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Asymptotic Expansion of the L^2 Norms of the Solutions to the Heat and Dissipative Wave Equations on the Heisenberg Group

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ASYMPTOTIC EXPANSION OF THE L^2-NORMS OF THE SOLUTIONS TO THE HEAT AND DISSIPATIVE WAVE EQUATIONS ON THE HEISENBERG GROUP

by

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ABSTRACT

ASYMPTOTIC EXPANSION OF THE L^2-NORMS OF THE SOLUTIONS TO THE HEAT AND DISSIPATIVE WAVE EQUATIONS ON THE HEISENBERG GROUP

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Motivated by the recent work on asymptotic expansions of heat and dissipative wave equations on the Euclidean space, and the resurgent interests in Heisenberg groups, this dissertation is devoted to the asymptotic expansions of heat and dissipative wave equations on Heisenberg groups. The Heisenberg group, $\mathbb{H}^n$, is the $\mathbb{R}^{2n+1}$ manifold endowed with the law

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + \frac{1}{2}(xy' - x'y)),$$

where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Let $v(t, z)$ and $u(t, z)$ be solutions of the heat equation, $v_t - \mathcal{L}v = 0$, and dissipative wave equation, $u_{tt} + u_t - \mathcal{L}u = 0$, over the Heisenberg group respectively, where $\mathcal{L}$ is the sub-Laplacian. To overcome the Heisenberg group setting, we first establish the Group Fourier transform for an integrable function on the space. The Fourier transform together with the Plancherel formula, help us to obtain the following expansions for $\|u(t, z)\|_{L^2(\mathbb{H})}$ and $\|v(t, z)\|_{L^2(\mathbb{H})}$ as $t \to \infty$,

$$\|u(t, \cdot)\|_{L^2(\mathbb{H})} \sim \sum_{n=0}^{N-1} b_n t^{-n-2} + O(t^{-N-2}), \qquad \|v(t, \cdot)\|_{L^2(\mathbb{H})} \sim \sum_{n=0}^{N-1} c_n t^{-n-2} + O(t^{-N-1}),$$

where b_n and c_n only depend on the initial conditions.

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1 Introduction

The heat equation describes the evolution in time of the density of a quantity such as heat. This equation is fundamental to the field of partial differential equations and has applications in many fields of mathematics including spectral geometry, differential geometry, and even probability theory. Understanding the heat equation on different spaces, such as the Heisenberg group, could prove to be useful in other fields of mathematics or applications. First, let's recall the heat equation on the Euclidean space. Let $u(t,x)$ be the weak solution of the heat equation

$$u_t - \Delta u = 0, t \geq 0, x \in \mathbb{R}^{\mathbb{N}},$$

with initial condition

$$u(0,\cdot) = u_0 \in L^2(\mathbb{R}^N).$$

Dr. Volkmer has a paper[4] that finds the asymptotic expansion of the squared L^2-norm, $\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} |u(t,x)|^2 dx$, as $t \to \infty$ in the Euclidean case. Similarly, he also finds an asymptotic expansion of $\|v(t,\cdot)\|^2_{L^2(\mathbb{R}^N)}$, where $v(t,x)$ is the solution to the dissipative wave equation

$$v_{tt} + v_t - \Delta v = 0, \quad t \geq 0, x \in \mathbb{R}^N,$$

with initial conditions

$$v_t(0,\cdot) = v_0 \in L^2(\mathbb{R}^N), \quad v(0,\cdot) = v_1 \in L^2(\mathbb{R}^N).$$

The main techniques used in Volkmer's paper involve the Fourier transform's relation with differentiation and Taylor series expansions. He goes further to find the asymptotic expansion of the difference of these solutions. Given suitable conditions on the inital values, this result leads to the cancellation of the leading terms in the expansion. This explains the diffusion phenomenon for linear hyperbolic waves.

The Heisenberg group has an important role in quantam mechanics. The representation of the Heisenberg group acting on $L^2(\mathbb{R}^N)$, known as the Schrödinger representation, is the action of the exponentiated position and momentum operators. One of the main results arriving from this is the Stone-Von Nuemann Theorem. This states that the Schrödinger representation is the unique (up to unitary equivalence) strongly continuous unitary representation of the Heisenberg group with non-trivial center.

In this thesis, we will be focusing on how the Heisenberg group acts on the space of square integrable functions, in particular the solutions to the heat and dissipative wave equations. We ask if we can find asymptotic expansions of these equations on the Heisenberg group with the sub-Laplacian, which we will define later. We are interested in if the behavior of these equations are consistent with the Euclidean case. In particular, will they still have the same power in their leading terms? If so, would we still have cancelation of the leading terms when considering the difference?

The main obstacles to overcome in this setting while considering these questions is its adaption of the Fourier transform and Plancherel formula. The Fourier transform in the Euclidean case converts differentiation in the space variable to scalar multiplication. This property can convert partial differential equations into ordinary differential equations that are much simpler to solve. While the adaption of the Fourier transform on the Heisenberg group doesn't convert differentiation to something as nice as scalar multiplication, it does convert the sub-Laplacian in our equations into the Harmonic oscillator. This operator has a useful eigenfunction system. In the Euclidean case, the Plancherel formula shows that the L^2-norm of a function is equal to the L^2-norm of its Fourier transform. While the Plancherel formula on the Heisenberg group doesn't give equality, it does give a relationship with the Hilbert-Schmidt norm of the Fourier transform operator. This allows us to take advantage of the eigenfunction system of the Harmonic oscillator. While we don't get to consider the cancelation of the leading terms in this thesis, we do confirm that the leading terms have the same power and is consistent with the results in the Euclidean case.

In what follows, we will state the organization of this thesis and summarize the main results. In Chapter 2 we introduce the Heisenberg group denoted by $\mathbb{H}^n$. That is the $\mathbb{R}^{2n+1}$ manifold endowed with the law

$$(x, y, s)\dot{(}x', y', s') = (x + x', y + y', s + s' + \frac{1}{2}(xy' - x'y)).$$

While we explore properties of the Heisenberg group in $2n + 1$ dimensions in the first section, we restrict ourselves to 3 dimensions for the rest of the thesis for simplicity. While the details still need to be shown, a similar proof will work for higher dimensions. In this thesis we mention the definition of the group Fourier transform on the Heisenberg group that appears in most texts using the Schrödinger representation. However, it is more beneficial for us to use a slightly different definition using a kernel mentioned later in the thesis. It can be shown with some clever substitutions that these definitions are equivalent[1]. With some restrictions on $f \in L^2(\mathbb{H})$, we can determine the expansion of $\|T_\lambda^* h_k\|^2$ as $\lambda \to 0$, where $\{T_\lambda\}_{\lambda \in \mathbb{N}\setminus\{0\}}$ is the group Fourier transform defined by f and h_k is the kth hermitian polynomial. This becomes essential to our work later when we use the Plancherel formula.

In Chapter 3 we let $u(t, z)$ be the solution to the heat equation on the Heisenberg group; namely, u satisfies

$$\partial_t u(t, z) - \mathcal{L}u(t, z) = 0$$

for all $t > 0$, $z \in \mathbb{H}$, $\mathcal{L}$ is the sub-Laplacian, and with the initial condition

$$u(0, z) = u_0(z).$$

The group Fourier transform converts the differentiation given in the original equation using properties analogous to the Euclidean case discussed in Chapter 2. This yields an ordinary differential equation that we are able to solve. Afterwards, we use the modified Plancherel formula to evaluate the norm.

The first section of that chapter is used to determine the asymptotic equivalence of the solution. We take advantage of our definition of the group Fourier transform involving the kernel and use an approximate identity to deduce a necessary convergence. By the virtue of Watson's lemma, this leads to the asymptotic equivalence

$$\lim_{t\to\infty} \|u(t,\cdot)\|^2_{L^2(\mathbb{H})} \sim \frac{|Q|}{8t} \qquad \text{as } t \to \infty,$$

where $Q = \int_{\mathbb{H}} u_0(x,y,s)dxdyds$. The second section is used to determine the asymptotic expansion of the solution,

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{H})} = \sum_{n=0}^{N-1} b_n t^{-n-2} + O(t^{-N-2}) \qquad \text{as } t \to \infty,$$

where b_n is dependent only on the initial condition for all nonegative integers n satisfying $n \le N-1$. This proof mainly consists of summing the expansions found in the first section over all the hermitian polynomials, which give an important orthogonal basis of $L^2(\mathbb{R})$, to determine the entire expansion.

In Chapter 4 we let $u(t,z)$ be the solution to the dissipative wave equation,

$$\partial_t^2 u(t,z) + \partial_t u(t,z) = \mathcal{L}u(t,z)$$

with initial conditions

$$u(0,x,y,s) = u_0(x,y,s), \quad \partial_t u(0,x,y,s) = u_1(x,y,s),$$

where $u_0, u_1 \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$. Again, we solve by first using the group Fourier transform to convert the equation to an ordinary differential equation.

The first section covers the asymptotic equivalence. We use a similar strategy as the solution to the heat equation. Utilizing the convergency shown in Chapter 3 and Watson's lemma, we are able to show the asymptotic equivalence,

$$\lim_{t\to\infty} \|u(t,\cdot)\|^2_{L^2(\mathbb{H})} \sim \frac{|Q|}{8t} \qquad \text{as } t \to \infty,$$

where $Q = \int\limits_{\mathbb{H}} (u_0(x,y,s) + u_1(x,y,s))dxdyds$. The second section of this chapter is used to determine the asymptotic expansion,

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{H})} = \sum_{n=0}^{N-1} c_n t^{-n-2} + O(t^{-N-2}) \qquad \text{as } t \to \infty,$$

where c_n is only dependent on the initial conditions for all non negative integers n such that $n \leq N-1$.

2 The Heisenberg Group

The Heisenberg group, $\mathbb{H}^n$, is the $\mathbb{R}^{2n+1}$ manifold endowed with the law:

$$(x,y,s)\dot{(}x',y',s') = (x+x', y+y', s+s' + \frac{1}{2}(xy' - x'y))$$

where $x, y, x', y' \in \mathbb{R}^n$ and $s, s' \in \mathbb{R}$. Given two elements, $x, y \in \mathbb{R}^n$, xy refers to the standard scalar product defined:

$$xy = \sum_{i=1}^{n} x_i y_i \quad \text{where} \quad x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n).$$

Dilation in this group is defined by

$$\lambda(x,y,s) = (\lambda x, \lambda y, \lambda^2 s), \quad \lambda \in \mathbb{R}.$$

We define dilation this way because we want it to be a homomorphism. If we dilate x and y linearly, then we have to dilate s quadratically. This is because we have a product in the definition of the operation.

Although we will not use it in this thesis, this group is commonly considered the matrix group with 1's along the diagonal. The x vector in the first row and the y vector in the last column. The s variable is the entry in the first row, last column. All other entries are 0.

2.1 Heisenberg Lie Algebra

The canonical basis of the Lie Algebra on $\mathbb{H}^n$ is given by the left-invariant vector fields:

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \qquad S = \partial_s,$$

where the canonical commutation relations are:

$$[X_j, Y_j] = S, \quad j = 0, 1, 2, ..., n,$$

$$[X_i, Y_j] = [X_i, S] = [Y_i, S] = 0, \quad i, j \in \mathbb{Z}, i \neq j.$$

Because of nice properties of the Lie algebra's structure, we can define integration over H^n in the natural way[1]:

$$\int_{\mathbb{H}^n} ...dxdyds = \int_{\mathbb{R}^{2n+1}} ...dxdyds = \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ...dxdyds.$$

The sub-Laplacian on the Heisenberg group $\mathbb{H}^n$, $\mathcal{L}$, is given by

$$\mathcal{L} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \sum_{j=1}^{n}\Big((\partial_{x_j} - \frac{y_j}{2}\partial_s)^2 + (\partial_{y_j} + \frac{x_j}{2}\partial_s)^2\Big).$$

2.2 Group Fourier Transform

In this section we will define the Group Fourier transform in a way that is most commonly seen. We will introduce another definition in the other sections. The definitions are equivalent as proved by Ruzhansky (2016).

We define the Schödinger representations, π_λ, as the representation of the group $\mathbb{H}^n$ acting on $L^2(\mathbb{R}^n)$ as

$$\pi_\lambda(x,y,s)h(w) := e^{i\lambda(s+\frac{1}{2}xy)}e^{i\sqrt{\lambda}yu}h(w+\sqrt{|\lambda|}x),$$

where $(x,y,s) \in \mathbb{H}^n$, $h \in L^2(\mathbb{R}^n)$, and $\sqrt{\lambda} = \operatorname{sgn}(\lambda)\sqrt{|\lambda|}$. The representation of π_λ acts on the canonical basis in the following way

$$\pi_\lambda(X_j) = \sqrt{|\lambda|}\partial_{w_j}, \quad \pi_\lambda(Y_j) = i\sqrt{\lambda}w_j, \quad \text{and} \quad \pi_\lambda(S) = i\lambda I.$$

We also note that

$$\pi^*_\lambda(x,y,s)h(w) = \pi_\lambda(-x,-y,-s)h(w) = e^{i\lambda(-s+\frac{1}{2}xy)}e^{-i\sqrt{\lambda}yw}h(w-\sqrt{|\lambda|}x).$$

We define the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ at π_λ as

$$T_\lambda h(u) = \mathcal{F}_{\mathbb{H}^n}(f)(\pi_\lambda)h(w) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^{2n+1}} f(x,y,s)\pi^*_\lambda(x,y,s)h(w-x)dxdyds.$$

T_λ is a Hilbert-Schmidt operator, that is, a bounded operator with finite Hilbert-Schmidt norm. That means given an orthonormal basis, $\{e_i\}_{i\in I}$ for $L^2(\mathbb{R}^n)$ with some indexing set I, we have

$$\|T_\lambda\|^2_{HS[L^2(\mathbb{H}^n)]} = \sum_{i\in I}\|T_\lambda e_i\|^2_{L^2(\mathbb{R}^n)}.$$

This gives us the Plancherel formula on the Heisenberg group as

$$\|f\|^2_{L^2(\mathbb{H}^n)} = \int\limits_{\lambda\in\mathbb{R}^*} |\lambda| \|T_\lambda\|^2_{HS[L^2(\mathbb{H}^n)]} d\lambda.$$

We have the following properties with the Lie algebra and group Fourier transform

$$T_\lambda(X_j) = \sqrt{|\lambda|}\partial_{w_j} T_\lambda, \quad T_\lambda(Y_j) = i\sqrt{\lambda} w_j T_\lambda, \quad T_\lambda(S) = i\lambda T_\lambda.$$

So, the group Fourier transform of the sub-Laplacian is

$$T_\lambda(\mathcal{L}) = |\lambda| \sum_{j=1}^{n} (\partial^2_{w_j} - w_j^2) = -|\lambda| H_u,$$

where $H_w := \sum_{j=1}^{n} (\partial^2_{w_j} - w_j^2)$ is the harmonic oscillator acting on $L^2(\mathbb{R}^n)$. It is important to note this because the harmonic oscillator is self-adjoint in $L^2(\mathbb{R}^n)$ and has a convenient system of eigenfunctions that we will take advantage of.

The eigenfunctions of H_u, $\{h_k\}_{k=1}^\infty$ form an orthonormal basis of $L^2(\mathbb{R}^n)$. The eigenfunctions are defined as [1]

$$h_k(w) := \prod_{j=1}^{n} P_{k_j} e^{-\frac{|w|^2}{2}},$$

where $P_m(\cdot)$ is the m-th order Hermite polynomial defined as

$$P_m(t) = c_m e^{\frac{|t|^2}{2}} (t - \frac{d}{dt})^m e^{-\frac{|t|^2}{2}}, \qquad t > 0, c_m = 2^{-\frac{m}{2}} (m!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}}.$$

The corresponding eigenvalues for each h_k are of the form

$$\sum_{j=1}^{n} (2k_j + 1), \qquad k = (k_1, k_2, ...) \in \mathbb{N}^n.$$

2.3 Norm of the Fourier Transform Operator

For the rest of this thesis, we will be working on $\mathbb{H}^1$ for simplicity. Let $f \in L^2(\mathbb{H})$, and let h_k for $k \in \mathbb{N}$ be defined as in the previous section but for one dimension. Our goal in this section is to find the asymptotic expansion of $\|T_\lambda^* h_k\|^2$ as $\lambda \to 0$. Let N be a fixed nonnegative integer and $k \in \mathbb{N} \setminus \{0\}$. We will define T_λ slightly differently in this section, but it can be shown that the definitions are equivalent[1]. Let p, q, and r be nonnegative integers satisfying $\max(p,q) + 2r \leq 2N$ and assume that

$$\int_{\mathbb{H}} |x|^p |y|^q |s|^r |f(x,y,s)| dxdyds < \infty.$$

Then we can introduce the following moments

$$M_{p,q,r} = \frac{1}{2\pi} \int_{\mathbb{H}} x^p y^q s^r f(x,y,s) dxdyds \qquad \text{if } \max(p,q) + 2r \leq 2N. \tag{2.1}$$

Define

$$F(x,\eta,\sigma) := \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y,s) e^{-i(y\eta + s\sigma)} dyds,$$

and

$$(T_\lambda h)(u) := \int_{\mathbb{R}} K_\lambda(u,v) h(v) dv$$

with the Hilbert-Schmidt kernel

$$K_\lambda(u,v) := \epsilon^{-1} F(\epsilon^{-1}(u-v), \operatorname{sgn}(\lambda)\frac{\epsilon}{2}(u+v), \lambda), \qquad \epsilon := \sqrt{|\lambda|}.$$

Note that the family of operators $\{T_\lambda\}$ is the group Fourier transform of f[1]. Let

$$\tilde{f}(x,y,s) := f(x,-y,-s),$$

and let $\tilde{T}_\lambda$ be the operators associated with $\tilde{f}$. Then

$$T_{-\lambda} = \tilde{T}_\lambda.$$

This shows that the case where $\lambda < 0$ can be reduced to $\lambda > 0$. Therefore, until Theorem 2.5, we will assume that $\lambda > 0$. Then

$$\overline{T_\lambda^* h_k(v)} = \frac{1}{\epsilon}\int F(\frac{u-v}{\epsilon}, \frac{\epsilon}{2}(u+v), \epsilon^2) h_k(u) du \tag{2.2}$$
$$= \int F(x, \epsilon(v + \frac{1}{2}\epsilon x), \epsilon^2) h_k(v+\epsilon x) dx. \tag{2.3}$$

Lemma 2.1. *For every $x, \eta, \sigma \in \mathbb{R}$, we have the Taylor expansion*

$$F(x,\eta,\sigma) = \sum_{q+2r<2N} \partial_\eta^q \partial_\sigma^r F(x,0,0) \frac{\eta^q}{q!}\frac{\sigma^r}{r!} + \sum_{q+2r=2N} Q_{q,r}(x,\eta,\sigma), \tag{2.4}$$

where

$$|Q_{q,r}(x,\eta,\sigma)| \le \frac{1}{2\pi}\int\int |y|^q |s|^r |f(x,y,s)| dyds \frac{|\eta|^q}{q!}\frac{|\sigma|^r}{r!}.$$

Proof. By assumption (2.1), the partial derivatives $\partial_\eta^q \partial_\sigma^r F$ exist for $q + 2r \le 2N$, and

$$|\partial_\sigma^q \partial_\sigma^r F(x,\eta,\sigma)| \le \frac{1}{2\pi}\int\int |y|^q |s|^r |f(x,y,s)| dyds.$$

In particular,

$$|\partial_\eta^q \partial_\sigma^r F(x,\eta,\sigma)| \le \frac{1}{2\pi}\int\int |y|^q |s|^r |f(x,y,s)| dyds.$$

First, we use the Taylor expansion of $\sigma \to F(x,\eta,\sigma)$ at $\sigma = 0$ with x, η fixed:

$$F(x,\eta,\sigma) = \sum_{r=0}^{N-1} \partial_\sigma^r F(x,\eta,0)\frac{\sigma^r}{r!} + Q_{0,N}(x,\eta,\sigma)$$

with

$$|Q_{0,N}(x,\eta,\sigma)| \le \frac{1}{2\pi}\int\int |s|^N |f(x,y,s)| dyds \frac{|\sigma|^N}{N!}. \tag{2.5}$$

Then we use the Taylor expansion of $\eta \to \partial_\sigma^r F(x,\eta,0)$ at $\eta = 0$. □

Substituting (2.4) into (2.3) we obtain

$$\overline{T_\lambda^* h_k(v)} = E(v,\lambda) + R_{k1}(v,\lambda), \tag{2.6}$$

where

$$E(v,\lambda) = \int \sum_{q+2r<2N} \partial_\eta^q \partial_\lambda^r F(x,0,0) \frac{\epsilon^q (v+\frac{1}{2}\epsilon x)^q}{q!} \frac{\epsilon^{2r}}{r!} h_k(v+\epsilon x) dx,$$

$$R_{k1}(v,\lambda) = \int \sum_{q+2r=2N} Q_{q,r}(x, \epsilon(v+\frac{1}{2}\epsilon x), \epsilon^2) h_k(v+\epsilon x) dx.$$

Lemma 2.2. *There is a constant C_{k1} (independent of λ) such that*

$$\|R_{k1}(\cdot,\lambda)\| \le C_{k1}\lambda^N \qquad \text{for } 0<\lambda<1.$$

Proof. Using (2.5) we have to show that the L^2-norm of the function

$$v \to \int\int\int |2v+\epsilon x|^q |y|^q |s|^r |f(x,y,s)||h_k(v+\epsilon x)| dxdyds$$

is bounded above as a function of $\lambda \in (0,1]$ where $q+2r = 2N$. We write $2v+\epsilon x = 2(v+\epsilon x) - \epsilon x$ and use the binomial formula. Then we see that it is sufficient to bound the

L^2-norm of the function

$$D(v,\lambda) = \int\int\int |x|^{q-i}|y|^q|s|^r|f(x,y,s)||v+\epsilon x|^i|h_k(v+\epsilon x)|dxdyds$$
$$= \int g(x)|v+\epsilon x|^i|h_k(v+\epsilon x)|dx,$$

where

$$g(x) = |x|^{q-i}\int\int |y|^q|s|^r|f(x,y,s)|dyds$$

and $i = 0, 1, ..., q$. By the Cauchy-Schwarz inequality,

$$\int |D(v,\lambda)|^2 dv \leq \Big(\int g(x)dx\Big)^2 \int |v|^{2i}|h_k(v)|^2 dv.$$

Since g is integrable by (2.1) and $v^{2i}h_k(v)^2$ is also integrable, this completes the proof. □

In the formula for E we substitute

$$(v+\frac{1}{2}\epsilon x)^q = \sum_{i=0}^{q}\binom{q}{i}\epsilon^i x^i 2^{-i} v^{q-i}.$$

Then E becomes a finite triple sum with indicies i, q, and r. The terms where $i+q+2r > 2N$ are small enough to be put in the remainder term which we will define below. In terms with $i+q+2r < 2N$, we use the Taylor expansion

$$h_k(v+\epsilon x) = \sum_{j=0}^{n-1}\frac{1}{j!}h_k^{(j)}(v)\epsilon^j x^j + L_{k,n}(x,v,\lambda)$$

with the remainder term in integral form

$$L_{k,n}(x,v,\lambda) = \frac{\epsilon^n x^n}{(n-1)!} \int_0^1 h_k^{(n)}(v+t\epsilon x)(1-t)^{n-1}dt, \tag{2.7}$$

where $n = 2N - i - q - 2r$. In this way we obtain

$$E(v,\lambda) = E_0(v,\lambda) + R_{k2}(v,\lambda) + R_{k3}(v,\lambda), \tag{2.8}$$

where each term is defined in the following way where i runs from 0 to q.

$$E_0(v,\lambda) = \sum_{i+j+q+2r<2N} \epsilon^{i+j+q+2r} M_{i+j,q,r}(-i)^{q+r} \frac{2^{-i}}{q!r!j!}\binom{q}{i} v^{q-i} h_k^{(j)}(v),$$
$$R_{k2}(v,\lambda) = \int \sum_{q+2r<2N\leq i+q+2r} \epsilon^{i+q+2r}\partial_\eta^q\partial_\lambda^r F(x,0,0)\frac{2^{-i}}{q!r!}\binom{q}{i} h_k(v+\epsilon x)dx,$$
$$R_{k3}(v,\lambda) = \int \sum_{i+q+2r<2N} \epsilon^{i+q+2r}\partial_\eta^q\partial_\lambda^r F(x,0,0)\frac{2^{-i}}{q!r!}\binom{q}{i} L_{k,n}(x,v,\lambda)dx.$$

Lemma 2.3. *There are constants C_{k2}, C_{k3} (independent of λ) such that*

$$\|R_{kj}(\cdot,\lambda)\| \leq C_{kj}\lambda^N \qquad \text{for } 0<\lambda\leq 1, j=2,3.$$

Proof. The estimate for the L^2-norm is shown using the same method as in Lemma 2.2, so it is omitted. Using (2.7), we can write

$$R_{k3}(v,\lambda) = \int_0^1 S(v,\lambda,t)dt,$$

where S is defined the same as R_{k3} except the integral in the definition of $L_{k,n}$ is not included. By the same method as in Lemma 2.2, we obtain

$$\int |S(v,\lambda,t)|^2 dv \leq C^2 \lambda^{2N}$$

with C independent of $t \in [0,1]$ and $\lambda \in (0,1]$. Then, by the Cauchy-Schwarz inequality,

$$\int |R_{k3}(v,\lambda)|^2 dv \leq \int_0^1 \int |S(v,\lambda,t)|^2 dv dt \leq C^2 \lambda^{2N}$$

which completes the proof. □

Theorem 2.4. *We have*

$$\overline{T_\lambda^* h_k(v)} = \sum_{m=0}^{2N-1} a_{k,m}(v)\epsilon^m + R_{k4}(v,\lambda), \tag{2.9}$$

where

$$a_{k,m}(v) = \sum_{p+q+2r=m} (-i)^{q+r} \frac{1}{r!} M_{p,q,r}(H_{p,q}h_k)(v),$$

and $H_{p,q}$ is the differential operator

$$(H_{p,q}h)(v) = \sum_{i=0}^{\min(p,q)} \frac{1}{2^i i!} \frac{1}{(p-i)!(q-i)!} v^{q-i} h^{(p-i)}(v).$$

We have

$$\|R_{k4}(\cdot,\lambda)\| \leq C_{k4}\lambda^N \qquad \text{for } 0 < \lambda \leq 1,$$

where $C_{k4} := C_{k1} + C_{k2} + C_{k3}$.

Proof. This follows from (2.6), (2.8), lemma 2.2, and lemma 2.3. □

Theorem 2.5. *We have*

$$\|T_\lambda^* h_k\|^2 = \sum_{n=0}^{2N-1} A_{k,n}\epsilon^n + R_{k5}(\lambda) \tag{2.10}$$

where

$$A_{k,n} = \sum_{m=0}^{n} \langle a_{k,m}, a_{k,n-m} \rangle,$$

and there is a constant C_{k5} *such that*

$$|R_{k5}(\lambda)| \le C_{k5}\lambda^N \qquad \textit{for } 0 < \lambda \le 1.$$

Proof. This follows immediately from Theorem 2.4. □

Since $a_{k,m}(-v) = (-1)^{k+m} a_{k,m}(v)$, we have $A_{k,n} = 0$ for odd n. By using (2.2) we have that Theorem 2.5 is also true for negative λ when we set

$$A_{k,n} = \sum (-i\mathrm{sgn})\lambda)^{q_1+r_1-q_2-r_2} \frac{M_{p_1,q_1,r_1}\overline{M}_{p_2,q_2,r_2}}{r_1! r_2!} \langle H_{p_1,q_1} h_k, H_{p_2,q_2} h_k \rangle, \tag{2.11}$$

where the sum is taken over all $p_1, q_1, r_1, p_2, q_2, r_2 \in \mathbb{N}_0$ that satisfy

$$p_1 + q_1 + 2r_1 + p_2 + q_2 + 2r_2 = n.$$

Lemma 2.6. *For every* $n = 0, 1, ..., 2N-1$, $A_{k,n} = O(k^{\frac{n}{2}})$ *as* $k \to \infty$. *Moreover, the constants* $C_{kj}, j = 1, 2, 3, 4, 5$ *can be chosen such that* $C_{kj} = O(k^N)$ *as* $k \to \infty$.

Proof. We apply the recursion formulas

$$h_k'(v) = \sqrt{\frac{k}{2}} h_{k-1}(v) - \sqrt{\frac{k+1}{2}} h_{k+1}(v),$$
$$v h_k(v) = \sqrt{\frac{k}{2}} h_{k-1}(v) + \sqrt{\frac{k+1}{2}} h_{k+1}(v).$$

It follows that

$$\int |v|^{2i} |h_k^{(j)}(v)|^2 dv = O(k^{i+j}) \qquad \text{as } k \to \infty.$$

Since every R_{kj} is the product of ϵ to some power, constants, and the integral above where $i + j \le N$, then this implies the statement of the lemma. □

3 Heat Equation

Consider the heat equation

$$\partial_t u(t,z) - \mathcal{L}u(t,z) = 0.$$

for all $t > 0$ and $z \in \mathbb{H}$ and with the initial condition

$$u(0,z) = u_0(z).$$

With the assumption that $u_0 \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$.

We will accomplish two goals in this chapter. We want to first find the behavior of the function $t \to \|u_t\|_{L^2(\mathbb{H})}$ as $t \to \infty$, where u is the solution of the heat equation. This also will give us the leading coefficient of the asymptotic expansion. The next goal will be to find the assymptotic expansion of the function. In order for us to accomplish that, however, we will need to make additional assumptions on our initial condition.

3.1 Asymptotic equivalence

Let the family of operators $\{T_\lambda\}$ be the group Fourier transform of u. First, we will find the behavior of the function $t \to \|u(t)\|_{L^2(\mathbb{H})}$ as $t \to \infty$, where u is the solution of the heat equation. We take the heat equation and apply the Fourier transform. This gives us

$$\begin{cases} \partial_t \hat{u}(t,\lambda) + \sigma_{\mathcal{L}} \hat{u}(t,\lambda) = 0, \\ \hat{u}(0,\lambda) = \hat{u}_0(\lambda). \end{cases}$$

Where $\sigma_{\mathcal{L}}(\lambda)$ is the symbol of $-\mathcal{L}$. This takes the form

$$\sigma_{\mathcal{L}}(\lambda) = |\lambda| H_w = |\lambda|(-\partial_w^2 + w^2)$$

Where $H_w = (-\partial_w^2 + u^2)$ is the harmonic oscillator acting on $L^2(\mathbb{R})$. Since the harmonic oscillator is self-adjoint in $L^2(\mathbb{R})$ and its system of eigenfunctions, $\{h_k\}_{k=1}^{\infty}$ is a basis in $L^2(\mathbb{R})$, we have an ordered set of positive numbers $\{\mu_k\}_{k=1}^{\infty}$ such that

$$H_w h_k(w) = \mu_k h_k(w).$$

The eigenfunctions are the Hermitian functions [3]

$$h_k(w) := (2^k k! \sqrt{\pi})^{-\frac{1}{2}} H_k(w) e^{-\frac{w^2}{2}},$$

where H_k is the kth Hermite polynomial defined as

$$H_k(w) := (-1)^n e^{w^2} \frac{d^n}{dx^n} e^{-w^2}.$$

The corresponding eigenvalues, $\mu_k = 2k+1$. For $(k,l) \in \mathbb{N} \times \mathbb{N}$, denote

$$\hat{u}_{k,l}(t,\lambda) = \langle \hat{u}(t,\lambda) h_l, h_k \rangle_{L^2(\mathbb{R})} = \langle T_\lambda h_l, h_k \rangle_{L^2(\mathbb{R})},$$

where T_λ is the group Fourier transform defined by the solution to the heat equation. Using this system of eigenvalues and eigenfunctions, this reduces our equation to

$$\begin{cases} \partial_t \hat{u}_{k,l}(t,\lambda) + |\lambda| \mu_k \hat{u}_{k,l}(t,\lambda) = 0, \\ \hat{u}_{k,l}(0,\lambda) = \hat{u}_{0,k,l}(\lambda). \end{cases}$$

Fixing $\lambda \in \mathbb{R}^*$ and $(k,l) \in \mathbb{N} \times \mathbb{N}$, we now solve the first order ordinary differential equation. This gives us the solution

$$\hat{u}_{k,l}(t,\lambda) = \hat{u}_0(\lambda)_{k,l} e^{-(2k+1)|\lambda| t}.$$

Before we state and prove our main theorem, we need two lemmas.

Lemma 3.1. *Let $g \in L^1(\mathbb{R})$ with $\int g(x)dx = 1$. For $\epsilon > 0$ define the convolution operator*

$$(S_\epsilon h)(y) := \int g_\epsilon(x)h(y-x)dx, \qquad g_\epsilon(x) := \frac{1}{\epsilon}g(\frac{x}{\epsilon}).$$

Then $S_\epsilon : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator with operator norm at most $M := \int |g(x)|dx$, and $S_\epsilon \to I$ strongly as $\epsilon \to 0$.

Proof. To prove the bound on the operator norm, we define $K : \mathbb{R}^2 \to \mathbb{R}$ as

$$K(x,y) := g_\epsilon(y-x).$$

Now we use the Schur test with

$$C_1 := \text{esssup}_{x\in\mathbb{R}} \int |K(x,y)|dy = M < \infty$$
$$C_2 := \text{esssup}_{y\in\mathbb{R}} \int |K(x,y)|dx = M < \infty$$

Therefore, the integral operator T defined by

$$(Th)(y) = \int K(x,y)h(x)dx$$

is a bounded linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and its operator norm satisfies $\|T\| \le (C_1C_2)^{\frac{1}{2}} = M$.

Now we have

$$\begin{aligned}(Th)(y) &= \int K(x,y)h(x)dx \\ &= \int g_\epsilon(y-x)h(x)dx \\ &= \int g_\epsilon(x)h(y-x)dx \\ &= (S_\epsilon h)(y).\end{aligned}$$

Therefore, $S_\epsilon : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator with operator norm at most $M := \int |g(x)|dx$.

Now we need to prove the convergence. Note

$$(S_\epsilon h)(y) - h(y) = \int g_\epsilon(x)(h(y-x) - h(y))dx.$$

This implies

$$\begin{aligned}\|S_\epsilon h - h\|^2_{L^2(\mathbb{R})} &\leq M \int\int |g_\epsilon(x)||h(y-x) - h(y)|^2 dxdy \\ &= M \int |g(w)|\Big(\int |h(y-\epsilon w) - h(y)|^2 dy\Big)dw.\end{aligned}$$

We know that $\int |h(y-\epsilon w) - h(y)|^2 dy \to 0$ as $\epsilon \to 0$. By Lebesgue's Dominated Convergence theorem, $\|S_\epsilon h - h\|_{L^2(\mathbb{R})} \to 0$ as $\epsilon \to 0$. □

Lemma 3.2. *Let $T_\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be an element of the group Fourier transform of an* *arbirtrary $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$. Let the constants Q and M be defined by*

$$Q := \int_{\mathbb{H}} f(x,y,s)dxdyds$$
$$M := \frac{1}{2\pi}\int_{\mathbb{H}} |f(x,y,s)|dxdyds.$$

Then the operator norm of T_λ is at most M, $T_\lambda \to \frac{Q}{2\pi}I$ strongly, and $T_\lambda^* \to \frac{\overline{Q}}{2\pi}I$ strongly as $\lambda \to 0$.

Proof. Define F and K_λ in the same way as the in the definition of the group Fourier transform of f. Then note that

$$\int |K_\lambda(u,v)|dv \le M, \qquad \int |K_\lambda(u,v)|du \le M.$$

By the Schur test, the operator norm T_λ is at most M. If we change $f(x,y,s)$ to $\overline{f}(-x,-y,-s)$, then we change T_λ to T_λ^*. Therefore, we only need to show that $T_\lambda \to \frac{Q}{2\pi}I$ strongly as $\lambda \to 0$. Since f is arbitrary, for simplicity we assume $Q = 2\pi$. For $\lambda \in \mathbb{R}^*$ we define the operator $S_\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as in lemma 3.1 by

$$(S_\lambda h)(y) := \int g_\epsilon(x)h(y-x)dx, \qquad g(x) := F(x,0,0), \qquad \epsilon := \sqrt{|\lambda|}.$$

By lemma 3.1, $S_\lambda \to I$ strongly as $\lambda \to 0$. Now we will show that $T_\lambda - S_\lambda \to 0$ strongly as $\lambda \to 0$. Let $h \in L^2(\mathbb{R})$. Recall that T_λ is the integral operator with kernel

$$K_\lambda(u,v) := \frac{1}{\epsilon}F(\frac{1}{\epsilon}(u-v), \operatorname{sgn}(\lambda)\frac{\epsilon}{2}(u+v), \lambda)$$

while S_λ is the integral operator with kernel

$$L_\lambda(u,v) = \frac{1}{\epsilon}F(\frac{u-v}{\epsilon}, 0, 0).$$

Therefore,

$$\|(T_\lambda - S_\lambda)h\|_{L^2(\mathbb{R})}^2 \le 2M \int\int |K_\lambda(u,v) - L_\lambda(u,v)|du|h(v)|^2 dv.$$

Since we have

$$|F(x,\eta,\lambda)| \le \frac{1}{2\pi}\int |f(x,y,s)|dyds$$

and that $(\eta,\lambda) \mapsto F(z,\eta,\lambda)$ is continuous, by Lebesgue's dominated convergence theorem,

$$\int |K_\lambda(u,v) - L_\lambda(u,v)|du \to 0$$

as $\epsilon \to 0$. Using another application of Lebesgue's dominated convergence, we have

$$2M\int\int |K_\lambda(u,v) - L_\lambda(u,v)|du|h(v)|^2 dv \to 0$$

as $\epsilon \to 0$. □

Theorem 3.3. *We have*

$$\|u(t)\|_{L^2(\mathbb{H})} \sim \frac{|Q|}{8t} \qquad \text{as } t \to \infty,$$

where

$$Q := \int_{\mathbb{H}} u_0(x,y,s)dxdyds.$$

Proof. Using the Plancherel formula on the Heisenberg group, we have

$$\begin{aligned}
\|u(t)\|^2_{L^2(\mathbb{H})} &= \int |\lambda| \sum_{k,l=0}^{\infty} |\hat{u}_{k,l}(t,\lambda)|^2 d\lambda \\
&= \int |\lambda| \sum_{k,l=0}^{\infty} |\hat{u}_{k,l}(\lambda)|^2 e^{-2(2k+1)|\lambda| t} d\lambda \\
&= \int |\lambda| \sum_{k=0}^{\infty} \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda| t} d\lambda.
\end{aligned}$$

Interchange the sum and integral and we have

$$\|u(t)\|^2_{L^2(\mathbb{H})} = \sum_{k=0}^{\infty} \int |\lambda| \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda| t} d\lambda.$$

By Lemma 3.2

$$\|T^*_\lambda h_k\|_{L^2(\mathbb{R})} \to \frac{|Q|}{2\pi} \qquad \text{as } \lambda \to 0$$

for every $k \in \mathbb{N}$. Also note that we have the bound

$$\|T^*_\lambda h_k\| \leq M \qquad \text{for } \lambda \in \mathbb{R}^*, k \in \mathbb{N},$$

where

$$M := \frac{1}{2\pi} \int_{\mathbb{H}} |u_0(x,y,s)| dx dy ds.$$

By Watson's Lemma we have the following for every $k \in \mathbb{N}$

$$\int |\lambda| \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda| t} d\lambda \sim \frac{2|Q|^2}{16\pi^2 (2k+1)^2 t^2} \qquad \text{as } t \to \infty.$$

Now we consider

$$\lim_{t\to\infty} t^2\|u(t)\|^2_{L^2(\mathbb{H})} = \lim_{t\to\infty} t^2 \sum_{k=0}^{\infty} \int |\lambda| \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda|t} d\lambda.$$

Since we have the bound

$$\int |\lambda| \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda|t} d\lambda \leq \frac{M^2}{2(2k+1)^2 t^2},$$

we can interchange the limit and the summation to have

$$\begin{aligned}\lim_{t\to\infty} t^2\|u(t)\|^2_{L^2(\mathbb{H})} &= \sum_{k=0}^{\infty} \lim_{t\to\infty} t^2 \int |\lambda| \|T^*_\lambda h_k\|^2_{L^2(\mathbb{R})} e^{-2(2k+1)|\lambda|t} d\lambda.\\ &= \sum_{k=0}^{\infty} \frac{2|Q|^2}{16\pi^2(2k+1)^2}.\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$

We have

$$\lim_{t\to\infty} t^2\|u(t)\|^2_{L^2(\mathbb{H})} = \frac{|Q|^2}{64}.$$

Therefore,

$$\|u(t)\|_{L^2(\mathbb{H})} \sim \frac{|Q|}{8t} \qquad \text{as } t \to \infty.$$

□

3.2 Asymptotic expansion

Theorem 3.4. *Let $u(t,z)$ be the solution to the Heat equation on the Heisenberg group. Let u_0 have the same assumptions as f in section 2.3. Let $N \in \mathbb{N}$. Then, as $t \to \infty$,*

$$\|u_{t,\cdot}\|^2_{L^2(H)} = \sum_{n=0}^{N-1} b_n t^{-n-2} + O(t^{-N-2}),$$

where

$$b_n = \frac{(n+1)!}{2^{n+1}} \sum_{k=0}^{\infty} \frac{B_{k,2n}}{(2k+1)^{n+2}},$$

and $B_{k,2n}$ is defined as $A_{k,n}$ in (2.11) but with terms corresponding to odd $q_1+r_1-q_2-r_2$ omitted and $sgn\lambda = 1$.

Proof. Consider

$$\phi_k(t) := \int_{-\infty}^{\infty} |\lambda| \|T^*_\lambda h_k\|^2 e^{-2(2k+1)|\lambda|t} d\lambda, \tag{3.1}$$

If we substitute (2.10) in (3.1) we see that the terms with odd $q_1 + r_1 - q_2 - r_2$ cancel. Therefore, we have

$$\phi_k(t) = 2\int_0^{\infty} \lambda \sum_{n=0}^{N-1} B_{k,2n} \lambda^n e^{-2(2k+1)\lambda t} d\lambda + \int_{-\infty}^{\infty} |\lambda| R_{k5}(\lambda) e^{-2(2k+1)|\lambda|t} d\lambda,$$

where R_{k5} is defined as in Theorem 2.5. The operator norms T^*_λ are bounded by a constant M. Then we obtain

$$\phi_k(t) = 2\sum_{n=0}^{N-1} B_{k,2n} \frac{(n+1)!}{(2(2k+1)t)^{n+2}} + R_{k6}(t), \tag{3.2}$$

where

$$|R_{k6}(t)| \le 2C_{k5} \frac{(N+1)!}{(2(2k+1)t)^{N+2}} + M^2 \frac{1 + 2(2k+1)t}{2(2k+1)^2 t^2} e^{-2(2k+1)t}.$$

Now consider

$$\phi(t) := \sum_{k=0}^{\infty} \phi_k(t)$$

which is the quantity we are interested in. By Lemma 2.6, adding the equations (2.13) from $k = 0$ to infinity, we obtain the expansion

$$\phi(t) = \sum_{n=0}^{N-1} b_n t^{-n-2} + O(t^{-N-2}) \qquad \text{as } t \to \infty,$$

where

$$b_n = \frac{(n+1)!}{2^{n+1}} \sum_{k=0}^{\infty} \frac{B_{k,2n}}{(2k+1)^{n+2}}.$$

□

4 Dissipative Wave Equation

Now we consider the Dissipative wave equation.

$$\partial_t^2 u(t,z) + \partial_t u(t,z) = \mathcal{L}u(t,z)$$

with initial conditions

$$u(0,x,y,s) = u_0(x,y,s), \qquad \partial_t u(0,x,y,s) = u_1(x,y,s).$$

We assume that

$$u_0, u_1 \in L^1(\mathbb{H}) \cap L^2(\mathbb{H}).$$

Similarly to the previous sections, we define

$$F_j(x,\eta,\lambda) := \frac{1}{2\pi} \int\int u_j(x,y,s) e^{-i(y\eta+s\lambda)} dyds \quad \text{for } j = 0, 1.$$

For $\lambda \in \mathbb{R} \setminus \{0\}$ we define the integral operators $T_{j,\lambda} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(T_{j,\lambda}h)(u) := \int K_{j,\lambda}(u,v)h(v)dv.$$

Where $K_{j,\lambda}$ is the kernel

$$K_{j,\lambda}(u,v) := \frac{1}{\sqrt{|\lambda|}} F_j(\frac{u-v}{\sqrt{|\lambda|}}, \operatorname{sgn}(\lambda)\frac{\sqrt{|\lambda|}(u+v)}{2}, \lambda).$$

The family of operators $\{T_{j,\lambda}\}$ is the group Fourier transform of u_j for $j = 1, 2$.

4.1 Asymptotic equivalence

First, we will find the behavior of the function $t \to \|u(t)\|_{L^2(\mathbb{H})}$ as $t \to \infty$, where u is the solution of the dissipative wave equation. This also will give us the leading coefficient of the asymptotic expansion. We take the dissipative wave equation and apply the Fourier transform. This gives us

$$\begin{cases} \partial_t^2 \hat{u}(t,\lambda) + \partial_t \hat{u}(t,\lambda) + \sigma_{\mathcal{L}} u(t,\lambda) = 0, \\ \hat{u}(0,\lambda) = \hat{u_0}(\lambda), \\ \partial_t \hat{u}(0,\lambda) = \hat{u_1}(\lambda), \end{cases}$$

Where $\sigma_{\mathcal{L}}(\lambda)$ is the symbol of $-\mathcal{L}$. This takes the form

$$\sigma_{\mathcal{L}}(\lambda) = |\lambda| H_w = |\lambda|(-\partial_w^2 + w^2).$$

Where $H_w = (-\partial_w^2 + w^2)$ is the harmonic oscillator acting on $L^2(\mathbb{R})$. Let $\{S_\lambda(t)\}_{\lambda \in \mathbb{R}\setminus\{0\}}$

be the group Fourier transform of $(x, y, s) \mapsto u(t, x, y, s)$. Set

$$\hat{u}_{k,l}(t, \lambda) := \langle S_\lambda(t) h_l, h_k \rangle,$$

$$\hat{u}_{j,k,l}(\lambda) := \langle T_{j,\lambda} h_l, h_k \rangle,$$

Where, $\{h_k\}_{k=0}^{\infty}$ is the same orthonormal basis of Hermite functions as the heat equation in $L^2(\mathbb{R})$ with eigenvalues, $\mu_k = 2k + 1$ [3]. Appying this to our equation, we have

$$\begin{cases} \partial_t^2 \hat{u}_{k,l}(t, \lambda) + \partial_t \hat{u}_{k,l}(t, \lambda) + \sigma_{\mathcal{L}} \hat{u}_{k,l}(t, \lambda) = 0, \\ \hat{u}_{k,l}(0, \lambda) = \hat{u_0}_{k,l}(\lambda), \\ \partial_t \hat{u}_{k,l}(0, \lambda) = \hat{u_1}_{k,l}(\lambda), \end{cases}$$

for every $k, l \in \mathbb{N}$ and every $\lambda \in \mathbb{R} \setminus \{0\}$. The solution to this equation is

$$\hat{u}_{k,l}(t, \lambda) = (\hat{u_0}_{k,l}(\lambda) + \hat{u_1}_{k,l}(\lambda))\omega(t, |\lambda|(2k + 1)) + \hat{u_0}_{k,l}(\lambda)\partial_t \omega(t, |\lambda|(2k + 1)),$$

where

$$\omega(t, w) := e^{\frac{-t}{2}} \frac{\sinh(\frac{t}{2}\sqrt{1 - 4w})}{\frac{1}{2}\sqrt{1 - 4w}} = e^{\frac{-t}{2}} \frac{\sin(\frac{t}{2}\sqrt{4w - 1})}{\frac{1}{2}\sqrt{4w - 1}}.$$

We can also write ω in the form

$$\omega(t, w) = g_1(t, w) - g_2(t, w),$$

where

$$g_1(t, w) := \frac{1}{\sqrt{1 - 4w}} e^{-\frac{t}{2}(1 - \sqrt{1 - 4w})}, \qquad g_2(t, w) := \frac{1}{\sqrt{1 - 4w}} e^{-\frac{t}{2}(1 - \sqrt{1 - 4w})}.$$

Therefore, we have

$$S_\lambda^*(t)h_k = \omega(t, |\lambda|(2k+1))a_k(\lambda) + \partial_t \omega(t, |\lambda|(2k+1))b_k(\lambda),$$

where

$$a_k(\lambda) := (T_{0,\lambda}^* + T_{1,\lambda}^*)h_k, \quad b_k(\lambda) := T_{0,\lambda}^* h_k.$$

Before we go any further, we will need the following Lemma.

Lemma 4.1. *This is a slight variation of [[4], Lemma 3.1]. Let $n \in \mathbb{N}$ and $t > 0$. Then*

$$|\partial_t^n \omega(t,\xi)| \leq \begin{cases} 2|\xi|^{\frac{n-1}{2}} e^{-\frac{t}{2}} & \text{if } |\xi| \geq \frac{34}{100} \\ \frac{10}{3}(\frac{4}{5})^n e^{-\frac{t}{5}} & \text{if } \frac{4}{25} \leq |\xi| \leq \frac{34}{100} \end{cases}$$

Moreover, for $|\xi| \leq \frac{4}{25}$,

$$|\partial_t^n g_1(t,\xi)| \leq \frac{5}{3} 5^{-n}, \qquad |\partial_t^n g_2(t,\xi)| \leq \frac{5}{3} e^{-\frac{4}{5}t}$$

where $\omega, g_1,$ and g_2 are defined as above.

Proof. This proof is found in Volkmer (2019). □

Since $a_k(\lambda), b_k(\lambda) \in L^2(\mathbb{R})$ for all k, then $S_\lambda^*(t)h_k \in L^2(\mathbb{R})$ for all $t > 0$. The Hilber-Schmidt norm $\|S_\lambda(t)\|_{HS[L^2(\mathbb{R})]} = \|S_\lambda^*(t)\|_{HS[L^2(\mathbb{R})]}$ satisfies

$$\|S_\lambda(t)\|_{HS[L^2(\mathbb{R})]}^2 = \sum_{k=0}^{\infty} \|S_\lambda^*(t)h_k\|^2.$$

We use the Plancherel formula for the group Fourier transform and we have

$$\|u(t,\cdot)\|_{L^2(\mathbb{H})}^2 = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) := \sum_{k=0}^{\infty} \int |\lambda| \omega(t, |\lambda|(2k+1))^2 \|a_k(\lambda)\|^2 d\lambda,$$

$$I_2(t) := \sum_{k=0}^{\infty} \int |\lambda| \partial_t((\omega(t, |\lambda|(2k+1))^2) \mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle_{L^2(\mathbb{R})} d\lambda,$$

$$I_3(t) := \sum_{k=0}^{\infty} \int |\lambda| (\partial_t \omega(t, |\lambda|(2k+1)))^2 \|b_k(\lambda)\|^2 d\lambda,$$

Now we will state and prove the following lemmas that help evaluate each term. Then we will state and prove our main theorem.

Lemma 4.2. *We have*

$$\lim_{t\to\infty} t^2 I_1(t) = \frac{|Q|^2}{64}$$

where

$$Q := \int_{\mathbb{R}^3} (u_0(x, y, s) + u_1(x, y, s)) dx dy ds.$$

Proof. We will first focus on the positive part of the integral. We will determine the behavior of

$$J_k(t) := \int_0^{\infty} \lambda \omega(t, \lambda(2k+1))^2 \|a_k(\lambda)\|^2 d\lambda$$

as $t \to \infty$ for fixed $k \in \mathbb{N}$. We use the same method as the proof of [[4] Theorem 3.2]. Using lemma 4.1, we have the following

$$J_k(t) = \int_0^{\frac{4}{25(2k+1)}} \lambda \frac{e^{-t(1-\sqrt{1-4\lambda(2k+1)})}}{1-4\lambda(2k+1)} \|a_k(\lambda)\|^2 d\lambda + O(e^{-\frac{2t}{5}}).$$

For now we will just focus on the integral. Substitute $w = \lambda(2k+1)$ and we have

$$\frac{1}{(2k+1)^2}\int_0^{\frac{4}{25}} w\frac{1}{1-4w}e^{-t(1-\sqrt{1-4w})}\|a_k(\frac{w}{2k+1})\|^2 dw.$$

Let $w = z(1-z)$ where $0 \leq z \leq \frac{1}{5}$. Substitute and note the following

$$\begin{aligned}1-\sqrt{1-4w} &= 1-\sqrt{1-4(z(1-z))}\\ &= 1-\sqrt{4z^2-4z+1}\\ &= 1-\sqrt{(1-2z)^2}\\ &= 2z.\end{aligned}$$

Therefore, after substitution we have

$$\frac{1}{(2k+1)^2}\int_0^{\frac{1}{5}} z\frac{(1-z)}{(1-2z)}e^{-2tz}\|a_k(\frac{z(1-z)}{2k+1})\|^2 dz.$$

We need the last substitution $j = 2z$ and we have

$$\frac{1}{4(2k+1)^2}\int_0^{\frac{2}{5}} j\frac{1-\frac{j}{2}}{(1-j)}e^{-tj}\|a_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2 dj.$$

By Lemma 3.2 $\|a_k(\lambda)\|$ converges to $\frac{|Q|}{2\pi}$ as $\lambda \to 0$. By Watson's lemma, we have the asymptotic equivalence

$$\int_0^{\frac{2}{5}} j\frac{1-\frac{j}{2}}{(1-j)}e^{-tj}\|a_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2 dj. \sim \sum_0^{\infty}\frac{g^n(0)\Gamma(n+2)}{n!t^{n+2}},$$

where $g(j) = \frac{1-\frac{j}{2}}{(1-j)}\|a_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2$

It follows that

$$\lim_{t\to\infty} t^2 J_k(t) = \frac{|Q|^2}{16\pi^2(2k+1)^2}.$$

Treating the integral $\int\limits_{-\infty}^{0}$ similarly, we have

$$\lim_{t\to\infty} t^2 \int |\lambda|\omega(t,\lambda(2k+1))^2\|a_k(\lambda)\|^2 d\lambda = \frac{|Q|^2}{8\pi^2(2k+1)^2}.$$

Given that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$

this provides us with

$$\lim_{t\to\infty} t^2 I_1(t) = \frac{|Q|^2}{64},$$

given that we can justify the interchange of the limit and the sum. Note that

$$|\omega(t,w)| \leq 2e^{-tw} \qquad \text{for } t>0, 0<w\leq \frac{4}{25},$$

and

$$\|a_k(\lambda)\| \leq M := \frac{1}{2\pi}\int\limits_{\mathbb{R}^3} (|u_0(x,y,s)+u_1(x,y,s)|)dxdyds.$$

Since we are taking a limit as $t\to\infty$, we will assume $t>1$ and have the following.

$$\begin{aligned}
\int\limits_{0}^{\frac{4}{25(2k+1)}} \lambda\omega(t,\lambda(2k+1))^2\|a_k(\lambda)\|^2 d\lambda &\leq 4M^2 \int\limits_{0}^{\frac{4}{25(2k+1)}} \lambda e^{-2\lambda(2k+1)t} d\lambda \\
&\leq 4M^2 \int\limits_{0}^{\infty} \lambda e^{-2\lambda(2k+1)t} d\lambda \\
&= \frac{M^2}{(2k+1)^2 t^2}.
\end{aligned}$$

For $k \in \mathbb{N}$, we define

$$A_k := \int_0^\infty |\lambda| \|a_k(\lambda)\|^2 d\lambda.$$

Then

$$\sum_{k=0}^\infty A_k < \infty.$$

$$\begin{aligned}
\int_{\frac{4}{25(2k+1)}}^\infty \lambda\omega(t,\lambda(2k+1))^2 \|a_k(\lambda)\|^2 d\lambda &\leq \int_{\frac{4}{25(2k+1)}}^\infty \lambda 16 e^{-\frac{2t}{5}} \|a_k(\lambda)\|^2 d\lambda \\
&\leq 16 A_k e^{-\frac{2t}{5}} \\
&\leq 64 A_k \frac{1}{t^2}.
\end{aligned}$$

Therefore,

$$t^2 \int_0^\infty \lambda\omega(t,\lambda(2k+1))^2 \|a_k(\lambda)\|^2 d\lambda \leq \frac{M^2}{(2k+1)^2} + 64 A_k.$$

Arguing similarly for $\lambda < 0$ we find

$$t^2 \int_{-\infty}^\infty \lambda\omega(t,\lambda(2k+1))^2 \|a_k(\lambda)\|^2 d\lambda \leq \frac{2M^2}{(2k+1)^2} + 128 A_k.$$

The right hand side is independent of t and

$$\sum_{k=0}^\infty \Big(\frac{2M^2}{(2k+1)^2} + 128 A_k \Big) < \infty.$$

Therefore by Tannery's theorem, the estimate justifies the interchange of the sum and limit. This proves the lemma. □

Lemma 4.3. *We have*

$$\lim_{t\to\infty} t^2 I_2(t) = 0.$$

Proof. Similar to Lemma 4.2, we will determine the behavior of

$$L_k(t) := \int_0^\infty \lambda \partial_t(\omega(t, \lambda(2k+1))^2)\mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle d\lambda.$$

Again by Lemma 3.1, we have

$$L_k(t) = \int_0^{\frac{4}{25(2k+1)}} \lambda \partial_t(g_1(t, \lambda(2k+1))^2)\mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle d\lambda.$$

By Cauchy-Schwartz and lemma 2.2, for all $\lambda \in [0, \frac{4}{25}]$ there exists M such that $\mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle \leq \|a_k(\lambda)\|\|b_k(\lambda)\| \leq M$. Therefore,

$$L_k(t) \leq M \int_0^{\frac{4}{25(2k+1)}} \lambda \partial_t(g_1(t, \lambda(2k+1))^2) d\lambda.$$

Now substituting $\rho = \lambda(2k+1)$ we have

$$\begin{aligned} L_k(t) &\leq \frac{M}{(2k+1)^2} \int_0^{\frac{4}{25}} \rho \partial_t(g_1(t, \rho)^2) d\rho \\ &= \frac{M}{(2k+1)^2} \int_0^{\frac{4}{25}} \rho \frac{1 - \sqrt{1-4\rho}}{1-4\rho} e^{-t(1-\sqrt{1-4\rho})} d\rho \end{aligned}$$

Similarly to Lemma 4.2, we let $\rho = z(1-z)$ where $0 \leq z \leq \frac{1}{5}$. Therefore, we have

$$\begin{aligned} &= \frac{M}{(2k+1)^2} \int_0^{\frac{1}{5}} z(1-z)(1-2z) \frac{2z}{(1-2z)^2} e^{-2zt} dz \\ &= \frac{M}{(2k+1)^2} \int_0^{\frac{1}{5}} z^2 \frac{2(1-z)}{(1-2z)} e^{-2zt} dz \end{aligned}$$

Now substitute $j = 2z$ and we have

$$= \frac{M}{4(2k+1)^2} \int_0^{\frac{1}{10}} j^2 \frac{(1-\frac{j}{2})}{(1-j)} e^{-jz} dj$$

By Watson's lemma we have the following asymptotic equivalence

$$\int_0^{\frac{1}{10}} j^2 \frac{(1-\frac{j}{2})}{(1-j)} e^{-jz} dj \sim \sum_0^{\infty} \frac{g^n(0)\Gamma(n+3)}{n! t^{n+3}},$$

where $g(j) = \frac{1-\frac{j}{2}}{1-j}$. It follows that $\lim_{t\to\infty} t^2 L_k(t) = 0$. Treating the integral $\int_{-\infty}^{0}$ similarly, we have

$$\lim_{t\to\infty} t^2 \int \lambda \partial_t(\omega(t, \lambda(2k+1))^2) \mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle d\lambda = 0.$$

Given that we can justify the interchange of the limit and the sum, this provides us with $\lim_{t\to\infty} t^2 I_2(t) = 0$ since $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$. We can justify the interchanging using the exact same technique as in lemma 4.2, so we will omit it. □

Lemma 4.4. *We have*

$$\lim_{t\to\infty} t^2 I_3(t) = 0.$$

Proof. The proof is similar to the proof of Lemma 4.3. But, we will write the details since we will refer to this proof when we find the expansion. We will determine the behavior of

$$H_k(t) := \int_0^{\infty} \lambda(\partial_t w(t, \lambda(2k+1))^2 \|b_k(\lambda)\|^2 d\lambda$$

as $t \to \infty$ for fixed $k \in \mathbb{N}$. Using lemma 4.1, we have the following

$$\begin{aligned} H_k(t) &= \int_0^{\frac{4}{25(2k+1)}} \lambda(\partial_t g_1(t,\lambda(2k+1)))^2 \|b_k(\lambda)\|^2 d\lambda \\ &= \int_0^{\frac{4}{25(2k+1)}} \lambda\Big(\frac{-1+\sqrt{1-4\lambda(2k+1)}}{2\sqrt{1-4\lambda(2k+1)}} e^{-\frac{t}{2}(1-\sqrt{1-4\lambda(2k+1)})}\Big)^2 \|b_k(\lambda)\|^2 d\lambda. \end{aligned}$$

Doing similar substitutions as in the previous sections, we have

$$\frac{1}{2(2k+1)^2} \int_0^{\frac{2}{5}} j^3 \frac{(1-\frac{j}{2})}{1-j} e^{-jt} \|b_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2 dz.$$

By Watson's lemma, we have the following asymptotic equivalence

$$\int_0^{\frac{2}{5}} j^2 \frac{(1-\frac{j}{2})}{1-j} e^{-jt} \|b_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2 dz \sim \sum_0^{\infty} \frac{g^n(0)\Gamma(n+3)}{n! t^{n+4}},$$

where $g(j) = \frac{1-\frac{j}{2}}{1-j} \|b_k(\frac{j(1-\frac{j}{2})}{2(2k+1)})\|^2$. It follows that $\lim_{t\to\infty} t^2 H_k(t) = 0$. Treating the integral $\int_{-\infty}^{0}$ similarly, we have

$$\lim_{t\to\infty} t^2 \int \lambda(\partial_t w(t,\lambda(2k+1))^2 \|b_k(\lambda)\|^2 d\lambda = 0.$$

Given that we can justify the interchange of the limit and the sum, this provides us with $\lim_{t\to\infty} t^2 I_3(t) = 0$ since $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$. We can justify the interchanging using the exact same technique as in lemma 4.2, we will omit it. □

Theorem 4.5.

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^2(\mathbb{H})} \sim \frac{|Q|}{8t} \qquad \textit{as } t \to \infty$$

Proof. The proof follows from Lemma's 4.2, 4.3, and 4.4. ☐

4.2 Asymptotic expansion

In order to find the Asymptotic expansion, it suffices to find the expansions of I_1, I_2, and I_3 from the previous section. Let $u(t, z)$ be the solution to the Dissipative wave equation on the Heisenberg group. Let u_0 and u_1 have the same assumptions as f in section 2.3. Let $N \in \mathbb{N}$.

Lemma 4.6. *We have the expansion for* $I_1(t)$

$$I_1(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{B_{k,2n} G_{n,q}(n+q+1)!}{2^{n+1}(2k+1)^{n+2} t^{n+q+2}} + O(t^{-N}).$$

where $B_{k,2n}$ *is defined as* $A_{k,n}$ *in (2.11) with respect to* $(T^*_{0,\lambda} + T^*_{1,\lambda})$ *but with terms corresponding to odd* $q_1 + r_1 - q_2 - r_2$ *omitted and* $sgn\lambda = 1$, $G_{n,q}$ *is the qth term in the Taylor series expansion of* G_n *defined as*

$$G_n(j) = \frac{(1 - \frac{j}{2})^{n+1}}{1 - j}.$$

Proof Consider

$$\phi_k(t) := \int_{-\infty}^{\infty} |\lambda| w(t, |\lambda|(2k+1))^2 \|(T^*_{0,\lambda} + T^*_{1,\lambda})h_k\|^2 d\lambda. \tag{4.1}$$

Similarly to Theorem 3.4, we substitute (2.10) in (4.1) and see that the terms with odd $q_1 + r_1 - q_2 - r_2$ cancel. Therefore, we have the following.

$$\phi_k(t) = 2 \int_0^{\infty} \lambda w(t, \lambda(2k+1))^2 \sum_{n=0}^{N-1} B_{k,2n} \lambda^n d\lambda + \int_{-\infty}^{\infty} |\lambda| w(t, |\lambda|(2k+1))^2 R_{k5}(\lambda) d\lambda, \tag{4.2}$$

where R_{k5} is defined as in Theorem 2.5. First we will evaluate the term on the right.

$$\int_{-\infty}^{\infty} |\lambda| w(t, |\lambda|(2k+1))^2 R_{k5}(\lambda) d\lambda \leq 2C_{k5} \int_{0}^{\infty} \lambda^{N+1} w(t, \lambda(2k+1))^2 d\lambda$$

By Lemma 4.1 and substituting $\rho = \lambda(2k+1)$ we have

$$\begin{aligned}\frac{2C_{k5}}{(2k+1)^{N+2}} \int_{0}^{\frac{4}{25}} \rho^{N+1} w(t,\rho)^2 d\rho &\sim \frac{2C_{k5}}{(2k+1)^{N+2}} \int_{0}^{\frac{4}{25}} \rho^{N+1} g_1(t,\rho)^2 d\rho \\ &= \frac{2C_{k5}}{(2k+1)^{N+2}} \int_{0}^{\frac{4}{25}} \rho^{N+1} \frac{1}{1-4\rho} e^{-t(1-\sqrt{1-4\rho})} d\rho.\end{aligned}$$

Similarly to Lemma 4.2, we let $\rho = z(1-z)$ where $0 \leq z \leq \frac{1}{5}$. Substitute and we have

$$\frac{2C_{k5}}{(2k+1)^{N+2}} \int_{0}^{\frac{1}{5}} z^{N+1} \frac{(1-z)^{N+1}}{(1-2z)} e^{-2zt} dz.$$

Let $2z = j$, substitute and we have

$$\frac{2C_{k5}}{2^{N+2}(2k+1)^{N+2}} \int_{0}^{\frac{2}{5}} j^{N+1} \frac{(1-\frac{j}{2})^{N+1}}{(1-j)} e^{-jt} dj.$$

It follows from Watson's lemma that

$$\frac{2C_{k5}}{2^{N+2}(2k+1)^{N+2}} \int_{0}^{\frac{2}{5}} j^{N+1} \frac{(1-\frac{j}{2})^{N+1}}{(1-j)} e^{-jt} dj = O(t^{-N-2}).$$

Now we will evaluate the left term of (4.2).

$$\begin{aligned}
&2\int_0^\infty \lambda w(t,|\lambda|(2k+1))\sum_{n=0}^{N-1} B_{k,2n}\lambda^n d\lambda + O(t^{-N-2})\\
=&2\sum_{n=0}^{N-1} B_{k,2n}\int_0^\infty \lambda^{n+1} w(t,\lambda(2k+1))^2 d\lambda + O(t^{-N-2}).
\end{aligned}$$

Using the same substitutions as before, we have

$$=\sum_{n=0}^{N-1}\frac{2B_{k,2n}}{2^{n+2}(2k+1)^{n+2}}\int_0^{\frac{2}{5}} j^{n+1}\frac{(1-\frac{j}{2})^{n+1}}{(1-j)}e^{-jt}dj + O(t^{-N-2})$$

Define G_n for $n\in\mathbb{N}$ as

$$G_n(j)=\frac{(1-\frac{j}{2})^{n+1}}{(1-j)}.$$

Note that G_n is C^∞ near 0. So, we denote its Taylor expansion at $j=0$ as

$$G_n(j)=\sum_{q=0}^{\infty} G_{n,q}j^q + O(j^N) \qquad \text{as } j\to 0.$$

Therefore, we have

$$\begin{aligned}
&\sum_{n=0}^{N-1}\frac{B_{k,2n}}{2^{n+1}(2k+1)^{n+2}}\int_0^{\frac{2}{5}} j^{n+1}\frac{(1-\frac{j}{2})^{n+1}}{(1-j)}e^{-jt}dj + O(t^{-N-2})\\
=&\sum_{n=0}^{N-1}\frac{B_{k,2n}}{2^{n+1}(2k+1)^{n+2}}\int_0^{\frac{2}{5}} j^{n+1}G_n(j)e^{-jt}dj + O(t^{-N-2})
\end{aligned}$$

Substitute $\eta = jt$ and we have

$$\sum_{n=0}^{N-1} \frac{B_{k,2n}}{2^{n+1}(2k+1)^{n+2}t^{n+2}} \int_0^{\frac{2t}{5}} \eta^{n+1} G_n(\frac{\eta}{t}) e^{-\eta} d\eta + +O(t^{-N-2})$$

$$= \sum_{n=0}^{N-1} \frac{B_{k,2n}}{2^{n+1}(2k+1)^{n+2}t^{n+2}} \sum_{q=0}^{\infty} \frac{1}{t^q} \int_0^{\frac{2t}{5}} \eta^{n+q+1} G_{n,q} e^{-\eta} d\eta + O(t^{-N})$$

$$= \sum_{n=0}^{N-1} \frac{B_{k,2n}}{2^{n+1}(2k+1)^{n+2}t^{n+2}} \sum_{q=0}^{\infty} \frac{1}{t^q} G_{n,q}(n+q+1)! + O(t^{-N}),$$

Therefore, we have

$$\phi_k(t) = \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{B_{k,2n} G_{n,q}(n+q+1)!}{2^{n+1}(2k+1)^{n+2}t^{n+q+2}} + O(t^{-N}).$$

Sum over all k and we have our final result

$$I_1(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{B_{k,2n} G_{n,q}(n+q+1)!}{2^{n+1}(2k+1)^{n+2}t^{n+q+2}} + O(t^{-N}).$$

Lemma 4.7. *We have the expansion for $I_2(t)$*

$$I_2(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{-\widetilde{C}_{k,2n} G_{n,q}(n+q+2)!}{2^{n+1}(2k+1)^{n+2}t^{n+q+3}} + O(t^{-N}).$$

where the constants $\widetilde{C}_{k,2n}$ are defined in terms of the real part of the inner products using lemma 2.4 and $G_{n,q}$ is defined the same as in lemma 4.6.

Proof. Consider

$$\phi_k(t) := \int_{-\infty}^{\infty} |\lambda| \partial_t (w(t, |\lambda(2k+1))^2) \mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle d\lambda.$$

We have that

$$\begin{aligned}\mathrm{Re}\langle a_k(\lambda), b_k(\lambda)\rangle &= \mathrm{Re}\langle (T^*_{0,\lambda} + T^*_{1,\lambda})h_k, T^*_{0,\lambda}h_k\rangle \\ &= \mathrm{Re}\langle \overline{(T^*_{0,\lambda} + T^*_{1,\lambda})h_k}, \overline{T^*_{0,\lambda}h_k}\rangle\end{aligned}$$

Using Theorem 2.4 where $M^{1,0}_{p,q,r}$ denotes the moment with respect to the sum of the initial conditions and M^0 denotes the moment with respect to just the initial condition u_0, we have

$$\begin{aligned}\mathrm{Re}\langle \overline{(T^*_{0,-\lambda} + T^*_{1,-\lambda})h_k}, \overline{T^*_{0,\lambda}h_k}\rangle &= \mathrm{Re}\langle \sum_{m=0}^{2N-1} a_{k,m}\sqrt{|\lambda|}^m, \sum_{j=0}^{2N-1} b_{k,j}\sqrt{|\lambda|}^j\rangle \\ &= \sum_{m=0}^{2N-1}\sum_{j=0}^{2N-1} |\lambda|^{\frac{m+j}{2}}\mathrm{Re}\langle a_{k,m}, b_{k,j}\rangle \\ &= \sum_{m,j=0}^{2N-1} |\lambda|^{\frac{m+j}{2}} C_{k,m,j},\end{aligned}$$

where $C_{k,m,j} = \mathrm{Re}\langle a_{k,m}, b_{k,j}\rangle$. It is important to note that $C_{k,m,j}$ does not depend on λ. Note that if $m+j$ is odd, then $C_{k,m,j} = 0$. Combining like terms and reindexing and we have

$$\sum_{m,j=0}^{2N-1} |\lambda|^{\frac{m+j}{2}} C_{k,m,j} = \sum_{n=0}^{N-1} \widetilde{C}_{k,2n}|\lambda|^n,$$

where $\widetilde{C}_{k,n} = \sum C_{k,m,j}$ summing over all m and j such that $m+j=n$. Therefore,

$$\phi_k(t) := \sum_{n=0}^{N-1} \widetilde{C}_{k,2n} \int_0^\infty 2\lambda^{n+1}\partial_t(w(t,\lambda(2k+1))^2)d\lambda.$$

Using Lemma 3.1 and substitutions similar to Lemma 4.3, we have

$$\sum_{n=0}^{N-1} \frac{-\widetilde{C}_{k,2n}}{2^{n+1}(2k+1)^{n+2}} \int_0^{\frac{2}{5}} j^{n+2}\frac{(1-\frac{j}{2})^{n+1}}{(1-j)}e^{-jt}dj.$$

Therefore, we have

$$\sum_{n=0}^{N-1} \frac{-\widetilde{C}_{k,2n}}{2^{n+1}(2k+1)^{n+2}} \int\limits_0^{\frac{2}{5}} j^{n+2} \frac{(1-\frac{j}{2})}{(1-j)} e^{-jt} dj$$
$$= \sum_{n=0}^{N-1} \frac{-\widetilde{C}_{k,2n}}{2^{n+1}(2k+1)^{n+2}} \sum_{q=0}^{\infty} G_{n,q} \frac{(n+q+2)!}{t^{n+q+3}},$$

where $G_{n,q}$ is defined the same as in Lemma 4.6. Therefore, we have

$$\phi_k(t) = \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{-\widetilde{C}_{k,2n} G_{n,q} (n+q+2)!}{2^{n+1}(2k+1)^{n+2} t^{n+q+3}} + O(t^{-N}).$$

Sum over all k and we have our final result

$$I_2(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{-\widetilde{C}_{k,2n} G_{n,q} (n+q+2)!}{2^{n+1}(2k+1)^{n+2} t^{n+q+3}} + O(t^{-N}).$$

□

Lemma 4.8. *We have the expansion for* $I_3(t)$

$$I_3(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} \sum_{q=0}^{\infty} \frac{D_{k,2n} G_{n,q} (n+q+3)!}{2^{n+1}(2k+1)^{n+3} t^{n+q+4}} + O(t^{-N}),$$

where $D_{k,2n}$ *is defined as* $A_{k,n}$ *in (2.11) with respect to* $T^*_{0,\lambda}$ *but with terms corresponding to odd* $q_1 + r_1 - q_2 - r_2$ *omitted,* $sgn\lambda = 1$*, and* $G_{n,q}$ *is defined the same as in Lemma 4.6.*

Proof. Consider

$$\phi_k(t) := \int\limits_{-\infty}^{\infty} |\lambda| (\partial_t w(t, |\lambda|(2k+1)))^2 \|T^*_{0,\lambda} h_k\|^2 d\lambda \tag{4.3}$$

Similarly to Theorem 3.4, we substitute (2.10) in (4.5) and see that the terms with odd

$q_1 + r_1 - q_2 - r_2$ cancel. Therefore, we have the following

$$\phi_k(t) = 2\int_0^\infty \lambda(\partial_t w(t,\lambda(2k+1)))^2 \sum_{n=0}^{N-1} D_{k,2n}\lambda^n d\lambda + \int_{-\infty}^{\infty} |\lambda|(\partial_t w(t,|\lambda|(2k+1)))^2 R_{k5}(\lambda)d\lambda, \tag{4.4}$$

where R_{k5} is defined as in Theorem 2.5. First, we will evaluate the term on the right.

$$\int_{-\infty}^{\infty} |\lambda|(\partial_t w(t,|\lambda|(2k+1)))^2 R_{k5}(\lambda)d\lambda \leq 2C_{k5}\int_0^\infty \lambda^{N+1}(\partial_t w(t,\lambda(2k+1)))^2 d\lambda$$

By Lemma 4.1 and substituting $\rho = \lambda(2k+1)$ we have

$$\frac{2C_{k5}}{(2k+1)^{N+2}}\int_0^{\frac{4}{25}} \rho^{N+1}(\partial_t w(t,\rho))^2 d\rho \sim \frac{2C_{k5}}{(2k+1)^{N+2}}\int_0^{\frac{4}{25}} \rho^{N+1}(\partial_t g_1(t,\rho))^2 d\rho$$

$$= \frac{2C_{k5}}{(2k+1)^{N+2}}\int_0^{\frac{4}{25}} \rho^{N+1}\Big(\frac{-1+\sqrt{1-4\rho}}{2\sqrt{1-4\rho}}\Big)^2 e^{-t(1-\sqrt{1-4\rho})}d\rho$$

Let $\rho = z(1-z)$ where $0 \leq z \leq \frac{1}{5}$ and substitute

$$\frac{2C_{k5}}{(2k+1)^{N+2}}\int_0^{\frac{1}{5}} z^{N+1}(1-z)^{N+1}\frac{4z^2}{4(1-2z)^2}(1-2z)e^{-2zt}dz$$

$$= \frac{2C_{k5}}{(2k+1)^{N+2}}\int_0^{\frac{1}{5}} z^{N+3}\frac{(1-z)^{N+1}}{1-2z}e^{-2zt}dz$$

Using Watson's lemma again, we have that the above is $O(t^{-N-4})$. Now we will evaluate the left term of (4.6).

$$
\begin{aligned}
&2\int_0^\infty \lambda(\partial_t w(t,\lambda(2k+1)))^2 \sum_{n=0}^{N-1} D_{k,2n}\lambda^n d\lambda + O(t^{-N-4})\\
=&2\sum_{n=0}^{N-1} D_{k,2n}\int_{0\infty} \lambda^{n+1}(\partial_t w(t,\lambda(2k+1)))^2 d\lambda + O(t^{-N-4}).
\end{aligned}
$$

Using the same substitutions as in lemma 4.6 we have

$$
\sum_{n=0}^{N-1} \frac{D_{k,2n}}{2^{n+1}(2k+1)^{n+2}} \int_0^{\frac{2}{5}} j^{n+3}\frac{(1-\frac{1}{2}j)^{n+1}}{1-j}e^{-jt}dj + O(t^{-N-4}),
$$

where $G_{n,q}$ is defined the same as in lemma 4.6. Therefore,

$$
= \sum_{n=0}^{N-1} \frac{D_{k,2n}}{2^{n+1}(2k+1)^{n+3}t^{n+3}} \sum_{q=0}^{\infty} \frac{1}{t^{q+1}} G_{n,q}(n+q+3)! + O(t^{-N}).
$$

Therefore, we have

$$
\phi_k(t) = \sum_{n=0}^{N-1}\sum_{q=0}^{\infty} \frac{D_{k,2n}G_{n,q}(n+q+3)!}{2^{n+1}(2k+1)^{n+3}t^{n+q+4}} + O(t^{-N})
$$

Sum over all k and we have our result

$$
I_3(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{N-1}\sum_{q=0}^{\infty} \frac{D_{k,2n}G_{n,q}(n+q+3)!}{2^{n+1}(2k+1)^{n+3}t^{n+q+4}} + O(t^{-N})
$$

□

Theorem 4.9. *Let $u(t,z)$ be the solution to the Dissipative wave equation on the Heisenberg group. Let u_0 and u_1 have the same assumptions as f in section 2.3. Let $N \in \mathbb{N}$. Then, as $t \to \infty$,*

$$\|u_{t,\cdot}\|^2_{L^2(H)} = \sum_{k=0}^{\infty}\sum_{n=0}^{N-1}\sum_{q=0}^{\infty} \frac{B_{k,2n}G_{n,q}(n+q+1)!}{2^{n+1}(2k+1)^{n+2}t^{n+q+2}}$$
$$+ \sum_{k=0}^{\infty}\sum_{n=0}^{N-1}\sum_{q=0}^{\infty} \frac{-\widetilde{C}_{k,2n}G_{n,q}(n+q+2)!}{2^{n+1}(2k+1)^{n+2}t^{n+q+3}}$$
$$+ \sum_{k=0}^{\infty}\sum_{n=0}^{N-1}\sum_{q=0}^{\infty} \frac{D_{k,2n}G_{n,q}(n+q+3)!}{2^{n+1}(2k+1)^{n+3}t^{n+q+4}}$$
$$+ O(t^{-N-1})$$

Where everything is defined the same as in Lemmas 4.6, 4.7, and 4.8.

Proof. The proof is an immediate consequence of Lemmas 4.6, 4.7, and 4.8. □

5 Conclusion and Future Work

Now that we have both expansions, it is important to note that the leading term in the expansions have the same power. This shows that it is consistent with the Euclidean case, which was to be expected. From here one could follow the same route as done in Volkmer's paper on the Euclidean case. That is finding an expansion of $\|u-v\|_{L^2(\mathbb{H})}$ as $t \to \infty$ where u and v are the solutions to the heat and dissipative wave equations on the Heisenberg group respectively.

What might be of more interest is extending this concept to other equations. In Ruzhansky and Tokmagambetov's paper [3], they find the solutions to the linear damped wave equation, $\partial_t^2 u + b\partial_t u - \mathcal{L}u + mu = 0$, on the Heisenberg group in the same way we do. One could find the asymptotic equivalence and expansion in a similar way done in this thesis.

Another problem that can be asked is finding expansions of these equations on different Lie groups. The Lie groups would need to have similar properties as the Heisenberg. They would need their own version of Lie algebra, group Fourier transform, and Plancherel formula.

If those properties are similar enough, one would use very similar strategies as in this paper to find the expansions.

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6 Appendix

Lemma 6.1 (Watson's Lemma). *Let $0 < a \leq \infty$ be fixed. assume $\phi(x) = x^j g(x)$, where $g(x) \in C^\infty$ on some neighborhood of $x = 0$, with $g(0) \neq 0$, $j > -1$. Suppose also that, $|\phi(x)| < Ke^{bx}$ for all $x > 0$ where k, b are independent of x, or*

$$\int_0^a |\phi(x)| dx < \infty.$$

Then, for all $t > 0$,

$$\Big| \int_0^a e^{-tx} \phi(x) dx \Big| < \infty,$$

And that we have the asymptotic equivalence

$$\int_0^a e^{-tx} \phi(x) dx \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0) \Gamma(j+n+1)}{n! t^{j+n+1}}, \qquad (t > 0, t \to \infty).$$

Proof. This proof is found in Miller (2006). □

Lemma 6.2 (Schur Test). *Let K be a measurable function on $\mathbb{R}^2$ that satisfies the mixed-norm conditions*

$$C_1 := esssup_{x \in \mathbb{R}} \int |K(x,y)| dy < \infty,$$
$$C_2 := esssup_{x \in \mathbb{R}} \int |K(x,y)| dy < \infty.$$

Then the integral operator T defined by

$$(Th)(x) = \int K(x,y) h(y) dy$$

is a bounded linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and its operator norm satisfies

$$\|T\| \leq (C_1 C_2)^{\frac{1}{2}}.$$

Proof. This proof is found in Halmos (1978). □

Theorem 6.3 (Fubini's Theorem)**.** *Suppose X and Y are σ-finite measure spaces, and suppose that $X \times Y$ is given the product measure. If*

$$\int_{X\times Y} |f(x,y)| d(x,y) < \infty$$

,

then

$$\int_X \Big(\int_Y f(x,y)dy \Big) dx = \int_Y \Big(\int_X f(x,y)dx \Big) dy = \int_{X\times Y} f(x,y)d(x,y).$$

Proof. This proof is found in Cohn (1980). □

Theorem 6.4 (Tannery's Theorem)**.** *Let $S_n = \sum_{k=0}^{\infty} a_k(n)$ and suppose that $\lim_{n\to\infty} a_k(n) = b_k$. If $|a_k(n)| \leq M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$, then $\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} b_k$.*

Proof. Tannery's theorem is an immediate consequence of Lebesgue's dominated convergence theorem applied to the sequence space l^1. □

CURRICULUM VITAE

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Education

University of Wisconsin-Milwaukee August 2016-present (Dec 2020 expected)
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