sequence for twelve of the fifteen participants. Cecilia echoed a common sentiment when she shared:

I know. I know I placed it at the end. It's, it's another one of those weird patterns. I think it's challenging because at first I did not see clear groupings. As I look at it more carefully now I guess I do see groupings but initially I did not. I think that the pattern is interesting and the groupings are more difficult to see at first. They are more clumped together.

Participants evidenced adjusting the number of dots in the patterns as a reasonable way to meet and advance children's subitizing growth. This was clear as each explained what pattern they developed and reasoned its placement in the sequence (see Table 4.4). For example, each of the five participants who intentionally created a pattern to place at the beginning of the Set 2 sequence developed familiar, easily recognizable patterns. Karolyn's explanation for the four pattern makes that clear, "It is a dice pattern and a smaller number and it is pretty similar to the five dot pattern (Pattern C)." In addition, four of the five patterns were less than five showing the understanding that a smaller quantity, in a familiar arrangement, would be easier to subitize.

Though each of the six patterns developed and placed somewhere in the middle of the Set 2 sequence were unique, they were each composed of five dots. The participants supported their arrangement by suggesting that it would help scaffold between two patterns they identified as particularly challenging for the young children. In general, that meant placing their pattern between Pattern B and Pattern D or offering it as a scaffold to Pattern E. For example, Kayla explained that her five pattern "could help them [the children] get ready for Pattern E." Marie used her pattern to scaffold between Pattern B and Pattern D (see Figure 4.37). She reasoned:

So I just always try to look for what makes sense first. Like what groups make sense or what patterns make sense. Like in this pattern (points to Pattern D), the groups make sense to me. I see the three and I see the two versus this one (points to Pattern B) where I see the three but not in a conventional way. I see the one and then would need to join it with the other one (dot) to get the total of 2. Then I need to put all that together to get 5. That is A LOT of thinking for a five-year old! This one (Pattern B) just kinda throws me off completely. The other ones I know what I would do. It is just a little tricky. That's why I would put my pattern here between them.

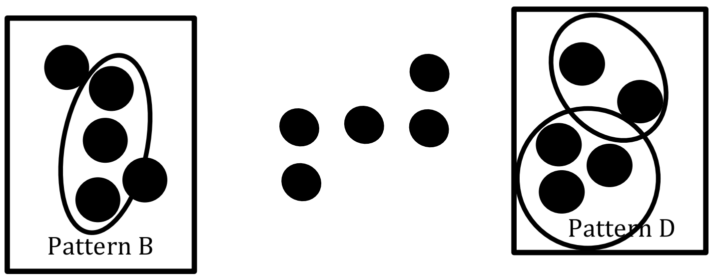


Figure 4.37. Marie shared that her dot pattern (middle image), placed between Pattern B and Pattern D, would give children experience managing those challenging patterns.

Finally, the patterns created and placed at the end of the Set 2 sequence were all greater than five. Participants prefaced these patterns with phrases such as "this is a challenge," "if they are ready for it," and "they can handle the others than I think they can try this one." They understood if they move to a quantity greater than five, the pattern can be become much more challenging. To help the children ease into these larger quantities they presented more "organized patterns" (Jaeden) so the children could more easily find groups and then focus on composing them to name the whole.

Each of the fifteen participants engaged their understanding of levels of sophistication as they discussed the order of their cards. (See Appendix H for the order developed by each participant.) Fourteen participants placed both easy patterns (A and C) in either the first or second spot. Nine of the participants placed either of the middle difficulty patterns (E or F) in the third or fourth spot. This demonstrated that in a general sense the participants could distinguish between easy patterns and medium difficulty patterns. It was the participants' unfamiliarity with the harder patterns, namely Patterns B, Pattern D, and Pattern E that sparked some disequilibrium. Participants were intrigued by those patterns as they deviated from the more familiar arrangements of Pattern A, Pattern C, and Pattern F.

Karolyn expressed this sentiment as she pondered her struggle with how to manage Pattern E, Pattern B, and Pattern D (see Figure 4.38). She stated:

> Well they kind of have no organization to them. They are in an odd pattern. They are not like a traditional pattern. And the one that has thrown me off a little bit is Pattern D. Pattern B and pattern D are very similar so I'm not quite sure how to organize them. I have them at the end. I don't know which should come first and which should come second I guess pattern D is a little bit more spaced out then Pattern B so maybe that one would come first?" So to see the patterns in parts and then need to put them back together again is so much more challenging that just seeing a familiar pattern, like Pattern C. Also, the unfamiliar patterns really make you pause and think about quantity.

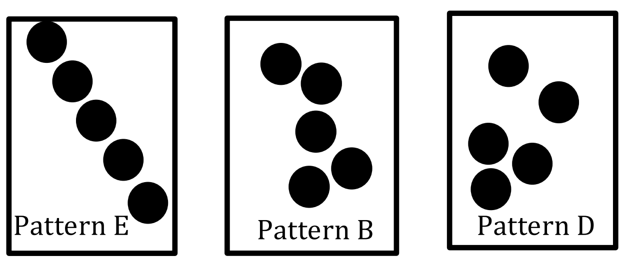


Figure 4.38. These three patterns caused each participant to consider how arranging the same five dots differently increase complexity and sophistication of thought.

Participants provided evidence of pedagogical content knowledge, specifically Knowledge of Content and Students (KCS) (Ball et al., 2008; Sztajn et al., 2012) as they discussed how varying the amount and arrangement of dots might affect young children's reasoning of quantity. The understanding of mathematical content combined with an understanding of students as applied to this study considers a teacher's knowledge of the various levels of the learning trajectory as children progress from less to more sophisticated ways of thinking. The participants demonstrated highly sophisticated KCS for the targeted levels of the subitizing trajectory as they discussed how the amount of dots and the arrangement of dots in a pattern could be used to advance children on the subitizing trajectory.

Theme 2: Centers Instructional Decisions on Children's Thinking

> *If they tell me "I see five!" after looking at Pattern C that tells me that...I don't know! I, I guess I don't really know what that tells me. Like, it's a right answer...(thinking) but do I know if they know it because they have seen it on a dice? Does that tell me that they have been exposed to patterns? Do they really know what five is? But if the child says, "Oh, I saw four and I saw one and I know that's five." That tells me a little more like they see it as a whole, and they also see the groups and see how it's put together. A child may be able to decompose and recompose a particular quantity but if we added one more dot, how challenging, or how much more challenging, does that particular pattern or that particular quantity become? So I guess I'm trying to think about it in steps and also try my best to understand their level and know where they are with their thinking.*

---Karolyn, EC Special Education Major

In the opening quote, Karolyn questioned what a child's response to a dot pattern reveals about their understanding of quantity and how she might evaluate that thinking. She pondered that though a child might respond with the correct answer, is a correct answer sufficient? Does that correct response reveal an understanding of quantity or do they give the right answer because they have seen a dot arrangement so often and simply know how many? Does part whole thinking (e.g., five is composed of a part of four and a part of one) take priority? She is centering her decisions on children's thinking.

Instruction based on learning trajectories requires the teacher to place student thinking at the center of instructional decision-making (Sztajn et al., 2012). When children's thinking serves as the starting point for instructional decisions we find that the act of teaching is simultaneously developmentally appropriate (NAEYC, 2009) and intentional (Epstein, 2012). When teachers mediate their instructional decisions through learning trajectories they engage their pedagogical content knowledge and position themselves to be uniquely responsive to children's developing capabilities as doers and learners of mathematics.

Below, I offer evidence of study participants' willingness to (a) honor young children's mathematical thinking, (b) employ strategies to elicit and understand children's thinking, and (c) provide next steps based on children's thinking intended to advance their mathematical understanding. These sub-themes explore ways in which study participants' enacted their PCK in order to promote children's movement along the subitizing trajectory.

Sub-theme 1: Honoring children's thinking. A standard way to prompt children to subitize is to show them a dot arrangement for three to five seconds and then ask, "How many dots did you see? How did you see them?" In the methods class we referred to this activity as Dot Pattern Flash. When asked how participants might use Set 1 dot patterns with the fictitious five-year olds, fourteen of the participants shared they would engage their children in Dot Pattern Flash as a way to initially investigate the children's subitizing ability and would begin by asking, "How many dots do you see? How do you know?" Justine expanded on that idea when she shared:

> I would start with 'Flash' with the Set 1 cards. I would ask them 'How did you see it?' If they can explain how they saw it, and talk about how they saw it to me, then I would know they understand that quantity. Doing an activity like that would help me to assess the group to see how much they knew already.

Amalie, a 3K teacher for a private childcare agency, added:

> I would probably start with Dot Pattern Flash. That would give me an idea of who understands it [quantity]. Who maybe doesn't. And then I could take it from there.

> That would help me figure out what games we might want to play or other activities that we could do with these.

Amber suggested, "I would just flash the pattern so they can like see [quantity] and like subitize. You know, see the group of numbers. But I don't think they could do that right away. You kinda need to scaffold them into it." These comments attest to the participants' interest in their children's thinking, intentionally facilitating a learning opportunity to allow children to share their thinking, and using that thinking to launch learning experiences.

Karina was the sole participant that would not initially use "Dot Pattern Flash" with her three-, four-, and five-year old students with identified learning needs in mathematics. She was adamant that her children were not ready to have the dot image taken away after a few seconds look. She explained:

> So what I do right now, is I show a quantity on the dot pattern card, then I asked them to show me how many they saw on their five frame. (See Figure 4.39.) This is a big step up! We actually started this activity by using some of the pre-printed five frames that are filled in, from one of the games that we played in [the methods] class. So I would hold it up, and then they would make it on the five frame. But the one thing that I did was I never took it away. My kids would kinda forget what they were looking at. It's really important that I leave it there for them to look at and think about. I know that sounds really easy, but this is really hard for them right now. I keep my focus on quantity.

By leaving the card visible and asking her children to make the same quantity that they see on the card on the five frame Karina intentionally centered her instruction on her children's current ability to manage quantity.

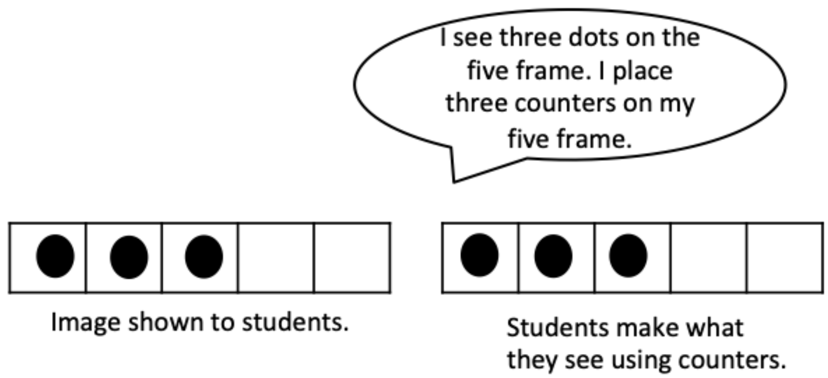


Figure 4.39. Karina shows how she worked to understand her children's thinking using a pre-filled five frame and asking her children to make what they saw.

Centering instruction decisions on children's thinking is one of the central components of instruction guided by learning trajectories. To that end, learning trajectory levels are described suing children's thinking. These descriptions are helpful as they provide teachers the needed information to match a child's developmental level with instructional tasks.

Jaeden highlighted the need to know her children's current level of understanding when asked to create a pattern to add in to the Set 2 sequence. She explained:

> So before I draw anything [a new pattern] I guess I have to think about where my kids are. I mean if they are understanding "five" I would add in a pattern that is above five. But if the kids were struggling with five I would probably give them a dot pattern of four.

Many participants were keen to begin with a pattern they believed would be familiar to the children. This belief influenced the order of the Set 2 dot cards and specifically how participants viewed the children's ability to interpret the quantity on subsequent cards. Flora made this explicit when she shared:

> I think with the scattered patterns [Pattern B and Pattern D] are going to take them longer to recognize. They're not really familiar with those more scattered patterns. It would be easier for them to say, "I see this number" (she motions to Pattern C.) They can go back

to something they've already seen before and something they can talk about, so that's why I would start with something that they already know and then move forward from there.

Each of the three participants seeking bilingual certification, Amber, Flora, and Marisol, stated they would most certainly begin with Dot Pattern Flash as a way to engage children's thinking of quantity. Each agreed they would ask children "How many dots do you see?" and each agreed on what they not do at least initially, which was to ask children "How do you see them?" Central to the discussion were their personal experiences as English Language Learners.

Amber, who arrived in this country at the age of five, knowing no English and receiving limited English language support during her elementary years, stated:

> Like if you write 2 [the symbol] I would have no idea how to read that or say it but if I see two dots then I know what to call those dots. I like the different representations. They make it easier for someone that doesn't know the language. For a while I did not know the language and math was a foreign language to me, too.

Flora, who emigrated from Mexico and attended bilingual schools throughout her K-12 education, discussed similar ideas when she shared:

> I would start first with just saying the number names with the children and then we could begin to move forward to using the patterns. I want them to be able to recognize the symbolic representation for the numbers and to be able to name them because without those names they might not even have anything to say when I ask them how many dots do they see. I would slowly move forward with the different patterns. But, I guess it could work both ways. Like I could start with the dot pattern I suppose. But I feel like if I would just start with the dot pattern it would be more difficult because I feel like... I guess it [the dot pattern] would be a better visual representation because they would

> understand what this symbol means and it might be easier to transfer from the dot pattern to the numbers, but I really don't know. I guess actually I would just want to use them together.

Finally, Marisol, a native Puerto Rican, teaching five-year-olds in a language immersion program, was already actively using the dot patterns with her students at the time of the interview. She shared:

> Right now, when we are together as a whole group I flash the dot cards. The first time that I did this with them I did not want them to tell me quickly how many dots they saw. I wanted them to count the dots one by one and then tell me how many there were on each card. I even let them use their fingers and count the patterns. For the first two weeks that we used the dot patterns I just asked them how many did they see. If they needed to count, I let them. And now already this week, I am giving them a three second look. And I tell them "Give me a thumbs up when you're ready." Then I ask them, "Who knows how many dots there are?" And, you know, they are they're doing very well. But I did not begin immediately with "How do you see them?" This week we're going to begin with "How do you see them?" And I'm going to ask them now explain to me how they see the dots in the patterns.

The three prospective early childhood bilingual teachers were only ones to place a specific focus on learning number words first as a precursor to exploring children's understanding of quantity. This was unique to the interview data. Amber, Flora, and Marisol all agreed, they would delve into their children's understanding of quantity only after they were confident the children knew the names of the numbers. I can only speculate that these three participants may have experienced a time when they did not have the needed language skills or vocabulary to express what they knew. Children's thinking surfaced as important to each of them and it seems they be providing an avenue to ensure children have a foundation from which to share their thinking.

Study participants reinforced the idea that responding to what we learn when children share their thinking lies at the heart of effective teaching and meaningful learning. Karina shared that figuring all that out takes time and focus. She continued:

> I don't want them to get overwhelmed if they don't have some of the earlier skills in place. It's not productive. You know for example, if they can correctly make the pattern with counters but they think that there are ten when there is only five then they have no concept of the meaning of the quantity, yet.

The purpose of this first sub-theme was to highlight study participants' awareness of and interest in honoring young children's thinking. Acknowledging the importance of children's thinking is foundational to effective teaching and meaningful learning. Centering the work of teaching on children's thinking entails skill in both interpreting and eliciting student thinking. Eliciting young children' mathematical thinking presents a formidable challenge for a variety of reasons. Study participants' strategies for eliciting thinking are evidenced in the next sub-theme.

Sub-theme 2: Strategies to elicit children's thinking. Responding appropriately to children's thinking requires that children share their thinking. As children's language skills are still growing asking them to simply explain their thinking provides a narrow and many times inadequate window into their cognitive processes. Study participants engaged in two explicit strategies intended to elicit and help them understand children's thinking. First, they posed questions intended to engage children in a mathematical discussion. Second, they engaged a variety of mathematical representations to help children express their thinking in ways other than words.

Participants were asked at the beginning of the interview if they would use the Set 1 dot patterns with their students, each responded with yes. Figure 4.40 displays the Set 1 dot patterns.

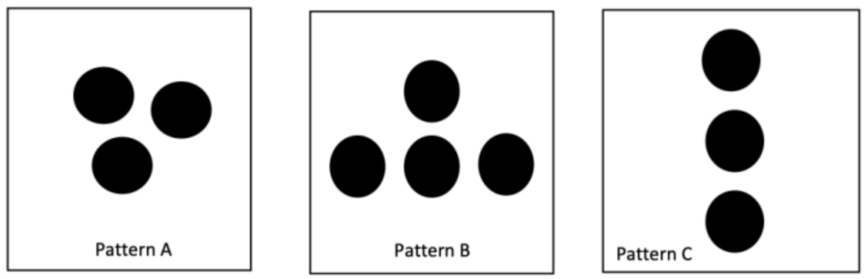


Figure 4.40. Set 1 dots patterns used to begin the interview and set the stage for subitizing.

When asked how they might use those patterns with their students, the common entry point was to ask, “How many dots do you see?” and “How do you see them?” When asked why those questions are important Karolyn replied, “Because our intention is that they be able to tell us what the whole quantity is and how they see that quantity.” Kayla provided additional insight when she shared:

> If I don’t ask “How did you see it?” I think that kids might just memorize the words for counting like one, two, three. They might not understand that those words actually mean something. Like they stand for an amount and the number one stands for the quantity of one. I don't want them to repeat what they hear. I want them to understand it and I want to know their thinking. That’s why I ask, “How did you see it?”

Crystal, Karolyn, and Marie suggested a specific instructional framework called a number talk to provide a context for the “How many?” and “How do see them?” questions. A number talk is a five-minute routine intentionally structured to actively engage children’s understanding of quantity. When asked why she would include those questions in a number talk, Crystal shared:

> Well, I would want them to talk about it [how they see the dots] and a number talk would help with that. I would listen to if they are able to explain their thinking. I think I would also ask them how are the patterns different and how they are the same.

Jaeden expands upon the questions she would ask during a number talk as she refers to Set 1 Pattern B. (See Figure 4.41.) She explained:

> During the number talk, if I showed them this pattern of four I would question them like "Oh, you saw two and two. Did you put these two together or did you put these two together?" or "Oh, you see three! What about that one? What are you going to do with that one dot over there?" (She points out the quantities she mentions as she poses the questions.) I would ask them "How did you see that?" a lot. I also might ask them "How did your friends see that pattern?" and "How might that compare or contrast with how you saw it?" I think that's really important and I would expect them to tell me.

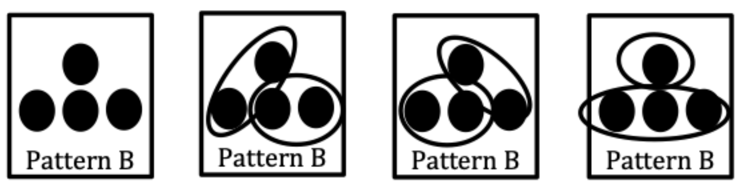


Figure 4.41. Jaeden refers to Pattern B as she states the questions she would her students.

In addition to asking questions to elicit and understand children's thinking study participants explored various ways for children to show their thinking. Two common suggestions included matching a dot pattern with its corresponding symbolic representation and using counters to recreate the pattern and at times transfer those counters to a five or ten frame. Figure 4.42 displays both suggestions. The photograph on the left shows a numeral card of three placed to the left of the dot pattern showing three. Flora suggested pairing the dot pattern with the symbol as a way "to help children understand what the symbols mean."

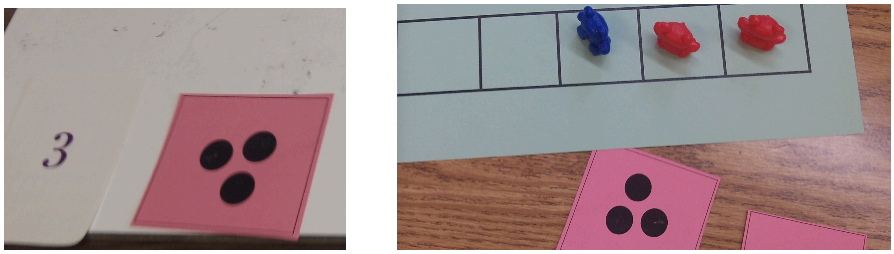


Figure 4.42. An example of matching a dot pattern with its symbolic representation and transferring the number of dots onto a five frame using teddy bear counters.

Karaleen suggested, "I think I would start by giving them the cards [with three dots] (see Figure 4.43) and I would ask them to fill in the ten frame. That way they can show me what that looks like."

The photograph on the right (see Figure 4.42) shows that same dot pattern of three represented on a five frame using teddy bear counters. Kayla suggested, "I would want them [the children] to find the number that matches it so they realize that the number, the written number three, actually has meaning to it." Transferring the quantity on the dot pattern to the five frame, according to Sasha, "helps children see three. Then we can see that we are anchoring it [three] to the quantity of five. So they see they need two more to get to five.

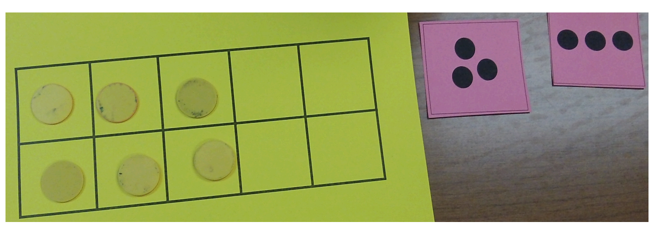


Figure 4.43. Karaleen places two different arrangements of three next to the ten frame. She place three counters on top to show the first group of three and three counters on the bottom to show the second group of three.

Karina acknowledged the important mathematical work children engage in as they see the same quantity in different formats. She explains her intentional structure and why moving among various representations of the same quantity is so challenging.

> I show them the pre-made five frame cards and then I asked them to show me that same quantity on their own five frame. Then after we did that for a while I would show them the dot pattern and then they would have to make that same quantity on their five frame. Now keep in mind that we did dot patterns at the beginning of the school year. So they're familiar with those cards. So now we take those dot patterns and I show them the cards and I asked them to show that quantity on their five frame. And that is really hard for them. Because then if they are counting, because they don't see it yet [subitize], they have to count them and remember how much they counted and then place that same amount on the five frame. So there is a lot happening. Cardinality for sure, in particular if they aren't subitizing.

Amalie and Marisol suggested that children could draw the pattern they are briefly shown. Amalie reasoned that if children struggled to explain their thinking with words, or were unsuccessful transferring the quantities to a five frame having the children create the pattern through a drawing might help. She suggested, "…if they can't figure it out, then I think I would probably have them draw it. That would make them see the pattern and how many dots there are." Marisol added:

> The other thing that I would do with them is let them draw what they saw. So I would give them a small white board and a marker. They could tell me how they saw them, for example I see two and one more, and then I would want them to draw the pattern make a circle around the two that they see and the one that they find on top.

Acknowledging that children can demonstrate their thinking in a variety of ways was important to Karina. As a teacher of young children with identified learning needs in mathematics she addressed how she used a variety strategies to allow children to express their thinking. Karina shared:

> Well, I think that sometimes the verbal response [to "How many do you see?"] might not always match with what the kids actually see. I think that sometimes the kids do see the correct amount but end up verbalizing incorrectly for whatever that reason may be. I would probably flash that pattern again and ask them, "Can you show me what you see?" So I would probably use counters, even using their fingers. That's fine with me because they're so young.

Responding to questions, drawing the dot patterns, and using manipulative to recreate patterns highlighted participants' awareness of young children's mathematical thinking and how to leverage thinking to advance children's learning. In the final section I provide evidence supporting the instructional decisions participants identified to help children explore a misconception.

Sub-theme 3: Interpreting and engaging with children's mathematical thinking. The interview protocol engaged participants in addressing a misconception in student thinking. They were told that the fictitious group of five-year olds offered a variety of answers to the question "How many dots do you see?" for Pattern F. (See Figure 4.44.) Participants were asked how they might respond to the children's answers of eight, or nine, or ten dots in Pattern F.

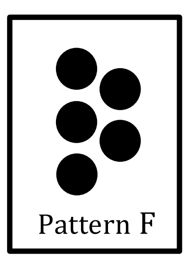


Figure 4.44. Pattern F was used to prompt participants' thinking regarding children's wrong answer to "How many dots do you see?"

After listening to the scenario each of the participants took an immediate interest in pursuing what provoked the wrong answer. After describing an initial instructional strategy they would used to investigate the answer of "ten," I asked participants to describe how they would follow up if that initial instructional strategy did not result in an accurate answer of "five." The participants subsequently offered a first, and in all cases but one, a second follow up strategy to explore the children's misconceptions. Table 4.5 displays the thirteen instructional strategies suggested by participants, at what point the strategy was discussed during the scenario, initial, follow up one, and follow up two, and the number of times the strategy was suggested. (See Appendix I for a more detailed description of participants' initial strategies and follow up strategies for investigating children's misconceptions with Pattern F.)

Participants suggested a variety of instructional strategies for investigating the challenges posed by Pattern F. The thirteen strategies were further grouped under six broad categories: (1) Ask how and listen; (2) Count to find out; (3) Show me; (4) I show you and relate to ten; (5) Passive engagement of students; and (6) Keep subitizing.

Table 4.5. Participants' instructional strategies for investigating Pattern F misconceptions, at what point it was suggested, and frequency of use.

<table>
<tr><th>Instructional Strategy</th><th>Initial</th><th>Follow Up One</th><th>Follow Up Two</th><th>Subtotal</th><th>Total</th></tr>
<tr><td>Ask how and listen.
Teacher asked, "How did you see ten?" Teacher listens to children.</td><td>5</td><td>6</td><td>0</td><td>11</td><td>11</td></tr>
<tr><td>Count to find out.
Teacher asked students to count the dots.</td><td>1</td><td>4</td><td>2</td><td>7</td><td rowspan="2">12</td></tr>
<tr><td>Count to find out.
Teacher counted the dots. Said, "Five."</td><td>1</td><td>0</td><td>0</td><td>1</td></tr>
</table>

Count to find out. Match counters to dots and count.	0	0	4	4	
Show me. Teacher asked, "Show me. Make it with counters."	1	2	0	3	6
Show me. Teacher asked, "Show me. Draw what you see."	2	1	0	2	
Show me. Teacher offered an open invitation to children. "Show me what you see."	1	0	0	1	
I show you and relate to ten. Teacher used a five or ten frame.	2	0	2	4	4
Passive engagement of students. Teacher prompted, "Look again."	0	2	0	2	3
Passive engagement of students. Teacher drew Pattern F on board following children's instructions.	1	0	0	1	
Keep subitizing. Teacher offered a smaller quantity to subitize.	0	0	2	2	4
Keep subitizing. Teacher showed class Pattern A and then returned to Pattern F.	1	0	0	1	
Keep subitizing. Teacher reoriented dot pattern cards.	0	1	0	1	

Ask how and listen. The most popular instructional strategy, *Teacher asked, "How did you see ten?" Children explain. Teacher listens*. I place this strategy under broader category of *how why and listen.* Eleven participants stated they would use this strategy, five as an initial strategy and four as a first follow up strategy. Participants suggested their next instructional steps would "depend on their [the children's] answers" (Marisol).

Three participants articulated why they would begin by asking children to tell them how they saw ten. Karaleen, the experienced 5K teacher, laughed out loud as she listened to the scenario. She shared:

> That sure sounds familiar! The first thing that I probably would do is ask them *why* do they think there are ten dots on that card? You know I might just ask them, "How do you see ten?" I would let them look at it for a while and then I would listen to what they have to say. I need to know their thinking.

Kayla stated she would begin this way because, "I would…try to gauge where they were getting that number of dots from. Like where did they see ten dots? Like what patterns did they see?" Amber supported her decision by explaining:

> I would ask them, "Where did that number come from? Where did you see it? How did you see it?" I think that just saying, just telling them, that it is wrong is not the right way. I feel like if I tell them like, 'No, its five.' They are not learning because they did not get to count or check it themselves.

Count to find out. Three strategies fall under what I refer to as *count to find out*. They include: (a) *Teacher asked students to count the dots; (b) Match counters to dots and count;* and *(c) Teacher counted the dots and said, "Five."* Twelve participants suggested counting to find out how many. Two would use this as an initial instructional strategy, four as the first follow-up strategy, and six as the second follow up strategy. For example, Marie shared, "You saw ten? I am going to show it to you again. Let's count carefully." She suggests that if children are losing one-to-one correspondence as they count she would place one bear on each dot and then transfer the bears to a number path (see Figure 4.45.). She explained:

I would certainly use the counters. Something they can move around. I would set one counter on each dot…Then maybe I would bring in the number path and transfer the bears to the number path. I might say we have five bears here. Each is sitting on its own dot. I would transfer them over to the number path and say "See, here they are and we can count them again, one, two, three, four, and five." That might help them see what five means and that it is not the same as ten.

Figure 4.45. Marie places one bear counter on each dot and the transfers the dots to the number path with the goal of helping reinforce the concept of five.

Though counting is an efficient approach to finding how many and does help children name quantity, counting does divert from the goal of the subitizing trajectory, which is to see groups, not count by ones. In addition, participants did not identify their intention for why they would have the children count. I conjecture participants prioritized getting the right answer over the goal of the trajectory which was subitizing.

Show me. Three instructional strategies fall under what I refer to as *show me*. They include: (a) *Show me. Make it with counters; (b) Show me. Draw what you see;* and, *(c) Teacher offered an open invitation to "Show me what you see."* Six participants suggested they would encourage children to show them where they see ten dots, four as an initial strategy and two as a first follow up strategy. Justine, who offered an open invitation for children to show her how they see ten conjectured:

> They would probably come up and point to me where they see the ten. Hopefully they would correct themselves and say like, "I see two here and three here." Or, they would count then "one, two, three, four, five." and say "Oh!! That is five." But they need to figure that out. (Figure 4.46 displays the two ways children might discover their own errors as they show Justine what they saw.)

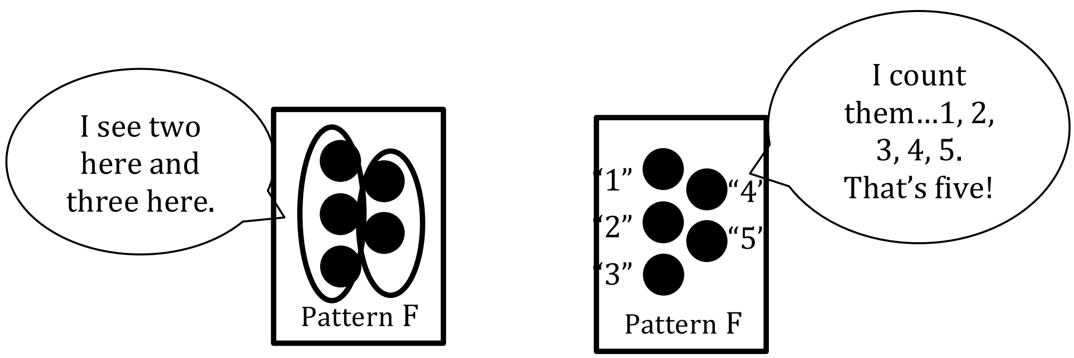


Figure 4.46. Justine demonstrates how the children might her “how many” in hopes of them self-correcting.

Karina, who teaches young children with learning disabilities in mathematics, explained that she would have the children make that pattern with counters after stating they saw ten. She explained:

> I think that sometimes the kids do see the correct amount but verbalize incorrectly for whatever that reason may be. So I would probably ask the kids to make the pattern using the counters. If the kid made it, I would probably be like, “ Oh, wow! Good job. How many is that?” And then you kind of know where they are, if they, if they're counting wrong or really don’t know how many.

Amber suggested, “I think I would have them show me by making the pattern. If I told them to make the pattern…it would really make them see the five.” Figure 4.47 displays Amber’s example of what she means when she says she would ask children to make the pattern.

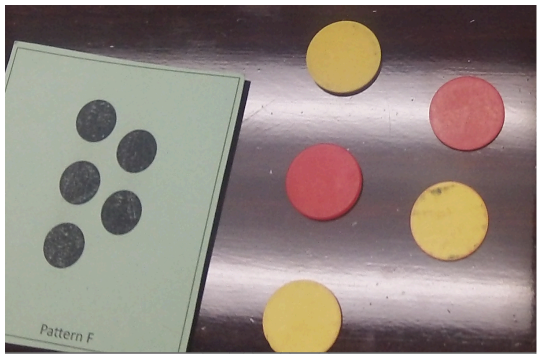
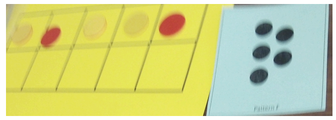


Figure 4.47. Amber shows how children might use counters to make Pattern F as a way to prove there are five dots in the pattern and not ten.

Instead of making the pattern with counters Karaleen would have her student draw what they see on a white board. Karaleen shared, “I guess I don't have a real reason why I would ask them to draw the pattern. I just think, I guess I'm just so used to having kids come up and show their work.” Sasha, who would also have the children draw what they see, explained that after the children drew Pattern F she would ask, “How do you see ten? Can you show me?”

I show you and relate to ten. Four participants suggested the strategy I refer to as *I show you and relate to ten*. Cecilia, Flora, Jaeden, and Karaleen each modeled the quantity of five on the five frame or the ten frame to show the children there are five. Cecilia and Jaeden as an initial strategy (see Figure 4.48) and Jaeden and Karaleen as the second follow up strategy.



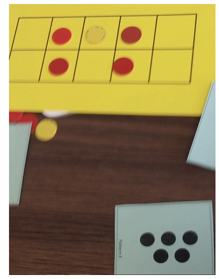


Figure 4.48. Cecilia (on the left) and Jaeden (on the right) each model the five dots of Pattern F on the ten frame.

Explaining her decision for why she would place the dots on the frame, Cecilia shared:

> I guess if they thought that they saw ten, I might pull out a ten frame. That might help them see kind of how much ten is in a different way. I guess maybe I could use the counters and put the counters on the ten frame. If they are familiar with the ten frame, they would know that only half of the ten frame would be filled, so it can't possibly be ten.

Flora and Karaleen transferred counters to the ten frame for their second follow up strategy and echoed a similar rationale. Both stated they would place one counter on each dot and then put those counters on the ten frame. Flora followed, "I think the ten frame would help them [the children] see the number five differently."

Passive engagement of students. Three participants engaged children using *passive engagement strategies meaning the strategy did not actively engage children's reasoning and thinking.* Mandisa suggested she would draw what the children tell her as they explain how they saw ten dots. She shared her rationale, "I would probably start by writing down what they say, definitely. I guess I would want to record it so I can see what they're saying. It would be important for me to break apart what they're saying." Both Crystal and Kayla suggested they would give children a second look. Kayla suggested she "would flash the pattern again, quickly, and have them explain how they saw it [Pattern F].” Crystal would let the children “look at the card again and recount.”

Keep subitizing. Four participants highlighted strategies I refer to as *keep subitizing.* They include: (a) *Teacher offered a smaller quantity to subitize;(b) Teacher showed Pattern A and then returned to Pattern F;* and, (c) *Teacher reoriented dot pattern cards.* Amalie, Marisol, and Karolyn offered instructional strategies that focused on subitizing. Amalie suggested that if Pattern F proved challenging, she would revisit Pattern A and then attempt Pattern F again. An additional change would be to rotate Pattern A and Pattern E so the dots were horizontal. She explained:

> I would flash Pattern A and if the children were able to tell me five what I would probably do next is then flash Pattern F and see if they might adjust their thinking. I see Pattern A and Pattern F as very similar. The difference is the two dots in Pattern F are shifted over from where they are in Pattern A. They are basically the same pattern. And I would probably skip Pattern B and Pattern D as I try to figure out what's going on. So if I still get weird answers for Pattern F after I have flashed Pattern A in the original way with the dots going up and down, I might turn Pattern A and Pattern F on their side to see if that would help. (See Figure 4.49.)

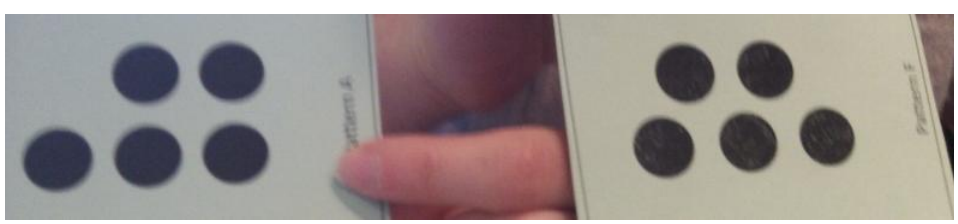


Figure 4.49. Amalie suggests rotating Pattern A and Pattern F so they have similar orientations as a strategy to support subitizing.

Marisol and Karolyn both recommended a smaller quantity for the children to subitize. Marisol suggested, “What I might do depending on their answers is go back to smaller quantities in the other patterns I use with them. Or, I might even try a different type of pattern to see if that would help." Moving to a smaller quantity to subitize was Karolyn's second follow up strategy. She explained, "I guess I think I might need to bring them [the children] back down to a smaller quantity. Maybe five is just too much for them right now."

Examining the Pattern F scenario provides insight into participants' pedagogical content matter knowledge as they made instructional decisions. Matching instructional decisions to a child's current level of development is a centerpiece of learning trajectory-based instruction and participants addressed that idea in a number of ways. First, participants acknowledged children are capable of mathematical thinking and positioned it as a priority to their decision-making. Second, participants engaged a variety of strategies to better understand children's thinking and to be responsive to their thinking. Finally, participants shared thirteen different ways they would respond to children's thinking. The vast majority (twelve out of thirteen) aligned with the goal of the subitizing trajectory. The evidence in this final theme suggests knowledge of a learning trajectory may be of particular importance to nurturing prospective early childhood teachers' skill with developmentally appropriate and intentional instruction meant to advance young children's mathematical thinking.

Summary

This chapter identified five themes that emerged from the fifteen interviews conducted with ECE prospective teachers. All participants at the time of the study were seeking state teacher certification and intended to teach young children in either public or private schools or in childcare settings. Five participants were full-time early childhood teachers. Ten participants worked with young children through part-time employment at local day care centers and field experiences as part of ECE program requirements.

This qualitative, phenomenological study provided rich, descriptive data needed to investigate how fifteen prospective early childhood teachers' mathematical knowledge needed for teaching and early mathematics learning trajectory understanding impacted the intentionality of decision-making. Figure 4.50 displays the five emergent themes based on the analysis of the data collected: (a) SMK Theme 1: Demonstrates an Understanding of Subitizing, (b) SMK Theme 2: Recognizes and Validates the Importance of Subitizing for Young Children, (c) SMK Theme 3: Articulates Learning Trajectory Progression Through Dot Arrangements, (d) PCK Theme 1: Demonstrates an Awareness of the Developmental Nature of Children's Mathematical Thinking, and (e) PCK Theme 2: Centers Instructional Decisions on Children's Thinking.

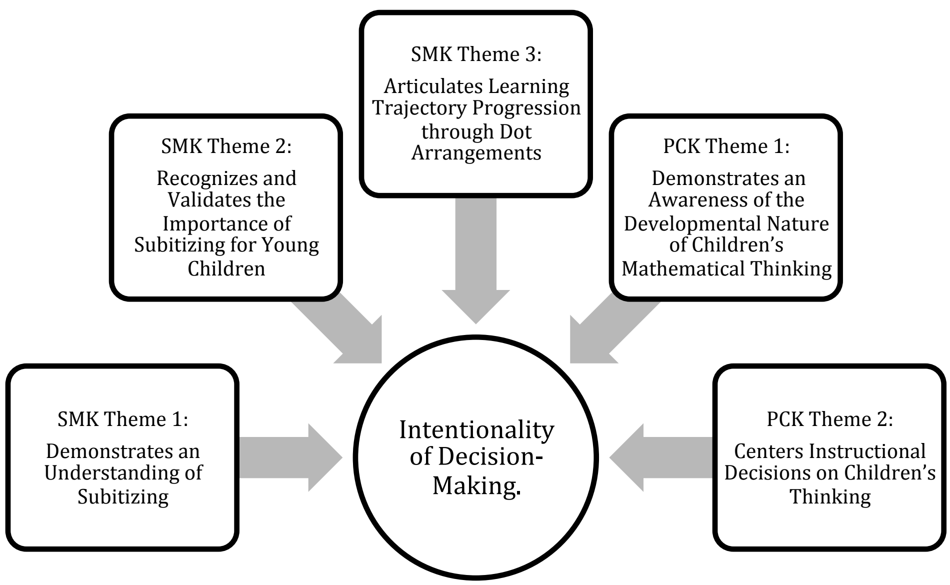


Figure 4.50. This diagram displays the five emergent themes of this study as contributing to the phenomena of intentionality of decision-making.

The first SMK theme highlighted participants' understanding of conceptual and perceptual subitizing. They demonstrated their understanding as they discussed how children would name *how many* and articulated their goals for using the Set 1 dot pattern cards with young children. They acknowledged small quantities (1-5) as opportunities for perceptual subitizing. They also acknowledge the arrangement and spacing of items mattered and even an arrangement as small as four dots could be conceptually subitized if organized appropriately. Though almost half of the participants could not independently remember the term *subitizing* each of them successfully described both types through references back to the Set 1 and Set 2 dot patterns. A deep and nuanced understanding of subitizing equipped the prospective ECE teachers in this study to be uniquely responsive to children's understanding of quantity. This responsiveness was evidenced in the questions they asked as the fictitious children engage in subitizing experiences, the mathematics they chose to highlight, and the depth of thinking they expected their young learners to share.

The second SMK theme illustrated participants' understanding of the important and unique niche subitizing skills and abilities occupy in young children's mathematical development. Understanding the meaning of numbers surfaced as an important touchstone for why subitizing is important to develop with young learners. Three sub-themes identified the role subitizing plays in helping children understanding cardinality and part-whole relationships. In addition, launching early addition and subtraction ideas while conceptually subitizing more complex patterns were key to the theme. For these reasons, subitizing is highlighted as a key component to the mathematics programming in early childhood classrooms (Nguyen et al., 2016). Understanding the short- and long-term benefits of strong subitizing skills supported intentionality of instructional decision-making in two ways. First, this knowledge ensured the study participants clearly understood why they were engaging in subitizing work with young learners. Second, this knowledge helped study participants be aware of and knowingly attend to the key understandings in children's thinking, specifically for subitizing, the shift from conceptual to perceptual subitizing.

The third SMK theme provided insight into participants' general knowledge regarding early mathematics learning trajectories and specifically the subitizing trajectory. Participants ordered the six Set B cards, thoughtfully addressed Pattern E, which in their eyes, did not promote subitizing and created a pattern and rationalized its placement in the Set 2 sequence. Overall, each of the fifteen participants attended to the mathematical big idea of the trajectory, that being subitizing, and enacted their knowledge of subitizing and the subitizing trajectory. Knowledge of an early mathematics learning trajectory helps teachers approach mathematics instruction in a "coherent, planful manner" (Epstein, 2014, p. 130) supporting intentional teaching of mathematics. Learning trajectories provide guidance to ECE teachers as they:

- design instructional experiences that ensure children encounter mathematical concepts in depth and in a logical sequence;
- plan for developmentally appropriate instructional "next steps" for all levels of learners; and
- make decisions about when to intervene and provide more focused and purposeful instruction.

The first PCK theme addressed the participants' capacity to recognize the developmental nature of children's subitizing skill and ability. Collectively, their comments acknowledged children's mathematical growth as developmental in nature amount. Each participant took ownership of facilitating children's growth. To varying degrees each participant relied on their understanding of the developmental path of the learning trajectory to intentionally nurture that growth. Participants enacted their PCK as they acknowledged the cognitive steps needed to take by children to support such development and explored a variety of teacher-initiated strategies intended to nurture that growth. When the ECE teachers viewed children's mathematical growth as developmental it contributed to intentionality in several ways. The ECE teachers appeared to:

- ensure meaningful engagement with key mathematical ideas;
- understand when a child needs more time and opportunity with a particular concept;
- monitor growth and recognize gaps children's in knowledge;
- position young children as mathematically competent and capable.

The second PCK theme revealed participants' willingness to prioritize children's thinking as they make instructional decisions. Participants' responses revealed their intense interest in children's thinking and data analysis surfaced numerous ways participants would elicit and respond to children's thinking. Those responses, mediated by an understanding of subitizing and the subitizing learning trajectory, suggested study participants viewed their instructional decisions as impacting children's growth along the trajectory. Prioritizing young children's thinking contributes to intentionality in important ways. ECE teachers appeared to center instructional decisions on children's thinking when they probed children's thinking around how they saw the various dot patterns and explored a variety of strategies for eliciting and responding to children's thinking. This information supported study participants to share how they would intentionally scaffold classroom interactions to meet and further develop children's mathematical thinking.

The five themes provide insight into participants' intentionality to make instructional decisions meant to advance young children's growth on the subitizing trajectory. Chapter Five offers a discussion of the findings as they relate to the research questions: In what ways do early mathematics learning trajectories support early childhood prospective teachers' preparation to become effective teachers? What understandings do early childhood prospective teachers have regarding the subitizing trajectory? Do early childhood prospective teachers draw upon their knowledge of learning trajectories as they make instructional decisions? The study's conclusions and limitations are also shared. The chapter ends with recommendations for future research and early childhood teacher education programs as they look to improve mathematics education courses for early childhood preservice.

CHAPTER FIVE

DISCUSSION AND CONCLUSIONS

The purpose of this phenomenological study was to understand the intentionality of early childhood teachers' decision-making meant to advance young children's mathematics learning. The research study was guided by the following questions:

- Central Research Question: In what ways do learning trajectories inform prospective early childhood teachers' instructional decisions in ways that are likely to advance student learning on the subitizing trajectory?
- Attendant Question 1: What understandings do early childhood pre-service teachers have regarding the subitizing trajectory?
- Attendant Question 2: Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?

Chapter 5 begins by revisiting the study's conceptual framework and its relationship to the identified themes and sub-themes. Then the study's findings are discussed with connections to the extant literature. The study's conclusions are presented next in relationship to the research questions. The chapter then addresses the limitations of this study and presents suggestions for future research. Finally, the chapter presents implications for future research and recommendations for early childhood preservice teacher education.

A semi-structured interview protocol was used to conduct interviews with fifteen prospective early childhood teachers. The protocol featured four stimulus texts. The first prompted participants' thinking in the mathematical content of the interview, subitizing. The second explored participants' understanding of the developmental progression of the subitizing learning trajectory. The third prompted participants to share strategies they would use to respond to an error in student thinking. The fourth encouraged participants to explain and justify an instructional decision by asking them to create and add in a dot pattern to an established sequence of dot patterns. The data gathered from these stimulus texts informed the five themes and each theme in turn provided insight into the phenomena of intentional decision-making.

Conceptual Framework and Related Themes

The conceptual framework shown in Figure 5.1 grounded the analysis of data and framed the findings of this study. The framework suggests prospective early childhood teachers' knowledge of mathematics learning trajectories and their developing mathematical knowledge needed for teaching (MKT) unite to support intentional instructional decisions that facilitate young children's mathematical growth.

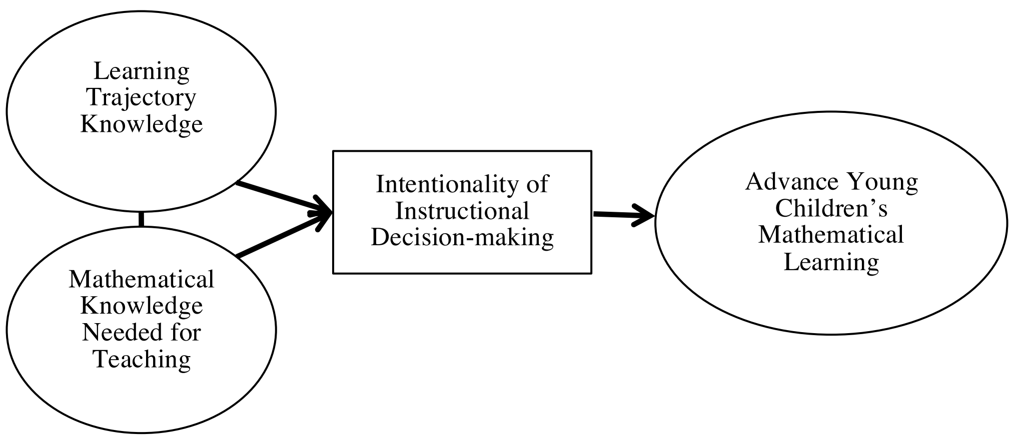


Figure 5.1. The conceptual framework for this study.

Ball et al. (2008) suggest MKT as a framework for examining teachers' knowledge for teaching mathematics. Two broad categories comprise MKT—subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Subject matter knowledge includes knowledge of mathematical concepts, structures, and procedures. Specifically, SMK assumes three sub-categories of knowledge: (1) mathematical knowledge used in any setting, not necessarily in the setting of teaching; (2) mathematical knowledge and skills needed uniquely by teachers; and, (3) an awareness of how mathematical ideas grow in complexity and sophistication overtime. Pedagogical content knowledge is an "amalgam of knowledge that combines the knowing of content with the knowing of students and pedagogy" (Ball et al., 2008, p. 398). It is mathematical knowledge that is quintessentially unique to teaching. Specifically, PCK assumes three sub-categories: (1) knowledge that combines knowing about students and knowing about mathematics; (2) knowledge of how to design instruction to ensure learning mathematics with understanding; and, (3) knowledge of how to use instructional materials in ways that advance student learning of mathematics. PCK illuminates the ways in which teachers relate what they know about teaching to what they know about the content they teach.

As I analyzed the data and identified themes, a clear distinction emerged categorizing participants' knowledge as either SMK or PCK. Table 5.1 displays the five themes and related sub-themes as they relate to subject matter knowledge and pedagogical content knowledge. Within the category of subject matter knowledge themes emerged that suggested participants demonstrated an understanding of subitizing, validated the importance of subitizing, and articulated knowledge of a developmental progression of subitizing. Within the category of pedagogical content knowledge themes emerged that suggested participants demonstrated an awareness of the developmental nature of children's mathematical thinking and centered instructional decisions on both children's thinking and the developmental progression. The five themes and their relationship to the two broad categories of MKT provided insight into the phenomena of intentional teaching of mathematics.

Table 5.1 Mathematical Knowledge Needed for Teaching and Related Themes and Sub-Themes

Mathematical Knowledge Needed for Teaching	Theme	Related Sub-theme
Subject Matter Knowledge	Demonstrates an Understanding of Subitizing	N/A
	Recognizes and Validates the Importance of Subitizing for Young Children	Sub-theme 1: Subitizing helps children understand number as quantity Sub-theme 2: Subitizing engages children's reasoning about cardinality Sub-theme 3: Subitizing lays the foundation for early addition and subtraction
	Articulates Learning Trajectory Progression Through Dot Arrangements	Sub-theme 1: Order matters Sub-theme 2: Pattern E—Honoring Subitizing and Working Within the Progression Sub-theme 3: Rationale for "added in" pattern
Pedagogical Content Knowledge	Demonstrates an Awareness of the Developmental Nature of Children's Mathematical Thinking	Sub-theme 1: Acknowledgement of developmental growth in mathematics Sub-theme 2: Amount and Arrangement of Dots Impact Growth in Subitizing
	Centers Instructional Decisions on Children's Thinking	Sub-theme 1: Honoring Children's Thinking Sub-theme 2: Strategies to Elicit Children's Thinking Sub-theme 3: Interpreting and Engaging with Children's Mathematical Thinking

For this study, I view SMK as settling squarely on a teacher's understanding of the big idea of the learning trajectory, in this case, subitizing. Two additional components include knowledge of why subitizing is important to current and future learning, and knowledge of how subitizing grows in sophistication and complexity according to the subitizing learning trajectory. For this study, PCK initiates from a developmental perspective of mathematics learning that is dependent upon experience and opportunities. In addition, I view PCK as knowledge of activities, tools, and math talk used to advance children on the subitizing trajectory, knowledge of how to elicit children's subitizing abilities and use children's thinking as starting points for instruction, and knowledge of how to respond to children's thinking in ways that aligns with their developmental level on the subitizing trajectory.

Discussion of the Findings

The rising status of early childhood mathematics education has placed a spotlight on what mathematics ECE teachers should know and how they should be trained to teach. As a result, national associations (e.g., AMTE, 2017; NAEYC, 2012; NAEYC and NCTM 2010; NRC, 2008) have called for ECE teachers to possess a deep understanding of important mathematical content for young children and to teach that content in ways that are "intellectually meaningful and respectful of the needs of young children" (Parks & Wager, 2015, p. 125). To date, limited research has been done to influence and inform the teaching of mathematics methods courses for prospective early childhood teachers (Moss et al., 2016; Parks & Wager, 2015). The results of this current study suggest the development of subject matter knowledge and pedagogical content knowledge within prospective early childhood teachers are necessary components for providing mathematically rich and developmentally appropriate instruction in the early years and that learning trajectory knowledge may facilitate a fluid and seamless interaction between both categories of MKT.

Discussion of Subject Matter Knowledge Findings

This study found that prospective teachers demonstrated an in-depth understanding of subitizing. Though many participants could not remember the word *subitizing*, each demonstrated their understanding of the concept throughout the course of the interview. They accurately discussed perceptual subitizing as the ability to immediately visualize and name the number of objects in a collection of four or fewer objects. Conceptual subitizing was discussed as participants identified opportunities for children to recognize smaller quantities and quickly combine them to find the whole. They recognized that the same amount of dots in distinct arrangements could prompt very different thinking in children. When creating a dot pattern to add in to their pre-established dot sequence participants commented that they did not want the dots too far apart or jump too fast to a larger quantity, otherwise children would likely need to count them.

In accordance with prior literature (e.g., Kaufman et al., 1949; LeCorre et al., 2006; Saltzman & Garner, 1948; Taves, 1941) the spatial arrangement and spacing of objects in the collection were identified as factors affecting the difficulty of subitizing tasks. Finally, study participants were adamant that if children were counting dots on the cards to find "how many" they were not subitizing. Indeed, many went to great lengths to ensure children were prompted and supported to subitize. They offered patterns with less dots, used counters to make "seeing groups" more explicit, transferred the quantity to a ten frame, or compared and contrasted two different patterns of the same quantity. Overall, this finding suggests prospective teachers are capable of deeply understanding important mathematics they will be expected to teach, a component of subject matter knowledge.

In a related finding, study participants not only understood subitizing, they articulated its importance to the early childhood mathematics classroom and why children needed extensive opportunities to subitize. They viewed subitizing as foundational to young children's early development of cardinality and related ideas of part-whole understanding, and beginning ideas for joining and separating. The importance of these ideas as the building blocks of mathematics through elementary, middle, and high school and beyond is extensively supported in the research literature (Baroody et al., 2006; Clements, 1999; Gallistel & Gelman, 1991; Hannula & Lehtinen, 2005; Nguyen et al., 2016; Sarama & Clements, 2009). This finding suggests these prospective teachers will confidently approach subitizing opportunities with a sense of purpose and intention as they see the short- and long-term benefits of this work.

In regards to SMK it appears early childhood PSTs are able to develop an in-depth awareness and understanding of young children's mathematics-related developmental milestones. For this study, this knowledge appeared to be mediated by their understanding of the subitizing learning trajectory (Clements & Sarama, 2014). The early childhood PSTs engaged their subject matter knowledge when they ordered dot arrangements or created new patterns for the sequence that mirrored the progression of the subitizing trajectory. This knowledge seemed to aid these teachers to select learning experiences and tasks and pose questions that were uniquely responsive to the developing needs of the fictitious class of five-year olds featured in this study. Future research could investigate the degree of carry over of these skills when these teachers are engaged in teaching actual children.

The depth of knowledge regarding subitizing shared by study participants was surprising. Subitizing is one small portion of the early childhood methods class yet something about subitizing resonated with them. For many subitizing was not just a novel word, but also a novel concept. They were intrigued when it was introduced in class and delighted in their own subitizing abilities. They were excited when they noticed children in their care or children in classroom settings that were engaged in subitizing during free-play or games. This suggests that they will recognize subitizing work as they move into more formal teaching experiences and knowingly support children to subitize during teacher-initiated, child-initiated, or guided-play learning opportunities. Arguably, SMK lays the foundation for intentional teaching, as it appears to be the knowledge teachers engage as they expand on the children's play through strategically placed questions, or during intentionally planned teacher-guided learning experiences. This research further suggests that deep SMK of early years mathematics, grounded in an understanding of early mathematics learning trajectories, is the launch point for intentional pedagogical choices that honor the coherence of mathematics and lay the foundation for developmentally appropriate instruction.

Discussion of Pedagogical Content Knowledge Findings

Evidence from this study suggests prospective ECE teachers view mathematics growth in young children through a developmental lens and that an understanding of an early mathematics learning trajectory supports that perspective. Though study participants were acquainted with the developmental nature of children's learning through previous university courses they found its application to mathematics teaching and learning extremely novel. Conceivably, their understanding of the subitizing trajectory helped to demystify the stages children move through as they acquire mathematics concepts and skills. Indeed, a teacher's understanding of the sequence and pace children's development and learning typically follow is a core component of developmentally appropriate practice (NAEYC, 2009). This finding suggests learning trajectory knowledge may guide early childhood PSTs to introduce concepts and skills in a coherent way and to scaffold children's progress from each idea and ability to the next.

Participants frequently mentioned learning trajectories or progressions as they considered how they might know when children might be ready for a more complex dot arrangement. For example, Flora mentioned she would consult a learning trajectory to "figure out how to get [the children] to the next level and keep on improving." Karolyn commented she would check with the learning trajectory "to help figure out where to go next and if I should be worried about where they are now." Karaleen offered, "I would also go back to how children grow developmentally and think about where the children are and where they need to go." These representative comments suggest early childhood PSTs' may benefit from anchoring their perspectives concerning children's developmental growth in mathematics to learning trajectory knowledge. In support, NAEYC (2009) suggests a developmental approach to teaching "requires both meeting children where they are and enabling them to reach goals that are both challenging and achievable" (p. xii). Indeed, learning trajectory knowledge has extensive support as a tool for supporting impactful instructional decisions (Bobis et al., 2005; Epstein, 2014; Mojica, 2010; NAEYC, 2009; NRC, 2009; Sarama & Clements, 2009; Seo & Ginsburg, 2004; Schoenfeld & Stipek, 2011; Stipek, 2019). This finding suggests early childhood PSTs may reference learning trajectories in the future as they identify goals for children's learning and development and are intentional in helping children achieve those goals.

Further, this study suggests that the early childhood PSTs appeared to be capable of planning and structuring meaningful mathematics learning opportunities tailored to specific needs of young children, knowledge particular to PCK. Specific instances underscore this finding and include when study participants: (a) justified the order of a set of dot arrangements; (b) offered ways for children to share their thinking beyond verbal explanations; and (c) articulated strategies for eliciting and responding to an error in children's thinking. These instances suggest early childhood PSTs honor and respect children's thinking and are keenly interested in, and capable of, holding children's thinking at the center of instructional decision-making. Mojica (2010) found elementary PSTs' knowledge of a learning trajectory enhanced their ability to leverage student thinking to advance learning and make instructional decisions. The current study suggests early childhood PSTs are capable of this as well and will likely carry these strategies forward in their teaching.

Study results suggest participants engaged their PCK when asked to interpret and respond to children's thinking as they investigated children's thinking, in general, and misconceptions children held for an uncommon arrangement of five (Pattern F), specifically. Evidence supporting this finding included six broad categories of participant-generated strategies. Two strategies purposefully engaged the children by asking them to explain or show how they saw the pattern. Two strategies were more teacher-directed and included comparing the pattern of five to a quantity of ten and telling the children to count to find out. A fifth strategy included asking the children to simply "look again" and a sixth strategy found the teacher actively adapting their instructional approach to keep the children subitizing. Each of the strategies with the exception of "count to find out" appeared to keep children engaged with subitizing to varying degrees. As articulated by Clements and Sarama (2014) teaching with learning trajectories demands that instructional decisions keep children engaged in the big idea of trajectory, which was supported by five of the six strategies. This finding acknowledges early children PSTs were able to support children to grow as subitizer. That the prospective teachers worked within the trajectory was a particularly salient finding as it reflects a deep and nuanced understanding of subitizing.

As part of the "keep subitizing" category, three strategies surfaced that would encourage children to actively subitize. They included: (a) offering a smaller quantity of dots to subitize; (b) returning to a more familiar pattern; and (c) reorienting the dot pattern to offer children another perspective. These are sophisticated instructional suggestions for three reasons. First, they keep children engaged in subitizing. Second, they honored children's agency when they gently shifted the responsibility of working through a misconception back to the child. Third, they provided a light adult scaffold with the intent of progressing the child toward the learning goal. Current research (Wiesburg, Hirsh-Pasek, Golinkoff, Kittredge, & Klahr, 2016) suggest light adult scaffolding focuses children "toward the pedagogical goal without usurping child autonomy" (p. 178) and ensures instruction is scaffolded to meet children's readiness for learning. I believe this requires highly developed and nuanced PCK and SMK. What is perhaps most significant is that these suggestions came from three participants that were either teaching preschool children while enrolled in the methods course (Amalie and Marisol) or at the time of the study (Karolyn). For that reason, each had some experience facilitating subitizing work with young children. This suggests that early childhood PSTs pedagogical content knowledge might benefit from focused opportunities to implement instruction guided by learning trajectories in supervised clinical settings.

Each finding from this study suggests that participants' instructional decisions were closely aligned to the developmental progression of children's subitizing skills as outlined by the subitizing trajectory (Clements & Sarama, 2014). Study findings suggest that learning trajectory knowledge can help beginning teachers be prepared for the range of student understanding they may likely encounter and the kinds of pedagogical responses that are likely to help children advance their learning. Finally, study findings suggest that learning trajectory knowledge "gives teeth" to the widely adopted practice of developmentally appropriate instruction (NAEYC, 2009) as they delineate how student learning actually progresses and identify "key steps forward" (Daro et al., 2011, p. 12) that are in line with young children's developmental pathways.

Conclusions

This study examined the impact of the confluence of early learning trajectory knowledge and mathematical knowledge for teaching on the intentionality of instructional decision-making in prospective early childhood teachers. In an effort to begin expanding the research base on the potential use of early mathematics learning trajectories in pre-service teacher education, this investigation sought to answer the following research questions:

- Central Research Question: In what ways do learning trajectories inform prospective early childhood teachers' instructional decisions in ways that are likely to advance student learning on the subitizing trajectory?
- Attendant Question 1: What understandings do early childhood pre-service teachers have regarding the subitizing trajectory?
- Attendant Question 2: Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?

A critical need exists for new knowledge and resources to guide and facilitate efforts to promote young children's mathematics learning and increase equity and excellence in mathematics achievement. Although important research has been conducted in recent years, much remains much to be learned about how to best prepare prospective teachers to facilitate meaningful and effective mathematics learning opportunities. I offer these conclusions as contributions to the emerging research regarding the use of learning trajectories in mathematics education university coursework for prospective early childhood teachers.

Central Research Question: In what ways do learning trajectories inform prospective early childhood teachers' instructional decisions in ways that are likely to advance student learning on the subitizing trajectory?

Conclusion 1: This study found that learning trajectory knowledge, in concert with MKT, sparked study participants' curiosity to investigate and understand children's thinking. This resulted in multiple participant-generated strategies to elicit children's mathematical thinking and intentional use of children's thinking as a launch-point for instruction.

Conclusion 2: Study participants held a developmental view of children's learning as purported by learning trajectory research. This resulted in a concerted and intentional effort to prioritize children's thinking during instructional decision-making meant to advance learning in line with developmentally appropriate next steps.

Conclusion 3: Study participants' knowledge of the subitizing trajectory provided a foundation for the development of a cohesive and connected mathematics learning experience for young children. An understanding of the subitizing trajectory and the short- and long-term benefits of subitizing skill resulted in instructional decisions that were both mathematically appropriate and particularly responsive to children's needs.

Attendant Question 1: What understandings do prospective early childhood teachers have regarding the subitizing trajectory?

Conclusion: Study participants' revealed a complex and nuanced understanding of the three components (mathematical goal, developmental progression, and instructional tasks) of the subitizing learning trajectory. Participants utilized their understanding of the learning trajectory to (1) guide their understanding of how children's understanding of subitizing develops over time as they mature and grow; (2) validate that not all children will learn at the same pace; and (3) recognize instructional tasks as a key means of support for developing student understanding.

Attendant Question 2: Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?

Conclusion: Study results revealed participants engaged in a cycle of intentional instructional decision-making highlighting an intricate relationship between subject matter knowledge, pedagogical content knowledge, and learning trajectory knowledge (see Figure 5.2).

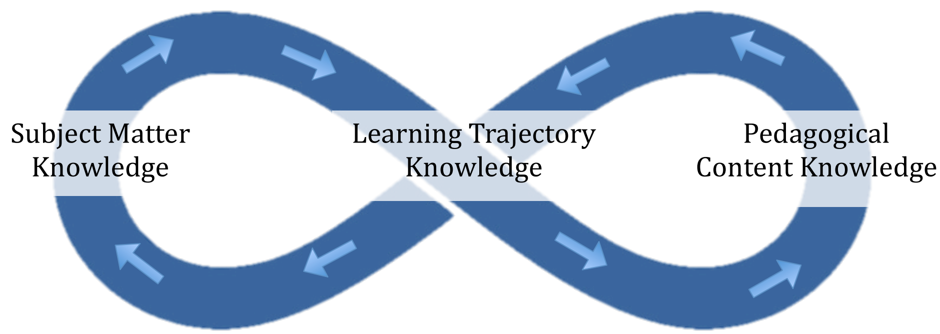


Figure 5.2. The cycle of instructional decision-making identified in this study.

Figure 5.2 suggests a cycle of instructional decision-making that engages both subject matter knowledge and pedagogical content knowledge and is mediated by learning trajectory knowledge. While engaged in this cycle prospective teachers appear to rely heavily on SMK to identify the mathematical big idea of the trajectory, understand its importance to early mathematics learning, and recognize key shifts in children's cognition. Teachers then rely on their PCK as they select and implement intentional tasks, interpret and respond to children's mathematical thinking, and modify learning opportunities. Learning trajectory knowledge appears to serve as the filter allowing prospective teachers to continually draw upon both domains interactively in order to demonstrate intentionality.

In reviewing how this cycle unfolds with prospective teachers, it is important to focus on how SMK and PCK intersect during the act of teaching and the intermediary role of the learning trajectory. As early childhood PSTs engage in and examine tasks and activities during the mathematics methods class and begin to consider using the activities with children they engage their PCK. Their knowledge of the activities is then filtered through the learning trajectory as they ponder whether to use the activity or how to best implement it with children. To make the instructional decision most likely to advance student learning, they must continue through the cycle to access their SMK. This allows them to target the mathematics of the activity while remaining attentive to key shifts in children's cognition. They then cycle back through the learning trajectory as they evaluate children's capabilities and engagement and consider adjustments to either the mathematics or the pedagogical approach. This synergistic cycle continues throughout the teaching act with learning trajectory knowledge serving as a filter for intentional and developmentally appropriate instructional decisions.

Limitations of the Study

This study has some limitations. All of the participants were enrolled in an early childhood education program at the same university. I was the instructor for both class sections of the early childhood mathematics methods course and was therefore both the teacher and the researcher. Since I designed and implemented the study, and collected and analyzed the data, my own theoretical perspectives and prior experiences influence the results. In order to convey to the reader what these potential biases and assumptions might be, I disclosed my own theoretical perspective in Chapter 3. In order to minimize this bias I engaged epoche (Moustakas, 1994), or bracketing, in an effort to set aside my own prejudgments regarding data analysis and the results of this study.

Learning trajectory research and specific learning trajectories, including the subitizing trajectory, were intentionally incorporated into the early childhood mathematics methods class sessions. In addition, a course requirement asked prospective teachers to conduct a diagnostic interview with one child for a variety of mathematical understandings and to place the child at the appropriate developmental on the appropriate learning trajectories. Therefore, the participants in this study had previous exposure to learning trajectory research and the subitizing trajectory used in this study. Since the content and requirements of the methods course may be unique, the findings cannot be generalized to a wider population of prospective early childhood teachers.

Future Research

Previous research on teachers' uses of learning trajectories during instruction suggest they may assist teachers in focusing on their students' mathematical thinking (Edgington, 2012; Wilson, 2009; Wilson et al., 2017), provide a framework for instructional decisions (Bobis et al., 2005; Mojica 2010; Wickstrom 2014), and improve learning outcomes (Clements et al., 2016). Mojica (2010) and Wilson (2009) found as prospective and inservice elementary teachers made sense of trajectories they deepened their MKT, thus enhancing their ability to select developmentally appropriate instructional tasks, engage in more focused classroom discussions, and make better use of students' responses to further learning.

In this study, I explored if an understanding of a specific learning trajectory when partnered with one's MKT supported intentional decision-making that would advance children's mathematical thinking. Further research should explore experiences prospective teachers might need to further their understanding of learning trajectories and to connect that understanding to realizing children's potential as doers of mathematics. Assessing children's abilities in relation a research-based developmental trajectory has the potential to illuminate children's interests and capabilities in mathematics, negating the common practice of underestimating children's mathematics abilities.

In addition, it might prove insightful to replicate this study with practicing ECE teachers who have not had opportunity to learn about and utilize learning trajectory research in their instruction. A study of this type might illuminate the impact of daily classroom experiences teaching mathematics on a teacher's SMK and PCK and what factors influence the intentionality of their day-to-day instructional decisions. Do those instructional decisions honor children's thinking and meet children at their developmental level? Equally insightful would be a follow-up study with the fifteen participants of this current study. What impact might their learning trajectory knowledge have on their instructional decisions or how they interpret and respond to children's thinking now that they are in the field? To what degree do they draw on learning trajectory knowledge as they plan for and implement mathematics instruction?

Finally, future research should focus on using learning trajectories in teacher preparation and professional development in a broader sense. Children's mathematical thinking does not progress in isolated trajectories. Indeed, children's mathematical thinking naturally flows from one idea to the next implying they move very naturally from one related trajectory to another. In turn, throughout the course of the interview study participants moved among the big ideas of subitizing, counting, and composing with ease though presumably without really knowing. Exploring how to best support teachers as they listen to and engage with the broad array of children's thinking is needed as teachers cannot be expected to track progress on multiple trajectories simultaneously.

Recommendations for Early Childhood Teacher Education

One implication of this study is that early mathematics learning trajectories are critical components in the early childhood mathematics methods course and offer numerous benefits to prospective teachers' understanding of mathematics and high-quality mathematics teaching. Each of the future teachers in this study demonstrated an ability to understand an early mathematics learning trajectory. In addition, many relied on their understanding of the progression of children's thinking articulated in the subitizing trajectory to make intentional and developmentally appropriate instructions meant to advance children's mathematical thinking. Moreover, this study shows that including learning trajectories in an early childhood mathematics methods course deepened participants' understanding of a mathematical big idea, in this case, subitizing. Therefore, one implication is that early mathematics learning trajectories should be a part of early childhood mathematics methods courses.

Another implication of this study is that teacher education programs should develop and provide prospective teachers with structures that allow them to enact teaching based on learning trajectory knowledge. Centering one's instructional decisions on children's thinking requires a deep understanding of how children think mathematically, a wide array of instructional strategies, and a keen awareness of developmental milestones. I suggest these three components form the core of intentional mathematics teaching at the early childhood level and as such should be the foundation for early childhood mathematics methods courses. I further suggest intentional teaching develops overtime and requires multiple opportunities to listen to and learn from the thinking of actual children. Therefore, I conjecture additional support could come in the form of supervised field experiences that purposefully engage prospective teachers in assessing, planning, and implementing instructional experiences around selected learning trajectories. This would help prospective teachers tune in to and appreciate the nuances of children's mathematical musings and quandaries during playtime or structured learning opportunities so as to recognize and capitalize on authentic teachable moments.

One last important finding of this study was though prospective teachers attended to the progression of children's thinking as outlined in the subitizing learning trajectory they tended to fairly quickly move children off the subitizing trajectory. This occurred when student participants suggested they would ask children to count the dots when they incorrectly subitized. The fact that this occurred is not surprising though I believe it signals the need for prospective teachers to develop a deeper understanding of what it is needed to keep a child moving forward on a specific trajectory. To that extent I suggest prospective teachers have extended experiences working with children under the guidance of a mentor teacher well versed in early mathematics learning trajectories. Pre- and post-coaching conversations could prepare prospective teachers to best address young children's responses in such a way that stays true to the mathematical goal of the trajectory and intentionally meets a child at their developmental level. Therefore, if an instructional decision moves a child's thinking to another trajectory, it is done with intention and purpose.

At minimum, a model for preparing prospective teachers to use to engage in responsive teaching should include the following components: focused study of the mathematical big idea of the trajectory, exploration of tasks related to the trajectory, introduction of the learning trajectory as a tool to understand and monitor children's thinking, use of video exemplars of children at various levels of the trajectory, experience working with early learners on learning trajectory related tasks, and implementing diagnostic interviews and structured reflection on those experiences.

Concluding Comments

A critical need exists for new knowledge and resources to guide and facilitate efforts to promote young children's math learning and increase equity and excellence in math achievement. Although important research has been conducted in recent years, much remains to be learned about how to prepare prospective teachers to deliver effective mathematics learning opportunities that are intentional, developmentally appropriate, and engaging. This study contributes to the knowledge base of how prospective teachers utilize learning trajectory knowledge to support intentional decision-making.

Early childhood education has risen to the top of the national policy agenda with recognition that ensuring educational success and attainment begins in the earliest years of schooling (Ginsburg et al, 2008; Hachey, 2013; Purpura, Baroody, Lonigan, 2013). Indeed, the National Research Council (2001, p. 6) stated:

> Young children show a remarkable ability to formulate, represent, and solve simple mathematical problems and to reason and explain their mathematical activities. They are positively disposed to do and to understand mathematics when they first encounter it.

According to Daro and colleagues (2011) in order to ensure all children realize their potential as learners and doers of mathematics, the norms of practice should move towards a model where teachers continually (1) seek evidence on whether children are on track to learn what they need to; (2) track indicators of what problems they might be having; and, (3) respond pedagogically to that evidence in ways that keep students on track, or get back on track, when necessary. Central to this model is a teacher's understanding of how specific concepts, like subitizing, develop over time. Findings from this study indicate that learning trajectory knowledge has the potential to be used as tool to prepare teachers to shift the model suggested by Daro and colleagues.

Learning trajectories outline the range of student understanding teachers may likely encounter in response to relatively well-specified instructional experiences and the kinds of pedagogical responses that are likely to help advance children's mathematical reasoning. They describe the interim goals that children should meet as they progress toward an understanding of a mathematical topic. Teachers must not only understand the mathematics they are expected to teach (Ball et al., 2008) and understand how students learn that mathematics, they must be skilled in using content-focused instructional pedagogies to advance the mathematics learning of each and every student (Forzani, 2014).

Trajectories involve hypotheses about the order and nature of the steps in the growth of students' mathematical understanding, and the nature of the instructional experiences that might support them in moving step by step toward the goals of school mathematics. To that end, the key to successful use of learning trajectories lies not in just understanding each of the three components of a learning trajectory but in understanding how the components work together and must be used in concert to engage and support young children's learning and thinking about mathematics.

Improving early mathematics learning requires teachers to know the content, understand children's thinking, engage in pedagogical practices that support learning, and see themselves as capable mathematics teachers. University methods classes should ensure prospective teachers understand the subject matter of early mathematics education, have insight into children's mathematical thinking and learning, can assess individual children's knowledge of mathematics, can think critically about teaching and teach effectively, and who ultimately enjoy early mathematics education and transfer the feeling to the children they will teach. Teachers' MKT paired with the knowledge of the learning trajectory unite to ensure intentionality of instructional decision-making that help children progress toward more sophisticated ideas.

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Appendix A Subitizing Learning Trajectory

Learning Trajectory Developmental Levels for

"Recognizing Number and Subitizing"

The ability to recognize number values develops over the course of several years and is a foundational part of number sense. Beginning at about age 2, children begin to name groups of objects. The ability to instantly know how many are in a group, called subitizing, begins at about age 3. By age 8, with instruction and number experience, most children can identify groups of items and use place values and multiplication skills to count them.

Level	Level Name	Description
1	Small Collection Namer	The first sign of a child's ability to subitize occurs when the child can name groups of one to two, sometimes three. For example, when shown a pair of shoes, this young child says, "Two shoes."
2	Nonverbal Subitizier	The child can name the value of a small collection (one to four objects) only briefly, the child can put out a matching group nonverbally, but cannot necessarily give the number name telling how many. For example, when four objects are shown for only two seconds, then hidden, child makes a set of four objects to "match.
3	Maker of Small Collections	The child can nonverbally make a small collection (no more than five, usually one to three) with the same number as another collection. For example, when shown a collection of three, makes another collection of three.
4	Perceptual Subitizer to 4	Progress is made when a child instantly recognizes collections up to four when briefly shown and verbally names the number of items. For example, when shown four objects briefly, says "four.
5	Perceptual Subitizer to 5	The child instantly recognizes briefly shown collections up to five and verbally names the number of items. For example, when shown five objects briefly, says "five."
6	Conceptual Subitizer to 5	The child can verbally label all arrangements to five shown only briefly. For example, a child at this level would say, "I saw 2 and 2 and so I saw 4."
7	Conceptual Subitizer to 10	The child can verbally label most briefly shown arrangements to six, then up to ten, using groups. For example, a child at this level might say, "In my mind, I made two groups of 3 and one more, so 7.
8	Conceptual Subitizer to 20	The child can verbally label structured arrangements up to twenty, shown only briefly, using groups. For example, the child may say, "I saw three 5s, so 5, 10, 15."
9	Conceptual Subitizer with Place Value and Skip Counting	The child is able to use skip counting and place value to verbally label structured arrangements shown only briefly. For example, the child may say, "I saw groups of tens and twos, so 10, 20, 30, 40, 42, 44, 46 . . .
10	Conceptual Subitizer with Place Value and Multiplication	The child can use groups, multiplication, and place value to verbally label structured arrangements shown only briefly. At this level a child may say, "I saw groups of tens and threes, so I thought, five tens is 50 and four 3s is 12, so 62 in all."

Source: Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Appendix B Course Syllabus

Catalog Description:

Methods and curriculum for facilitating the learning of mathematics with children ages 3-8. Emphasis on number concepts, problem solving, and intuitive geometry. Notes: Prereq: jr st, admis to School of Educ, Math 176 with grade of C or better, & CURRINS 302(P); or cons instr.

Course Objectives:

The purpose of this course is to provide opportunities for you to develop an emerging knowledge base in the teaching and learning of mathematics at the early childhood level that encompasses both theoretical and practical pedagogies. We will explore topics aligned to the Common Core State Standards for Mathematics (CCSSM). The emphasis is on engaging young learners (ages 3–8) through problem-solving instructional approaches with questioning and inquiry strategies. You will become acquainted with instructional and assessment strategies, materials, learning environments, and linguistic and non-linguistic representations used in teaching mathematical concepts to children.
Through this course, you are expected to:

1. Reflect on and evaluate one's own beliefs, conceptions, strengths, and weaknesses regarding mathematics and the teaching and learning of mathematics, and develop a personal philosophy and approach to teaching mathematics informed by current research and recommendations.
2. Develop mathematical knowledge for teaching (MKT) and pedagogical content knowledge (PCK) for students in ages 3 – 8 in the domains of Counting & Cardinality, Operations & Algebraic Thinking, and Number & Operations in Base Ten through familiarity with the CCSSM mathematical practices & standards.
3. Gain skills in using formative assessment practices through analysis of student mathematical work and performances to plan appropriate instruction.
4. Understand ways to support all children's learning of mathematics including the mathematical learning of English Language Learners and students with exceptional needs and abilities.
5. Select, plan, adapt, implement, and evaluate instructional activities and prompts that emphasize mathematical problem solving, reasoning, communication, and understanding to develop mathematical content knowledge.

High-quality educational practices recognize that all students bring distinct strengths to the classroom and teachers assume responsibility for the learning and growth of each child. Of central importance to all segments of this course is a commitment to learn ways to work effectively with diverse populations of children, to ensure equal educational opportunity for all children, and to use recent research to guide educational practice. For instruction to be effective, educators must know a great deal about students, mathematical development, and learning processes.

Required Course Materials and Electronic Communication Expectations

- Van de Walle, J. A., Lovin, L.H., Karp, K., & Bay-Williams, J.M. (2014). *Teaching student centered mathematics: Developmentally appropriate instruction for grades Pre-K-2.* Boston, MA: Pearson.
- *Common Core State Standards for Mathematics*, pgs. 1-26 and pgs. 85-90. This document is accessible via D2L or at http://www.corestandards.org/the-standards/mathematics
- *Progressions for the Common Core State Standards in Mathematics*. This is accessible via D2L.
- Selected articles, handouts, and Web sites. You must utilize the internet to download articles and handouts from D2L.

General Course Expectations

As a developing teacher of mathematics, you are expected to:

- Be well-prepared for each class session by completing all assigned readings and tasks prior to each class.
- Conduct oneself in a professional and collaborative manner during each class session.
- Use your assigned LU email account regularly for course information and instructor communication.
- Devote numerous hours to professional reading, including assigned readings and self-selected readings for

professional growth to establish a knowledge base for teaching mathematics.
- Type all written assignments using standard guidelines, unless otherwise stated.

Credit Hour Policy

This is a three-credit course, so the expected time commitment from students is approximately 144 hours (3 credits x 48 hours per credit earned). Students will spend 37.5 hours in class and approximately 107 hours preparing for class and completing assignments and group work. This means that instructors will plan for students to spend about 7 hours per week completing coursework outside of class. Although the above breakdown will vary by student and by week, my expectation is that students will spend approximately 20% of the time participating in class sessions and completing in-class tasks and discussion reflections, 40% of the time reading, studying, accessing and participating in D2L, and completing homework tasks; 20% of the time completing Ages 4-6 Child Interview and Ages 7-9 Child Interview; and 20% the time preparing for and taking exams.

Grading

Your grade for this course will be determined as follows:

Requirements	Percent of Grade
Class Attendance	15
Class Participation	5
Homework Tasks	15
Child Interview Ages 4-6	15
Basic Fact and Equality Interview Ages 7-9	10
Learning Log and Two Summative Reflections	10
Mid-Term Exam #1; Mid-Term Exam #2; Final Exam	30 (10 each)

Grades will be assigned on the following scale:

A = 93–100%	A– = 90–92%	B+ = 87–89%
B = 83–86%	B– = 80–82%	C+ = 77–79%
C = 73–76%	C– = 70–72%	D+ = 67–69%
D = 63–66%	D– = 60–62%	F = 0–59%

Course Requirements and Assignments

Assignments should be word processed unless otherwise stated in the syllabus or in-class. Each assignment should be presented in a neat, organized, and clear manner, utilizing headings as appropriate. The body of all papers should be typed using Arial 12 and double spaced. Other styles and sizes may be used for headings, emphasis, or illustrations. Keep a copy (hard copy or electronic) of assignments for your record keeping purposes in case questions or discrepancies arise.

All assignments are due at the beginning of class on the date specified in the syllabus whether or not you attend class. If you are absent, you must submit the assignment by the beginning of the class session (9:30 a.m.) via email or in the instructor's university mailbox. Late homework will not be accepted. Child Interviews will be penalized by 10% for each day late.

No extra credit assignments will be granted or rewrite of assignments allowed. However, the instructor reserves the right to require a rewrite of an assignment if it does not meet minimum expectations. The final score of the rewritten assignment cannot exceed 90% of the total points for the assignment.

Attendance (15%)

You are expected to attend class regularly and on time. Attendance will be taken and tardiness and early departures noted. Two late arrivals and/or early departures will count as one full absence. Email the course instructor if you are going to be absent prior to the start of class. You start with 15 attendance points. Each missed class will result in a loss of 5 points. For example if you miss one class your attendance points for the semester goes from 15 total points to 10 total points. As further illustration, if you were at 100% for the course and miss one class the highest semester grade you could earn is 95%. If you miss two classes you will lose 10 points. As further illustration, if you were at 100% for the course and missed two classes the highest semester grade you could earn is 90%. Missing three or more classes will result in no attendance points earned for the semester. Excused absences include a medical issue under a doctor's care for oneself or an immediate family member, a death in the immediate family, or a religious observance. Excused absences must be accompanied by written documentation. Emails do not count as written documentation of your absence.

Note: The following policy does not supersede the above course attendance policy.

Early Childhood Education Attendance Policy: In the ECE Program we believe strongly that the development of strong relationships supports us in creating productive learning communities, a model that we hope you take with you into your professional work as a classroom or childcare teacher. All of the program/professional courses are interactive and discussion based. Missing class means missing the interaction and the collaborative construction of knowledge. It is difficult if not impossible to make this up. Thus, we take attendance and participation in class very seriously. So, for any ECE program class, **more than two absences** will require you to submit a letter of explanation to the Early Childhood Committee, and may result in the requirement to retake the course. More than three absences may result in a failing grade in the course.

Class Participation, Class Tasks, and Discussion Reflections (5%)

You are expected to participate in discussions and small group work in a professional manner that contributes to the engagement and learning of all class members toward course goals. Coming to class without completed homework results in a deduction of two of the three participation points for that class period. Class experiences provide the opportunity to participate in dialogue that is crucial to the learning process. The reflection on what others share is an important aspect to your learning in this course. You are expected to contribute to and complete in-class tasks, such as written reflections, mathematical tasks, and analysis of student work or video segments. Impromptu (unannounced) quizzes will be administered during class time. Impromptu quizzes will incorporate readings and class discussions and will be administered at the beginning of class. No make-ups for impromptu quizzes will be permitted.

Learning Log & Two Summative Reflections (10%)

Throughout the semester you will keep a log of key points and critical content from each class period. You will be provided a form to record class session reflections. If you prefer, you may use your computer or electronic device to type your log of reflections.

You will submit your log and two papers synthesizing your learning at different points throughout the semester. Each synthesis paper should be 2-2.5 pages in length (double-spaced) and should describe changes in your understandings, shifts in thinking, attitudes, beliefs about children, and/or teaching practices in relationship to mathematics.

Child Interview on Number Knowledge (Ages 4-6 15%) & Child Interview on Basic Fact and Equality Knowledge (Ages 7-9 10%)

The purpose of this assignment is to provide a structure for you to acquire information about students' mathematical thinking and to develop your listening and questioning strategies. It should provide you insight into students' thinking strategies, ways students represent problems, and students' mathematical language. You will be expected to make arrangements to interview 2 children. One that is between the ages of 4 and 6 and one that is between the ages of 7 and 9. It would be best if you arrange for at least an hour with each child, broken into 15 or 20 minute sessions. You can space these over a couple days or plan some fun activities to intersperse with the questions, including the reading of a children's book related to mathematics and a mathematical game. More information will be provided.

Homework Tasks (15%)

All completed homework tasks are due at the beginning of each class session; late homework will not be accepted. Have access to either a hard copy or an electronic copy of your homework during class sessions. If you are missing class, you may post your homework in D2L prior to the class session to receive credit for it.

Homework will not be accepted if it is submitted after the beginning of class. Each week 3 to 4 homework tasks will be assigned. Two points will be deducted for each missing task.

This course uses LU D2L to provide classroom resources, articles, homework tasks, and your grades. You access D2L through an Internet browser using your LU ID and Password. Homework tasks and articles will be posted within 24 hours after a class session. It is your responsibility to access and download the tasks, articles, and readings. DO NOT wait until the night before class to start your homework. The required article readings for the semester will also be posted on D2L for you to download.
In general, homework will be assigned each week and tasks will consist of items from the following list:
1. Assigned readings from the textbook, articles, and handouts.
2. Reflections and summaries on readings.
3. Analysis of children's work samples or video segments.
4. Examine mathematics education Web sites for research, resources, and support.
5. Other tasks assigned during class to be completed outside of class.

Exams (Midterm Exam #1, 10%; Midterm Exam #2, 10%; Final Exam, 10%)
The Mid-Term Exams will be allotted one hour and will be given at either the beginning or end of a class session on the day indicated on the course schedule. The final examination will be cumulative. Exams are given on the day scheduled, no make-up exams allowed for unexcused absences. Arrangements for excused absences must be discussed in advance of the exam.
The exams will include short answer prompts, student work analysis, lesson and activity critiques, and analysis of assessment tasks. The examinations will ask you to apply what you have learned in class as well as demonstrate that you know and understand the subject matter, math tasks, and course readings (i.e., textbook, articles, and handouts).

Early Childhood Education Policies

Grades in Professional Sequence Courses: A grade of C or better must be earned in this course in order to fulfill the professional requirements of the Early Childhood Certification program. This course cannot be taken for credit/no credit.
ECE Program Attendance Policy: In the ECE Program we believe strongly that the development of strong relationships supports us in creating productive learning communities, a model that we hope you take with you into your professional work as a classroom or childcare teacher. All of the program/professional courses are interactive and discussion based. Missing class means missing the interaction and the collaborative construction of knowledge. It is difficult if not impossible to make this up. Thus, we take attendance and participation in class very seriously. So, for any ECE program class, more than two absences will require you to submit a letter of explanation to the Early Childhood Committee, and may result in the requirement to retake the course. More than three absences may result in a failing grade in the course.

Early Childhood Technology Policy: We realize that cell phones and laptop computers are a mainstay of student life for many of you. Should you find it necessary to bring a cell phone to class, please turn it off before class begins. If you are expecting an emergency call, please notify the instructor before class and step out into the hallway to take your call. Text messaging during class is **strictly** prohibited, and will result in the loss of your participation points for that day. Taking class notes on your laptop is allowed, but engaging in web surfing, checking email, or completing work for other classes is not. Again, these activities will result in the loss of your participation points for that class period.

Accommodations for students with disabilities: If you have an identified disability that may affect your performance in this class, schedule an appointment with me (no later than the third week of class) so that provisions can be made to ensure that you have an equal opportunity to meet all the requirements of the course.

Accommodations for religious observances: Students will be allowed to complete requirements that are missed because of a religious observance.

Academic misconduct: The university has a responsibility to promote academic honesty and integrity and to develop procedures to deal effectively with instance of academic dishonesty. Students are responsible for the honest completion and representation of their work, for the appropriate citation of sources, and for respect of others' academic endeavors. Please note that any use of another source, whether in print or on-line, should be cited appropriately. The ECE program takes academic misconduct very seriously and **will** pursue sanctions in the event of an occurrence.

University Policies and Procedures:

- *Students with disabilities.* Notice to these students should appear prominently in the syllabus so that special accommodations are provided in a timely manner.
- *Religious observances.* Accommodations for absences due to religious observance should be noted.
- *Students called to active military duty.* Accommodations for absences due to call-up of reserves to active military duty should be noted.
- *Incompletes***.** A notation of "incomplete" may be given in lieu of a final grade to a student who has carried a subject successfully until the end of a semester but who, because of illness or other unusual and substantiated cause beyond the student's control, has been unable to take or complete the final examination or to complete some limited amount of term work.
- *Discriminatory conduct (such as sexual harassment*). Discriminatory conduct will not be tolerated by the University. It poisons the work and learning environment of the University and threatens the careers, educational experience, and well-being of students, faculty, and staff.
- *Academic misconduct.* Cheating on exams or plagiarism are violations of the academic honor code and carry severe sanctions, including failing a course or even suspension or dismissal from the University.
- *Complaint procedures.* Students may direct complaints to the head of the academic unit or department in which the complaint occurs. If the complaint allegedly violates a specific university policy, it may be directed to the head of the department or academic unit in which the complaint occurred or to the appropriate university office responsible for enforcing the policy.
- *Grade appeal procedures*. A student may appeal a grade on the grounds that it is based on a capricious or arbitrary decision of the course instructor. Such an appeal shall follow the established procedures adopted by the department, college, or school in which the course resides or in the case of graduate students, the Graduate School. These procedures are available in writing from the respective department chairperson or the Academic Dean of the College/School.
- Selected Academic and Administrative Policy 24.5, Firearms and Dangerous Weapons Policy

Reference List of Required Article Readings

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Huinker, D., (2010). Proficiency with basic facts it's so much more than knowing the answer. *Wisconsin Teacher of Mathematics.*

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National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematics success for all*. Reston, VA: Author.

Ross, S. (1989). Parts, wholes, and place value. *Arithmetic Teacher,* 36*(6),* 47-51.

Russell, S. (2000). Developing computational fluency with whole numbers. *Teaching Children Mathematics,* 7*(3),* 154-158.

Tentative Course Schedule -- Subject to change.

Session Number	**Topics**	**Readings** To be completed PRIOR to class.	**Assignments** Due at the beginning of class.
Session 1	1.Number Talks 2.Strands of Mathematical Proficiency 3.Becoming a Teacher of Mathematics for Young Learners: 4.Review of Syllabus	Course Syllabus	
Session 2	K-2 Operations and Algebraic Thinking: Counting and Cardinality (K.CC.1 – K.CC.6) • Know number names and the count sequence. • Count to tell the number of objects. 1) Recognize small quantities without needing to count. Teaching for Understanding: • Relational vs. Instrumental Understanding	Van de Walle Chapter 1 Van de Walle Chapter 8 p. 100-106 Key Shifts in the Common Core State Standards for Mathematics	Weekly Homework Introduce Student Interview Assignment Ages 4-6
Session 3	Developing number sense by building number relationships The foundation of fluency: • Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from. (K.OA.3 & K.OA.4)	Van de Walle Chapter 8 p. 107-116 Clements, Sarama, & DiBase (2002) Huinker (2011) Huinker (2012)	Weekly Homework Child Interview (Ages 4-6) Checkpoint: Know the student you will interview. Bring a draft of the Background Information on the child to class.
Session 4	Developing Meanings for the Operations: ▪ Addition & Subtraction Problem Types (K.OA.2, 1.OA.1, 2.OA.1) • Modes of representation and representational competence	Van de Walle Chapter 2 p. 12-26 Van de Walle Chapter 9 p. 126-130 Marshall, Superfine, & Canty (2010)	Weekly Homework Child Interview (Ages 4-6) Checkpoint: Bring draft of interview results for rote counting, counting objects, and subitizing to class.
Session 5	** **Midterm Exam #1**** Basic Fact Thinking Strategies: Addition • Reasoning strategies • Developmental Levels of Thinking (K.OA.1, 1.OA.6, 2.OA.2)	Van de Walle Chapter 9 p. 130-134 Van de Walle Chapter 10 p. 153-163 Buchholz (2004)	Weekly Homework Exam #1

Session 6	Basic Fact Thinking Strategies: Subtraction • Reasoning strategies • Developmental Levels of Thinking (K.OA.1, 1.OA.6, 2.OA.2)	Van de Walle Chapter 10 p. 163-174 Van de Walle Chapter 9 p. 134-139 Kling (2011) Bay-Williams & Kling (2014)	Weekly Homework Due: Learning Log Synthesis Paper #1
Session 7	Mastering Basic Facts & Assessing basic fact fluency (K.OA.1, 1.OA.6, 2.OA.2) Algebraic Foundations: Equality & Properties of the Operations (1.0A.7, 1.OA.8) Introduce **Student Interview Assignment Ages 7-9**	Van de Walle Chapter 10 p. 167-174 Kling & Bay-Williams (2014) Van de Walle Chapter 13 p. 225-226 (read); 227-229 (skim); 230-242 (read) Falkner, Levi, & Carpenter (1999)	Due: Child Interview Report (Ages 4-6) Introduce Student Interview Assignment Ages 7-9
Session 8 Oct 27	K-2 Number and Operations in Base Ten • Work with numbers 11–19 to gain foundations for place value. (K.NBT.1) • Understand place value. (1.NBT.2) (K.NBT.1, 1.NBT.2, 2.NBT.1)	Van de Walle Chapter 8 p. 117-120	Weekly Homework
Session 9	K-2 Number and Operations in Base Ten • Understand place value. (2.NBT.1) (K.NBT.1, 1.NBT.2, 2.NBT.1)	Van de Walle Chapter 11 p. 175-202 Ross (1989)	Weekly Homework Child Interview (Ages 7-9) Checkpoint: Bring draft results of basic fact strategies to class.
Session 10	K-2 Number and Operations in Base Ten • Use place value understanding and properties of operations to **add**. -Special Strategies and General Methods (1.NBT.4 & 2.NBT.5, 2.NBT.5, 2.NBT.7)	Van de Walle Chapter 12 p. 203-215 NBT Progressions Document p. 2-4 Bobis (2007)	Weekly Homework Child Interview (Ages 7-9) Checkpoint: Bring draft results for Equality to class.
Session 11	K-2 Number and Operations in Base Ten • Use place value understanding and properties of operations to **subtract**. -Special Strategies and General Methods (1.NBT.6, 2.NBT.5, 2.NBT.7)	Van de Walle Chapter 12 p. 213-219 Huinker, Freckmann, & Steinmeyer (2003)	Weekly Homework Exam #2

Session 12	A commitment to access and equity: Teaching strategies that support diverse learners Mathematical Representations: Visual Models -Focus on Tape Diagrams	Van de Walle Chapter 5 p. 54-69 NCTM (2014) Use and connect mathematical representations p. 24-29 NCTM (2014) Access and equity p. 59-63	Weekly Homework Child Interview (Ages 7-9) Due
Session 13	Exploring Early Fraction Concepts Equal Sharing Tasks (1.G.2, 2.G.3, 3.G.2) Unit Fractions (3.NF.1)	Van de Walle Chapter 14 p. 251-268	Weekly Homework
Session 14	Laying the Foundation for Multiplication and Division Multiplication: Concepts and Basic Fact Strategies (2.OA.4, 3.OA.5, 3.OA.7, 3.MD.7a-d) Course/Instructor Evaluation	Van de Walle Chapter 9 p. 139-152 Kinzer & Stanford (2013) OA Progressions Grade 3 p.22-25	Weekly Homework Due: Learning Log Synthesis Paper #2
Thursday	**Final Exam** 10:00 a.m. -12:00 p.m.		Cumulative final exam.

Appendix C Interview Protocol

For Interviewer Use Only—Not for Distribution to Study Participants

Interview #________________

Date________________

Checklist immediately prior to interview:

- Confirm room availability 30 minutes prior to interview
- Confirm room layout/lighting/seating
- Test audio recorder
- Materials: dot arrangement cards, white board, dry erase marker, eraser, rekenrek, five-frame, ten-frame, red/yellow counters, counting bears, number path
- Bring two writing utensils
- Print two copies of interview questions
- For participants: water, book provided as a thank you.

Opening:

Script: Thank you for participating in this research study and for agreeing to this interview. Today's interview will consist of one opening question and one teaching scenario situated in a kindergarten class. I may include follow-up questions for added clarity or depth as we work our way through the scenario.

Your responses will remain confidential and you are free to end the interview at any time. I would like your permission to record the interview to ensure I accurately document your responses. If at any time, you wish to take a break or stop the recording, please let me know. Although the research findings from this interview may be published, no identifier information will be included to connect you with the findings.

Participants who complete the face-to-face interview will receive "It Make Sense: Using Ten Frames to Build Number Sense" or "Fluency with Flexibility" as a thank you. Your responses will help inform early childhood mathematics education program here at our university. In addition, your responses will contribute to the broader mathematics education community about what knowledge is needed by early childhood prospective teachers to be well-prepared beginning teachers of mathematics.

Please know that your participation is voluntary and you may stop at any time. This study will involve minimal risk and discomfort and your responses and participation will remain confidential. At this time, I would like to remind you of your written consent to participate in this study. I am the investigator and we both have signed and dated the consent to participate forms, certifying that we agree to continue this interview.

Your participation is completely voluntary and you are free to end the interview or withdraw participation at any time. Do you have any questions or concerns before we begin? Then with your permission, we will begin the interview.

PRESS PLAY ON AUDIO RECORDING DEVICE!

Opening Questions:
Thank you for taking time to meet with me. Your interview will contribute to research I am doing to help improve early childhood prospective teacher mathematics education.
Let's get started with a few background questions.

First, could you tell where you are in your undergraduate program? When do you expect to student teach and graduate? Do you have a grade that you are interested in teaching once you graduate?
Thanks!

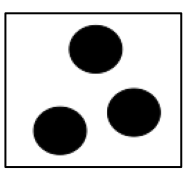
Next, can you tell me about some experiences you had which helped you decide you wanted to be a teacher?
Thank you.

Now, I am going to share with you a scenario from a kindergarten classroom.

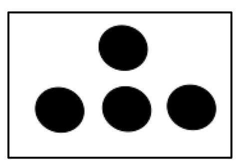
Scenario #1 (Subitizing Trajectory)

Let's imagine that you are student teaching in a kindergarten classroom. It is early October and your teacher asks that you to begin to take over mathematics. You sit together and begin to go through the manual. The second lesson begins with a series of dot patterns such as these. Your teacher comments, "You know, the kids last year really liked these dot patterns. How about if you start here next week?" (Briefly show Pattern A, Pattern B, and Pattern C).

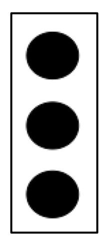
Pattern A



Pattern B



Pattern C



- Would you use these dot patterns with your kindergarten students? Tell me why?
  Purpose of the question: Does participant recognize this an activity to prompt subitizing? Is the participant able to correctly name the skill of subitizing and explain what it looks like? Does the participant name subitizing as foundational to children's early number sense?

- How might you use these with your kindergarten students? Thank you! Are they any other ways you might use this with your students?

Purpose of the question: Can the participant identify one instructional strategy (e.g., dot pattern flash) that would prompt subitizing in young children? Does she address how she might prompt children to explain their thinking?

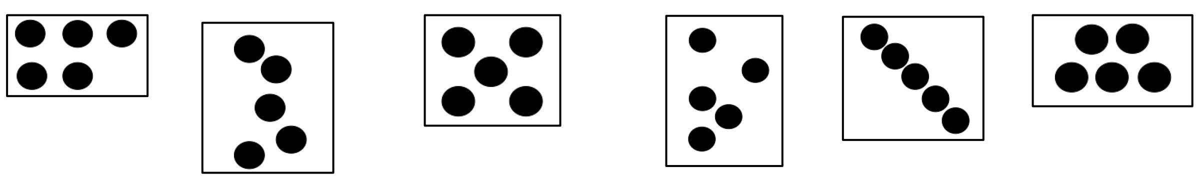
- What responses do you expect from the children with these dot patterns?

  Purpose of the question: Can she anticipate a variety of student responses (e.g., I just know it is 3 because it looks like a triangle. I counted one by one. I know it is 4 because I see three and one more) and acknowledge the varying levels of sophistication in each response? Do her responses demonstrate an understanding learning trajectories, in particular knowledge of the subitizing learning trajectory?

- What do the responses suggest?

  Purpose of the question: Is the participant able to verbalize the difference between "seeing quantity" and "counting by ones?" Do her responses suggest knowledge of the subitizing learning trajectory?

(Returning back to the scenario.) As you continue to look through the manual you see that on one of the lessons offers the following 6 dot patterns. (Place the 6 patterns in the order below on the table in front of the participant.)



Pattern A Pattern B Pattern C Pattern D Pattern E Pattern F

- Can you place them in order as to how you might use them with your kindergartners? Explain for me why you placed them in that order? (Encourage participants to give rationale for each of their choices.)

  Purpose of the question: Will the participant order the patterns from easier to more challenging patterns and provide justification for her decisions? Will she place them in order according to the trajectory even though we have not made this order explicit in XXX 330? (The order according to the learning trajectory (Clements & Sarama, 2014) is: C, A, F, E, B, D or C, A, F, E, D, B)

- What would you hope to hear from students that tell you that they are ready to move to the next pattern?

  Purpose of the question: Does she mention both conceptual and perceptual subitizing either formally or informally? Does the rationale provided indicate application of mathematical knowledge needed for teaching, in particular Knowledge of Students and Knowledge of Content?

- (Select a dot pattern from the middle of the sequence that the participant created in the above question.) What if you flashed this pattern to one of your kindergarten students and they responded with the incorrect amount. How might you investigate this wrong answer to find out where their troubles lie? You want them to continue to engage with "dot pattern flash" how could you adjust the activity to meet them where they are?

Purpose of the question: Does the participant draw upon their understanding of learning trajectories by either decreasing the quantity of the dot pattern or exploring early counting skills and abilities (one to one correspondences, rote counting, cardinality) and thus referencing a different learning trajectory? What tools might they suggest to help elicit thinking and understanding from the child? What rationale do they provide for their instructional decisions?

- What if you showed a student Pattern F and that student gave a non-sensible does that tell you about their understanding? How would you follow-up?

  Purpose of the question: This question can be used if the participant did not offer a coherent or clear response to the previous question. It can be helpful in exploring whether or not the participant attends to developmental learning trajectories and has an idea of how to back up questions or tasks to closely explore young children's understanding.

- If you were to suggest a pattern to include in this collection, what would it be, where would you place it, and why? What different responses might you anticipate getting from your students? How would those responses help you decide if it is an appropriate next step?

  Purpose of the question: What theoretical constructs inform the participant's thinking as she recommends next steps? Is her recommended next step appropriate for the progression of the subitizing learning trajectory?

**You made some very thoughtful decisions throughout this whole scenario. What information (do you have about young children, mathematics, etc.) helped you make those decisions?

Thank you for taking time to share your thinking with me! May I come back to you with further questions in the event I have them?

Is there anything else that you would like me to know regarding any of the information you have shared with me or any portion of the interview? Is there any thing you would like to expand upon or add?

Do you have any feedback for me about any part of this interview?

Please pick a book as a thank you for your time.

Appendix D Rationale for Interview Questions

This is a compilation of the interview questions, supporting literature, and the research question being informed.

Interview Question	Research Base for Interview Question	Research Question Being Informed
Would you use these dot patterns with your kindergarten students? Tell me more about your thinking.	Ball et al., 2008 Clements, 1999 Douglas, 1925 Freeman, 1912 Sztajn et al., 2012	In what ways do learning trajectories support prospective early childhood teachers' preparation to become effective teachers?
How might you use these dot patterns with your kindergarten students?	Clements, 1999 Huinker, 2011 Markovits & Hershkowitz, 1997 Risden, 1986 Sztajn et al., 2012	In what ways do learning trajectories support prospective early childhood teachers' preparation to become effective teachers?
What responses do you expect from the children with these dot patterns?	Ball et al, 2008 Baroody et al., 2006 Clements, 1999 Carper, 1942 Sarama & Clements, 2009 Sztajn et al., 2012	Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?
What do the responses suggest?	Ball et al, 2008 Clements, 1999 Fitzhugh, 1978 Huinker, 2011 Risden, 1986 Sztajn et al., 2012	Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?
The dots are in different arrangements, show me how you would use these 6 dot patterns? Explain. Why?	Ball et al, 2008 Clements, 1999 Sarama & Clements, 2009 Sztajn et al., 2012	What understandings do prospective early childhood teachers have regarding the subitizing learning trajectory?
Can you place them in order as to how you might use them with your kindergartners? Explain for me why you placed them in that order?	Ball et al., 2008 Beckwith & Restle, 1966 Brownwell, 1928 Sarama & Clements, 2009 Sztajn et al., 2012	Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?
How do you think the students will tell you that they are ready to move from one pattern to the next pattern?	Ball et al., 2008 Brownwell, 1928 Clements 1999 Sarama & Clements, 2009 Sztajn et al., 2012	What understandings do prospective early childhood teachers have regarding the subitizing learning trajectory?
What if you flashed this pattern to one of your kindergarten students and they responded with the incorrect	Ball et al., 2008 Sarama & Clements, 2009 Sztajn et al., 2012	What understandings do prospective early childhood teachers have regarding the subitizing learning trajectory?

amount. How might you try to find out where their troubles lie?		
If you were to suggest a pattern to include in this collection, what would it be, where would you place it, and why?	Ball et al., 2008 Sarama & Clements, 2009 Sztajn et al., 2012	In what ways do learning trajectories support prospective early childhood teachers' preparation to become effective teachers?
What different responses might you anticipate getting from your students?	Ball et al., 2008 Sarama & Clements, 2009 Sztajn et al., 2012	In what ways do learning trajectories support prospective early childhood teachers' preparation to become effective teachers?
How would those responses help you decide if it is an appropriate next step?	Ball et al., 2008 Sarama & Clements, 2009 Sztajn et al., 2012	Do prospective early childhood teachers draw upon their knowledge of learning trajectories as they make instructional decisions?

Appendix E Pilot Study Coding Manual

Parent Codes and Child Codes

Identity as Student of Mathematics

- Characteristics of Ineffective Teachers of Mathematics (K-12)
- Characteristics of Effective Teachers of Mathematics (K-12)
- Feelings About Self As a Student of Mathematics (K-12)
- Ownership of Mathematics Performance (K-16)

Identity as a Teacher of Mathematics of Young Children

- Pedagogical Choices: "Bad" Teacher
- Pedagogical Choices: "Good" Teacher
- Goals As A Future Teacher of Mathematics
- Developing Beliefs About How Teachers Should "Be “As a Math Teacher

Decisions that Meet and Advance Mathematical Understanding of Young Children

- Evidence of Learning Trajectory-Based Instruction
- Decisions Peripherally Related to the Mathematics of the LT
- EC Posts’ beliefs about children
- Developing Pedagogical Content Knowledge

Appendix F This Study's Coding Manual

Coding manual used for first cycle inductive codes for this study.

Inductive Code	Description of Code
Defined subitizing either informally or formally.	I assigned this code to data that identified participants' ability to (1) recognize dot patterns as prompting subitizing, and (2) discuss subitizing as quickly seeing quantity.
Stated the difference between perceptual and conceptual subitizing.	I assigned this code to data that identified participants' noticing of perceptual (see and naming quantities as a whole) and conceptual subitizing (seeing groups and quickly combining to name the whole.)
Identified why subitizing is important.	I assigned this code to data that identified participants' statements regarding the short- and long-term benefits of subitizing abilities for young learners of mathematics.
Focused on quantity.	I assigned this code to data that identified participants' noticing of the importance of children's understanding of quantity.
Articulated how subitizing skills grow over time.	I assigned this code to data that identified participants' understanding of subitizing abilities and the developmental nature of how those skills grow with experience.
Provided rationale for order of dot patterns that mirrored developmental growth.	I assigned this code to data that identified participants' reasons for the order of the Set 2 dot cards. The developmental nature of children's mathematical thinking was articulated in these passages.

Awareness of developmental growth in math.	I assigned this code to data that identified instances of participants' recognition that growth in young children mathematics is developmental in nature.
Addressed Pattern E in a way that revealed why subitizing is important.	I assigned this code to data that identified participants' initial discomfort with Pattern E. Initially this code was used to identify instances where participants were uncertain about using Pattern E as its linear arrangement did not support subitizing.
Kept a focus on understanding quantity.	I assigned this code to data that identified instances where participants wanted children's focus to be on naming quantity, and not counting the dots in the arrangements. This was most prevalent with Pattern E.
Adjusts the pattern to a smaller quantity.	I assigned this code to data that identified instances where participants managed a child's incorrect response to "how many" by stepping back to a smaller quantity that the child would likely successfully subitize.
Asks the child to count the dots.	I assigned this code to data that identified instances where participants managed a child's incorrect response to "how many" by asking them to "count to find out." Theoretically moving them off the subitizing trajectory.
Prompted child to "show me why you think there are ten"	I assigned this code to data that identified instances where participants managed a child's incorrect response by asking the child to show them how they see ten dots in pattern of five.
Told them children they were wrong.	I assigned this code to data that identified instances where participants managed a child's incorrect response by telling the child their thinking was wrong.

Introduced new representation to support understanding of quantity.	I assigned this code to data that identified instances where participants managed a child's incorrect response by introducing a new representation or tool to see quantity such as a five frame, a ten frame, a number card, or counters.
Organized proposed instruction around student readiness.	I assigned this code to data that identified instances where participants stated they would informally assess children's current level of understanding before they began instruction.
Articulated strategies to elicit student thinking.	I assigned this code to data that identified instances where participants' suggestions to prompt young children to share or explain their thinking.
Used student reasoning as a starting point for instruction.	I assigned this code to data that identified instances where participants articulated they would start instruction by eliciting children's thinking.
Justified pattern based on developmental nature of subitizing trajectory.	I assigned this code to data that identified instances where participants' created a pattern to include with Set 2 that followed developmental pathway of the subitizing trajectory.
Based new pattern on children's potential subitizing skill.	I assigned this code to data that identified instances where study participants' discussed the inclusion of the new pattern through the lens of children's existing subitizing abilities and appropriate next steps.
Discussed development of the pattern in light of children's thinking.	I assigned this code to data that identified instances where participants shared the dot pattern they created was informed by children's thinking.

Personal thoughts about teaching mathematics to young children	I assigned this code to data that identified instances where participants shared personal beliefs throughout the course of the interview. Some examples included their perspectives on the content, their own interpretation of developmental growth, children's abilities as learners of mathematics, and their feelings (positive or negative) about mathematics and teaching mathematics.

Appendix G Analytic Memo Sample

Analytic memo documenting participant-generated dot patterns.

why?

Follows LT: yes.

This task ~~[illegible]~~ provided additional insights into understanding of subitizing trajectory

#1 Karina

Quantity ↑

[I'm thinking that I want to add a 6 in here.] For a challenge. So this would not be for a lot of my students, but maybe break it down so there are two rows, actually I'm going to give a few options. So I would go with this one, first, which is the basic pattern from the die. Or even something to the extent of this. so it would be challenging, but it pushes the concept of three, which I like because, we are seeing two quantities of three. So I like this idea that I have here because they're going to see this (pointing to the quantity of three) and then they're going to see this (pointing to another quantity of three). And if I'm going to introduce 6 to my students, they're going to know three first. I mean -ish. Then this can push us into, How many do you see all together? And I like that. (322-334)

Challenge → perceptual to 3. → 3 and 3 is 6 conceptual

connects to own students

intentional → challenge
→ leads to conceptual subitizing.

purpose: pushes the concept of 3.

#2 Jaeden

Pattern A

5 and one more is 6.

Quantity ↑

Follows LT: yes

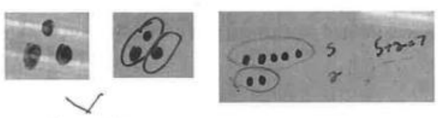
see lines 265-268

[Well I would start here because the pattern is organized.] Kind of my standard thing. And the kids could see three and three. they could also see the group of four at the top and then add one more on for 5 and then add one more on for 6. And that reminds me of pattern A from before. So if they could find that pattern inside of this new pattern then they would know that it's five and one more and that would get them to 6. (277-28) If they were comfortable with six, and they knew that this was 6 right away, I would add one more. I would make it obvious that it is another one that is being added on. that way they could see that it's the same pattern of six. but it is just one more. (297-299)

intentional → organized
→ add on one more to get to 7.

purpose: scaffold from 5 to 6.

#3 Karaleen



→ I think for myself, if they were struggling with the five, I would probably go with lower numbers. [assuming that they are struggling I would probably do something smaller like three.] This would help me focus on grouping. We could find the smaller groups inside of the three. (220-222)

missed the idea of perceptual subitizing

intentional

→ I like to incorporate this work with the ten frame. So I would give them this pattern with the five on top and 2 on the bottom because it will be familiar. So I would use the ten frame in the dot patterns together.so I see here that I have a total of 7, but what I like about this is that we can talk about groups. So I can see the five here and I can see the two here and if I wanted to I could bring in a number sentence. (writes 5 + 2 = 7) So my work around this would eventually lead the kids ~~two~~ number sentences, and Define the different groups that make up the different quantities. And I would want them to be able to add them to get the correct answer. (234-240)

does this move them forward? or is this a depth of assessment

funneling

Quantity ↓

Follows LT: yes.

intentional → move back and move forward
→ organization of dots according to structure of 5.

talk about groups

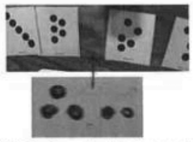
purpose: strugglers? → get engaged
next step? → groups → number sentence and addition

were they advancing knowledge?

#4 Mandisa

Quantity 5

1.



This is inserted between Pattern F and Pattern D

Follows LT: yes

So you have that triangle again and I think they would be able to see that. The two then would be in a straight line. I think then that would help them more easily identify that as a group of 2. Yeah I think this help pull those groups of parts so they see the two groups the group of 3 and then the group of 2. (189-192)

I hope that they would say 3. I think it would be cool if they would say I saw three in the triangle And then I would hope that they would say and two more. I also would want them to be able to tell me how many dots they see altogether. (197-199)

pattern is built around those that came before

purpose: more easily identify 2 and 3

prerequisite.

#5 Marisol

Pattern that was added in: (The created pattern is place between C and A)

The pattern that I created would be very similar than to pattern A. And it would help them in recognizing the similarities between the patterns. They can see the four in the pattern that I created and then they can see the four in pattern A. If they play a lot of dice games they might recognize pattern A is 6 with one missing. And that would still help us see that there are many ways to look at it and still get to 5. (303-306)

- pattern is built around those that came before.
- connection between the patterns

prerequisite:

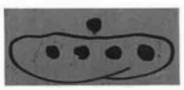
Quantity 5

Follows LT: yes

purpose: see 4 and 1 like dice

recognize similarities between patterns (Pattern C is the prereq)

#6 Sasha



This is the pattern that I would do. So maybe if they already understood ~~for~~ four I could just do this. So if they already understood for, but this pattern is 4 and just one more. I think I would put this pattern at the beginning of the sequence. that way they would just see the four and the one. And I think it would be nice to do it right after the pink card with the ~~tee~~ and the one and then right before the recognizable dot pattern of 5. (259-263)

Quantity 5

Follows LT:

purpose: See 4 and 1 before Pattern C

prerequisite: understand four.

#7 Amalie

Added between E and B (at the end

So I would add a pattern between B pattern and pattern E. I would make it look something like this. So it gives the center diagonal, but then it also groups the first dot and the last dot. So then they have an easier transition to the diagonal. So then when you move them into a full linear pattern or a diagonal like we see in pattern E they can still shift the dots into a pattern that they've seen before and subitize the quantity. (293-297)

That's a similar pattern to what they've seen and where there is go out where they're going. So when they get to that last pattern, pattern E, it's not such a shock. (304-305)

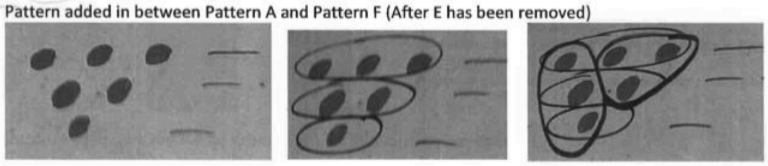
Quantity: 5

LT?: yes

purpose: nice transition to Pattern E

#8 Cecilia

Pattern added in between Pattern A and Pattern F (After E has been removed)



I know that this is not 5. It's 6. And I think I would put it here in between these two (places the board inbetween Pattern A and Pattern F) because it's a little bit more clumped together. I think it's pretty easy to see these different groupings. I can see where it would be more challenging. I can also see where you could get lots of different answers. (309-312)

Quantity: 6

inline w/ LT: yes - ultimately placed at end.

purpose: see groups in many ways

No. I guess, Now I'm trying to decide if I would put it after pattern F instead of before because it does have that extra dot. I don't know, I guess I would still put it before, just because (pause) I think it's just an easy way to see many different groupings. (324-326)

Another way that you could see it would be like this with the two groups of three. (She circles the three in a diagonal on the left, and then the remaining three and the upper right hand side which form a triangle.) I guess I'm still struggling with where I might place it. I'm trying to figure out of adding that one extra dot makes it that much more challenging. So maybe, I guess I'm not sure yet. I never really thought one dot would make that big of a difference. But now that I'm thinking about it it is a step up. (335-339)

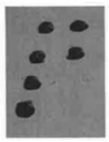
interesting aspects:

- come to the realization that is was a hard pattern

#9 Flora

Added pattern and how children might see it:

2nd time this pattern has appeared. :)

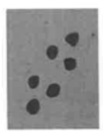


I would actually put it at the end. We are moving one number above 5, because as you can see I have 6 in this pattern. I think that they would see you before here or they would see the four here and then they could add the to another spot to make 6. Going back to the idea of having the pattern be more organized that way they will see more clearly the different groupings that they can make. (417-420)

Quantity: 6
in line: Yes
purpose: surface groups of 4 and 2
interesting: wants pattern organized. not scattered.

#10 Crystal

Pattern added how she would expect children to name to total quantity.



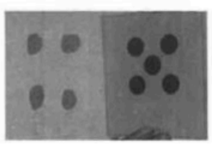
So here is one three and another three and I would put it at the end. (330)
Researcher Note: very shy and had a hard time explaining her thinking so I did not push.

interesting

Quantity: 6
in line: yes
purpose: see 3 and 3

#11 Karolyn

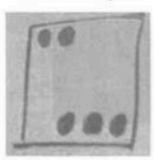
Suggesting patterns for the dot sequence (424)



I think I would have shown up four pattern. Like the kind that you would see on the die. (424-425) I would have placed that before we even did the first pattern of 5 because first, it's a smaller numbe and it is pretty similar to the 5 dot pattern that is coming up because you are just missing the one, and I hope that they would be able to recall that the dot is missing.[00:33:35] They would also need to know that four and one more is five. So I think I would do that to get started. (430-43)

Quantity: 4
in line w/ LT: yes
purpose: introduce 4 before 5.
interesting aspect: Hope they would remember 5 and use it to get to 4.

And another pattern (430-435)



Next, I remember us having a pattern in class that was like two or three because I think, I don't know if that would be easier or more challenging because it is more spaced out...(433-435) (Where might you place it?) Ummm....Possibly, maybe, like right after pattern D. (Pause.) I don't know I guess this pattern could be a little bit more challenging because the dots are so spaced apart so your eyes would need to see one group and then see the other group and then put them together to get the total. They would really need to capture those two images (of 2 and 3) in their minds. I would hope that over time as we do more dot patterns they would become more efficient and more fluent so maybe this one would it be too hard for them. I hope that they would know that it's fine because they could look for those different groups. I remember this type of. Pattern from that game where we had to match different patterns together. I think it actually was like Find the Same Amount or something like that and that's where that dot pattern came from. (441-449)

Quantity: 5
LT: Yes
purpose: see 2 distinct groups + combine for 5.
interesting: if dots are spread apart is it more easier or more challenging.

#12 Kayla

Adding in the first dot pattern (290-293)

I would probably add something like this because I don't really see, I see a lot of threes and twos but I don't see you four all by itself. Unless you look at pattern A and you see the four in pattern A I would probably added in between pattern F and pattern B somewhere in there. (290-293)

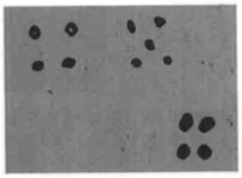
interesting.

I thought that fit in nicely with to this idea of a line of three and a line of four. And now there's this line of four with the one on the side which would help them see another way for 5. This could also help them get ready to think a little bit more carefully I guess for Pattern E. Because in Pattern E they see all the dots in a straight line so this pattern that I just created has four in a straight line and one more. So moving from Pattern, or moving from this pattern that I just created, to Pattern E might not be as big of a challenge as I thought it was. (300-305)

Quantity: 5
in line w/ LT: yes
purpose: highlight 4 and prep for Pattern E.

#13 Amber

Adding in a pattern of 4 before Pattern C. First example shows the dots more spread out than the second pattern.



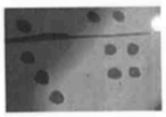
I guess that maybe before Pattern C I would include a 4 pattern like you see on a die. (355-356)

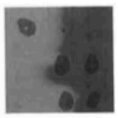
Then they can learn the 4 pattern and know that the next one (Pattern C) we are just adding the one dot in the middle. If they know the 4 -- I might even place them closer together – (363-364)

Yes. I think that some kind of 4 would be really helpful. Either the really close one or the one that is a little more open. I am not set on which one. (368-369)

quantity: 4
in line w/ LT: yes
purpose: lead in to 5.

#14 Justine

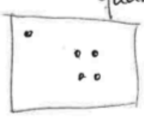




I would pick...I really like the dice patterns. So would go back to that. I would start with the three....actually. I would not start with the 5. I would start with 1, 2, 3 and 4. (Draws the patterns as seen on the die for each pattern.) (213-215)

Yeah...in the middle...any other patterns I would choose? I mean, just this one...I really like the 4 and 5. I remember in the video the child said "This one outside should go in the middle to make 5." (makes a pattern of 4 like on a die and places one dot off to the top right above the 4). (221-223)

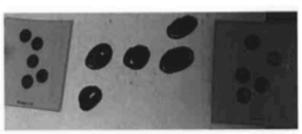
in line w/ LT: yes
purpose: work w/ smaller quantities

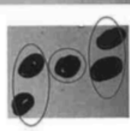


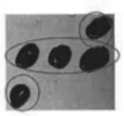
four and one more is five.

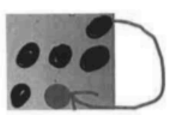
#15 Marie

Adding in dot patterns:









I am looking here between there. I am looking at the 3 that are easy to visualize (#15 points to Patterns C, E, and F) and then this one (Pattern B) where the dots are very random and then this one (points to Pattern D). I guess B and D could go in either order. But what would I put there, between them, is the question. (Thinks for about 10 seconds.) ~~Ummmm...I have no idea!!~~ (Starts drawing.) I would put 1, 2, 3, 4, 5 (draws out a pattern shown below). I think that what I have is kinda the same thing as (Pattern B) but rotated. It is just rotated a little bit but I think that groups are more clear. But now they can show me that they see the 2, 1, and 2. They can see the 1, 3, and 1 or they could move one or move this guy over to make the 4 and 1 and see that it looks the same. (314-321)

This one just kinda throws me off (Pattern B) completely. The other ones I know what I would do--it is just a little tricky. That is why I would put this one here between them.So, looking at this, is there a correct way of like setting these up -- is there a correct way of dot patterns of exposing different numbers? Like would you do like one solid group, then move to 2 groups, then move to 3? Is there are "real" way to do this? (329-333)

Quantity: 5
in line w/ LT: yes
purpose: make groups more clear.
interesting: B is random
D and B can go in any order.

Appendix H Participants' Order of Set 2 Dot Patterns

Order of Set 2 dot patterns as established by each participant.

	Participant	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	Correct order?	Teaching math during methods semester?
1	Karina	C	A	F	E	D	B	Yes	Yes[1]
2	Jaeden	C	A	F	E (Omitted)	B	D	Yes	No
3	Karaleen	C	A	E	F	D	B	Yes	Yes[3]
4	Mandisa	C	A	E	F	D	B	Yes	Yes
5	Marisol	C	A	F	E	C	B	Yes	Yes[2]
6	Sasha	C	E	D	A	F	B	No	No[4]
7	Amalie	A	C	D	F	B	E	No	Yes
8	Cecilia	C	D	A	E	F	B	No	No
9	Flora	E	C	A	F	D	B	No	No
10	Crystal	C	D	F	A	E	B	No	No
11	Karolyn	C	A	F	E (Omitted)	D	B	Yes	No[5]
12	Kayla	C	A	F	E (Omitted)	B	D	Yes	No
13	Amber	C	A	F	E (Omitted)	D	B	Yes	No
14	Justine	C	F	E	A	D	B	No	No[5]
15	Marie	C	A	F	E (Omitted)	B	D	Yes	No[6]

[1]K3-K5 EC SPED full time teacher record
[2]Full-time K5 teacher
[3]K5 paraprofessional (previous full-time teacher)
[4]SP ED paraprofessional during methods semester; K3-K-5 EC SPED full time teacher record at time of interview
[5]Student teaching at time of interview. No experience teaching math during methods semester.
[6]In-home childcare provider (3-year old twins)

Appendix I Participant-generated Strategies

Participant's instructional strategies for investigating student misconceptions of Pattern F.

Initial Strategy and Outcome	Participant	Follow Up Strategy #1	Follow Up Strategy #2
Teacher used questions to explore student thinking: • Where did you see ten? • How did you see ten? • Explain to me where the ten is. Outcome: Children still believe Pattern F has ten dots.	Amber	*Count the dots.* "Maybe if we counted [the dots] together and looked at how the pattern flows but noticing the three and then the two."	*Use counters to make the pattern.* "If I told them to make the pattern, they can count it, but having them make it...would really make them see the five."
	Crystal	*Ask children to look again.* "I think they would look at the card again and recount."	*Count the dots.* "I guess we need to go back and start with step one and work on one-to-one correspondence. I would have them count the dots."
	Flora	*Count the dots.* "They would probably count the dots one by one to show me how many they saw."	*Translate to ten frame.* "I would put one counter on each dot and then put those on the ten frame. I think the ten frame would help them just to see the number five differently here."
	Karaleen	*Listen to children.* "I am listening for how they might be grouping those dots that are in that pattern."	*Translate to ten frame.* "Put the counters on the ten frame so they can see… and decompose that pattern to get down to smaller quantities."
	Kayla	*Look again.* "I would flash the pattern again quickly, and have them explain how they saw it."	*Draw the pattern.* "Have them draw what they saw. Then they could explain why they think that it's ten or how they saw it as ten."
Teacher asked students, "Show me. Draw what you see." Outcome: Children still believe Pattern F has ten dots.	Karolyn	*Use counters to make the pattern.* "I guess they could also represent the pattern using manipulatives, each student doing it on their own and ask, "Does	*Decrease quantity to subitize.* "I guess I think that I might need to bring them back down to a smaller quantity. Maybe five is just too much for them

		this represent your thinking?"	right now."
	Sasha	*Ask questions so children explain their thinking.* "Can you show me? Can you show me where you saw them? Can you show me you thinking?"	*Count the dots.* "Each time they touch a dot they would say a number. I actually have a student [that keeps counting]. And this is what I've been trying to have him do."
Teacher modeled the quantity of five on a ten frame. Outcome: Children still believe Pattern F has ten dots.	Cecilia	*Count the dots.* "Let's take a minute and let's count the dots."	*Compare five to ten.* "If they are familiar with the ten frame, they would know that only half of the ten frame would be filled, so it can't possibly be ten."
	Jaeden	*Ask questions so children explain their thinking.* "How can there be ten dots on that card if the ten frame is not filled up?"	*Reference ten frame.* "See, there's a three and there's a two. That's not ten."
Teacher asked students, "Show me. Make the pattern with counters." Outcome: Children still believe Pattern F has ten dots.	Karina	*Ask questions so children explain their thinking.* "How many is that? And then you kinda know…if they're counting, or if they just tell you."	*Practice one-to-one correspondence.* "If they still think it is ten after counting, like if they keep counting after five, I would take a whole step back, and focus on one-to-one correspondence."
Teacher flashes Pattern A and then returns to Pattern F. Outcome: Children still believe Pattern F has ten dots.	Amalie	*Change orientation of patterns.* "If I still get weird answers for Pattern F after I have flashed Pattern A in the original way with the dots going up and down, I might turn Pattern A and Pattern F on their side to see if that would help."	*Build and count.* "If I'm still getting weird answers then I would probably have them build it. And after they build that I would probably have them count it."

Open invitation for children to "show me what you see." Outcome: Children still believe Pattern F has ten dots.	Justine	*Count the dots.* "I would ask them to come up and point to me where they see the ten. Hopefully they would correct themselves and say like 'I see two here and three here.' Or, they would count then 'one, two, three, four, five' and say Oh!! That is five.'"	*One-to-one match and count.* "So I just put one bear counter on each dot maybe and then after he matched them up I would ask them to count them again, like 'one, two, three, four, five.'"
Teacher drew Pattern F on the board to record children's thinking following directions from the children. Outcome: Children still believe Pattern F has ten dots.	Mandisa	*Ask children to explain their thinking.* "From what I drew I would have them tell me how they saw the ten."	A second follow-up was not provided.
Teacher counted the dots on the card for the children. Tells them there are five. Outcome: Children still believe Pattern F has ten dots.	Marisol	*Ask questions so children explain their thinking.* "I would ask them things like 'how did you see this as ten?' or 'Where did you see the ten dots?'"	*Decrease quantity to subitize & Offer a familiar pattern.* "I might do depending on their answers is go back to smaller quantities in the other patterns that I use with them. Or I might even try a different type of pattern to see if that would help them."

Curriculum Vitae

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EDUCATION

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Graduate Teaching Assistant, Curriculum and Instruction — 2014 – 2017
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Classroom Teacher, Grade 5, Golda Meir School — 2006 – 2008
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Teacher-in-Residence, Milwaukee Mathematics Partnership — 2004 – 2006
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Milwaukee Public Schools & University of Wisconsin–Milwaukee

Classroom Teacher, Grade 3, Milwaukee Spanish Immersion School 1996 – 1998
Milwaukee Public Schools, Milwaukee, WI

Student Support Teacher, Burleigh Elementary School 1994 –1996
Elmbrook School District, Brookfield, WI,

Classroom Teacher, Grade 3, Milwaukee Spanish Immersion School 1988-1994
Milwaukee Public Schools, Milwaukee, WI

Classroom Teacher, Grade 3-4, Franklin Pierce Elementary 1986-1988
(renamed Riverwest Elementary), Developmental Bilingual
Milwaukee Public Schools, Milwaukee, WI

SUMMARY OF UNIVERSITY TEACHING EXPERIENCE

Graduate Teaching Assistant
2014-2017 University of Wisconsin-Milwaukee, Milwaukee, WI

- Taught two sections of Teaching Mathematics: Early Childhood each semester for prospective regular and special education teachers for 6 semesters.
- Facilitate the development of a mathematics teacher identity in prospective teachers.
- Guided the enactment of equitable teaching practices and supported prospective teachers in identifying long-and short-term consequences of inequities in the mathematics classroom.
- Grounded course content in Wisconsin Academic Standards for Mathematics, young children's mathematical thinking, and early childhood mathematics learning trajectories.
- Supervised student teachers in Middle Childhood Early Adolescence Program.

Teacher-in-Residence, Milwaukee Mathematics Partnership
2004-2008
Milwaukee Public Schools & University of Wisconsin–Milwaukee, Milwaukee, WI

- Teamed with university mathematician and mathematics educator to design, revise, and implement curriculum for Geometry for Elementary Education Majors.
- Designed and delivered conference presentations to disseminate grant initiatives and research highlights at local, state, and national conferences.
- Collaboratively designed and delivered monthly professional development for Math Teacher Leaders and Assessment Pilot Committee.

Teacher-in-Residence, Milwaukee Partnership for Teacher Quality
2000-2002
Milwaukee Public Schools & University of Wisconsin–Milwaukee, Milwaukee, WI

- Instrumental in the development and implementation of Professional Urban Teaching Seminar for Middle Childhood through Early Adolescence (MCEA) students. Co-instructed with university faculty for 4 semesters.
- Provided a vital link for MCEA students between university theory and urban classroom experience. Assist students to plan and implement first formal lessons in elementary urban classrooms, including classroom observations and formative feedback.
- Supported MCEA student to critically examine pre-conceived beliefs about students' language, gender, race, or socio-economic status and their influence on inequitable educational outcomes.
- Member Milwaukee Partnership Academy Implementation Team. Supported implementation of MPS Balanced Literacy Initiative.

Graduate Teaching Experience

University of Wisconsin–Milwaukee, Milwaukee, Wisconsin
Co-Instructor, *Pathways to Teacher Leadership in Mathematics* Project
CURRINS 705 Research in Schools and Communities, Spring 2016
CURRINS 624 Instructional Trajectories for Fraction Concepts and Operations, Summer 2015
CURRINS 626 Principles and Practices of Teaching Algebraic Reasoning, Summer 2014

Co-Instructor, *Strong Start Mathematics* Project
CURRINS 626 Principles and Practices of Teaching Algebraic Reasoning, Summer 2016
CURRINS 725 Improving Teaching and Learning with Classroom-Based Assessments, Spring 2017
CURRINS 730 Mathematics in Early Childhood and Elementary Education, Summer 2017

Undergraduate Teaching Experience

University of Wisconsin – Milwaukee, Milwaukee, WI
1999-present
CURRINS 330 Teaching Mathematics: Early Childhood Mathematics (10 semesters)
CURRINS 331 Teaching Mathematics: Grades 1–6 (10 semesters)
CURRINS 332 Teaching of Mathematics: Grades 6-8 (Fall 2005) (Co-instructor Dr. Kepner)
Middle Childhood Early Adolescence Professional Urban Teaching Linking Seminar (1999-2002)
MATH 277 Geometry for Elementary Education Majors (Co-taught with Mathematics Department Faculty) (Fall 2004; Fall 2005)
CURRINS 561: Issues in Elementary and Middle School Mathematics (Special Education Focus) (Spring, 2002)

Other

Guest Lecturer: Middle Childhood Early Adolescence Student Teaching Seminar (2000-2002) Developed and delivered workshops focused on teaching of conflict resolution skills to elementary students and proactive and positive classroom management strategies.

SUMMARY OF PK-12 PROFESSIONAL EXPERIENCE

Mathematics Consultant **2017-Present**

Wisconsin Department of Public Instruction, Madison WI

- Provide leadership for Pre-K-Grade 8 mathematics education across the state of Wisconsin to ensure all students are college and career ready.
- Collaborate with statewide educational organizations and leaders to realize Wisconsin's vision for student success in mathematics.
- Lead the development of Instructional Guide for Equitable Teaching of Mathematics a project that collaboratively engages statewide mathematics teacher leaders. Anticipated completion date: August 2019
- Designed, piloted, and revised two Math Professional Development Modules anchored on early mathematics learning trajectories intended to support high-quality early childhood mathematics instruction. Prepared materials for dissemination and trained statewide educational partners on content and facilitation of modules.
- Develop and facilitate professional development to identify and employ instructional practices in mathematics within a culturally responsive and equitable multi-level system of support.
- Statewide Coaching for Equity Collaborative. Under the guidance of Dr. Decoteau Irby. Member of Mathematics Leadership Team.
- Establish and maintain effective working relationships with stakeholders inside and outside the Department.

K-8 Mathematics Teaching Specialist **2010-2014**

Mequon-Thiensville School District, Mequon, Wisconsin,

- Coordinated and facilitated implementation of K-8 mathematics instructional program for regular and special education students for 3 elementary schools and 2 middle schools.
- Developed structure and content for district K-8 Response to Intervention Program (RtI) for Mathematics. Trained 27 teachers to facilitate mathematics interventions.
- Successfully transitioned K-8 faculty to the Wisconsin Academic Standards for Mathematics.
- Founded and facilitated the K-8 Mathematics Leadership Team which served as the driving force for mathematics reform and improvement.
- Developed and delivered all mathematics-related professional development for K-8 staff.
- Developed structure and content for district K-8 Response to Intervention Program (RtI) for Mathematics. Trained 27 teachers to facilitate mathematics interventions.

Mathematics Teaching Specialist, Milwaukee Mathematics Partnership
2008-2010
Milwaukee Public Schools, Milwaukee, WI

- Collaborated with a team of 9 mathematics specialists, university professors and additional support staff to develop and deliver monthly training for 140 district Math Teacher Leaders.
- Supported cohort of 15 MTLs and their respective school communities in successfully realizing school-based and district math improvement goals.
- Designed and delivered monthly training for Math Teacher Leaders.
- Developed and delivered custom math professional development opportunities for cohort schools, district leaders, and support staff.

GRANT RELATED EXPERIENCE

Hedges, M. (Project Director/Lead Instructor). *Starting students strong in mathematics: Strengthening teacher mathematical knowledge and instruction in grades K-3*. (Project Period: 2015-2018). Principal Investigators: D. Huinker, G. Pinter, & A. Ford. U.S. Department of Education Mathematics and Science Partnership Program, ESEA Title II, Part B (Grant No. WI151201), $756,814.

Hedges, M. (Instructor). *Pathways to teacher leadership in mathematics: Building capacity for excellence in mathematics teaching and learning*. (Project Period: 2014–2017). Principal Investigators: D. Huinker & M. Steele. Wisconsin ESEA Improving Teacher Quality Program (WEITQ), U.S. Department of Education, ESEA Title II, Part A (Grant No. 14-0333), $379,097.

Hedges, M. (Instructor). *Core mathematics partnership: Building mathematical knowledge and high-leverage instruction for student success*. (Project Period: 2015-2016). Principal Investigators: D. Huinker, K. McLeod, & K. Ebbers. U.S. Department of Education Mathematics and Science Partnership Program. ESEA Title II, Part B (Grant No. WI140707), $717,929.

Hedges, M. (Instructor). *Leadership for the common core in mathematics: Developing teacher leaders for transitioning instruction and assessment to the Common Core State Standards for Mathematics*. (Project Period: 2012–2014). Principal Investigator: D. Huinker & K. McLeod. Wisconsin ESEA Improving Teacher Quality Program, U.S. Department of Education, ESEA Title II, Part A, $238,088.

PUBLICATIONS

Hedges, M. (2014). Classroom talk: A tool for powerful teaching and learning. *Wisconsin Teacher of Mathematics, 65*(2), 19-22.

Hedges, M., Brekke, J., Clark, N., Cuirlik, B., Cutter, E., Grube-Thur, S., Lock, M., & Maleszewski, J. (2012). *Mathematical reasoning and representation: Modeling with tape diagrams. A resource guide developed by teachers for teachers.* Developed by the Mequon-Thiensville Mathematics Leadership Team. Freely distributed.

Hedges, M., Huinker, D., & Steinmeyer, M. (2005). Unpacking division to build teachers' mathematical knowledge. *Teaching Children Mathematics*, 11, 478-483.

Huinker, D. & Hedges, M. (2015, April). *Comparison of student and teacher perceptions of mathematics teaching practices*. Paper presented at the annual meeting of the National Council Teachers of Mathematics Research Conference, Boston, MA.

Post, L., Pugach, M., Harris, S., & Hedges, M. (2006). The Teachers-in-Residence program: Veteran urban teachers as teacher leaders in boundary spanner roles. In Howey K. & Zimpher, N. (Eds). *Boundary Spanners* (pp. 213 – 236). Washington, DC: American Association of State Colleges and Universities and National Association of State Universities and Land-Grant Colleges.

NATIONAL PROJECT INVOLVEMENT

Scholarly Inquiry and Practices Conference for Mathematics Education Methods
Fully-funded Participant
September/October 2015
Kennesaw State University Research and Service Foundation
Kennesaw State University, Atlanta, GA.

Developing Teachers' Mathematical Knowledge for Teaching Institute
Fully-funded Participant
June 2004
Center for Proficiency for Teaching Mathematics
University of Michigan, Ann Arbor, MI

PROFESSIONAL DEVELOPMENT WORK

Mathematics Institute of Wisconsin**,** Waukesha, WI
Wisconsin Statewide Mathematics Initiative, Since 2012
Co-developed and facilitated professional development modules for:

- Operations and Algebraic Thinking (OA) Domain (10 three-hour sessions)
- Number and Operations in Base Ten (NBT) Domain (10 three-hour sessions)
- New Facilitator Training (27 hours total)

Professional Development Provided to Districts and Schools
Provided K-8 professional development in mathematics, including unpacking standards, developing assessments aligned to standards, facilitating math-focused classroom discourse, and understanding student mathematical thinking for the following:

- Greendale School District, Greendale, Wisconsin, 2014-2016
- Big Foot Area School District, Walworth, Wisconsin, 2016
- Divine Mercy Catholic School, South Milwaukee, Wisconsin, 2016
- Wauwatosa School District, Wauwatosa, Wisconsin, 2014-2015

AWARDS

Chancellor's Graduate Student Award Fellowship 2015-2016; 2016-2017
Robert Kuehneisen Teachers for A New Era Scholarship 2016-2017

NATIONAL WEBINAR

Huinker, D., Hedges, M., & Cutter, L. (2013, March). *Moving to common practice with the common core: Building district capacity to transform mathematics classrooms*. Invited webinar presented through the MSPnet Academy, National Science Foundation, Math and Science Partnership program.

PRESENTATIONS

Selected National Presentations

Douglas-Meyer, M. **& Hedges, M.** (2019, April) More or Less? Using Learning Progressions to Support Student Understanding of Comparison. National Council of Supervisors of Mathematics. San Diego, CA.

Hedges, M. & Huinker, D. (2016, November) *Emerging Leadership Actions of Elementary Teachers in an Urban Mathematics Education Graduate Program.* Poster presented at the Psychology of Mathematics Education-North America Conference, Tuscon, AZ.

Huinker, D., Steele, M. & **Hedges, M.** (2016, January) A *Graduate Model for Developing Regional Leaders in Urban Mathematics Education.* Association of Mathematics Teacher Educators Annual Meeting, Irvine, CA.

Hedges, M. (2015, October) *Deepening Representational Competence Through the Use of Linear Models for Single-Digit Addition and Subtraction Kindergarten—Grade 2.* Poster presented at the Scholarly Inquiry and Practice for Mathematics Education Conference, Atlanta, GA.

Huinker, D. & **Hedges, M.** (2015, April). *Comparison of Student and Teacher Perceptions of Mathematics Teaching Practices.* National Council of Teachers of Mathematics Research Conference, Boston, MA.

Schefelker, B., Laughlin, C., & **Hedges, M.** (2015, April) *Making Sense of the K-5 OA Domain: Using the Progressions to Support Classroom Instruction*, National Council of Supervisors of Mathematics Annual Meeting, Boston, MA.

Hedges, M. & Schefelker, B. (2014, April) *Balancing Conversations in Mathematics Classrooms*, National Council of Supervisors of Mathematics Annual Meeting, New Orleans, LA.

Hedges, M., Laughlin, C., Schefelker, B. (2012, April) *Building Collegiality Between Regular Education and Special Education Teachers to Meet the Needs of Struggling Students.* National Council of Supervisors of Mathematics Annual Meeting, Philadelphia, PA.

Hedges, M., Laughlin, C., Schefelker, B. (2012, April) *Building Collegiality Between Regular Education and Special Education Teachers to Meet the Needs of Struggling Students.* National Council of Supervisors of Mathematics Annual Meeting, Philadelphia, PA.

Schefelker, B., Laughlin, C., & **Hedges, M.** (2012, April) *Tapping Into Potential: Developing a Culture that Supports a Growth Mindset and Fosters Perseverance in Mathematics.* National Council of Supervisors of Mathematics Annual Meeting, Philadelphia, PA.

Hedges, M. & Schefelker, B. (2011, April) *Representations: Getting at the Heart of Understanding.* National Council of Teachers of Mathematics Annual Meeting, Indianapolis, IN.

Laughlin, C., **Hedges, M.,** & Schefelker, B. (2011, April) *Absolute to Relative Thinking: The Road to Proportional Reasoning*. National Council of Teachers of Mathematics Annual Meeting, Indianapolis, IN.

Schefelker, B., Laughlin, C., & **Hedges, M.** (2011, April) *Developing Mathematics PD Sessions: Planning Conversations and Instructional Decisions that Lead to Improved MKT in District Leaders*. National Council of Supervisors of Mathematics, Indianapolis, IN.

Bedford, P., **Hedges, M.,** & Schefelker, B. (2010, April) *Building Effective Relationships that Lead to Instructional Change in Mathematics Classrooms.* National Council of Supervisors of Mathematics Annual Meeting, San Diego, CA.

Hedges, M. & Schefelker, B. (2010, April). *Get Them Into the Ball Park: Using Estimation As A Means To Help Students Determine Reasonableness.* National Council of Teachers of Mathematics Annual Meeting, Washington, DC.

Hedges, M. & Schefelker, B. (2010, April) *Using repeating patterns to think functionally.* National Council of Teachers of Mathematics Annual Meeting, San Diego, CA.

Schefelker, B, Marchillo, J. & **Hedges, M.** (2010, April) *Effective Conversations That Lead To Effective Instruction: Using A Portfolio Process to Structure Teaching and Learning of Mathematics.* National Council of Supervisors of Mathematics Annual Meeting, San Diego, CA.

Schefelker, B. & **Hedges, M.** (2008, April) *Making Sense of Division.* National Council of Teachers of Mathematics Annual Meeting, Salt Lake City, UT.

McLeod, K., Kepner, H., **Hedges, M.,** & Schefelker, B. (2008, January) *Mathematics Fellowships: Providing Mathematics Content to Middle-grades Teachers.* Association of Mathematics Teacher Educators Annual Conference, Tulsa, OK.

Schefelker, B, **Hedges, M.,** Buckholt, P., & Buckholt, J. (2007, March) *Supporting Special Education Teachers: Exploring Alternate Algorithm Strategies to Expand Student Experiences with Mathematics.* National Council of Supervisors of Mathematics Annual Meeting, Atlanta, GA.

Laughlin, C., **Hedges, M.,** & Huinker, D. (2007, March) *Content Knowledge for Leadership: District Model to Strengthen and Assess Mathematics Teacher Leaders' Understanding of the Big Ideas of Algebra.* Presentation at the National Council of Supervisors of Mathematics, Atlanta, GA.

Hedges, M., Kepner, H., Luck, G., McLeod, K. & Pruske, L. (2007, January) *Instruction for Mathematical Knowledge for Teachers of Elementary and Middle Grades.* Association of Mathematics Teacher Educators Annual Conference, Irvine, CA.

Huinker, D. Pruske, L, & **Hedges, M.** (2007, January) *Capturing Growth in Teacher Mathematical Knowledge: An Inquiry into Elementary and Middle School Teacher Understanding of Algebraic Reasoning and Relationships.* Association of Mathematics Teacher Educators Annual Conference, Irvine, CA.

Harris, S., **Hedges, M.,** Steinmeyer, M. (2006, April) *Dad, Mom, Sister, Brother, Dog: There Has to Be a Better Way to Divide.* National Council of Teachers of Mathematics Annual Meeting, St. Louis, MO.

Huinker, D., **Hedges, M.,** McLeod, K., & Bay-Williams, J. (2006, January) *Decompressing Teacher's Mathematical Knowledge: The Case of Division.* Association of Mathematics Teacher Educators Conference, Tampa, FL, January 26, 2006.

Selected State Level Presentations

Hedges, M., Douglas-Meyer, M., & Schefelker, B. (2019). Beyond Circle, Triangle, Square: Developing Geometric and Spatial Thinking in Young Learners Through Playful Pedagogy. Pre-Conference Workshop. Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Huinker, D. **Hedges, M.**, Douglas-Meyer, M., Schefelker, B., DeWindt, B., & Robinson, D. (2018, May) Preconference Session *Learning Trajectories: Understanding and Using Young Children's Mathematical Pathways to Guide Intentional Teaching.* Pre-Conference Workshop. Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Robinson, D. & **Hedges, M.** (2017, October) *Using Culturally Relevant Pedagogy to Improve Students Mathematical Identities and Achievement in Elementary Classrooms.* Mathematical Proficiency for Every Student Conference. Wisconsin Dells, WI.

Hedges, M. & Schefelker, B. (2016, May) *Developing Linear Thinking: A Progression from PK-K-Grade 2.* Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Hedges, M. & Schefelker, B. (2015, December) *Scaffolding Thinking: Putting Students' Visual Representations to Work in the Primary Mathematics Classroom*, Mathematical Proficiency for Every Student Conference, Oconomowoc, WI.

Hedges, M. & Cutter, E. (2015, May) Using *Tape Diagrams to Grow Conceptual Understanding and Math Practices in Students*, Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Hedges, M. & Schefelker, B. (2013, May). *Multiplicative Comparison Problems*, Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Hedges, M. & Schefelker, B. (2013, May) *What the Heck is a Rekenrek?* Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Schefelker, B., **Hedges, M.,** & Mooney, M. (2011, May) Using Properties to Reason through Tough Questions. Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Hedges, M. Schefelker, B & Mooney, M. (2011, May) As the World Terns: Reason Abstractly and Quantitatively. Wisconsin Mathematics Council Annual Meeting, Green Lake, WI.

Harris, S., **Hedges, M.,** & Rahming, B. (2005, January) *High Quality Math Instruction for All: Milwaukee Public Schools Comprehensive Mathematics Framework.* Wisconsin Department of Public Instruction New Wisconsin Promise Conference, Madison, WI.

Selected Milwaukee Area Presentations

Huinker, D., Kepner, H., & **Hedges, M.** (2006, March) *Teacher Preparation and Professional Development: The Mathematical Knowledge Teachers Need.* 18th Annual University of Wisconsin-Milwaukee School of Education Research Conference, Milwaukee, WI.

Ancel, F., **Hedges M.,** & Kepner, H. (2005, June) *Math 277: Geometry for Elementary Teachers*. Presentation at the Milwaukee Mathematics Partnership National Science Foundation site visit, Milwaukee, WI.

Moyer, J., **Hedges, M.,** & Kepner, H. (2004, August) Developing Teachers' Mathematical Knowledge for Teaching Examination of the work from the Center for Proficiency in Teaching Mathematics 2004 Summer Institute, University of Michigan. Milwaukee, WI.

PROFESSIONAL AFFILIATIONS

Association of Mathematics Teacher Educators

National Council of Teachers of Mathematics

National Council of Supervisors of Mathematics

North American Chapter of the International Group for the Psychology of Mathematics Education

Wisconsin Mathematics Council

RESEARCH AND TEACHING INTERESTS

Learning trajectory-based instruction
Equitable mathematics teaching practices
Elementary mathematics teacher preparation
Early childhood mathematics teacher preparation
Mathematical knowledge needed for teaching
Early childhood and elementary mathematics education

STATE OF WISCONSIN PROFESSIONAL CERTIFICATIONS

T001 Teacher	Professional Educator
1088 Teacher	Elementary/Middle Level Education
1365 Spanish	Grades Prekindergarten – 8
1400 Mathematics	Middle Childhood to Early Adolescence