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Algebraic Relations Via a Monte Carlo Simulation

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ALGEBRAIC RELATIONS VIA A MONTE CARLO SIMULATION

by

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ABSTRACT

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The conjugation action of the complex orthogonal group on the polynomial functions on $n \times n$ matrices gives rise to a graded algebra of invariants, $\mathcal{P}(M_n)^{O_n}$. A spanning set of this algebra is in bijective correspondence to a set of unlabeled, cyclic graphs with directed edges equivalent under dihedral symmetries. When the degree of the invariants is $n+1$, we show that the dimension of the space of relations between the invariants grows linearly in n. Furthermore, we present two methods to obtain a basis of the space of relations; we construct a basis using an idempotent of the group algebra $\mathbb{C}[S_n]$ referred to as Young symmetrizers, and we propose a more computationally efficient method for this problem using a Monte Carlo algorithm.

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1 Introduction

The goal of this work is to understand the relations between invariant polynomials of a specific group action on a finite dimensional representation. We present results about the dimension of the space of relations, denoted $\mathcal{REL}$, and further discuss two methods for obtaining a basis of this space. First, we construct a basis of relations by using Young symmetrizers, however, this can be computationally expensive. Thus we propose a more efficient method to determine a basis of $\mathcal{REL}$ by employing a Monte Carlo algorithm.

In Chapter 3 we discuss our main results about the space of relations between the invariants of the conjugation action of the complex orthogonal group, $O_n(\mathbb{C})$, on the polynomial functions on $n \times n$ matrices, denoted $\mathcal{P}(M_n)$, given by:

$$g \cdot f(x) = f(g^T x g)$$

where $x \in M_n, f \in \mathcal{P}(M_n)$ and $g \in O_n(\mathbb{C})$.

We have that $\mathcal{P}^d(M_n)$, the homogeneous polynomials of degree d, are a finite dimensional representation of $O_n(\mathbb{C})$. Therefore, we have a graded structure on $\mathcal{P}(M_n)$,

$$\mathcal{P}(M_n) = \bigoplus_{d \geq 0} \mathcal{P}^d(M_n)$$

and thus a graded algebra of invariant polynomials, $\mathcal{P}(M_n)^{O_n}$. It is shown in [10], that this algebra is generated by functions of the form $Tr(x^{a_1}(x^T)^{a_2}x^{a_3}(x^T)^{a_4} \cdots x^{a_{m-1}}(x^T)^{a_m})$ for $x \in M_n$. Products of these functions span the set of invariants, however, they are not linearly independent if the degree of the polynomials is greater than n.

In Chapter 3 we give a precise definition of the space $\mathcal{REL}$ as the kernel of a specific map. We then prove the following theorem regarding the linear growth of the dimension of $\mathcal{REL}$:

Theorem 1 *Let n be a positive integer. The dimension of the space of relations, $\mathcal{REL}_{n+1}$, between the degree $n+1$ invariants of the O_n conjugation action on $\mathcal{P}(M_n)$ is equal to*

$$dim(\mathcal{REL}_{n+1}) = \begin{cases} \frac{n}{2}+1 & n \text{ even} \\ \frac{n+3}{2} & n \text{ odd} \end{cases}$$

The following table illustrates this idea; the columns are indexed by n, and the rows by the degree, d, of the invariants.

Dimension of $\mathcal{REL}$

$d \backslash n$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0
3	2	2	0	0	0	0	0	0
4	5	3	3	0	0	0	0	0
5	5	7	4	3	0	0	0	0
6	9	13	12	5	4	0	0	0
7	9	21	21	14	6	4	0	0
8	14	33	48	30	19	7	5	0
9	14	51	75	67	39	21	8	5

Table 1.1: Table of values of the space of relations. Columns give the dimension of the defining representation, rows give the degree of the invariants.

The space of zeros in the table represents what we define as the *stable range*, where the degree of the invariant polynomials is smaller than n. In this space there are no relations between the invariants [15]. The first time relations arise is just outside of the stable range, when the invariants are degree $n+1$. The highlighted diagonal gives the dimension of the space of relations between the degree $n+1$ invariants, $\mathcal{REL}_{n+1}$, which we see grows linearly in n.

In order to prove Theorem 1, we rely on correspondences between the elements of $\mathcal{P}(M_n)^{O_n}$ and several different spaces. In Section 3.1.1 we illustrate a useful bijection between the invariant polynomials and Necklace diagrams, which are unlabeled, cyclic graphs embedded in the plane. These graphs are equivalent under dihedral symmetries, and this characteristic corresponds to properties of the trace of a matrix.

For example, the following figure represents a Necklace diagram corresponding to an invariant polynomial of degree 5:

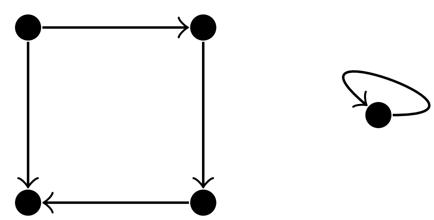


Figure 1.1: Necklace diagram corresponding to a quintic invariant.

Furthermore, we present a bijection from $\mathcal{P}(M_n)^{O_n}$ to a set of fixed-point free involutions in Section 3.1.2, and a correspondence from the space of invariants to a double coset of the symmetric group $S_{2(n+1)}$, in Section 3.1.3.

We use these correspondences to prove the main result regarding the dimension of $\mathcal{REL}_{n+1}$, and we present this proof in Section 3.3.

Next, we discuss two ways to determine a basis of $\mathcal{REL}$. In Section 3.4.2 we explicitly describe the relations using idempotents of the group algebra, $\mathbb{C}[S_n]$, called Young symmetrizers. However, the size of these elements grows exponentially in n, and thus it becomes computationally overwhelming to obtain a basis using this method as n increases.

Therefore, in Chapter 4, we propose a new approach to explicitly define a basis of the relations that is more efficient and does not rely on Young symmetrizers. This algorithm is designed using a Monte Carlo simulation that repeatedly generates random matrices to give numerical values to the invariants, which allows us solve a linear system of equations and thus recover a basis of relations.

All of our code is written in Python and Sage [12], and can be found in Chapter 5.

2 Preliminaries

In this section we will set up common notation, and introduce important concepts and theorems that will be used throughout the subsequent discussion of our main results. We give some basic notions of representation theory, and the construction of the symmetric tensors and symmetric algebra. Furthermore, we describe Young diagrams and Young symmetrizers within the context of these ideas and explain a few examples. Finally, we discuss details of Schur-Weyl duality and several related theorems.

We will be working over the field of complex numbers, $\mathbb{C}$, specifically concerning the following matrix groups:

Definition 1 *The complex general linear group of dimension n, denoted $GL_n(\mathbb{C})$, is the set of $n \times n$ invertible matrices with complex entries under the group operation of matrix multiplication.*

Definition 2 *The complex orthogonal group of dimension n, denoted $O_n(\mathbb{C})$, is the group of $n \times n$ orthogonal matrices with complex entries under the group operation of matrix multiplication. The orthogonal matrices are solutions to the equation $XX^T = Id$, where Id is the identity matrix.*

Remark: We will use the convention that $O_n(\mathbb{C}) = O_n$, and $GL_n(\mathbb{C}) = GL_n$ for more succinct notation.

Next, we define the construction of the group algebra.

Definition 3 *Let G be an arbitrary group with operation $*$. The group algebra $\mathbb{C}[G]$ is the set of all linear combinations of finitely many elements of G with complex coefficients k_g,*

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} k_g g \mid k_g = 0 \text{ for all but finitely many } g \right\} \tag{2.1}$$

equipped with a natural addition and multiplication.

Remark: We will work with the space $\mathbb{C}[M_n]$, which is a $\mathbb{C}$-algebra of polynomial functions on the $n \times n$ matrices, M_n. There is a graded $\mathbb{C}$-algebra structure on this space:

$$\mathbb{C}[M_n] = \bigoplus_{d=0}^{\infty} \mathbb{C}[M_n]_d \tag{2.2}$$

where $\mathbb{C}[M_n]_d$ denotes the subspace of homogeneous polynomials of degree d. For clarity, we will use the following definition and notation for this graded algebra of polynomials.

Definition 4 *Let $d \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{P}(V)$ denote the algebra of polynomial functions on vector space V. Then we have a graded module structure on $\mathcal{P}(V)$, that is $\mathcal{P}(V) = \bigoplus_{d \geq 0} \mathcal{P}^d(V)$ where $\mathcal{P}^d(V)$ is the subspace of homogeneous polynomials of degree d on V.*

Next, we describe another algebra called the *tensor algebra*, and two important related spaces.

Definition 5 *The tensor algebra $T(V)$ of vector space V is the algebra of tensors on V where multiplication is defined to be the tensor product.*

Construction of T(V): We define V as a vector space over arbitrary field, F. Then for $n \in \mathbb{N}$, the n^{th} tensor power of V is the tensor product of V with itself n times:

$$T^n V = \otimes^n V = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$$

together with a bilinear multiplication,

$$(\otimes^k V) \otimes (\otimes^p V) \to \otimes^{k+p} V$$

We therefore have a graded algebra structure on $T(V)$:

$$T(V) = \bigoplus_n T^n V = \otimes^0 V \oplus \otimes^1 V \oplus \otimes^2 V \cdots$$

We can use the tensor algebra $T(V)$ to define several useful spaces, namely the symmetric algebra $\mathcal{S}(V)$, and the symmetric tensors $Sym(V)$.

Definition 6 *The symmetric algebra, denoted $\mathcal{S}(V)$, is a quotient of the tensor algebra $T(V)$:*

$$\mathcal{S}(V) = {}^{T(V)}\diagup_{(v \otimes w - w \otimes v,\ v, w \in V)}$$

where $(v \otimes w - w \otimes v)$ is the ideal of differences generated for all $v, w \in V$.

Essentially, $\mathcal{S}(V)$ may be thought of as the polynomial ring over F in inderterminates in V (without choosing coordinates) that are a basis for V. We again have a graded algebra structure on $\mathcal{S}(V)$:

$$\mathcal{S}(V) = \bigoplus_k S^k(V) = S^0(V) \oplus S^1(V) \oplus S^2(V) \cdots$$

where S^k denotes the k^{th}-symmetric power of vector space V, that is, the vector subspace of the symmetric algebra $\mathcal{S}(V)$ consisting of degree k elements.

Definition 7 *The space of symmetric tensors, $Sym(V)$, consists of tensors T, of order n that are invariant under the natural permutation action of the symmetric group S_n on the tensor factors:*

$$T(v_1, v_2, \ldots, v_n) = T(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})$$

for $\sigma \in S_n$.

We emphasize the difference between $\mathcal{S}(V)$ and $Sym(V)$; the symmetric algebra $\mathcal{S}(V)$ is a *quotient* of the tensor algebra, while the space of symmetric tensors $Sym(V)$ is a *subspace* of $T(V)$.

There is a useful map between $\mathcal{S}(V)$ and $Sym(V)$ called the *symmetrization map*:

$$\phi : \mathcal{S}(V) \longrightarrow Sym(V) \tag{2.3}$$

$$v_1\dot{v}_2 \cdots v_k \longrightarrow \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \tag{2.4}$$

For example, given $\mathcal{S}(V) = \bigoplus_{n=0}^{\infty} S^n(V)$, we may have that $S^m(V) \otimes S^k(V)$ is not symmetric for some $m, k \in \mathbb{N}$, thus we *symmetrize* via the map 2.3 :

$$S^m(V) \otimes S^k(V) \to S^{m+k}(V)$$

where for $u \in S^m(V)$ and $w \in S^k(V)$ we have

$$Sym(u \otimes w) = \left(\frac{1}{(m+k)!} \sum_{\sigma \in S_{m+k}} \sigma \right) (u \otimes w)$$

We will use the previous definitions in the context of representation theory. Thus, we introduce the concept of a representation, and state several theorems that will be necessary for understanding our subsequent results.

Definition 8 *Let n be a positive integer. Then a matrix representation of a group G on a vector space V over field K is a group homomorphism:*

$$\rho : G \to GL_n$$

Equivalently, for each element $g \in G$, we assign a matrix $\rho(g)$, such that

- $\rho(e) = I_n$ *(the $n \times n$ identity matrix), and*
- $\rho(gh) = \rho(g)\rho(h)\ \forall g, h \in G$

The parameter n is called the degree or dimension of the representation.

Remark: It is common for V to be referred to as the representation when the homomorphism mapping is clear.

We note that every group has a trivial representation, given by mapping every element of the group to I_n, the identity matrix. As our future discussion largely focuses on representations of the symmetric group, S_n, we recall Cayley's theorem which states that every finite group is isomorphic to a permutation group. Then we remark that it can be shown that every finite group has a permutation matrix representation. Next, we define what it means for a representation to be irreducible.

Definition 9 *A subspace W of V that is invariant under the group action is called a subrepresentation.*
A representation is called irreducible if it contains no non-trivial subrepresentations.

The following theorem describes the setting in which a finite-dimensional representation can be constructed from irreducible subrepresentations.

Theorem 2 ***(Maschke's Theorem)*** *Every representation of a finite group, G, over a field F with characteristic not dividing the order of G is a direct sum of irreducible representations.*

Furthermore, Schur's Lemma describes the maps between two irreducible representations of a group G.

Theorem 3 ***(Schur's Lemma)*** *Let V, W be vector spaces over $\mathbb{C}$, and let ρ_V, ρ_W be irreducible representations of G on V and W, respectively. Then,*

- *If $V \ncong W$ then there are no nontrivial G-linear maps between them.*
- *If $V = W$ and $\rho_V = \rho_W$, then the only nontrivial G-linear maps are the identity and scalar multiples of the identity.*

Next, we describe a combinatorial object called a Young tableau, which gives us a convenient and succinct way to describe the representations of two groups that are of high importance in our work, namely, the symmetric group S_n, and the general linear group GL_n.

Discussion of Young Tableaux [2], [18]: For a given finite group, G, the number of its conjugacy classes is equal to the number of irreducible representations of G. For the symmetric group on n letters, S_n, we have a correspondence between its irreducible representations and a combinatorial object called a Young Diagram.

First, recall that there is a canonical bijection between the conjugacy classes $\mathscr{C}$, in S_n and the partitions, P_n, of n:

$$\varphi : \mathscr{C} \longrightarrow P_n$$
$$\sigma \longrightarrow \lambda$$

where for $\sigma \in S_n$, the corresponding partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ of n is given by the lengths of the cycles in a decomposition of σ as a product of disjoint cycles.

Then there is a correspondence between this partition λ and a finite collection of boxes called a Young Diagram; where the total number of boxes in the diagram is n, and the number of boxes in each row is given by λ_i of the partition.

For example, the partition of 10 given by $10 = 5 + 3 + 2$ gives the Young diagram shown in Figure 2.1.

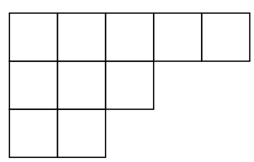


Figure 2.1: Example of a Young Diagram

Definition 10 *A Young tableau, T, is a Young diagram, D, with a filling that assigns a positive integer to each box of D, where the number of times each integer appears in a tableau is called the weight of the T.*

- *A filling is **semi-standard** if entries weakly increase along each row and strictly increase down each column.*

- *A filling is **standard** if entries are a bijective assignments of* $\{1, 2, \ldots, |D|\}$.

Examples of each of these tableaux are shown in Figure 2.2, for the Young Diagram given by partition $\lambda = (5, 3, 1)$. In Figure 2.2a, the tableau has weight $(1, 4, 2, 2, 1)$ as there is one 1, four 2's, two 3's, two 4's, and one 5 in the diagram. Similarly the tableau in Figure 2.2b has weight (1,1,1,1,1,1,1,1,1,1).

1	2	2	4	5
2	2	3		
3	4			

(a) Example of a Semi-standard Young Tableau.

1	2	3	4	5
6	7	8		
9	10			

(b) Example of a Standard Young Tableau.

Figure 2.2: Examples of fillings of Young Diagrams

The following definition describes a specific type of Young tableau that will be of particular importance to the discussion in Section 3.2 of Littlewood-Richardson numbers.

Definition 11 *Let* λ *be a partition of* $n + m$*, and* μ *a partition of* m*. A skew shape is a pair of partitions* (λ, μ)*, denoted* λ/μ*, such that the Young diagram of* λ *contains the Young diagram of* μ*.*

The skew diagram of λ/μ *is the set of* n *squares that belong to* λ *but not to* μ*.*

A standard skew tableau of shape λ/μ *is a filling of the skew diagram with the integers from* 1 *to* n *in increasing order in each row and column. If we allow repetitions of numbers and the rows are weakly increasing, then this filling gives a semi-standard skew tableau.*

The following figure gives examples of skew tableau.

		1	2
	3	4	
5	6		

$\lambda/\mu = (4, 3, 2)/(2, 1)$

(a) Standard skew tableau

		1	1
	2	2	
1	3		

$\lambda/\mu = (4, 3, 2)/(2, 1)$

(b) Semi-standard skew tableau

The following theorem gives a bijective correspondence between permutations in S_n and pairs of standard Young tableau of the same shape. Details can be found in [13], [2].

Theorem 4 *(**Robinson-Schensted Correspondence**) Let $\sigma \in S_n$. Given a permutation in two-line notation,*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix}$$

where $\sigma_i = \sigma(i)$. The Schensted algorithm constructs a sequence of ordered pairs (P_i, Q_i) of standard Young tableaux of the same shape:

$$(P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n)$$

where the Q_i are called insertion tableaux, P_i are called recording tableaux, and $P_0 = Q_0$ is the empty tableaux.

The Schensted algorithm creates a bijection between the permutation and the tableaux by simultaneously constructing the recording tableaux P and the insertion tableaux Q defined above. We start from the empty tableaux P_0 and Q_0, then insert each element of σ in turn to get a standard Young tableaux. At the same time, in the sequence of tableau Q, we record the order in which new cells are created in P.

A useful result of this correspondence tells us that if σ is an involution, then the recording and insertion tableaux are equal, $P = Q$.

The following theorems connect these ideas of group algebras, Young diagrams, and irreducible representations. The proofs are not provided here but can be found in [6], [3]. In particular, the Schur-Weyl duality theorem will be used in the proof of our result in Section 3.

Discussion of ($\mathbf{GL_n}$, $\mathbf{GL_m}$) duality [6], [17]: For $m, n \in \mathbb{Z}^+$, consider the tensor product $\mathbb{C}^n \otimes \mathbb{C}^m$. Then the standard $GL_n(\mathbb{C})$ action on $\mathbb{C}^n$ given by $g.v = g^{-1}v$ for $g \in GL_n$, $v \in \mathbb{C}^n$, induces an action on $\mathbb{C}^n \otimes \mathbb{C}^m$. Similarly this holds for the $GL_m(\mathbb{C})$ action on $\mathbb{C}^m$.

It is clear that these actions commute, and thus we construct a $GL_n \times GL_m$ action on the space $\mathbb{C}^n \otimes \mathbb{C}^m$:

$$(g,h).v = g^{-1}v_1 \otimes h^{-1}v_2$$

for $v = v_1 \otimes v_2 \in \mathbb{C}^n \otimes \mathbb{C}^m$ and $(g,h) \in GL_n \times GL_m$. Furthermore, we can define an action on the algebra of degree d homogeneous polynomial functions on $\mathbb{C}^n \otimes \mathbb{C}^m$, denoted $\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^m)$ by

$$(g,h).f(v \otimes w) = f(g^{-1}v \otimes h^{-1}w) \tag{2.5}$$

for $(g,h) \in GL_n \times GL_m, (v \otimes w) \in \mathbb{C}^n \otimes \mathbb{C}^m$, and f a degree d homogeneous polynomial function on $\mathbb{C}^n \otimes \mathbb{C}^m$. (GL_n, GL_m)-duality describes the decomposition of this space of functions into irreducible representations.

Theorem 5 ***($(\mathbf{GL_n}, \mathbf{GL_m})$-duality)*** *Under the action of $GL_n \times GL_m$ on $\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^m)$ given in equation 2.5, the algebra $\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^m)$ has the following decomposition into irreducible $GL_n \times GL_m$ modules:*

$$\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{\substack{\lambda: \\ |\lambda|=d \\ \ell(\lambda) \leq min(n,m)}} F_n^\lambda \otimes F_m^\lambda \tag{2.6}$$

The proof of this theorem can be found in [6].

If we consider a similar situation to above, but furthermore let the symmetric group S_n act on the right of a k-dimensional tensor space, then we develop the following concept of Schur-Weyl duality.

Discussion of Schur-Weyl duality: [6], [3] Consider the tensor space given by

$$\otimes^k(\mathbb{C}^n) = \underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k \text{ times}}$$

Then we have that the symmetric group on k letters, S_k, has a natural right action on this space by permuting the tensor factors:

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k)\sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}$$

where $\sigma \in S_k$, and $v_i \in \mathbb{C}^n$. Additionally, the general linear group GL_n acts on $\otimes^k(\mathbb{C}^n)$ on the left by simultaneous matrix multiplication:

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k$$

where $g \in GL_n$ and $v_i \in \mathbb{C}^n$. Then it is clear that $g((v_1 \otimes v_k)\sigma) = (g(v_1 \otimes v_k))\sigma$, that is, the two actions commute. Furthermore, the spans of the images of S_k and GL_n in the endomorphism group $End(\mathbb{C})^{\otimes k}$ are centralizers of each other. Schur-Weyl duality describes the multiplicity-free decomposition of this space as a representation of the group $S_k \times GL_n$.

Theorem 6 ***(Schur-Weyl duality)*** *We have the decomposition*

$$\otimes^k(\mathbb{C}^n) \simeq \bigoplus_{\substack{\lambda:|\lambda|=k \\ \ell(\lambda)\leq n}} F_n^\lambda \otimes Y_k^\lambda \tag{2.7}$$

as a representation of $S_k \times GL_n$, where Y_k^λ is indexed over all irreducible representations of S_k, and each F_n^λ is an irreducible representation of GL_n. These irreducible representations are associated to Young Diagram, λ, with k boxes and number of nonzero rows $\leq n$.

The proof of this theorem can be found in [6].

Below is a special case of the Cartan-Helgason theorem (details can be found in [6]).

Theorem 7 *Let F^λ denote the irreducible finite dimensional representation of the general linear group GL_n with highest weight λ. Then*

$$\dim(F^\lambda)^{O_n} = \begin{cases} 1 & \text{if } \lambda \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

In the next chapter we use the definitions and theorems detailed in this section in order to discuss a new result about the relations between a specific set of invariant polynomials under an action of the complex orthogonal group.

3 Main Result

In the following sections we develop our main results regarding the space of relations between the invariant polynomials under the conjugation action of O_n on the polynomial functions on $n \times n$ matrices. In Section 3.1 we construct the graded ring of O_n invariants under this action. We then illustrate a correspondence between elements of the invariant ring and unlabeled cyclic graphs called Necklace diagrams. Furthermore, there is an injection between these diagrams and fixed-point free involutions, which we describe in Section 3.1.2. Our goal is to understand the space of relations between invariants under the O_n conjugation action. We state and prove a theorem about the dimension of this space in Section 3.3, and in Section 3.4 give a construction for a basis of this space. The final result in Chapter 4 presents a computationally efficient way to recover relations between invariants via a Monte Carlo algorithm.

3.1 O_n-invariant polynomials

Consider the complex general linear group, GL_n, and the polynomial functions on $n \times n$ matrices, $\mathcal{P}(M_n)$ (Definition 4). There is a conjugation action of GL_n on $\mathcal{P}(M_n)$ defined by

$$g \cdot f(x) = f(g^{-1}xg)$$

for $g \in GL_n, x \in M_n$, and $f \in \mathcal{P}(M_n)$.

We can see that $Tr(x)$ is invariant under this action, since

$$g \cdot Tr(x) = Tr(g^{-1}xg) = Tr(gg^{-1}x) = Tr(x)$$

It can be shown, [10], that $Tr(x^k)$ is invariant for $k \in \mathbb{N}, k \leq n$, and these functions generate $\mathcal{P}(M_n)^{GL_n}$. Furthermore, there are no relations between these invariants. Thus, we restrict the adjoint action of GL_n to the action of the complex orthogonal group, O_n, on

$\mathcal{P}(M_n)$ defined by

$$g \cdot f(x) = f(g^{-1}xg)$$

for $g \in O_n, x \in M_n$ and $f \in \mathcal{P}(M_n)$. However, since $g \in O_n$, this action is equivalent to the following:

$$g \cdot f(x) = f(g^T xg) \tag{3.1}$$

The homogeneous polynomials of degree d, denoted $\mathcal{P}^d(M_n)$, are a finite dimensional representation of O_n. Therefore, we have a graded algebra structure,

$$\mathcal{P}(M_n) = \bigoplus_d \mathcal{P}^d(M_n)$$

and a similar grading of the algebra of invariants, $\mathcal{P}(M_n)^{O_n}$.

The polynomial invariants under the GL_n action are also invariant under the orthogonal group, but there are additional invariants under the action of O_n. It is shown in [10], that $\mathcal{P}(M_n)^{O_n}$ is generated by traces of monomials in x and x^T,

$$Tr(x^{a_1}(x^T)^{a_2}x^{a_3}(x^T)^{a_4}\cdots x^{a_{m-1}}(x^T)^{a_m}) \tag{3.2}$$

for $x \in M_n, a_i \in \mathbb{Z}^+$.

Products of these polynomial functions span the set of invariants. When the degree of the invariants is less than or equal to n, there are no relations between the polynomials. We call this space the stable range, defined below.

Definition 12 *For $d, n \in \mathbb{N}$, we define the ordered pair (d, n) to be in the **stable range** if $d \leq n$.*

If the degree, d, of the polynomials is greater than the dimension of the defining representation, the spanning set of invariants is not linearly independent. We look at the first occurrence of relations between invariants, which happens when the invariant polynomials have degree $d = n + 1$. The main result in Section 3.3 gives the dimension of space of relations between the degree $n + 1$ invariants under the conjugation action of O_n described above.

In the following section we discuss a useful correspondence between the graded invariant algebra $\mathcal{P}(M_n)^{O_n}$ and unlabeled cyclic graphs called Necklace Diagrams.

3.1.1 Necklace Diagrams

We develop a bijective correspondence between an unlabeled cyclic graph with oriented edges and the polynomial generators of the invariant ring $\mathcal{P}(M_n)^{O_n}$.

Definition 13 *Let $m \in \mathbb{N}$. We embed an unlabeled, directed, cyclic graph with d nodes in the plane and centered at the origin; each edge is given an orientation of clockwise or counterclockwise determined by choosing an arbitrary edge, E, and noting the direction traveled to subsequent edges.*

These graphs, called ***Necklace diagrams*** *and denoted N_m, have the following structure:*

- *An edge may join a node to itself*
- *At most two edges may join two different nodes*

Furthermore, diagrams are considered equivalent under dihedral symmetries of rotation and reflection.

An example of two equivalent necklace diagrams in N_4 is shown in Figure 3.1

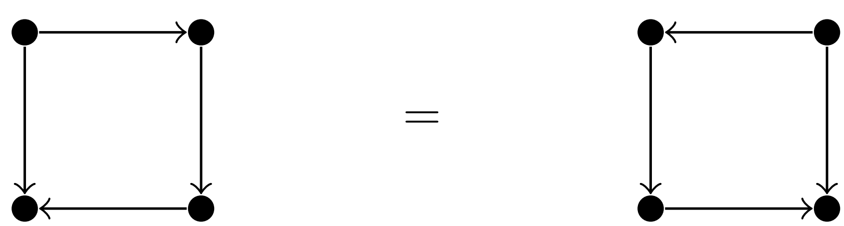


Figure 3.1: Graph of $Tr(x^3x^T)$ or equivalently, $Tr(x^2x^Tx)$, $Tr(x(x^T)^3)$, etc.

Then we have the following correspondence, [15], between N_m and the generators of $\mathcal{P}(M_n)^{O_n}$. Consider diagram N_m, and let $\{E_1, E_2, \ldots, E_m\}$ be the set of edges in N_m. Let $f : M_n \to \mathbb{C}$ be given by $f(x) = Tr(\prod_{i=1}^{m} f_i(x))$, where

$$f(x_i) = \begin{cases} x & \text{if } E_i \text{ is directed clockwise} \\ x^T & \text{if } E_i \text{ is directed counterclockwise} \end{cases} \tag{3.3}$$

It is clear that $f(x)$ is a function of the form described in 3.2, namely, the trace of monomials in x and x^T. Since the trace of a matrix is invariant under cyclic permutation, this correspondence is independent of the choice of initial edge. Furthermore, as $Tr(x) = Tr(x^T)$, our assertion of counterclockwise and clockwise is arbitrary.

Let $\mathscr{D}_d$ denote the set of (not necessarily connected) necklace diagrams in which each connected component is in N_m, where there are d total nodes in the set of diagrams. That is, define

$$\mathscr{D}_d = \{N_{m_i} | \sum_i m_i = d\}.$$

An example of an element in $\mathscr{D}_7$ is in figure 3.2 . Then we have a similar correspondence to 3.3 between $\mathscr{D}_d$ and the generators of $\mathcal{P}(M_n)^{O_n}$ given by the following:

$$\text{For } f : M_n \to \mathbb{C}, \text{ let } \quad f(x) = \prod_{j=1}^{d} g_j(x) \quad \text{where} \quad g_j(x) = Tr(\prod_{i=1}^{m} f_i(x)) \tag{3.4}$$

and the function $f_i(x)$ is the same as described in Equation 3.3 .

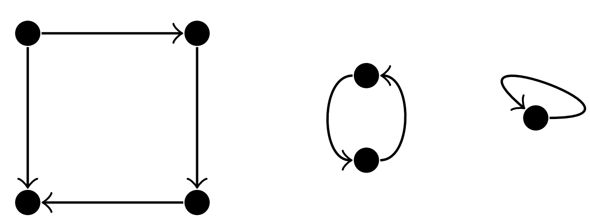


Figure 3.2: Diagram of $Tr(x^3x^T)Tr((x^T)^2)Tr(x))$, an example of an element in $\mathcal{D}_7$.

Let $\mathcal{G}_{n,d}$ denote the set of degree d polynomial functions on M_n, which are the functions given by $f(x)$ in Equation 3.4 , that is,

$$\mathcal{G}_{n,d} = \left\{Tr(x^{a_1}(x^T)^{a_2}\cdots)Tr(x^{b_1}(x^T)^{b_1}\cdots)\cdots | x \in M_n, \sum a_i + \sum b_j + \cdots = d\right\}$$

Then we have a one-to-one correspondence between elements of $\mathcal{G}_{n,d}$ and elements of the set $\mathscr{D}_d$. That is,

$$|\mathscr{D}_d| = |\mathcal{G}_{n,d}|$$

For example, consider M_2, and degree 3 polynomial functions on M_2. Then we have $\mathcal{G}_{2,3} = \left\{Tr(x^3), Tr(x^2x^T), Tr(xx^T)Tr(x), Tr(x^2)Tr(x), Tr(x)^3\right\}$, and thus the following injection into $\mathscr{D}_3$ via the function described in Equation 3.4 :

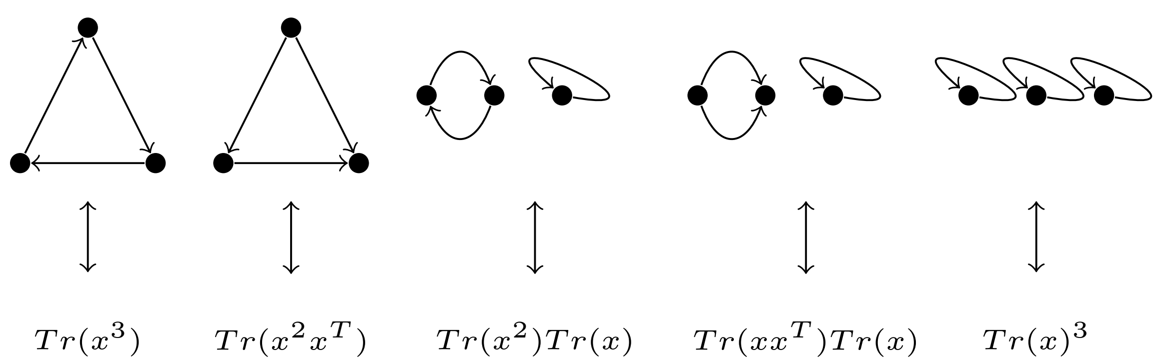


Figure 3.3: Injection between $\mathcal{G}_{2,3}$ and $\mathscr{D}_3$

Therefore, we adopt the following definition.

Definition 14 *The* ***degree of an invariant*** *of the O_n action on the polynomial functions on $n \times n$ matrices is equal to the number of nodes of the corresponding set of Necklace diagrams $\mathscr{D}_d$.*

It can be shown, [15], that in the stable range where the degree of the invariants, d, is less than or equal to n, we have that

$$|\mathcal{G}_{n,d}| = \dim \mathcal{P}(M_n)^{O_n}$$

That is, in the stable range where $d \leq n$, the dimension of the space of invariants under this O_n action is exactly equal to the number of necklace diagrams $\mathscr{D}_d$. As a result, there are no relations between the invariants in this setting. In Section 3.3 we present a new result about the dimension of the invariant space outside of the stable range, where relations between the invariants arise. To aid in this discussion, it is useful to note a correspondence between Necklace diagrams and a set of fixed-point free involutions.

3.1.2 Fixed-Point Free Involutions

There is a one-to-one correspondence between the set of involutions on S_n without fixed-points and the invariants of $\mathcal{P}(M_n)^{O_n}$ which are described by the Necklace diagrams (Definiton 13).

Definition 15 *An* ***involution*** *on a set A is a permutation, σ, of A, where $\sigma(\sigma(x)) = x \ \forall x \in A$.*

A ***fixed-point free involution*** *is an involution on A which does not contain any 1-cycles.*

Thus, by definition, the set of all fixed-point free involutions on set A is the set of all transpositions of elements in A. Let I_n denote the set of all fixed-point free involutions on the set $\{1, 2, \ldots, n\}$. Then by [15], we have a bijective correspondence

$$\Theta : \mathscr{D}_d \longrightarrow I_{2d} \tag{3.5}$$

given in the following way.

Let $B \in \mathscr{D}_d$, and label each edge in B with an element of $\{1, 2, \ldots, d\}$ such that each element is only used once. Then consider the set $A = \{1, 2, \ldots, 2d\}$ and arbitrary element

$k \in A$. We next construct $\sigma \in I_{2d}$ such that σ is an involution, and gives a bijection between the desired sets.

First consider $1 \leq k \leq d$. For the edge labeled k in necklace diagram B, if the tail of the directed edge is attached to the tail of directed edge i then we let $\sigma(k) = i$. If in contrast the tail of directed edge k is attached to the head of edge i then we define $\sigma(k) = i + d$.

If $d+1 \leq k \leq 2d$, then we define σ similarly. If the head of directed edge $k-d$ is attached to the the tail of edge i then we let $\sigma(k) = i$. If the head of edge $k - d$ is attached to the head of edge i, then define $\sigma(k) = i + d$.

It is shown in [15] that this gives a bijection between I_{2d} and $\mathscr{D}_d$, so here we illustrate this idea with an example.

Let $d = 8$, and consider the set I_{16} of fixed-point free involutions on the set $\{1, 2, \ldots, 16\}$ and element (1 13)(2 9)(3 12)(4 5)(6 7)(8 16)(10 11)(14 15). We construct the bijection described above on the labeled necklace diagram $B \in \mathscr{D}_8$ shown in Figure 3.4 .

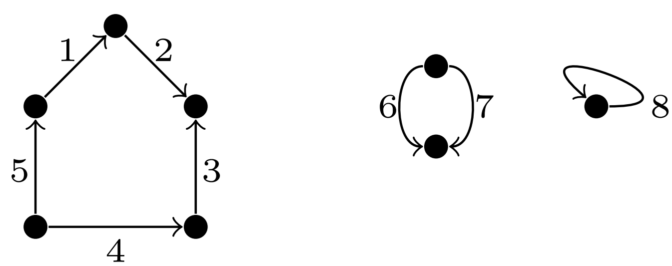


Figure 3.4: Labeled element $B \in \mathscr{D}_8$.

In Figure 3.5 we construct a new graph with 16 labeled nodes by making a row of nodes labeled from the set $\{1, 2, \ldots, 8\}$, and above them make a row of nodes labeled from the set $\{9, 10, \ldots, 16\}$. Then, we connect each node with its image under the involution we chose from I_{16}.

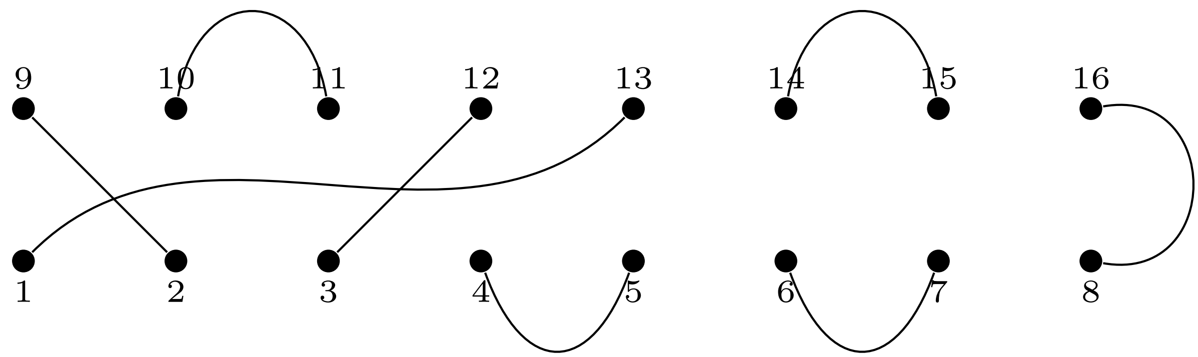


Figure 3.5: Edges drawn between nodes via the involution $\in I_{16}$.

Next, as shown in Figure 3.6 , we draw a directed edge from each node i to node $i + 8$, for $i \in \{1, 2, \ldots, 8\}$.

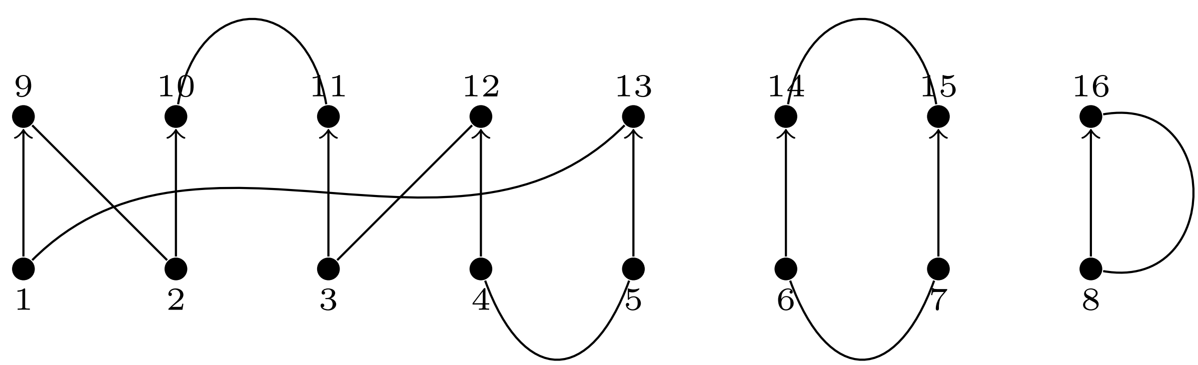


Figure 3.6: Arrows drawn from node j to node $j + d$.

Finally, we identify the pairs of nodes in the graph when an arrow connects them, and collapse the remaining edges. For example, using Figure 3.6 , we connect nodes 4 and 12, and 12 and 3, and so forth. The resulting graph, Figure 3.7, is a necklace diagram in $\mathscr{D}_8$, in which each edge is labeled by a pair $(i, i + 8)$. We note that without loss of generality we can assert that each edge be simply labeled by i, and thus we arrive at our initial diagram B.

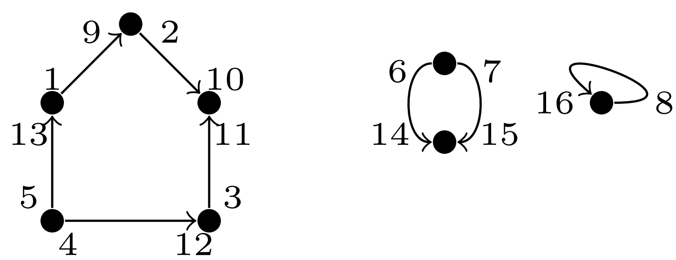


Figure 3.7: Collapsed edges.

Thus, we are free to consider the degree d elements of $\mathcal{P}(M_n)^{O_n}$ as products of traces of x and x^T, or as Necklace diagrams, or as fixed-point free involutions of the set $\{1, 2, \ldots, 2d\}$. In the next section, we describe an important correspondence between these fixed-point free involutions, I_{2d}, and a set of double cosets of the symmetric group.

3.1.3 Another Useful Bijection

Consider the symmetric group S_{2n}, and note the following inclusion:

$$\Delta S_n \subseteq S_n \times S_n \hookrightarrow S_{2n} \tag{3.6}$$

where ΔS_n denotes the diagonally embedded copy of S_n in $S_n \times S_n$, that is,

$$\Delta S_n = \{(\sigma, \sigma) |\, \sigma \in S_n\}.$$

Additionally, we consider $S_n \times S_n$, where the first copy of S_n is composed of permutations of the set $\{1, 2, \ldots, n\}$, and the second copy of S_n is permutations of the set $\{n+1, n+2, \ldots, 2n\}$. Then we have an embedding of $S_n \times S_n$ into S_{2n}, and thus we can consider ΔS_n and $S_n \times S_n$ as subgroups of S_{2n}.

We define the following element of S_{2n} using disjoint cycle notation:

$$\tau = (1\ \ 2)(3\ \ 4) \cdots (i\ \ i+1) \cdots (2n-1\ \ 2n)$$

and we define H_n as the centralizer of τ in the group S_{2n},

$$H_n = \{\sigma \in S_{2n} | \, \sigma\tau = \tau\sigma\}.$$

Clearly, by definition τ is also an element of I_{2n}. Now, we can construct a set of double cosets of the group S_{2n} as follows:

$$(\Delta S_n)\backslash S_{2n}/H_n = \{(S_n)\sigma H_n | \, \sigma \in S_{2n}\}. \tag{3.7}$$

The ΔS_n conjugation action on I_{2n} is equivalent to the left action of ΔS_n on S_{2n}/H_n by:

$$\psi : S_{2n}/H_n \longrightarrow I_{2n} \tag{3.8}$$

$$\sigma H_n \longrightarrow \sigma\tau\sigma^{-1} \tag{3.9}$$

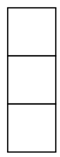
Then we have that the ΔS_n orbits in I_{2n} are exactly the double cosets $\Delta(S_n)\backslash S_{2n}/H_n$, and there is a bijective correspondence between these orbits and the set of Necklace diagrams, $\mathscr{D}_d$, detailed in [15]. We note this construction is used in our code to find the elements of $\mathcal{P}(M_n)^{O_n}$, which can be found in Section 5 .

In the following sections our goal is to understand the space of relations between the invariants, and in order to do this we rely on the combinatorial object called Littlewood-Richardson numbers.

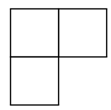
3.2 Littlewood-Richardson Numbers

There are several equivalent definitions of Littlewood-Richardson numbers, [6], [3], [9]. In this section we describe the definitions that will be of particular importance to our results. We view the Littlewood-Richardson numbers as the structure constants in the multiplication of Schur polynomials, and thus equivalently define them as coefficients in the decomposition of the induced representation of the symmetric group.

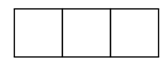
First, we know that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes; for example, we have that there are three conjugacy classes in S_3, and thus there are three irreducible representations of S_3. These can be described using the following Young diagrams:



(a) Trivial representation of S_3.



(b) Standard representation of S_3.



(c) Sgn representation of S_3.

The Littlewood-Richardson rule allows us to compute the multiplicities of the irreducible representations in such a decomposition of the finite symmetric group S_n. We will discuss this rule by introducing the connection between Young diagrams and Schur polynomials.

Definition 16 *Given partition λ of integer n, consider the semi-standard Young tableau, T, associated to λ and fix an upper bound, N, on the size of the entries in the tableau. Then the **Schur polynomial** is,*

$$s_\lambda(x_1, \ldots, x_N) := \sum_T x^T \tag{3.10}$$

where $x^T = \prod_{i=1}^{N} x_i^j$ such that j is the number of i's in tableau T.

The Schur polynomials are a basis of the space of symmetric polynomials, $f \in \mathbb{C}[x_1, \ldots, x_n]$, where f is invariant under any permutation σ of subscripts, for $\sigma \in S_n$.

We recall the following injection (details provided in Section 2):

$$\text{Conjugacy classes of } S_n \leftrightarrow \text{Cycle types of } \sigma \in S_n.$$

The cycle type of σ is the sequence of the lengths of cycles, say $\lambda_1, \ldots, \lambda_k$, in the decomposition of σ into disjoint cycles. Then since $\lambda_1 + \cdots + \lambda_k = n$, we have that the cycle type gives a partition of n.

Because the irreducible representations of S_n are in one-to-one correspondence with the conjugacy classes, we have that the irreducible representations are also in one-to-one correspondence with the partitions of n.

This correspondence is how Schur polynomials are related to representations of the symmetric group S_n. By the definition above, each Schur polynomial, s_λ, corresponds to a given partition $\lambda \vdash n$. Thus the s_λ corresponds to an irreducible representation of S_n.

The Littlewood-Richardson rule decomposes a representation into irreducible representations by taking the product of the corresponding Schur polynomials. Then we have the following definition,

Definition 17 *The product of two Schur polynomials s_μ and s_ν is,*

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{3.11}$$

where s_λ is a Schur polynomial, $|\mu| + |\nu| = |\lambda|$, and the resulting coefficients $c^\lambda_{\mu\nu}$ are defined to be the Littlewood-Richardson numbers.

We can equivalently define the Littlewood-Richardson numbers by inducing a representation from a subgroup of a symmetric group to the entire symmetric group.

Definition 18 *Let μ and ν be partitions of positive integers n and m, respectively. Then the module $Y_\mu \otimes Y_\nu$ is naturally an $(S_n \times S_m)$-module. Thus we can induce to an S_{n+m} representation,*

$$Ind^{S_{n+m}}_{S_n \times S_m} Y_\mu \otimes Y_\nu = \bigoplus_{\lambda \vdash n+m} c^\lambda_{\mu\nu} Y_\lambda \tag{3.12}$$

where the coefficients, $c^\lambda_{\mu\nu}$ are ***Littlewood-Richardson numbers****.*

The Littlewood-Richardson rule: Given both the above definitions, the Littlewood-Richardson rule states that the coefficients, $c^\lambda_{\mu\nu}$, count the number of skew semi-standard Young tableaux of shape λ/μ with weight ν, with the additional restriction that the concatenation of the reversed rows is a lattice word. It can be shown using character theory that

$c_{\mu\nu}^{\lambda} \neq 0$ when $\mu, \nu \subseteq \lambda$. That is, the Young diagrams of μ and ν must fit inside the Young diagram of λ in order to have nonzero coefficients.

A natural question arises about finding a method for constructing the irreducible representations from a partition. For this, we define an idempotent which depends on partition λ in the group algebra, $y_\lambda \in \mathbb{C}[S_n]$, such that $\mathbb{C}[S_n]y_\lambda$ is irreducible. The element y_λ is called a Young symmetrizer. A thorough discussion of Young symmetrizers is found in Section 3.4.1. We will use them in Section 3.4 in order to determine a basis of the space of relations between invariants under the O_n action on polynomials on M_n.

Using the bijections and theorems described in the previous sections, we are now ready to state and prove a theorem about the dimension of the space of relations between the polynomial invariants under the action of the complex orthogonal group on $\mathcal{P}(M_n)$.

3.3 The Dimension of the Space of Relations

Under the conjugation action of O_n on $\mathcal{P}(M_n)$, we know that there are no relations between the invariants in the stable range, where the degree of the polynomials is less than or equal to n. Therefore, we analyze a space outside of the stable range where there are relations. In this section, we present a result about the dimension of the space relations between the invariant polynomials of degree $d = n + 1$. We start by giving an example.

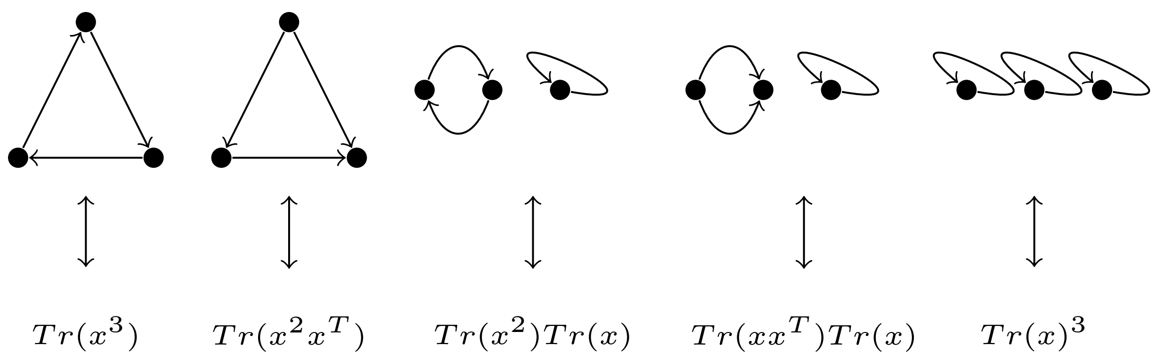
Example: Let $n = 2$. We consider invariant polynomials under the action of O_2; we know that there are no relations between the invariants if $d \leq 2$. Thus we take one step outside of the stable range and let $d = 3$, where we consider cubic invariant polynomials under the action of O_2 on $\mathcal{P}(M_2)$. Here,

$$dim(\mathcal{P}^3(M_2))^{O_2} = 5$$

The space consists of the following set of cubic invariants,

$$Tr(x^3),\ \ Tr(x^2x^T),\ \ Tr(x^2)Tr(x),\ \ Tr(xx^T)Tr(x),\ \ Tr(x)^3.$$

In the language of Section 3.1, these polynomials correspond to the set of Necklace diagrams $\mathcal{G}_{2,3}$, with 3 nodes and oriented edges. We recall Figure 3.3 and show it here again:



Bijection between Necklace diagrams and elements of $P^3(M_2)^{O_2}$.

These polynomials span the set of invariants, however, they are not linearly independent. There are actually two relations,

$$Tr(x)^3 - 3Tr(x)^2Tr(x) + 2Tr(x^3) = 0$$
$$2Tr(x^2x^T) - 2Tr(xx^T)Tr(x) + Tr(x)^3 - 3Tr(x^2)Tr(x) = 0$$

The primary result of this section describes how many relations there are between the degree $n+1$ invariants under the action of O_n. We start by giving some set up for the main theorem.

Consider the following inclusion map from the square n-dimensional matrices to the square

$(n+1)$-dimensional matrices:

$$\begin{aligned} M_n &\hookrightarrow M_{n+1} \\ x &\hookrightarrow \left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right)_{n+1} \end{aligned}$$

Then we have a surjection between the polynomial spaces:

$$\mathcal{P}[M_{n+1}] \twoheadrightarrow \mathcal{P}[M_n]$$

where we restrict the $(n+1) \times (n+1)$ dimensional matrix down to an $n \times n$ dimensional matrix. Recall, in Section 3.1 we discuss the invariants of this space under the conjugation action of the complex orthogonal group O_n. Under this action we have a surjection between the invariant rings:

$$\mathcal{P}[M_{n+1}]^{O_{n+1}} \twoheadrightarrow \mathcal{P}[M_n]^{O_n} \tag{3.13}$$

Remark: We note that $\mathcal{P}(M_n)$ is a $\mathbb{C}$-algebra of polynomial functions on M_n with a graded structure:

$$\mathcal{P}(M_n) = \bigoplus_d \mathcal{P}^d(M_n)$$

where $\mathcal{P}^d(M_n)$ denotes the subspace of homogeneous degree d polynomials, which are a finite dimensional representation of O_n. Thus we also have a graded structure on the invariant algebras,

$$\mathcal{P}(M_n)^{O_n} = \bigoplus_d \mathcal{P}^d(M_n)^{O_n}$$

Therefore if we fix the degree, $d = n+1$, of the invariant polynomials, we have a surjection between the algebras

$$\mathcal{P}^{n+1}(M_{n+1})^{O_{n+1}} \twoheadrightarrow \mathcal{P}^{n+1}(M_n)^{O_n} \tag{3.14}$$

Note that when $d = 1$, we have that $\mathcal{P}^1(M_n) = M_n^*$, which leads to the following identification:

$$\mathcal{P}(M_n) = \bigoplus_d \mathcal{P}^d(M_n) = \bigoplus_d \mathcal{S}^d(M_n^*) = \mathcal{S}(M_n^*)$$

where $\mathcal{S}(M_n^*)$ is the symmetric algebra (defined in Section 2) of the dual space M_n^* of polynomials on M_n.

Then if we again fix the degree $d = n+1$, we see that:

$$\mathcal{P}^{n+1}(M_n) = \mathcal{S}^{n+1}(M_n^*) = \mathcal{S}^{n+1}(M_n)^*$$

Furthermore, as we are working over the characteristic zero, algebraically closed field $\mathbb{C}$, we have that $\mathcal{S}^{n+1}(M_n) \simeq [\otimes^{n+1}(M_n)]^{\Delta S_{n+1}}$. Note that the symmetric tensors are invariant under the natural permutation action of the symmetric group on the tensor factors. Thus map 3.14 can be written as the surjective map:

$$[\otimes^{n+1}(M_{n+1})]^{O_{n+1}\times\Delta S_{n+1}} \twoheadrightarrow [\otimes^{n+1}(M_n)]^{O_n\times\Delta S_{n+1}} \tag{3.15}$$

where ΔS_{n+1} denotes the diagonally embedded copy of S_{n+1} in $S_{n+1}\times S_{n+1}$,

Next, we use the following decomposition of M_n in the above map 3.15. Consider the $Gl_n \times Gl_n$ action on M_n given by:

$$(g,h)\cdot x = gxh^T$$

for $x \in M_n$ and $(g,h) \in Gl_n \times Gl_n$. Restricting to the diagonal Gl_n action on this space gives:

$$(g,g)\cdot x = gxg^T$$

and thus under this action, we have a decomposition of M_n as follows,

$$M_n \simeq \mathbb{C}^n \otimes \mathbb{C}^n \tag{3.16}$$

We focus on the O_n decomposition of M_n under the adjoint action of Gl_n, where $x \to gxg^{-1}$. Under this action we have that

$$M_n \simeq (\mathbb{C}^n)^* \otimes \mathbb{C}$$

However, consider the map:

$$\begin{aligned} \mathbb{C}^n \to \quad & (\mathbb{C}^n)^* \\ v \mapsto \quad & \varphi : \mathbb{C}^n \to \mathbb{C} \\ & \qquad w \to v \cdot w \end{aligned}$$

where $v, w \in \mathbb{C}^n$ and $v \cdot w$ is the usual dot product. Since the dot product is invariant under the O_n action, we have that as an O_n- representation, $\mathbb{C}^n \simeq (\mathbb{C}^n)^*$. Thus, we are free to decompose M_n as in 3.16 using this property that O_n is self-dual.

Remark on the Brauer algebra [16]: Before we symmetrize and consider the ΔS_{n+1} action, we can decompose the even dimensional tensor space using the dual, $(\mathbb{C}^n)^*$, which gives the following,

$$[(\otimes^{n+1}\mathbb{C}^n)^* \otimes (\otimes^{n+1}\mathbb{C}^n)]^{O_n} \cong End_{O_n}(\otimes^{n+1}\mathbb{C}^n)$$

where the endomorphism group is defined to be the Brauer algebra [6], [16]. If we consider the O_n action on the tensor space $\otimes^k\mathbb{C}^n$ instead of the traditional general linear group action, then the Brauer algebra replaces the symmetric group in the decomposition of the space via Schur-Weyl duality.

Now, returning to our main discussion, using the decomposition of M_n described above, we can write map 3.15 as:

$$\zeta : [\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}} \twoheadrightarrow [\otimes^{2(n+1)}\mathbb{C}^n]^{O_n\times\Delta S_{n+1}} \tag{3.17}$$

We see that in the domain of this map, $[\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}}$, we are in the stable range where the degree of the invariants is equal to the dimension of the defining representation. In the codomain, $[\otimes^{2(n+1)}\mathbb{C}^{n}]^{O_{n}\times\Delta S_{n+1}}$, is where relations arise. Thus, the kernel, $\mathcal{REL}$, of this map

$$(0)\to\mathcal{REL}\to[\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}}\to[\otimes^{2(n+1)}\mathbb{C}^{n}]^{O_{n}\times\Delta S_{n+1}}$$

is exactly the space of relations between the degree $n+1$ invariants.

Definition 19 *Let $n\in\mathbb{N}$, and let ΔS_{n+1} denote the diagonally embedded copy of the symmetric group S_{n+1} in $S_{n+1}\times S_{n+1}$. Then the kernel of the map,*

$$\zeta:[\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}}\twoheadrightarrow[\otimes^{2(n+1)}\mathbb{C}^{n}]^{O_{n}\times\Delta S_{n+1}}$$

denoted $\mathcal{REL}$, is defined to be the space of relations between the invariants of the conjugation action of the complex orthogonal group on $\mathcal{P}(M_n)$.

Our goal is to understand $\mathcal{REL}$ as a vector space in order to compute its dimension, and determine a basis. The following theorem describes the dimension of this vector space when we are just outside of the stable range, that is, when the degree of the invariants is $n+1$.

Theorem 8 *Let n be a positive integer. The dimension of the space of relations, $\mathcal{REL}_{n+1}$, between the degree $n+1$ invariants of the O_n conjugation action on $\mathcal{P}(M_n)$ is equal to*

$$dim(\mathcal{REL}_{n+1})=\begin{cases}\frac{n}{2}+1 & n \text{ even}\\ \frac{n+3}{2} & n\text{ odd}\end{cases}$$

In order to determine the dimension of $\mathcal{REL}_{n+1}$, the proof will proceed as follows. First, we consider the projection:

$$\left[\otimes^{n+1}(M_n)\right]^{O_n}\longrightarrow\mathcal{P}^{n+1}(M_n)^{O_n}\tag{3.18}$$

where when we project to polynomial space, the tensors become symmetric. The relations in the polynomial algebra pull back into the tensor algebra, and they form an irreducible representation of $S_{2(n+1)}$ corresponding to the partition $[n+1, n+1]$. When this partition is restricted to $S_{n+1} \times S_{n+1}$, there is a multiplicity-free decomposition into irreducible representations $Y^\alpha \otimes Y^\alpha$ corresponding to size $n+1$ diagrams with at most two parts. Thus, to each α there exists a polynomial relation which corresponds to the S_{n+1} invariant where we embed S_{n+1} into $S_{n+1} \times S_{n+1}$ diagonally (see 3.1.3). This ΔS_{n+1} is the symmetric group that symmetrizes to go from the tensor algebra to polynomial space.

The linear growth of $\mathcal{REL}_{n+1}$ is shown in the highlighted diagonal of the following table:

Dimension of $\mathcal{REL}$

$d\backslash n$	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0
3	2	2	0	0	0	0	0	0
4	5	3	3	0	0	0	0	0
5	5	7	4	3	0	0	0	0
6	9	13	12	5	4	0	0	0
7	9	21	21	14	6	4	0	0
8	14	33	48	30	19	7	5	0
9	14	51	75	67	39	21	8	5

Here, the columns are indexed by the dimension of the defining representation, and the rows by the degree of the invariants in $\mathcal{P}^{n+1}(M_n)^{O_n}$. The data in this table again shows that there are no relations when we are in the stable range where the degree of the invariants is less than or equal to n, and it illustrates the linear behavior of the dimension of the space of relations.

Proof of Theorem 8 : Recall we are determining the dimension of the kernel, $\mathcal{REL}_{n+1}$,

$$\mathcal{REL}_{n+1} \to [\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}}$$

We have the following decomposition into irreducible representations via Schur-Weyl Duality, (Theorem 6):

$$\otimes^{2(n+1)}\mathbb{C}^{n+1} \simeq \bigoplus_{\substack{\mu:\, \mu\vdash 2(n+1)\\ \ell(\mu)\leq n+1}} F^{\mu}_{n+1}\otimes Y^{\mu}_{2(n+1)} \tag{3.19}$$

where the F^{μ}_{n+1} and $Y^{\mu}_{2(n+1)}$ are irreducible representations of Gl_{n+1} and $S_{2(n+1)}$, respectively, which are associated to Young Diagram μ with $2(n+1)$ boxes and number of nonzero rows $\leq n+1$. We compute the O_{n+1} invariants,

$$\begin{aligned}[\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}} &\simeq \bigoplus_{\mu}\left[F^{\mu}_{n+1}\otimes Y^{\mu}_{2(n+1)}\right]^{O_{n+1}}\\ &\simeq \bigoplus_{\mu}(F^{\mu}_{n+1})^{O_{n+1}}\otimes Y^{\mu}_{2(n+1)}\end{aligned}$$

By the Cartan-Helgason Theorem, (Theorem 7), we have that $dim(F^{\mu}_{n+1})^{O_{n+1}}$ is nonzero and equal to 1 only when the corresponding tableaux μ has all even parts. Thus, we let $\mu = 2\lambda$:

$$\bigoplus_{\substack{\mu:\, \mu=2\lambda\\ \mu\vdash 2(n+1)\\ \ell(\mu)\leq n+1}} (F^{\mu}_{n+1})^{O_{n+1}}\otimes Y^{\mu}_{2(n+1)} = \bigoplus_{\substack{2\lambda:\, 2\lambda\vdash 2(n+1)\\ \ell(2\lambda)\leq n+1}} (F^{2\lambda}_{n+1})^{O_{n+1}}\otimes Y^{2\lambda}_{2(n+1)} \tag{3.20}$$

This space consists of all Young diagrams, denoted 2λ, where $2\lambda\vdash 2(n+1)$, and $\ell(2\lambda)\leq n+1$. We know from Section 3.1.3 that this space of invariants corresponds to the set of fixed-point free involutions, thus we have that

$$dim\left(\bigoplus_{\substack{2\lambda\vdash 2(n+1)\\ \ell(2\lambda)\leq n+1}} (F^{2\lambda}_{n+1})^{O_{n+1}}\otimes Y^{2\lambda}_{2(n+1)}\right) = \frac{(2(n+1))!}{2^{n+1}(n+1)!} \tag{3.21}$$

Remark: The Robinson-Schensted correspondence, Theorem 4, associates a pair of standard Young tableaux, (P, Q) to a permutation. It is shown, [13], that if the permutation is an involution, then $P = Q$. Furthermore, due to a result of Schutzenberger [14], we have that the fixed-point free involutions correspond to Young diagrams with all even rows. Thus, the space described above in Equation 3.21 consists of all standard Young tableaux of shape 2λ, which again have all even rows.

Recall the kernel contained in this space,

$$\mathcal{REL} \subset \bigoplus_{\substack{2\lambda:\, 2\lambda \vdash 2(n+1) \\ \ell(2\lambda) \leq n+1}} (F_{n+1}^{2\lambda})^{O_{n+1}} \otimes Y_{2(n+1)}^{2\lambda}$$

is the space of relations between the invariants. Once again, we know that relations do not exist in the stable range. Therefore, relations occur when we violate the inequality $\ell(2\lambda) \leq n$.

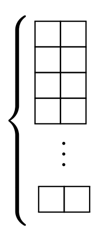
As such, $\mathcal{REL}$ is the space of irreducible representations corresponding to the Young diagrams in our space where $\ell(2\lambda) > n$,

$$\mathcal{REL} = \bigoplus_{\substack{2\lambda:\, 2\lambda \vdash 2(n+1) \\ \ell(2\lambda) > n}} (F_{n+1}^{2\lambda})^{O_{n+1}} \otimes Y_{2(n+1)}^{2\lambda} \subset \bigoplus_{\substack{2\lambda:\, 2\lambda \vdash 2(n+1) \\ \ell(2\lambda) \leq n+1}} (F_{n+1}^{2\lambda})^{O_{n+1}} \otimes Y_{2(n+1)}^{2\lambda}$$

since $\bigoplus_{\substack{2\lambda \vdash 2(n+1) \\ \ell(2\lambda) > n}} (F_{n+1}^{2\lambda})^{O_{n+1}} \otimes Y_{2(n+1)}^{2\lambda}$ is the kernel of the map:

$$\zeta : [\otimes^{2(n+1)} \mathbb{C}^{n+1}]^{O_{n+1} \times \Delta S_{n+1}} \twoheadrightarrow [\otimes^{2(n+1)} \mathbb{C}^{n}]^{O_{n} \times \Delta S_{n+1}}.$$

Now, since we have the restriction that the length of λ must be greater than n, we have only one option for the Young diagram, that is, $\ell(2\lambda) = n+1$,

$n+1$  $=: 2\lambda$

Thus we define Young diagram 2λ as the Young diagram pictured above, with two columns and $n+1$ rows. Here, we have that the dimension of the irreducible representation $Y_{2(n+1)}^{2\lambda}$ is equal to the number of standard Young tableaux of the column shape $(n+1) \times 2$, thus

$$dim\left(\bigoplus_{\substack{2\lambda\vdash 2(n+1)\\ \ell(2\lambda)>n}} (F^{2\lambda}_{n+1})^{O_{n+1}}\otimes Y^{2\lambda}_{2(n+1)}\right) = C_{n+1}$$

where C_{n+1} is a Catalan number.

The irreducible representation $Y^{2\lambda}_{2(n+1)}$ corresponds to partition $[n+1, n+1]$, and we want to restrict this partition to $S_{n+1}\times S_{n+1}$. As dicussed in Section 3.2, we can induce representations in the following way:

$$Ind^{S_{2(n+1)}}_{S_{n+1}\times S_{n+1}} Y^{\alpha}_{n+1}\otimes Y^{\beta}_{n+1} = \bigoplus_{2\lambda:\,2\lambda\vdash 2(n+1)} c^{2\lambda}_{\alpha\beta} Y^{2\lambda}_{2(n+1)} \tag{3.22}$$

Where the coefficients $c^{2\lambda}_{\alpha\beta}$ are the Littlewood-Richardson numbers discussed in Section 3.3. These coefficients count the number of skew semi-standard Young tableaux of shape $2\lambda/\alpha$ with weight β.

By Frobenius reciprocity for finite groups, we can restate Equation 3.22 in terms of restricting the representation,

$$Res^{S_{2(n+1)}}_{S_{n+1}\times S_{n+1}} Y^{2\lambda}_{2(n+1)} = \bigoplus_{\substack{\alpha\vdash n+1\\ \beta\vdash n+1}} c^{2\lambda}_{\alpha\beta} Y^{\alpha}_{n+1}\otimes Y^{\beta}_{n+1}$$

Thus we have the decomposition:

$$\bigoplus_{\substack{2\lambda:\,2\lambda\vdash 2(n+1)\\ \ell(2\lambda)=n+1}} (F^{2\lambda}_{n+1})^{O_{n+1}}\otimes Y^{2\lambda}_{2(n+1)} = \bigoplus_{\substack{2\lambda\\ \alpha\vdash n+1\\ \beta\vdash n+1}} (F^{2\lambda}_{n+1})^{O_{n+1}}\otimes c^{2\lambda}_{\alpha\beta}(Y^{\alpha}_{n+1}\otimes Y^{\beta}_{n+1})^{\Delta S_{n+1}}$$

The Littlewood-Richardson rule tells us $c^{2\lambda}_{\alpha\beta}\neq 0$ when the Young diagrams of α and β fit inside the Young diagram of 2λ. Furthermore, since $|2\lambda| = 2(n+1)$ and $|\alpha| = n+1$ and $|\beta| = n+1$ we must have that $\alpha=\beta$.

Alternately, we determine that $\alpha=\beta$ by considering homomorphisms between the two irreducible representations. We have that as representations, the S_{n+1} are self dual, and thus we can view the tensor of irreducible representations as an endomorphism group:

$$Y^{\alpha}_{n+1} \otimes Y^{\beta}_{n+1} \cong End(Y^{\alpha}_{n+1}, Y^{\beta}_{n+1})$$

By Schur's Lemma we see that there are no nonzero homomorphisms between distinct irreducible representations, and thus we must have that $\alpha = \beta$:

$$\bigoplus_{\substack{2\lambda \\ \alpha\vdash n+1 \\ \beta\vdash n+1}} c^{2\lambda}_{\alpha\beta} Y^{\alpha}_{n+1} \otimes Y^{\beta}_{n+1} = \bigoplus_{\substack{2\lambda \\ \alpha\vdash n+1}} c^{2\lambda}_{\alpha\alpha} Y^{\alpha}_{n+1} \otimes Y^{\alpha}_{n+1} \tag{3.23}$$

Now, the Littlewood-Richardson coefficients $c^{2\lambda}_{\alpha\alpha}$ correspond to the number of semi-standard fillings of tableaux of skew-shape $2\lambda/\alpha$ of weight α. Thus, since 2λ has two columns of length $n+1$, it is clear that we must have that each coefficient is equal to 1.

Recall that we are working in the space of relations, $\mathcal{REL}$, and thus we have that

$$\mathcal{REL}_{n+1} = \bigoplus_{\substack{\alpha:\, \alpha\vdash n+1 \\ c^{2\lambda}_{\alpha\alpha}=1 \\ 2\lambda\vdash 2(n+1) \\ \ell(2\lambda)=n+1}} (Y^{\alpha}_{n+1} \otimes Y^{\alpha}_{n+1})^{\Delta S_{n+1}} \tag{3.24}$$

So, to each α there exists a polynomial relation which corresponds to the irreducible representation Y^{α}_{n+1}.

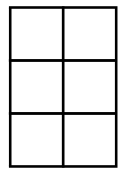
The number of α that satisfy this is precisely the dimension of the space of relations,

$$dim(\mathcal{REL}_{n+1}) = \begin{cases} \dfrac{n}{2} + 1 & n \text{ even} \\ \dfrac{n+3}{2} & n \text{ odd} \end{cases} \tag{3.25}$$

We proceed by induction on n, the degree of the invariants.

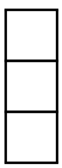
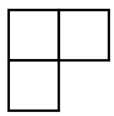
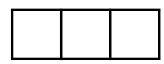
Case I: Let n be a positive, even integer; $n = 2m$ for some $m \in \mathbb{N}$.

Base case: Let $n = 2$. Then we are considering the dimension of the space of relations between the elements of $\mathcal{P}^3(M_2)^{O_2}$. Thus we look at all partitions, $\alpha \vdash 3$. We are concerned with the specific α that give relations in our space; these are the ones in which the Young diagrams corresponding to α fit exactly inside the column shaped Young diagram $(2, 2, 2)$:



Young diagram $(2, 2, 2)$

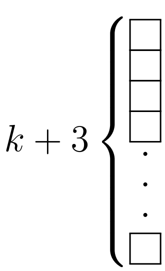
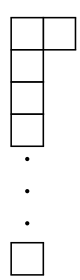
Below we show all of the possible partitions of α and their corresponding Young diagrams:

 $\leftrightarrow (1, 1, 1)$  $\leftrightarrow (2, 1)$  $\leftrightarrow (3)$

Clearly, only partitions $(1, 1, 1)$ and $(2, 1)$ will fit appropriately inside the Young diagram of column shape $(2, 2, 2)$. Thus, when $n = 2$, we have the dimension of the space of relations is equal to $2 = \frac{2}{2} + 1$.

Induction step: We assume the proposition holds for even integer $n = k$, that is, for positive even integer k, the dimension of the space of relations between elements of $\mathcal{P}^{k+1}(M_k)^{O_k}$ is equal to $\frac{k}{2} + 1$. We show this holds for $k + 2$.

Thus, consider the following two partitions $\alpha \vdash k + 2 + 1 = k + 3$ and their corresponding Young diagrams:

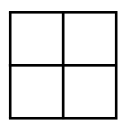
 $\leftrightarrow (\underbrace{1 + 1 + \cdots + 1}_{k+3 \;\; times})$  $\leftrightarrow (2 + \underbrace{1 + \cdots + 1}_{k+1 \;\; times})$

Then, it is clear that the Young diagram corresponding to partition $(1+\cdots+1)$ fits inside the diagram $(2,\ldots,2)$ of $2(k+1)$ boxes. Additionally, we know from our assumption that there are $\frac{k}{2}+1$ partitions of $k+1$ that satisfy our condition on the corresponding Young diagrams. Thus the total number of partitions that work is

$$\frac{k}{2}+1+1=\frac{k}{2}+\frac{2}{2}+1=\frac{k+2}{2}+1$$

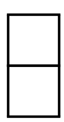
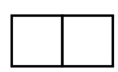
. Case II: Let n be an odd, positive integer; $n=2m+1$ for some $m\in\mathbb{N}$.

Base case: Let $n=1$. Then we are considering the dimension of the space of relations between the elements of $\mathcal{P}^2(M_1)^{O_1}$. Thus we look at all partitions, $\alpha\vdash 2$. We are again concerned with only the α which correspond to relations in our space, that is, the α in which their corresponding Young diagrams fit exactly into the column shaped Young diagram $(2,2)$:



Young diagram $(2,2)$

Below we show all the possible partitions of α and the corresponding Young diagrams:

 $\leftrightarrow (1,1)$  $\leftrightarrow (2)$

Both partitions $(1,1)$ and (2) fit appropriately inside the column shape $(2,2)$. Thus, when $n=1$, we have the dimension of the space of relations is equal to $2=\frac{1+3}{2}$.

Induction step: We suppose the proposition holds for odd integer $n=k$. Therefore we assume for odd integer k, the dimension of the space of relations between elements of $\mathcal{P}^{k+1}(M_k)^{O_k}$ is equal to $\frac{k+3}{2}$. We show this proposition holds for $k+2$.

Similar to Case I, we consider the following two partitions of $k+2+1=k+3$:

$$\underbrace{1+1+\cdots+1}_{k+3\ \ times}$$

$$2+\underbrace{1+\cdots+1}_{k+1\ \ times}$$

Then, we know from our assumption that there are $\dfrac{k+3}{2}$ partitions of the $k+1$ that satisfy our conditions. Thus the total number of partitions that work is

$$\frac{k+3}{2}+1=\frac{k+3}{2}+\frac{2}{2}=\frac{k+3+2}{2}.$$

which thus concludes the proof.

Thus we have determined the dimension of the space $\mathcal{REL}_{n+1}$ of relations between the invariants of degree $n+1$ under the conjugation action of O_n on $\mathcal{P}(M_n)$. In the next section we will discuss a basis for $\mathcal{REL}$ by using a combinatorial object described in Section 2 called a Young symmetrizer.

3.4 Finding a Basis of $\mathcal{REL}$

In the previous subsection we state and prove a theorem regarding the dimension of the space $\mathcal{REL}_{n+1}$. Here, we present a method to determine a basis for this space of relations by relying on a construction of elements from the group algebra $\mathbb{C}[S_{2(n+1)}]$, called Young symmetrizers. The proceeding section presents a thorough explanation of Young symmetrizers, and states several important theorems that we will use in order to obtain a basis of the space of relations.

3.4.1 A Discussion of Young Symmetrizers

Details of the concepts in this section can be found in Fulton and Harris, [3]. Let n, k be positive integers, and consider vector space $\mathbb{C}^k$. Then we can construct the tensor product space,

$$\otimes^n \mathbb{C}^k = \underbrace{\mathbb{C}^k \otimes \mathbb{C}^k \otimes \cdots \otimes \mathbb{C}^k}_{n \text{ times}}.$$

Recall we have a natural action of S_n on this space given by permuting the tensor factors, and this gives rise to a group algebra representation on $\otimes^n \mathbb{C}^k$,

$$\mathcal{E} : \mathbb{C}[S_n] \to End(\otimes^n \mathbb{C}^k)$$

where the group algebra consists of all complex-valued functions on S_n with the standard structure of a complex vector space and multiplication given by convolution. Let $\lambda \vdash n$, where $\lambda = (\lambda_1, \cdots, \lambda_r)$, for $r \leq k$. Then we have that λ corresponds to a Young diagram (details in Chapter 2). We create a standard Young tableau by filling the Young diagram with the numbers $1, \ldots, n$ such that the rows and columns strictly increase. Then, we can define the following:

$$P_\lambda = \{\sigma \in S_n | \sigma \text{ preserves each row of } \lambda\}$$
$$Q_\lambda = \{\sigma \in S_n | \sigma \text{ preserves each column of } \lambda\}$$

These subgroups of S_n define elements a_λ and b_λ in the group algebra, $\mathbb{C}[S_n]$:

$$a_\lambda := \sum_{\sigma \in P_\lambda} e_\sigma$$
$$b_\lambda := \sum_{\sigma \in Q_\lambda} \text{sgn}(\sigma) e_\sigma$$

where e_σ denotes the unit vector corresponding to σ.

Remark: We have that a_λ, b_λ are in $\mathbb{C}[S_n]$, but we can also view them as operators on the tensors, $\otimes^n\mathbb{C}^k$ by considering their image under the map $\mathcal{E}$.

To see what these elements do, note that the images of a_λ and b_λ in the endomorphism group, $End(\otimes^n\mathbb{C}^k)$ are

$$Im(a_\lambda) \simeq Sym^{\lambda_1}\mathbb{C}^k \otimes \cdots \otimes Sym^{\lambda_k}\mathbb{C}^k$$

$$Im(b_\lambda) \simeq \wedge^{\mu_1}\mathbb{C}^k \otimes \cdots \otimes \wedge^{\mu_m}\mathbb{C}^k$$

where $Sym^\lambda\mathbb{C}^k$ and $\wedge^\mu\mathbb{C}^k$ are the symmetric tensors and alternating tensors, respectively, and μ is the conjugate partition to λ.

By construction, the elements a_λ and b_λ are idempotents in the group algebra. They do not commute, however, their product is also idempotent, and is defined as the *Young Symmetrizer* [17], [18].

Definition 20 *The Young symmetrizer is*

$$y_\lambda := a_\lambda \cdot b_\lambda$$

corresponding to Young diagram λ.

We recall that Schur-Weyl duality (Theorem 6) gives a decomposition of the tensor space into a direct sum of minimal $GL_k \times S_n$ invariant subspaces:

$$\otimes^n\mathbb{C}^k \simeq \bigoplus_{\substack{\lambda:\lambda\vdash n \\ \ell(\lambda)\leq k}} F_k^\lambda \otimes Y_n^\lambda$$

where Y_n^λ are irreducible representations of S_n, and F_k^λ are irreducible representations of GL_k.

We note that any element, d, of $\mathbb{C}[S_n]$ gives an invariant subspace, $\mathbb{C}[S_n]d$, of $\mathbb{C}[S_n]$. However, the image of a Young symmetrizer (by right multiplication on $\mathbb{C}[S_n]$) is an invariant subspace which is irreducible under the action of $\mathbb{C}[S_n]$, and unique for each partition λ [3].

The following theorem tells us that the subspaces $\mathbb{C}[S_n]y_\lambda$ given by the Young symmetrizers are, in fact, irreducible representations of S_n, and every irreducible representation of S_n is of this form.

Theorem 9 *Given S_n, let λ be a partition of n. Define Y^λ as the subspace of $\mathbb{C}[S_n]$ spanned by the Young symmetrizer y_λ. Then:*

- *Y^λ is an irreducible representation of S_n*
- *If λ, μ are distinct partitions of n, then $Y^\lambda \ncong Y^\mu$*
- *The Y^λ account for all irreducible representations of S_n.*

Proof of this theorem can be found in Fulton and Harris [3].

Example: We give a simple example of finding a Young symmetrizer in $\mathbb{C}[S_3]$ with λ corresponding to partion $3 = 2 + 1$. We have a standard Young tableau:

1	2
3	

Then the permutation groups P_λ and Q_λ are defined as:

$$P_\lambda = \{1, (1\ 2)\}$$

$$Q_\lambda = \{1, (1\ 3)\}$$

thus we have,

$$a_\lambda = e_{id} + e_{(1\ 2)}$$

$$b_\lambda = e_{id} - e_{(1\ 3)}$$

If we let an element in $\otimes^3\mathbb{C}^k$ be given by $v_{1,2,3} := v_1 \otimes v_2 \otimes v_3$, then

$$a_\lambda v_{1,2,3} = v_{1,2,3} + v_{2,1,3} = (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes v_3,$$

which clearly spans $Sym^2\mathbb{C}^k \otimes Sym^1\mathbb{C}^k$. We have a similiar calculation for the b_λ. Then the Young symmetrizer, y_λ is,

$$y_\lambda = (e_1 + e_{(1\ 2)})(e_1 - e_{(1\ 3)}) = e_1 + e_{(1\ 2)} - e_{(1\ 3)} - e_{(1\ 3\ 2)}$$

Now, we have a right action of y_λ on each basis vector of the group algebra $\mathbb{C}[S_3]$. Instead of multiplying them all out, we use SageMath to generate the matrix,

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

with respect to the basis $\{1, (12), (13), (23), (123), (132)\}$. Thus the image $\mathbb{C}[S_n]y_\lambda$ of y_λ is the span of

$$e_1 + e_{(12)} - e_{(13)} - e_{(132)} \text{ and } -e_1 + e_{(13)} - e_{(23)} + e_{(123)}$$

By Theorem 9, we see that this gives a two dimensional irreducible representation of S_3, and this corresponds to the standard representation.

Because we can construct the irreducible representations of S_n in this way, we revisit our decomposition from Section 3.3 and construct a basis for the space of relations using Young symmetrizers.

3.4.2 Constructing a Basis

We show in Section 3.3 that the space of relations, $\mathcal{REL}_{n+1}$ is,

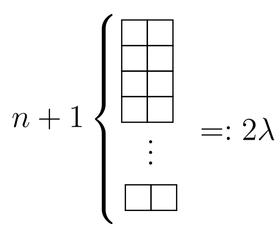
$$\mathcal{REL}_{n+1} = \bigoplus_{\substack{\alpha:\, \alpha \vdash n+1 \\ c^{2\lambda}_{\alpha\alpha}=1 \\ 2\lambda \vdash 2(n+1) \\ \ell(2\lambda)=n+1}} (Y^{\alpha}_{n+1} \otimes Y^{\alpha}_{n+1})^{\Delta S_{n+1}}$$

where, as vector spaces,

$$\bigoplus_{\substack{\alpha:\, \alpha \vdash n+1 \\ c^{2\lambda}_{\alpha\alpha}=1 \\ 2\lambda \vdash 2(n+1) \\ \ell(2\lambda)=n+1}} (Y^{\alpha}_{n+1} \otimes Y^{\alpha}_{n+1})^{\Delta S_{n+1}} \simeq \bigoplus_{\substack{2\lambda:\, 2\lambda \vdash 2(n+1) \\ \ell(2\lambda)=n+1}} (F^{2\lambda}_{n+1})^{O_{n+1}} \otimes Y^{2\lambda}_{2(n+1)}.$$

For each partition α of $n+1$, there exists a polynomial relation, and this relation corresponds to the Y^{α}_{n+1}.

Furthermore, recall that the dimension of the irreducible representation $Y^{2\lambda}_{2(n+1)}$ is equal to a Catalan number, the dimension of the vector space. A natural basis is in one-to-one correspondence with the standard Young tableau with two columns of length $n+1$. These tableaux are all the diagrams



with a standard filling (Definition 2.2b) of the numbers $[1, 2, \ldots, 2n]$ such that each row and column strictly increases. Let

$$T_{2\lambda} := \{T |\, T \text{ is a standard tableau of shape } 2\lambda.\}.$$

Then we know that each T in the set $T_{2\lambda}$ corresponds to a relation between the elements of

$\mathcal{P}^{n+1}(M_n)^{O_n}$. Additionally, since each element of $T_{2\lambda}$ is a standard tableau, it corresponds to an element of $\mathbb{C}[S_{2(n+1)}]$ called a Young symmetrizer (discussed in the previous subsection 3.4.1). We denote this element y_T.

Section 3.4.1 defines the y_T as a product of the row and column stabilizers of its associated Young tableau, and Theorem 9 states that each Young symmetrizer corresponds to an irreducible representation of the symmetric group.

Recall in Theorem 3 we determine the dimension of the space of relations between degree $n+1$ invariants under the conjugation action of O_n on polynomials on $n \times n$ matrices. By averaging over ΔS_{n+1}, we have the projection,

$$[\otimes^{n+1}(M_n)]^{O_n} \to \mathcal{P}^{n+1}(M_n)^{O_n}$$

between the space of orthogonally invariant tensors and the polynomial invariants under the O_n action. We discussed the following maps in Section 3.1.3,

$$S_{2(n+1)} \to S_{2(n+1)}/H_{n+1} \to \Delta S_{n+1} \backslash S_{2(n+1)}/H_{n+1}.$$

Recall, the cosets $S_{2(n+1)}/H_{n+1}$ are in one-to-one correspondence with the set $I_{2(n+1)}$, the fixed-point free involutions on $S_{2(n+1)}$. Furthermore, the double cosets in the above map are in bijective correspondence to a basis of the degree $n+1$ polynomial invariants under the orthogonal group action on $\mathcal{P}(M_n)$.

Thus we have the following invariant subspaces of the full group algebra:

$$\mathbb{C}[S_{2(n+1)}]^{\Delta S_{n+1} \times H_{n+1}} \hookrightarrow \mathbb{C}[S_{2(n+1)}]^{H_{n+1}} \hookrightarrow \mathbb{C}[S_{2(n+1)}] \tag{3.26}$$

where elements of $\mathbb{C}[S_{2(n+1)}]^{\Delta S_{n+1} \times H_{n+1}}$ are linear combinations of the permutations that correspond to the polynomial invariants, $\mathcal{P}^{n+1}(M_n)^{O_n}$.

Then, we consider the following maps:

$$\mathbb{C}[S_{2(n+1)}]^{\Delta S_{n+1}\times H_{n+1}} \xrightarrow{\mathcal{E}} [\otimes^{n+1} M_n]^{O_n\times\Delta S_{n+1}} \to \mathcal{P}^{n+1}[M_n]^{O_n}$$

where $\mathcal{E}$ takes elements of the group algebra invariant under the left ΔS_{n+1} action and the right H_{n+1} action into endomorphisms on tensors.

So, for arbitrary degree of invariants d, we have the projection:

$$\mathbb{C}[S_{2d}]^{\Delta S_d\times H_d} \twoheadrightarrow \mathcal{P}^d(M_n)^{O_n}$$

where if $d = n+1$, there exists a nonzero kernel which corresponds to the relations between degree $n+1$ invariants.

Thus we take a Young symmetrizer and conjugate $\tau = (12)(34)\cdots(2(n+1)-1\ 2(n+1))$ by each of its terms in order to write it as a linear combination of fixed-point free involutions. We can then determine which double coset, $\Delta S_{n+1}\backslash S_{2(n+1)}/H_{n+1}$, each term is in. Thus, we rewrite the Young symmetrizer using coset representatives for each of its terms. We denote this by $\widetilde{y_T}$, so that

$$\widetilde{y_T} \in \mathbb{C}[S_{2(n+1)}]^{\Delta S_{n+1}\times H_{n+1}}.$$

Now, the $\widetilde{y_T}$ form a spanning set for the relations between the degree $n+1$ invariants. In order to find a basis of relations, we look for a subspace of this vector space with dimension dictated by Theorem 3.

We give an example to illustrate this construction, but we keep the dimension small as the number of terms in the Young symmetrizers gets too large in much higher degrees.

Example: Consider the space $[\otimes^3 M_2]^{O_2}$, where the degree of the invariants is three. Then we have $dim(Y^{2\lambda}_{2(3)}) = 5$; there are five standard tableaux of shape $(2,2,2)$ shown in Figure 3.13.

1	2
3	4
5	6

1	2
3	5
4	6

1	4
2	5
3	6

1	3
2	5
4	6

1	3
2	4
5	6

Figure 3.13: All five standard Young tableaux of shape (2,2,2).

Then there are five corresponding Young symmetrizers, each with 288 components in the symmetric group S_6,

$$\begin{aligned} y_{\tiny\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}} = {} & [1,2,3,4,5,6] - [1,2,3,4,6,5] - [1,2,3,5,4,6] + [1,2,3,5,6,4] + [1,2,3,6,4,5] \\ & - [1,2,3,6,5,4] + [1,2,6,3,4,5] - [1,2,6,3,5,4] - [1,2,6,4,3,5] + [1,2,6,4,5,3] \\ & + [1,2,6,5,3,4] - [1,2,6,5,4,3] - \cdots + [6,5,4,3,2,1] \end{aligned}$$

$$\vdots$$

$$\begin{aligned} y_{\tiny\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}} = {} & [1,2,3,4,5,6] + [1,2,3,4,6,5] - [1,2,3,5,6,4] - [1,2,3,6,5,4] + [1,2,4,3,5,6] \\ & + [1,2,4,3,6,5] - [1,2,4,5,6,3] - [1,2,4,6,5,3] - [1,2,5,3,4,6] - [1,2,5,4,3,6] \\ & + [1,2,5,6,3,4] + [1,2,5,6,4,3] - \cdots + [6,5,4,3,2,1] \end{aligned}$$

In order write each permutation as a fixed-point free involution, we conjugate the element $\tau = (1\,2)(3\,4)(5\,6) \in S_6$ by every component of each y_T. Then we determine which double coset the fixed-point free involution is in, and thus replace each term with the appropriate double coset representative. Thus, each term of the Young symmetrizer is an element of

$\Delta S_3 \backslash S_6 / H_3$,

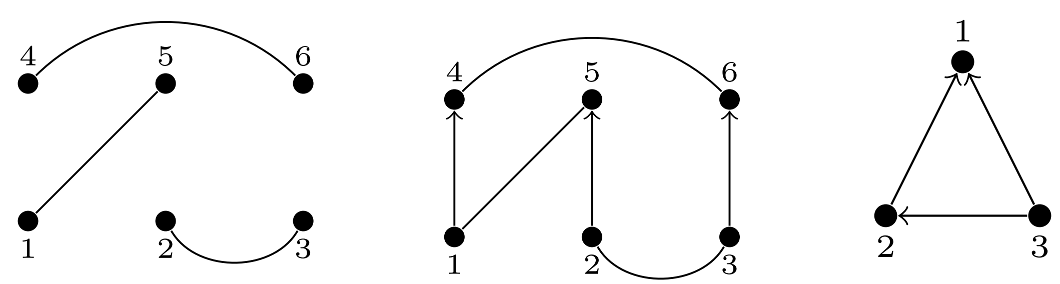
$$\widetilde{y}_{\begin{smallmatrix}1&4\\2&5\\3&6\end{smallmatrix}} = (15)(23)(46) - (14)(25)(36) - (12)(36)(45) + \cdots + (15)(23)(46)$$

$$\vdots$$

$$\widetilde{y}_{\begin{smallmatrix}1&2\\3&4\\5&6\end{smallmatrix}} = (14)(25)(36) + (12)(36)(45) - (12)(36)(45) - \cdots + (12)(34)(56)$$

Next we recall the correspondence between the elements of the double cosets, which are fixed-point free involutions, and elements of the invariant algebra, $\mathcal{P}^{n+1}(M_n)^{O_n}$. There is a bijection given in Map 3.5 between the set of involutions on $S_{2(n+1)}$ without fixed points, and the set of Necklace diagrams (Definition 13) with $n+1$ nodes. Furthermore, there is then a bijection between that set of Necklace diagrams and elements of $\mathcal{P}^{n+1}(M_n)^{O_n}$, detailed in Section 3.1.1.

For this example, we illustrate one case of these correspondences:



(a) Edges drawn between nodes via the involution $(15)(23)(46) \in I_6$.

(b) Connecting nodes to create a Necklace diagram.

(c) Collapsed edges; diagram corresponds to $Tr(x^2 x^T)$.

Thus, each component of the $\widetilde{y_T}$ corresponds to an invariant in $\mathcal{P}^3(M_2)^{O_2}$, so we combine like terms, set equal to zero, and arrive at the following relations:

$$Tr(x)^3 - 3Tr(x^2)Tr(x) + 2Tr(x^3) = 0$$
$$-Tr(x^2x^T) + Tr(xx^T)Tr(x) - Tr(x^2)Tr(x) + Tr(x^3) = 0$$
$$2Tr(x^2x^T) - 2Tr(xx^T)Tr(x) + Tr(x)^3 - Tr(x^2)Tr(x) = 0$$

Here, we note that our result in Theorem 8 dictates that there should only be two relations in this space. Clearly, only two of these equations are linearly independent. We can arbitrarily choose any two linearly independent relations as a basis of $\mathcal{REL}$.

We use Python and Sage to write the code for finding relations via this method of Young symmetrizers. Our code runs on the Sagemath cloud with 11 GB disk space, 5 GB of RAM and 1 core. However, the size of the Young symmetrizers grows exponentially and the calculations quickly become too RAM intensive as we increase the degree of the invariants. For example, in $[\otimes^5 M_4]^{O_4}$ there are 42 Young symmetrizers, each with $460,800$ components; the calculation to find the relations using the method discussed here takes a little over 43 hours to compute on this server.

In the next chapter, we discuss a much faster method for finding the relations between the invariants under the O_n action on $\mathcal{P}(M_n)$.

4 A Monte Carlo Method

In this chapter we introduce a new method for determining a basis for the space of relations, $\mathcal{REL}$.

We want to avoid the lengthy calculations involved in computing relations using Young symmetrizers, discussed in Section 3.4. We know that the kernel, $\mathcal{REL}$, of the map

$$\zeta : [\otimes^{2(n+1)}\mathbb{C}^{n+1}]^{O_{n+1}\times\Delta S_{n+1}} \twoheadrightarrow [\otimes^{2(n+1)}\mathbb{C}^{n}]^{O_{n}\times\Delta S_{n+1}}$$

consists of relations between the degree $n+1$ invariants under the complex orthogonal group action on the polynomial functions on $n \times n$ matrices.

We showed in Chapter 3 that $\mathcal{P}(M_n)^{O_n}$, the algebra of invariant polynomials under the conjugation action of O_n, is generated by elements of the form,

$$Tr(x^{a_1}(x^T)^{a_2}x^{a_3}(x^T)^{a_4}\cdots x^{a_M})$$

for matrix $x \in M_n$. Products of the above polynomials form a spanning set of the invariants, and they are not linearly independent when the degree of the monomials is greater than n.

We want to find a basis of $\mathcal{REL}$ using elements of the invariant algebra, $\mathcal{P}^{n+1}(M_n)^{O_n}$, and avoid computations with Young symmetrizers. Determining the relations between these polynomials requires solving the nonlinear equations:

$$\sum_{i=1}^{k} y_i Tr(x^{a_{i_1}}(x^T)^{a_{i_2}}\cdots) = 0 \tag{4.1}$$

where $x \in M_n$, $k = dim(\mathcal{P}^{n+1}(M_n)^{O_n})$, (recall that this is the number of double cosets $S_{n+1}\backslash S_{2(n+1)}/H_{n+1}$), and the y_i are constant coefficients in $\mathbb{C}$.

Solving these nonlinear equations can be computationally challenging. However, we introduce a method for finding relations that allows us to instead solve a $k \times k$ linear system of equations via a Monte Carlo algorithm.

Each of the invariants $Tr(x^{a_{i_1}}(x^T)^{a_{i_2}}\cdots)$ are constructed using a matrix $x \in M_n(\mathbb{C})$. By definition,

$$Tr(x^{a_{i_1}}(x^T)^{a_{i_2}}\cdots) \in \mathbb{C}.$$

Therefore, if we randomly generate a matrix $x \in M_n$ we can compute a numerical value for each of the k invariants in $\mathcal{P}^{n+1}(M_n)^{O_n}$. Each equation then becomes linear in the variables y_i:

$$\sum_{i=1}^{k} y_i \underbrace{Tr(x^{a_{i_1}}(x^T)^{a_{i_2}}\cdots)}_{\in\mathbb{C}} = 0.$$

Thus, if we generate k random matrices in M_n, and compute the values of each invariant in $\mathcal{P}^{n+1}(M_n)^{O_n}$, this gives k different linear equations.

Let $x_1, x_2, \ldots x_k$ denote the k randomly generated matrices in $M_n(\mathbb{C})$. We can then solve the $k \times k$ linear system for the y_i,

$$y_1 Tr(x_1^{a_{1_1}}(x_1^T)^{a_{1_2}}\cdots) + y_2 Tr(x_1^{a_{2_1}}(x_1^T)^{a_{2_2}}\cdots) + \cdots + y_k Tr(x_1^{a_{k_1}}(x_1^T)^{a_{k_2}}\cdots) = 0 \quad (1)$$

$$y_1 Tr(x_2^{a_{1_1}}(x_2^T)^{a_{1_2}}\cdots) + y_2 Tr(x_2^{a_{2_1}}(x_2^T)^{a_{2_2}}\cdots) + \cdots + y_k Tr(x_2^{a_{k_1}}(x_2^T)^{a_{k_2}}\cdots) = 0 \quad (2)$$

$$\vdots$$

$$y_1 Tr(x_k^{a_{1_1}}(x_k^T)^{a_{1_2}}\cdots) + y_2 Tr(x_k^{a_{2_1}}(x_k^T)^{a_{2_2}}\cdots) + \cdots + y_k Tr(x_k^{a_{k_1}}(x_k^T)^{a_{k_2}}\cdots) = 0 \quad (\mathrm{k})$$

and the solution is exactly the relations in the space $\mathcal{REL}_{n+1}$.

Example: We revisit the space $\mathcal{P}^3(M_2)^{O_2}$ of degree 3 invariants under the conjugation action of O_2 on M_2. We know that the dimension of the invariant space is 5, recall we have the following invariants:

$$Tr(x^3),\ \ Tr(x^2x^T),\ \ Tr(x^2)Tr(x),\ \ Tr(xx^T)Tr(x),\ \ Tr(x)^3$$

We construct a basis for this space using the Monte Carlo method described above. The relations can be described as solutions to the following the nonlinear equations in the invariants:

$$y_0Tr(x^3) + y_1Tr(x^2x^T) + y_2Tr(x^2)Tr(x) + y_3Tr(xx^T)Tr(x) + y_4Tr(x)^3 = 0 \tag{4.2}$$

where the y_i's are constant coefficients.

In order to avoid solving a complicated nonlinear system, we first repeatedly generate random complex 2×2 matrices and compute numerical values for each of the five invariant polynomials. By generating five random matrices, we produce five linear equations using the numerical values for the invariants. That is we generate,

$$\begin{aligned}
y_0Tr(x^3) + y_1Tr(x^2x^T) + y_2Tr(x^2)Tr(x) + y_3Tr(xx^T)Tr(x) + y_4Tr(x)^3 &= 0 && (1)\\
y_0Tr(x^3) + y_1Tr(x^2x^T) + y_2Tr(x^2)Tr(x) + y_3Tr(xx^T)Tr(x) + y_4Tr(x)^3 &= 0 && (2)\\
&\vdots\\
y_0Tr(x^3) + y_1Tr(x^2x^T) + y_2Tr(x^2)Tr(x) + y_3Tr(xx^T)Tr(x) + y_4Tr(x)^3 &= 0 && (5)
\end{aligned}$$

where each equation uses a different randomly generated 2×2 matrix to compute a numerical value for each of the invariants, and thus the equations are all linear in the y_i with coefficients in $\mathbb{C}$.

We then run code, (detailed in Section 5), to solve this 5×5 linear system, and the result is two linearly independent relations in the space $\mathcal{REL}$,

$$\begin{aligned}
Tr(x)^3 - 3Tr(x^2)Tr(x) + 2Tr(x^3) &= 0\\
2Tr(x^2x^T) - 2Tr(xx^T)Tr(x) + Tr(x)^3 - Tr(x^2)Tr(x) &= 0
\end{aligned}$$

Again, we use Python and Sage to write the code for finding relations via this Monte Carlo simulation method. Our code runs on the Sagemath cloud with 11 GB disk space, 5 GB of RAM and 1 core. However, this method is much faster at finding relations than the method of Young symmetrizers. For example, in the case of $[\otimes^5 M_4]^{O_4}$ which took over 43 hours to compute relations via Young symmetrizers, our new method takes just a bit over 2 minutes.

4.1 Classical Invariant Theory

Classical invariant theory involves finding polynomial functions that are invariant under the natural action of three linear groups on representations of a finite dimensional vector space: The general linear group, Gl_n, the orthogonal group, O_n, and the symplectic group, Sp_{2n}. The First Fundamental Theorems (FFT) of invariant theory give the generators for the ring of invariants under these group actions, and the Second Fundamental Theorems (SFT) give the relations between the invariants given in the FFT.

In this section we give a brief description of the classical cases, [11], and then give a few examples to illustrate that the Monte Carlo method described above in Section 4 recovers the relations described by the SFT for Gl_n, O_n, and Sp_2n.

Let V be a finite dimensional vector space over $\mathbb{C}$. Then for positive integer n, consider the following vector space:

$$V^p \oplus (V^*)^q := \underbrace{V \oplus \cdots \oplus V}_{\text{p times}} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{\text{q times}}$$

We note there is a useful canonical identification,

$$V^p \oplus (V^*)^q = M_{n\times p} \oplus M_{q\times n}$$

Then the representation of Gl_n on this space is given by

$$g(v_1, \ldots, v_p, \varphi_1, \ldots, \varphi_q) := (gv_1, \ldots, gv_p, g\varphi_1, \ldots, g\varphi_q)$$

where $(g\varphi_i)(v) = \varphi_i(g^{-1}v)$. The following FFT describes the invariants under this group action,

Theorem 10 ***(First Fundamental Theorem (FFT) for*** $\mathbf{Gl_n}$***)*** *Let* V *be a finite dimensional* $\mathbb{C}$*-vector space. The ring of invariants for the action of* $GL_n(V)$ *on* $V^p \oplus (V^*)^q$ *is generated by the invariants* $(i|j)$*:*

$$\mathbb{C}[V^p \oplus (V^*)^q]^{GL(V)} = \mathbb{C}[(i|j)|i = 1, \ldots, p, j = 1, \ldots, q]$$

Where $(i|j)$ *is a bilinear function on* $V^p \oplus (V^*)^q$ *given by*

$$(i|j) : (v_1, \ldots, v_p, \varphi_1, \ldots, \varphi_q) \to (v_i|\varphi_j) = \varphi_j(v_i)$$

Furthermore, the SFT gives the relations between these invariants,

Theorem 11 ***(Second Fundamental Theorem (SFT) for*** $\mathbf{GL_n}$***)*** *The set of all* $(n + 1) \times (n + 1)$ *minors is a minimal generating set for the ideal of polynomials vanishing on* $\mathcal{DV}_{k,m,n}$*, the space of all matrices in* $M_{k,m}$ *of rank at most* n*.*

Proofs of the FFT and SFT can be found in [11].

We can recover these relations by using the Monte Carlo algorithm described in Section 4. We give the following example of using this method to generate the relations described by the (SFT) for Gl_n. We consider a small n since in high dimensions where $n >> p$ and $n >> q$, the relations vanish.

Example: Let $n = 1$ and consider Gl_1. Then we have that Gl_1 acts on

$$V = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus (\mathbb{C}^1)^* \oplus (\mathbb{C}^1)^* \oplus (\mathbb{C}^1)^*$$

. Let $(a_1, a_2, b_1, b_2, b_3) \in V$, and $a_i * b_j = \mu_{ij}$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$.

The $[\mu_{ij}]_{2x3}$ matrix of invariants is:

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \end{bmatrix}$$

Using the Monte Carlo simulation, we generate random matrices to calculate numerical values for each element of the above matrix, and we look for quadratic relations. Thus we construct a linear system of equations using these numerical invariants, and our code gives the following solution:

$$(a_1b_1)(a_2b_2) - (a_1b_2)(a_2b_1) = 0$$
$$(a_1b_1)(a_2b_3) - (a_1b_3)(a_2b_1) = 0$$
$$(a_1b_2)(a_2b_3) - (a_1b_3)(a_2b_2) = 0$$

These are exactly the 2×2 minors of the above $[\mu_{ij}]$ matrix, which are the relations described in the SFT.

There are similar first and second fundamental theorems for the orthogonal and symplectic groups [11]. We don't state or prove them here, but we note that code for using a Monte Carlo simulation to recover the relations between the invariants under these group actions can be found in the following section.

5 Code

The following code finds the invariant polynomials of degree $n+1$ under the conjugation action of O_n on $\mathcal{P}(M_n)$, and gives relations between these invariants via two methods. First it gives and explicit basis using Young Symmetrizers, which is computationally expensive. The second method gives a basis by using the Monte Carlo method described in Section 4.

```
import numpy as np
import itertools
import string
from sage.groups.perm_gps.permgroup_element import string_to_tuples

def symgroup(n): #makes symmetric group object in sage and Gap
    return (SymmetricGroup(2*n), libgap.eval(’G:=SymmetricGroup(2*n)’))
def tau(n): #make the element Tau of the group
    list1 =[]
    for i in range(1, 2*n+1):
        list1.append(i)
    m = zip(*[iter(list1)]*2)
    return ’’.join(map(str, m))
def central(Tau, H, G): #find the centralizer of Tau in S_2n
    t = H(Tau)
    a = t.gap()
    return G.Centralizer(a)
def dcosets(n, H, q): #find double cosets, and their reps/sizes
    list2, list3, list4 = ([] for i in range(3))
    for i in range(1,n):
        list2.append((i, i+1))
    for i in range(n+1, 2*n):
        list3.append((i,i+1))
    A     = PermutationGroup(list2)
    B     = PermutationGroup(list3)
```

```
    list4 = list(map(list, zip(A,B)))
    p     = [inner for outer in list4 for inner in outer]
    A1    = PermutationGroup([p[i] * p[i+1] for i in range(0, len(p), 2)])
    B1    = H.subgroup(A1)
    u     = B1.gap()
    return (G.DoubleCosets(u,q), G.DoubleCosetRepsAndSizes(u,q).sage())
def duplicate(testList, n):
    return [elt for elt in testList for i in range(n)]
def ModCorresp(n, t, d): #Symmetric Group 2n mod centralizer of Tau
    list5, list6, list7, list8, list9, list10, list11, list12, list13, list14, list15, list16, list17 = ([] for i in range(13))
    for i in range(len(d)):
        list5.append(d[i][0])
    for i in range(len(list5)):
        list6.append(list5[i].to_permutation_group_element())
    for i in range(len(list5)):
        list7.append(((list5[i].to_permutation_group_element()).inverse()))
    for i in range(len(list6)):
        list8.append(list6[i]*t*list7[i])
    for i in range(len(list8)):
        list9.append(string_to_tuples(str(list8[i])))
    for sublist in list9:
        for item in sublist:
            list10.append(item)
    list11 = [x for y in list10 for x in y]
    list12 = [list11[i:i+2*n] for i in range(0, len(list11), 2*n)]
    x = duplicate(iter(list(string.ascii_lowercase)),2)
    L = [x for i in xrange(len(list8))]
    for i in range(len(list8)):
        dictionary = dict(zip(list12[i], L[i]))
        for value in dictionary.values():
            list13.append(value)
```

```
    list14 = [list13[i:i + 2*n] for i in range(0, len(list13), 2*n)]
    u = []
    for i in range(0,n):
        u.append((i, i+n))
    myorder = [x for y in u for x in y]
    for j in range(len(list8)):
        list15.append([list14[j][i] for i in myorder])
    for i in range(0, len(list8)):
        list16.append(''.join(map(str, list15[i])))
    for i in range(0, len(list8)):
        list17.append(",".join([list16[i][j:j+2] for j in range(0, len(
   list16[i]), 2)]))
    return list17, len(list8)
#############################################
##  Run for Monte Carlo method             ##
#############################################
eqnlist = []
n        = 2   #n = degree of invariants
n        = libgap.eval('n:=2')
H,G      = symgroup(n)
Tau      = tau(n)
t        = H(Tau)
q        = central(Tau, H, G)
c,d      = dcosets(n,H,q)
invariants, numinv = ModCorresp(n, t, d)
for x in range(0, numinv):
    list19 = []
    M      = MatrixSpace(CC, n-1, n-1).random_element()
    coeff  = var(','.join('y%s'%i for i in range(0, numinv)))
    for i in range(0, numinv):
        list19.append((np.einsum(invariants[i], M,M,M,M,M)))
    eqn = sum(coeff[w]*list19[w] for w in range(0, numinv))
    eqnlist.append(eqn)
```

```
solve([eqnlist[i]==0 for i in range(0, numinv)], coeff)
#############################################
##  Run for Young symmetrizer method        ##
#############################################
from sage.combinat.symmetric_group_algebra import e
eqnlist = []
n        =  2                # n = degree of invariants
n        = libgap.eval('n:=2')
H,G      = symgroup(n)
Tau      = tau(n)
t        = H(Tau)
q        = central(Tau, H, G)
c,d, B1      = dcosets(n,H,q,G)
invariants, numinv = ModCorresp(n, t, d)
doublecosets = []  # A list of all the elts of each double coest from c
for i in range(len(c)):
    doublecosets.append(c[i].List().sage())
doublecosetsffi = []
for elt in doublecosets:
    tempy = []
    for permu in elt:
        tempy.append(permu.to_permutation_group_element()*t*(permu.to_permutation_group_element().inverse()))
    doublecosetsffi.append(list(set(tempy)))
ST = list(StandardTableaux([2,2]));  #change this for size of tableaux
YS = []
for elt in ST:
    YS.append(e(elt)) #this finds the Young symmetrizer
YSlist=[]
for i in range(len(YS)):
    YSlist.append(list(YS[i]))
Y = []
for j in range(len(YSlist)):
```

```
    tempy = []
    for i in range(len(YSlist[0])):
        tempy.append(YSlist[j][i][0])
    Y.append(tempy)
Yffi = []
for elt in Y:
    tempy = []
    for permu in elt:
        tempy.append(permu.to_permutation_group_element()*t*(permu.to_permutation_group_element().inverse()))
    Yffi.append(tempy)
YScoset = []
YSsgn   = []
for lst in Yffi:  #assign coset rep to each component
    tempy = []
    for permu in lst:
        for i in range(len(c)):
            if permu in doublecosetsffi[i]:
                tempy.append(i+1)
    YScoset.append(tempy)
YScoset1 = []
for i in range(len(YSlist)):
    tempy = []
    for elt in YSlist[i]:
        tempy.append(elt[1])
    YSsgn.append(tempy)
YSSgnCosets = []
for i in range(len(YScoset)):
    YSSgnCosets.append([a*b for a,b in zip(YScoset[i],YSsgn[i])])
N1= []
for elt in YSSgnCosets: #gives relations, output is coefficients
    tempy = []
    for i in range(1, len(c)+1):
```

```
        tempy.append(elt.count(i)-elt.count(-i))
    N1.append(tempy)
```

Listing 5.1: Code to find the dimension and basis of REL. There is code for using both a Monte Carlo and Young symmetrizer approach.

The following code finds the invariants and relations under the three classical group actions of invariant theory, GL_n, O_n and Sp_{2n} using a Monte Carlo simulation. The results are consistent with those given by the first and second fundamental theorems of invariant theory.

```

###########################
## General Linear Group  ##
###########################
import random
n=3
m=2
p=(m*n+1)*(m*n)/2
b=[]
for x in range(0,p):
    l=[]
    k=[]
    Pnumbers = MatrixSpace(CC,1,n).random_element()
    Qnumbers = MatrixSpace(CC,1,m).random_element()
    coeff= var(','.join('y%s'%i for i in range(0,p)))
    for i in range(0,n):
        for j in range(0,m):
            l.append(Pnumbers[0,i]*Qnumbers[0,j])
    for i in range(len(l)):
        for j in range(len(l)):
            if i <= j:
                k.append(l[i]*l[j])
    z=sum(coeff[w]*k[w] for w in range(0,p))
    b.append(z)
```

```
solve([b[i]==0 for i in range(0,p)], coeff)

############################
## Orthogonal Group        ##
############################
import random
n=3
m=((n-1)*n)/2
p=((m+n)*(m+n+1))/2
b=[]
for x in range(0,p):
    l=[]
    k=[]
    numbers = MatrixSpace(CC,1,n).random_element()
    coeff= var(','.join('y%s'%i for i in range(0,p)))
    for i in range(0,n):
        for j in range(0,n):
            if i <= j:
                l.append(numbers[0,i]*numbers[0,j])
    for i in range(len(l)):
        for j in range(len(l)):
            if i <= j:
                k.append(l[i]*l[j])
    z=sum(coeff[w]*k[w] for w in range(0,p))
    b.append(z)
solve([b[i]==0 for i in range(0,p)], coeff)

############################
## Symplectic Group        ##
############################
import random
n=6
b=[]
```

```
m=((factorial((n**2-n)/2 + 1))/(factorial(2)*factorial((n**2-n)/2 -1)))
z= (n**2-n)/2
for x in range(0,m):
    l=[]
    p=[]
    MS=MatrixSpace(CC,2,n)
    A=MS.random_element()
    for i in range(0,n-1):
        for j in range(1,n):
            if j > i:
                B=A.matrix_from_rows_and_columns([0,1], [i,j])
                D=det(B)
                l.append(D)
    for i in range(0,z):
        for j in range(0, z):
            if i <= j:
                p.append(l[i]*l[j])
    coeff= var(','.join('y%s'%i for i in range(0,m)))
    k= sum(coeff[n]*p[n] for n in range(m))
    b.append(k)
solve([b[i]==0 for i in range(0,m)], coeff)
```

Listing 5.2: Code to find relations in the three classical invariant theory cases.

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Curriculum Vitae

Alison Elaine Becker

Education

University of Wisconsin-Milwaukee, Mathematics PhD

Advisor: Dr. Jeb Willenbring

University of Wisconsin-Milwaukee, Mathematics M.S

Advisor: Dr. Allen Bell

University of Wisconsin-Milwaukee, Mathematics B.S, Economics B.A.

Computer Languages

- Python — Proficient
- SQL — Proficient
- LaTeX — Fluent

Experience

Research Assistant

Advanced Opportunity Program (AOP) Fellow

University of Wisconsin-Milwaukee, Teaching Assistant

- **Sole Instructor for the following:**

  Math 231 - Calculus and Analytic Geometry I

  Math 211 - Survey in Calculus and Analytic Geometry

  Math 116 - Precalculus

  Math 108 - Algebraic Literacy II

  Math 098 - Algebraic Literacy I

  Math 105 - Introduction to College Algebra

Sole Instructor for Flipped Classroom:

Math 095 - Essentials of Algebra

Math 094 - Foundations of Elementary Mathematics

- **Discussion Instructor for the following:**

  Math 1132 - Calculus II

  Math 1131 - Calculus I

  Math 211 - Survey in Calculus and Analytic Geometry

- **Grader for the following:**

  Math 431 - Modern Algebra with Applications

- **Teaching Assistant for the following:**

  Math 341 - Introduction to the Language and Practice of Mathematics

- **Online learning systems used:**

  ALEKS, MyMathLab, WebAssign, Blackboard, D2L

STEM-Inspire WiscAMP Graduate/Faculty Mentor

STEM-Inspire WiscAMP (Wisconsin Alliance for Minority Participation) is a program aimed at improving the recruitment, retention, and graduation of underrepresented minority students in STEM fields at UW-Milwaukee

- Mentored 15 underrepresented minority students in STEM majors
- Held workshops for discussion of applications to graduate school
- Coordinated/supervised study nights
- Tutored students in math, statistics, and economics

Research Assistant - UWM Department of Educational Statistics

- Compiled and analyzed data from UW-Milwaukee WiscAMP program in order to submit a proposal to expand the program and continue to receive funding

Undergraduate Research - UWM Department of Physics

- Used several different surveys to search for new pulsars by analyzing data

McNair Post-Baccalaureate Achievement Senior Scholars Program

This is a federal TRIO program designed to prepare first-generation college students, students with financial need, and members of traditionally underrepresented groups for graduate education

- Mentor: Dr. James Peoples, Department of Economics (UW-Milwaukee)
- Research: *Technological Change and Labor Demand in the Telecommunications Industry Following the 1996 Telecommunications Act*

McNair Scholar Peer Mentor

- Mentored incoming McNair students on studying techniques, research methods and writing
- Tutored students in mathematics

McNair Post-Baccalaureate Achievement Scholars Program

- Mentor: Dr. Eric Key, Department of Mathematical Sciences (UW-Milwaukee)
- Research: *Stirling's Formula*

Library Assistant in Interlibrary Loan, UW-Milwaukee

Library Assistant in Processing Department, Marquette University

Service and Outreach

Association for Women in Mathematics- UW-Milwaukee Chapter

Chapter Co-Founder

Chapter President

Chapter Vice President

Mentor for the McNair Scholars Program

Mentor a student in the McNair Program at UW-Milwaukee interested in graduate school for mathematics

Ambassador for the Mathematics Program

Met with prospective graduate students for the department of Mathematical Sciences at UW-Milwaukee

UW-Milwaukee Udergraduate Research Symposium Judge

Oostburg Christian School Math Outreach

Spent the day at an elementary school giving math workshops to 4th and 5th grade students with the UW-Milwaukee AWM Chapter

Co-organized this event

Undergraduate Poster Judge- Joint Mathematics Meetings

Girls Empowered by Math and Science (GEMS)
Conference Workshop Presenter

GEMS is a conference at UW-Parkside to promote participation and opportunities to learn about STEM fields to middle-school girls

My workshop title: Roll the Dice on Disease- workshop on mathematical models as they apply to the spread of diseases

Marquette University Service Learning

Taught 3rd grade students elementary physics concepts at a Spanish Immersion school in Milwaukee

Papers

- Master's Thesis: *Results on n-absorbing Ideals of Commutative Rings*, May 2013 http://dc.uwm.edu/etd/793

Presentations/Talks

- *Finding Relations Between Invariants of Representations Using a Monte Carlo Method*, Joint Math Meetings (JMM)
- Algebra Seminar (UW-Milwaukee)
  - *Introduction to SageMath and Python*
  - *Using SageMath to code a Monte Carlo method*
  - *Finding Relations Between Invariants- Discussion of Classical Theory*
  - *Wiring Diagrams and Symmetric Tensors*
- *Technological Change and Labor Demand in the Telecommunications Industry Following the 1996 Telecommunications Act*, 2012 National McNair Conference, Lake Geneva WI
- *Stirling's Formula*

  University of California-Berkeley National McNair Conference

  The Ohio State University Summer Research Conference

  National McNair Conference, Lake Geneva WI

Awards and Scholarships

- Advanced Opportunity Program (AOP) Fellow
- 3 Minute Thesis Finalist Link to video of talk: `uwm.edu/graduateschool/3mt/`
- Teaching Recognition at University of Connecticut
- Ernst Schwandt Teaching Award (UW-Milwaukee)
- NSF Compass Scholar, UW-Milwaukee
- Dean's List
- Marden Award in Mathematics (UW-Milwaukee)

  Awarded for research on Stirling's Formula

Memberships

- Association for Women in Mathematics - Co-founder of UWM Chapter
- American Mathematical Society
- National Honor Society of Phi Beta Kappa
- Golden Key International Honour Society
- Honor Society of Phi Kappa Phi
- UW-Milwaukee Astronomy Club

Extracurricular Activities

- University Community Orchestra (Violist)
- William Lowell Putnam Contest