May 2022

Coarse Cohomology of the Complement and Applications

Arka Banerjee
University of Wisconsin-Milwaukee

COARSE COHOMOLOGY OF THE COMPLEMENT AND APPLICATIONS

by

Arka Banerjee

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in Mathematics

at

The University of Wisconsin-Milwaukee
May 2022

ABSTRACT

COARSE COHOMOLOGY OF THE COMPLEMENT AND APPLICATIONS

by

Arka Banerjee

The University of Wisconsin–Milwaukee, 2022
Under the Supervision of Professor Boris Okun

John Roe [15] introduced the notion of coarse cohomology of a metric space to study large scale geometry of the space. Coarse cohomology of a metric space roughly measures the way in which uniformly large bounded set in that space fit together. In the first part of this dissertation, we describe a joint work with Boris Okun that generalizes Roe's theory to define coarse (co)homology of complement of any given subspace in a metric space. Inspired by the work of Kapovich and Kleiner [12], we introduce a notion of a manifold like object in the coarse category (called coarse PD(n) space) and prove a coarse version of the Alexander duality for these spaces. In the second part of this dissertation, we generalize a Theorem of Roe [15] to compute coarse cohomology of the complement for many spaces by relating coarse cohomology of the complement with the Alexander–Spanier cohomology. In the final part of this dissertation, we introduce an equivariant version of coarse cohomology of the complement. We then use this theory to find obstruction to coarse embedding of a given space into any uniformly contractible n-manifold.

To

Maa and Baba

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

This dissertation would have never seen the light of day without the contributions of many important people. First and foremost, I would like to thank my advisor, Professor Boris Okun for his guidance, encouragement and advice throughout my time as his student.

I would like to thank my professors at UW-Milwaukee, especially Professor Craig Guilbault, Professor Ric Ancel, Professor Chris Hruska and Professor Jonah Gaster in the topology group for all I've learned from them and for the time they have given. I would also like to thank Professor Kevin McLeod, Professor Peter Hinow and Professor Ian Musson who taught me a lot of interesting mathematics. In addition, I would like to thank Professor Boris Okun, Professor Chris Hruska, Professor Craig Guibault, Professor Jonah Gaster and Professor Kevin McLeod for donating their time as a member of my thesis committee.

I would like to thank Kevin Schreve for helpful conversations during this work. I would also like to thank my professors from my B.Sc. and M.Sc years, especially, Professor Swapan Kumar Ghosh and Professor Kingshook Biswas whose courses were very influential in shaping the way I think about math.

I would like to thank my longtime friend Abhishek Mallick, without whose encouragement I might not have gotten into Math in my undergrad in the first place. I would like to thank all my college friends, especially, Gorapada Bera, Krishnendu Khan, Subhankar Dey and Sudipta Ghosh whose friendships and supports to this day are invaluable to get through the tough times. I want to thank my fellow graduate students at UWM for their help and friendship. Especially, I would like to thank Ashani Dasgupta, Cong He, Dan Gulbrandsen, Prayagdeep Parija, Sahadeb Upretee, Songpon Sriwongsa and Will Braubach for all the interactions I had with them.

Finally, I would like to thank my parents and sister for everything. Without their sacrifices, support and love, I would have never made it this far in any aspect of my life.

Chapter 1

Introduction

1.1 Overview

Coarse geometry is often thought of as the study of the "geometry at infinity" of a space, i.e. the study of structures that survive as one zooms infinitely far away from the space. John Roe defined the notion of coarse cohomology of a metric space that encodes information about the coarse geometry of that space. We now recall the definition of coarse cohomology from [15]. Let X be a metric space and X^{n+1} be the cartesian product of $(n+1)$-copies of X equipped with the sup metric. Let Δ be the diagonal $\{(x,\dots,x) \mid x \in X\} \subset X^{n+1}$. Let R be a ring and $|f|$ be the support of the function $f : X^{n+1} \to R$. The coarse cohomology of X with coefficients in R, denoted by $\mathrm{H\,X}^*(X;R)$, is the cohomology of the following cochain complex where d is the coboundary map.

$$\mathrm{C\,X}^n(X;R) := \{f : X^{n+1} \to R \mid |f| \cap N_r(\Delta) \text{ is bounded for all } r\}$$

$$df(x_0,\dots,x_n) := \sum_{i=0}^{n} (-1)^i f(x_0,..,\hat{x_i},..,x_n)$$

Roe proved that the coarse cohomology is an invariant of the coarse geometry. In particular, any two quasi-isometric metric spaces have isomorphic coarse cohomology. Part of this dissertation is motivated by the following question.

Question. Let X be a metric space and $A \subset X$. What should be the coarse cohomology of the complement of A? Or more generally, what large scale structure should the 'coarse complement' of A have?

Our primary motivation behind studying this question was to get a notion of a coarse configuration space of X. To get the configuration space of a space K in a topological setting, one starts with the space $K \times K - \Delta$ where $\Delta = \{(x,x) \mid x \in K\}$. This space, known as *deleted product* of K admits a free $\mathbb{Z}_2$ action by switching coordinates. The configuration space of K, denoted by $Conf(K)$, is the space $(K \times K - \Delta)/\mathbb{Z}_2$. $Conf(K)$ encodes many interesting properties of K. For instance, van Kampen showed that certain classes in $\mathrm{H}^n(Conf(K))$ obstruct topological embedding of K into $\mathbb{R}^n$. One would hope that an appropriate theory of the coarse configuration space should have analogous applications in the coarse category. The first step to understand coarse configuration space of X should be to understand an appropriate coarse analogue of $X \times X - \Delta$. That calls for a notion of coarse complement.

In the second chapter of this dissertation we describe a joint work with Boris Okun [1] where we introduce and study the notion of coarse (co)homology of the complement of a subspace in a metric space. We give a model space which encodes coarse geometric structure of the complement. Kapovich and Kleiner defined a notion of coarse analogue of n-manifolds, called coarse PD(n) spaces. We introduce a new approach to coarse PD(n) spaces using coarse cohomology in Roe's sense. Inspired by Kapovich and Kleiner's further work, we prove a version of coarse Alexander duality for these spaces and give a criterion for a space to be coarse PD(n) space.

In the third chapter of this dissertation we turn our attention to computation of coarse cohomology of complement. Roe showed that the coarse cohomology is isomorphic to the compactly supported Alexander-Spanier cohomology if the space is uniformly contractible and proper. Unfortunately, this theorem is not enough to compute coarse cohomology of the complement because model space of the coarse complement almost never satisfies these hypothesis even though the ambient space does. To deal with such spaces, we introduce a notion of boundedly supported cohomology and prove that coarse cohomology of many such spaces are isomorphic to the boundedly supported cohomology. As an application of our main theorem, we show that coarse cohomology of the complement can be computed in terms of Alexander-Spanier cohomology for many spaces. This chapter is based on the work in [2].

In the final chapter of this thesis, we introduce an equivariant version of coarse cohomology of the complement. We use this cohomology to define a notion of coarse cohomology of the configuration space of a metric space X. Following classical van Kampen obstruction theory, we identify a class of degree n in the coarse cohomology of configuration space of X that obstructs coarse embedding of X into any uniformly contractible $(n-1)$-manifold. The material in this chapter is from [3].

1.2 Preliminaries

In this section we fix some notations which we mainly adopt from [14]. Let (X, d) be a metric space. For $A \subset X$ denote $N_R(A) = \{x \in X \mid d(x, A) < R\}$. We will call such neighborhoods *metric neighborhoods* of A. We will say that A is *R-contained* in B, $A \overset{R}{\subset} B$, if $A \subset N_R(B)$. A is *coarsely contained* in B, $A \overset{c}{\subset} B$, if $A \overset{R}{\subset} B$ for some R. Two subsets are *coarsely equal*, $A \overset{c}{=} B$ if $A \overset{c}{\subset} B$ and $B \overset{c}{\subset} A$.

We will use the following observation about the intersection of metric neighborhoods in later chapters:

Lemma 1.2.1. *Let A, B and C be subsets of metric space X. If $C \overset{R}{\subset} A$ and $C \overset{r}{\subset} B$, then*

$$C \overset{R}{\subset} A \cap N_{R+r}(B).$$

Proof. If $x \in C$, then there exist $y \in A$ such that $d(x, y) < R$. Therefore $y \in A \cap N_{R+r}(B)$, and hence $x \in N_R(A \cap N_{R+r}(B))$ by the triangle inequality. $\square$

Recall from [14] the notion of *coarse intersection*: $A \overset{c}{\cap} B \overset{c}{=} C$ if for all sufficiently large R, $N_R(A) \cap N_R(B) \overset{c}{=} C$. The coarse intersection is not always well defined, it may happen that the coarse type of $N_R(A) \cap N_R(B)$ does not stabilize as R goes to infinity. However the notion "coarse intersection is coarsely contained in" is well defined. $A \overset{c}{\cap} B \overset{c}{\subset} C$ means that for any R, $N_R(A) \cap N_R(B) \overset{c}{\subset} C$. It is immediate from Lemma 1.2.1 that this condition is equivalent to the condition that for any R, $A \cap N_R(B) \overset{c}{\subset} C$.

The following Lemma shows that coarse containment and intersection behave as expected.

Lemma 1.2.2. *Let A, B, C and D be subsets of metric space X. If $A \overset{c}{\subset} B$, $A \overset{c}{\subset} C$, and $B \overset{c}{\cap} C \overset{c}{\subset} D$, then $A \overset{c}{\subset} D$.*

A subset C is *coarsely disjoint* from A if $A \overset{c}{\cap} C \overset{c}{=} *$, in other words if $A \cap N_R(C)$ is bounded for all R. Note that coarse disjointness is independent of the ambient space, it depends only on the metric on $A \cup C$.

We will write $C \overset{c}{\subset} B \overset{c}{-} A$ to mean that $C \overset{c}{\subset} B$ and $A \overset{c}{\cap} C \overset{c}{=} *$. One should think of the coarse difference $B \overset{c}{-} A$ as a collection of subsets, it rarely has a well defined coarse type in X. Nevertheless, the notion of coarse containment between differences is well defined. We will write $(D \overset{c}{-} C) \overset{c}{\subset} (B \overset{c}{-} A)$ to mean that $Y \overset{c}{\subset} D \overset{c}{-} C$ implies $Y \overset{c}{\subset} B \overset{c}{-} A$. Unwinding the definitions, this really means that any subset of any metric neighborhood of D which has bounded intersection with C is contained in some metric neighborhood of B and has bounded intersection with A.

Lemma 1.2.3. *$B \overset{c}{\cap} D \overset{c}{\subset} A$ if and only if $(D \overset{c}{-} A) \overset{c}{\subset} (D \overset{c}{-} B)$.*

Proof. The forward direction is obvious. To prove the converse, suppose $B \overset{c}{\cap} D \overset{c}{\not\subset} A$, then the intersection D and some metric neighborhood of B contains a sequence of points going away from A. This sequence is in $D \overset{c}{-} A$ but not in $D \overset{c}{-} B$. $\square$

Now we introduce some notation for various (co)chain complexes associated to X. Suppose, R be a ring. The basic complexes is the complex of all cochains with $d : \mathrm{C}^{n-1}(X; R) \to$

$\mathrm{C}^n(X;R)$ being the boundary maps as follows:

$$\mathrm{C}^n(X;R) = \{\phi : X^{n+1} \to R\}$$
$$(d\phi)(x_0,\ldots,x_n) = \sum_{i=0}^{n}(-1)^i\phi(x_0,\ldots,\hat{x}_i,\ldots,x_n)$$

It is an acyclic complex.

We will refer to points in X^{n+1} as *n-simplices*. $v(\sigma)$ will denote the set of vertices in a simplex σ. We will refer a continuous map $f : \Delta^n \to X$ as *n-singular simplex*. In what follows, we will need to measure distances between simplices of different dimensions. A convenient way to do this is to stabilize simplices by repeating the last coordinate, as follows. Let $i : X^{n+1} \to X^\infty$ denote the map $(x_0,\ldots,x_n) \mapsto (x_0,\ldots,x_n,x_n,x_n,\ldots)$. For a cochain ϕ define its stabilized support $|\phi| = \{i(\sigma) \mid \sigma \in X^{n+1} \text{ and } \phi(\sigma) \neq 0\} \subset X^\infty$. Equip X^∞ with the sup metric.

Let $\Delta = i(X)$ denote the diagonal of X^∞. For any subset $A \subset X$, let Δ_A denote $i(A) \subset i(X)$. Define the support of ϕ at scale r to be $|\phi|_r = |\phi| \cap N_r(\Delta)$.

With this notation, coarse cochain complex $\mathrm{C\,X}^*(X)$ can be wriitten as follows.

$$\mathrm{C\,X}^*(X;R) := \{\phi \in \mathrm{C}^*(X;R) \mid \forall r \quad |\phi|_r \overset{\mathrm{c}}{=} *\}$$

Let us now recall some other (co)chain complexes which we will use throughout this article. We will denote the support used by Alexander–Spanier by $\|\phi\| = Cl(|\phi|) \cap \Delta$. By identifying Δ with X, we think of $\|\phi\|$ as a subset of X. The following two definitions below use this support.

$\mathrm{C}^*_{\mathrm{c}}(X;R) = \{\phi \in \mathrm{C}^*(X;R) \mid \|\phi\| \text{ is compact}\}$ — compactly supported cochains.

$\mathrm{C}^*_0(X;R) = \{\phi \in \mathrm{C}^*(X;R) \mid \|\phi\| = \emptyset\}$ — locally zero cochains.

$\mathrm{C}^*_{as}(X;R) = \mathrm{C}^*(X;R)/\,\mathrm{C}^*_0(X;R)$ — the cochain complex for the Alexander–Spanier cohomology $\mathrm{H}^*(X;R)$.

$\mathrm{C}^*_{cas}(X;R) = \mathrm{C}^*_{\mathrm{c}}(X;R)/\,\mathrm{C}^*_0(X;R)$—the cochain complex for the compactly supported

Alexander–Spanier cohomology $\mathrm{H}_c^*(X;R)$.

$\mathrm{C}_*(X;R) := \{\sum_{i=0}^n c_i\sigma_i \mid c_i \in R \text{ and } \sigma_i \in X^{*+1}\}$

$\mathrm{C}_*^{lf}(X;R) := \{c \in \mathrm{C}_*(X;R) \mid \#\{|c| \cap B^{*+1}\} < \infty$ for any bounded set $B\}$—locally finite chains.

Chapter 2

Coarse cohomology of the complement

2.1 Introduction

In this chapter we introduce and study the coarse cohomology of the complement. We use this theory to recast the coarse Alexander duality theorems of Kapovich and Kleiner [12] into the framework of the coarse homology and cohomology theories of Roe [15, 16]. Recall that for A, a subcomplex of S^n, the classical Alexander duality is given by the isomorphisms $\mathrm{H}^*(S^n - A) \cong \mathrm{H}_{n-1-*}(A)$ and $\mathrm{H}_*(S^n - A) \cong \mathrm{H}^{n-1-*}(A)$. The results of [12] are similar in spirit in the context of subsets of coarse Poincaré duality spaces, but their actual statements are quite technical. Our main result is the following theorem (Theorem 2.6.2):

Theorem 2.1.1. *If X is a coarse PD(n) space, then for any $A \subset X$*

$$\mathrm{H\,X}^k(X - A) \cong \mathrm{H\,X}_{n-k}(A),$$

$$\mathrm{H\,X}_k(X - A) \cong \mathrm{H\,X}^{n-k}(A).$$

Our main innovation is in the definition of the left-hand sides of these formulas, and in fact in introduction of a notion of the coarse complement.

Naively, since in coarse geometry one is allowed to replace any subset with it's metric neighborhood, the coarse complement $X - A$ should be thought as the complement of (perhaps bigger and bigger) metric neighborhoods, $X - N_r(A)$. If A is bounded, this is indeed the right picture, as all these complements are coarsely equivalent to each other (and in fact to X). However for unbounded subsets it is not always the right notion. For example, If X is a Euclidean plane and A is a line in X, then $X - N_r(A)$ are still all coarsely equivalent to X, contrary to the idea that a line coarsely separates a plane, and therefore the coarse topology should change.

Instead we consider complements of *expanding* neighborhoods $X - N_f(A)$, where $N_f(A) = \bigcup_{x \in A} B(x, f(x))$ for proper functions $f : A \to (0, \infty)$. These complements form a directed system of metric spaces under inclusion (in the direction of slower growth of f), and we define coarse (co)chain complexes of the complement $X - A$ as appropriate limits of the usual coarse complexes of these spaces.

Our second observation, inspired by [14], is that almost all relevant information is captured by the notion of *coarse containment*. We say A is coarsely contained inside B if $A \subset N_r(B)$ for some r. In some sense coarse geometry is the study of the poset of the equivalence classes of subsets of X induced by this relation. From this point of view the coarse complement of A in X is the collection of classes of subsets which are coarsely disjoint from A. i.e. have bounded intersection with any metric neighborhood of A. This leads us to a very explicit description of the limiting complexes in terms of support, which are very similar to Roe's definition of the coarse cochains. This description is key to our proof the Alexander duality.

Moreover, this description allows us to identify the coarse type of complement of A in X with the coarse type of the metric space $(X/\bar{A}, d_A)$ whose metric comes from the pseudometric $d_A(x, y) = \min\{d(x, A) + d(y, A), d(x, y)\}$ on X. This identification will be helpful in computations of $\mathrm{HX}^*(X - A)$ in section 3.5.

Our definition of coarse $\mathrm{PD}(n)$ space is a slight generalization of the definition of Kapovich and Kleiner, in terms of controlled chain homotopy equivalence between coarse chain and cochain complexes. We do not require any local structure. The classical Poincaré duality maps on manifolds are given by the cap product with the fundamental class in one directions and the slant product with the orientation class in the other. We define a coarse version of these classes (called *orientation pair*) and prove a similar statement for coarse PD(n) spaces (Theorem 2.8.2):

Theorem 2.1.2. *X is a coarse PD(n) space if and only if it has an orientation pair.*

As a corollary, we obtain that uniformly acyclic topological manifolds are coarse PD(n) spaces, a similar statement is claimed in [12] with only a sketch of the proof.

Remark. This theorem, as well as the Alexander duality, indicate that the term "coarse PD(n) space" is perhaps a misnomer, a more appropriate name would be "coarse homology manifold".

Overview. This chapter is organized as follows. In Section 1.2, we introduce coarse language and state basic results about coarse containment. We also describe several variations of Alexander–Spanier cochains and the corresponding cohomology theories.

Section 2.2 contains our main definition of the coarse (co)homology of a complement. The main result there is Lemma 2.2.1, which gives an equivalent description of the cochain complex.

In Section 2.3, we discuss a more geometric version of our approach. In section 2.4 we show elements in $\mathrm{H\,X}^1(X - A; \mathbb{Z}_2)$ corresponds to different ways that A can coarsely separate X into.

In section 2.5, we turn our attention to coarse homology. Unlike the majority of the literature, which uses anti-Čech covers, we give a direct description of coarse chains. The same description was used by S. Hair in his PhD thesis [8]. We generalize this to complements. The main result here is that chains and cochains are dual of each other in the sense of taking $\mathrm{Hom}(\cdot, \mathbb{Z})$.

Finally, in Section 2.6 we introduce the notion of a coarse Poincaré duality space, and prove the main result of this paper, the Coarse Alexander Duality, Theorem 2.6.2.

In Section 2.7 and Section 2.8, we derive a criterion for a space to be coarse PD(n) that allows us to prove that any uniformly contractible n-manifold is a coarse PD(n) space.

2.2 Coarse cohomology of complements

Let $A \subset X$. A straightforward approach to define the coarse cohomology of the complement $X - A$ is to look at cochains on X which restrict to coarse cochains on complements of expanding neighborhoods of A, as follows. Given a proper function $f : A \to \mathbb{R}_{\geq 0}$, the corresponding expanding neighborhood of A is obtained by taking the union of balls of radius $f(x)$ centered at x as x varies through A

$$N_f(A) = \bigcup_{x \in A} B(x, f(x)).$$

Note that the balls of radius 0 are empty, so $N_f(A)$ might not be an actual neighborhood of A.

The complements of expanding neighborhoods $X - N_f(A)$ form a directed system under the inclusion, and we are interested in the limit in the direction of slower growth of f. In other words, we want the complement to be as big as possible.

Then the coarse chains and cochains on the complements of expanding neighborhoods $X - N_f(A)$ form a direct and an inverse systems, respectively. Define the coarse chains and cochains of the complement to be the (co)limits of these systems. Since the system of complements $X - N_f(A)$ is cofinal in the system of sets coarsely disjoint from A, we have

$$\mathrm{C\,X}_n(X - A) = \{c \in \mathrm{C}_n^{lf}(X) \mid \exists C \overset{\mathrm{c}}{\subset} X \overset{\mathrm{c}}{-} A \quad c \in \mathrm{C\,X}_n(C)\}$$
$$\mathrm{C\,X}^n(X - A) = \{\phi \in \mathrm{C}^n(X) \mid \forall C \overset{\mathrm{c}}{\subset} X \overset{\mathrm{c}}{-} A \quad \phi|_C \in \mathrm{C\,X}^n(C)\}$$

The usual (co)boundary operators make $\mathrm{C\,X}_*(X - A)$ and $\mathrm{C\,X}^*(X - A)$ into complexes, and we call their (co)homology *the coarse (co)homology of the complement* $\mathrm{H\,X}_*(X - A)$ and $\mathrm{H\,X}^*(X - A)$. Note that $(X - A)$ here is a part of notation, not a set-theoretic complement.

The next two Lemmas provide more explicit descriptions of these complexes.

Lemma 2.2.1.

$$\mathrm{C\,X}_n(X - A) = \{c \in \mathrm{C}_n^{lf}(X) \mid |c| \overset{c}{\subset} \Delta \overset{c}{-} \Delta_A\}$$
$$\mathrm{C\,X}^n(X - A) = \{\phi \in \mathrm{C}^n(X) \mid |\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A\}.$$

Proof. The condition $c \in \mathrm{C\,X}_n(C)$ for some $C \overset{c}{\subset} X \overset{c}{-} A$ implies that $|c| \overset{c}{\subset} \Delta_C$. Therefore $|c| \overset{c}{\subset} \Delta \overset{c}{-} \Delta_A$. Vice versa, if $|c| \overset{c}{\subset} \Delta \overset{c}{-} \Delta_A$, set $C = \bigcup_{i=0}^n p_i(|c|)$. Then $C \overset{c}{\subset} X \overset{c}{-} A$ and $|c| \subset C^{n+1}$, so $c \in \mathrm{C\,X}_n(C)$.

The condition $\phi|_C$ is coarse means that $|\phi| \cap C^{n+1}$ is coarsely disjoint from Δ_C for all $C \overset{c}{\subset} X \overset{c}{-} A$. Since we can replace C with its metric neighborhood, this is equivalent to $|\phi| \overset{c}{\cap} C^{n+1} \overset{c}{\cap} \Delta_C \overset{c}{=} *$, which we can rewrite as $|\phi| \overset{c}{\cap} \Delta_C \overset{c}{=} *$. The condition $C \overset{c}{\subset} X \overset{c}{-} A$ is equivalent to $\Delta_C \overset{c}{\subset} \Delta \overset{c}{-} \Delta_A$, so we have $(\Delta \overset{c}{-} \Delta_A) \subset (\Delta \overset{c}{-} |\phi|)$. By Lemma 1.2.3 this is equivalent to $|\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A$, as claimed. □

We consider a slight generalization of the previous (co)chain complexes. Let $A, B \subset X$.

$$\mathrm{C\,X}_*(B - A \subset X) = \{c \in \mathrm{C}_*^{lf}(X) \mid |c| \overset{c}{\subset} \Delta_B \overset{c}{-} \Delta_A\}$$
$$\mathrm{C\,X}^*(B - A \subset X) = \{\phi \in \mathrm{C}^*(X) \mid |\phi| \overset{c}{\cap} \Delta_B \overset{c}{\subset} \Delta_A\}.$$

We claim that the homology of these complexes does not depend on the ambient space X, it depends only on the metric on $B \cup A$.

Lemma 2.2.2. *The maps induced by inclusion* $i_* : \mathrm{C\,X}_*(B - A \subset B \cup A) \to \mathrm{C\,X}_*(B - A \subset X)$ *and* $i^* : \mathrm{C\,X}^*(B - A \subset X) \to \mathrm{C\,X}^*(B - A \subset B \cup A)$ *are chain homotopy equivalences.*

Proof. Choose an approximate projection $\pi : X \to B \cup A$ with the property that π restricts to id on $B \cup A$ and for all $x \in X$, $d(x, \pi(x)) < d(x, B \cup A) + 1$. This map induces a map on n-simplices $\pi : X^{n+1} \to (B \cup A)^{n+1}$, and we claim that it extends linearly to a well-defined map $\pi_n : \mathrm{C\,X}_n(B - A \subset X) \to \mathrm{C\,X}_n(B - A \subset B \cup A)$.

Indeed, if $c \in \mathrm{C\,X}_n(B - A \subset X)$, then there exists R such that $|c| \overset{R}{\subset} \Delta_B$. Since $\pi|_{N_R(\Delta_{B\cup A})}$ is $(R+1)$-close to the identity, it follows that $\pi_n(c)$ is a well defined chain in $\mathrm{C\,X}_n(B - A \subset B \cup A)$.

Next define

$$D_n(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, x_i, \pi(x_i), \dots, \pi(x_n)).$$

A similar argument shows that D_n extends linearly to a well defined map $D_n : \mathrm{C\,X}_n(B - A \subset X) \to \mathrm{C\,X}_{n+1}(B - A \subset X)$. Finally, we note that $\pi_* i_* = id$ and $i_* \pi_* - id = D_* \partial + \partial D_*$.

The approximate projection π induces a map $\pi^* : \mathrm{C}^*(B \cup A) \to \mathrm{C}^*(X)$, $\pi^*(\phi) = \phi\pi$. Thus we have $|\pi^*(\phi)| \cap N_R(\Delta_A) \subset (\pi|_{N_R(\Delta_A)})^{-1}(|\phi|)$ and $|\pi^*(\phi)| \cap N_R(\Delta_B) \subset (\pi|_{N_R(\Delta_B)})^{-1}(|\phi|)$. Since $\pi|_{N_R(\Delta_{B\cup A})}$ is close to identity, it follows that π^* restricts to a map $\pi^* : \mathrm{C\,X}^*(B - A \subset B \cup A) \to \mathrm{C\,X}^*(B - A \subset X)$. Similarly D^* defined by $D^*(\phi) = \phi D_*$ gives a chain homotopy between $\pi^* i^*$ and id. □

It follows that we can write unambiguously $\mathrm{H\,X}_*(B - A)$ and $\mathrm{H\,X}^*(B - A)$. Note that when $B = X$ or $A = \emptyset$ this agrees with the previous notation.

2.3 Coarse complement as a space

As we already mentioned, $X \overset{c}{-} A$ rarely has a coarse type of a subspace of X. However, it is still possible to compute the coarse cohomology of the complement as the coarse cohomology of a single space, but this space is not a subspace of X, but rather a quotient space.

We start by defining a pseudometric d_A on X by

$$d_A(x, y) = \min(d(x, A) + d(y, A), d(x, y)).$$

Lemma 2.3.1.

1. $N_r^d(A^{n+1}) = N_r^{d_A}(A^{n+1})$

2. $N_r^d(\Delta) \subset N_r^{d_A}(\Delta)$

3. $N_r^{d_A}(\Delta) \subset N_{2r}^d(\Delta \cup A^{n+1})$

Proof. (1) and (2) follow from the facts that $d_A(x, A) = d(x, A)$ and $d_A(x, y) \leq d(x, y)$.

To prove (3), let $\sigma \in N_r^{d_A}(\Delta)$. So we have $d_A(\sigma, x) < r$, for some $x \in \Delta$. If $d(x, A^{n+1}) \leq r$, then $d(\sigma, A) = d_A(\sigma, A^{n+1}) \leq d_A(\sigma, x) + d_A(x, A^{n+1}) < 2r$ and $\sigma \in N_{2r}^d(A^{n+1})$. If $d_A(x, A^{n+1}) > r$ then, since $d_A(\sigma, x) < r$, it follows from the definition of d_A that $d_A(x, \sigma) = d(x, \sigma)$ and $\sigma \in N_{2r}^d(\Delta)$. □

Much of the coarse metric theory works the same for pseudometric spaces. In particular, we have the usual definition of coarse (co)chains.

$$\mathrm{C\,X}_n(X, d_A) = \{c \in \mathrm{C}_n^{lf}(X, d_A) \mid |c| \overset{\mathrm{c}, d_A}{\subset} \Delta\}$$
$$\mathrm{C\,X}^n(X, d_A) = \{\phi \in \mathrm{C}^n(X) \mid |\phi| \overset{\mathrm{c}, d_A}{\cap} \Delta \overset{\mathrm{c}, d_A}{=} *\}.$$

Proposition 2.3.2.

$$\mathrm{C\,X}_n(X, d_A) = \mathrm{C\,X}_n(X - A)$$
$$\mathrm{C\,X}^n(X, d_A) = \mathrm{C\,X}^n(X - A).$$

Proof. Let $\phi \in \mathrm{C\,X}^n(X, d_A)$. Then $|\phi| \overset{\mathrm{c}, d_A}{\cap} \Delta \overset{\mathrm{c}, d_A}{=} *$. Since A^{n+1} is bounded in (X^{n+1}, d_A), this is equivalent to $|\phi| \overset{\mathrm{c}, d_A}{\cap} \Delta \overset{\mathrm{c}, d_A}{\subset} A^{n+1}$.

By Lemma 2.3.1 (1) and (2), we have $|\phi| \overset{\mathrm{c}, d}{\cap} \Delta \overset{\mathrm{c}, d}{\subset} A^{n+1}$. It follows that $(|\phi| \overset{\mathrm{c}, d}{\cap} \Delta) \overset{\mathrm{c}, d}{\subset} (A^{n+1} \overset{\mathrm{c}, d}{\cap} \Delta) = \Delta_A$. Hence, $\phi \in \mathrm{C\,X}^n(X - A)$.

Let $\phi \in \mathrm{C\,X}^n(X - A)$. Then $|\phi| \overset{\mathrm{c}, d}{\cap} \Delta \overset{\mathrm{c}, d}{\subset} \Delta_A$. It follows that $|\phi| \overset{\mathrm{c}, d}{\cap} (\Delta \cup A^{n+1}) \overset{\mathrm{c}, d}{\subset} A^{n+1}$. By Lemma 2.3.1 (1) and (3) this implies $|\phi| \overset{\mathrm{c}, d_A}{\cap} \Delta \overset{\mathrm{c}, d_A}{\subset} A^{n+1}$. Hence $\phi \in \mathrm{C\,X}^n(X, d_A)$.

Let $c \in \mathrm{C\,X}^n(X, d_A)$. since any bounded set in (X, d) is bounded in (X, d_A) we have

$c \in \mathrm{C}^{lf}_n(X)$. To prove $|c| \overset{c,d}{\subset} \Delta \overset{c,d}{-} \Delta_A$, first note that $|c| \overset{c,d}{\subset} \Delta \cup A^{n+1}$ by Lemma 2.3.1(3). Now $c \in \mathrm{C}^{lf}_n(X, d_A)$ implies $\#\{|c| \overset{c,d}{\cap} A^{n+1}\}$ is finite by Lemma 2.3.1 (1). So we have $|c| \overset{c,d}{\subset} \Delta$.

Let $c \in \mathrm{C\,X}^n(X - A)$. Since $|c| \overset{c,d}{\subset} \Delta$, by Lemma 2.3.1(2) we get $|c| \overset{c,d_A}{\subset} \Delta$. Now, $|c| \overset{c,d}{\cap} \Delta_A \overset{c,d}{=} *$ and $c \in \mathrm{C}^{lf}_n(X)$ implies $\#\{|c| \overset{c,d}{\cap} \Delta_A\}$ is finite. By Lemma 2.3.1 (1) we have $\#\{|c| \overset{c,d_A}{\cap} \Delta_A\}$ is finite i.e $c \in \mathrm{C}^{lf}_n(X, d_A)$. Hence $c \in \mathrm{C\,X}^n(X, d_A)$. □

Since $d_A(x, y) = 0$ if and only if $(x, y) \in \overline{A} \times \overline{A}$, the pseudometric d_A becomes a metric on the quotient space $X/\overline{A}$. One easily checks that the quotient map $q : (X, d_A) \to (X/\overline{A}, d_A)$ is a coarse equivalence, and therefore it induces an isomorphism on the coarse (co)homology. Using the previous proposition we obtain the following.

Proposition 2.3.3. *The quotient map $q : X \to X/\overline{A}$ induces isomorphisms* $\mathrm{H\,X}_*(X/\overline{A}) = \mathrm{H\,X}_*(X - A)$ *and* $\mathrm{H\,X}^*(X/\overline{A}) = \mathrm{H\,X}^*(X - A)$.

Remark. Another coarse model for $(X \overset{c}{-} A)$ can be obtained by the "coning off" construction, popular in the study of relatively hyperbolic groups. Here instead of taking a quotient by A one attaches a cone cA on A, and declares $d(c, a) = 1$ for all $a \in A$.

Next we address the following question: What kind of maps between X and Y induce a map between $\mathrm{H\,X}^*(Y - B)$ and $\mathrm{H\,X}^*(X - A)$? We begin with the following definitions.

Definition ([15])**.** Suppose X and Y are pseudo metric spaces. A map $f : X \to Y$ is called a *proper map* if $f^{-1}(B)$ is bounded in X whenever B is bounded in Y

Definition ([15])**.** A map $f : X \to Y$ is called a *coarse map* if f is proper and there exist a non-decreasing function $\rho : [0, \infty) \to [0, \infty)$ so that for any two points $x, y \in X$ the following holds.

$$d(f(x), f(y)) \le \rho(d(x, y))$$

Proposition 2.3.4 ([15])**.** *A coarse map $f : X \to Y$ induces a chain map $f^* : \mathrm{C\,X}^*(Y) \to \mathrm{C\,X}^*(X)$ by the usual formula*

$$(f^*\phi)(x_0, \ldots, x_n) := \phi(f(x_0), \ldots, f(x_n))$$

Proof. We need to check that $f^*(\phi) \in \mathrm{C\,X}^*(X)$ if $\phi \in \mathrm{C\,X}^*(Y)$. Suppose, $f_* : X^{*+1} \to Y^{*+1}$ is the induced map by f. Note that, $|f^*(\phi)| \cap N_r(\Delta) \subset (f_*)^{-1}(|\phi| \cap f_*(N_r(\Delta)))$. Since f is proper f_* is also proper with respect to the sup metrics. Hence it is enough to prove that $|\phi| \cap f_*(N_r(\Delta))$ is bounded. Since f is a coarse map, there exists an $s > 0$ such that $f_*(N_r(\Delta)) \subset N_s(\Delta)$. That implies $|\phi| \cap f_*(N_r(\Delta)) = |\phi| \cap N_s(\Delta)$ is bounded as $\phi \in \mathrm{C\,X}^*(Y)$. □

Proposition 2.3.2 suggests the following notion of coarse map in the coarse complement category.

Definition (Coarse map between coarse complements)**.** Suppose $A \subset X$ and $B \subset Y$. A map $f : X \to Y$ is called a *coarse map between the coarse complements of A and B*, if the induced map $f : (X, d_A) \to (Y, d_B)$ is a coarse map.

The following Proposition is now immediate from Proposition 2.3.2 and 2.3.4.

Proposition 2.3.5. *If $f : X \to Y$ is a coarse map between the coarse complements of A and B, then f induces a chain map $f^* : \mathrm{C\,X}^*(Y - B) \to \mathrm{C\,X}^*(X - A)$.*

2.4 Separations and $\mathrm{H\,X}^1(X - A; \mathbb{Z}/2)$

We now consider coarse cohomology with $\mathbb{Z}/2$-coefficients $\mathrm{H\,X}^*(X - A; \mathbb{Z}/2)$. It is the cohomology of the cochain complex

$$\mathrm{C\,X}^n(X - A; \mathbb{Z}/2) = \{\phi \in \mathrm{C}^n(X; \mathbb{Z}/2) \mid |\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A\}.$$

We will see that $\mathrm{H\,X}^1(X - A; \mathbb{Z}_2)$ measures the number of coarse components of the complement of A. To make this precise we need following definitions.

There is a simple interpretation of 1-cocycles with $\mathbb{Z}/2$-coefficients in terms of subsets of X. The power set 2^X is a boolean algebra, with multiplication given by the intersection and addition by the symmetric difference. The complementation map $C \to X - C$ is a free involution on 2^X. Let $\mathcal{S}$ be the quotient space, we will think of its elements as unordered pairs $\{C, X - C\}$, and refer to them as *separations* of X. Alternatively, since $X - C = X \triangle C$, one can think of $\mathcal{S}$ as a quotient of abelian groups $2^X/\{\emptyset, X\}$. As an abelian group 2^X is isomorphic to $\mathrm{C}^0(X; \mathbb{Z}/2)$, the isomorphism is given by taking supports and the map $C \to 1_C$. Thus we get a commutative diagram, where F is the induced isomorphism.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \{\emptyset, X\} & \longrightarrow & 2^X & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle |\cdot|}\uparrow\downarrow{\scriptstyle 1_{(\cdot)}} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathrm{C}^0(X; \mathbb{Z}/2) & \xrightarrow{d} & Z^1(X; \mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

More explicitly, $F(\{C, X - C\}) = d(1_C)$, and he inverse map is given by the following construction. A cocycle $\phi \in ZX_1(X; \mathbb{Z}_2)$ defines a relation on X: $x \sim y$ if $\phi(x, y) = 0$. Since ϕ is a cocycle, it is an equivalence relation with exactly two equivalence classes, i.e. a separation.

We now introduce coarse supports into the picture. Some of the following definitions are inspired by [13]. Let $A \subset X$. $C \subset X$ is a *coarse complementary component* of A if $C \overset{c}{\cap} (X - C) \overset{c}{\subset} A$. A coarse complementary component C of A is *shallow* if $C \overset{c}{\subset} A$, otherwise it is *deep*. Let $\mathcal{C}_A$ denote the set of all coarse complementary components of A. As before, each $C \in \mathrm{C}^*_A$ determines a separation $\{C, X - C\}$, we will refer to it as *separations of* X *with respect to* A.

The coarse r-boundary of C is

$$\partial_r C := \{x \in X - C \mid d(x, C) \leq r\}.$$

The following Lemma follows easily from Lemma 1.2.1.

Lemma 2.4.1. *C is a coarse complementary component of A if and only if $\partial_r(C) \overset{c}{\subset} A$ for all r.*

Since $\partial_r(C \cup B) \subset \partial_r(C) \cup \partial_r(B)$, $\mathcal{C}_A$ is closed under taking unions, and since it is closed under the complementation, it follows that $\mathcal{C}_A$ is a subalgebra of the boolean algebra 2^X and $\mathcal{SS}_A \subset \mathcal{S}_A$ are subgroups of $\mathcal{S}$.

Note that $1_C \in \mathrm{C\,X}^0(X - A; \mathbb{Z}/2)$ precisely when C is shallow, so F maps isomorphically $\mathcal{SS}_A$ onto $BX^1(X - A; \mathbb{Z}/2)$. Moreover $|d(1_C)| \overset{c}{\cap} \Delta_X = (C \times (X - C) \cup (X - C) \times X) \overset{c}{\cap} \Delta_X = \Delta_C \overset{c}{\cap} \Delta_{X-C}$, so F restricts to an isomorphism $F : \mathcal{S}_A \to ZX^1(X - A; \mathbb{Z}/2)$.

Thus we have:

Proposition 2.4.2. *F induces an isomorphism $\mathcal{S}_A/\mathcal{SS}_A \to \mathrm{H\,X}^1(X - A; \mathbb{Z}/2)$.*

For geodesic spaces we have the following interpretation of coarse complementary components:

Proposition 2.4.3. *If X is a geodesic space and $A \subset X$, then C is a coarse complementary component of A if and only if there exist r such that $C - N_r(A)$ is a union of path components of $X - N_r(A)$.*

Proof. ($\Rightarrow$) Let C be a coarse complementary component of A in X, then there exists r such that $\partial_1(C) \subset N_r(A)$. We claim that $C' = C - N_{r+1}(A)$ is a union of path components of $X - N_{r+1}(A)$.

Suppose $x \in C'$ and $y \in X - N_{r+1}(A)$ are connected by a path in $X - N_{r+1}(A)$. We need to show that $y \in C'$.

Choose points $\{x = x_0, \ldots, x_n = y\}$ on the path so that $d(x_i, x_{i+1}) < 1$ for all i. Suppose $x_i \in C'$, then $d(x_{i+1}, A) \geq d(x_i, A) - d(x_i, x_{i+1}) > r + 1 - 1 > r$, so $x_{i+1} \notin N_r(A)$, and therefore $x_{i+1} \notin \partial_1(C)$. On the other hand, $d(x_{i+1}, C) \leq d(x_i, x_{i+1}) < 1$. Combining these together we get $x_{i+1} \in C'$. So by induction $y \in C'$.

($\Leftarrow$) Suppose that for some r, $C' = C - N_r(A)$ is a union of path components of $X - N_r(A)$. We will show that C' is a coarse complementary component, and hence so is C. We claim that for any R, $\partial_R(C') \overset{R+r}{\subset} A$.

Let $x \in \partial_R(C')$. Then there exists $y \in C'$ such that $d(x, y) \leq R$. Since $x \notin C'$ and $y \in C'$, x and y are in different path components of $X - N_r(A)$, thus a geodesic between x and y must dip into $N_r(A)$: there exists z on the geodesic with $d(z, A) < r$. Note that $d(x, z) \leq d(x, y) \leq R$. Then by triangle inequality $d(x, A) \leq d(x, z) + d(z, A) < R + r$, as claimed.

□

2.5 Algebraic duality

We will show that chains and cochains with $\mathbb{Z}$ coefficients are dual to each other in the sense of taking $\mathrm{Hom}(\cdot, \mathbb{Z})$. This requires some algebraic preliminaries, most of which can be found in [7].

Let R be a commutative ring with 1. Consider a countable direct product $R^{\mathbb{N}}$. Let e_i denote the image in $R^{\mathbb{N}}$ of 1 in the i^{th} factor. Given an R-module H and homomorphism $\phi : R^{\mathbb{N}} \to H$, the support of ϕ is $|\phi| = \{i \in \mathbb{N} \mid \phi(e_i) \neq 0\}$. H is *slender* if for every homomorphism $\phi : R^{\mathbb{N}} \to H$, $|\phi|$ is finite. A ring R is slender if it is a slender module over itself.

The following is a classical result, going back to Specker.

Lemma 2.5.1 (cf. Corollary III.2.4 in [7])**.** *$\mathbb{Z}$ is slender.*

For the sake of completeness, we provide a proof.

Proof. Suppose the support of $\phi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ is infinite. We will build inductively an element of $\mathbb{Z}^{\mathbb{N}}$ of the form $v = \sum 2^{n_i} e_i$ with fast growing powers of 2. For each k, let $v = h_k + t_k$ be the splitting of v into the head of length $k - 1$ and the infinite tail. At each step k we already have h_k ($h_1 = 0$), so choose n_k so that $2^{n_k} > k + |\phi(h_k)|$ and $n_k > n_{k-1}$.

Let $k \geq |\phi(v)|$. Since $\phi(v) = \phi(h_k) + \phi(t_k)$, we have

$$|\phi(t_k)| \leq |\phi(v)| + |\phi(h_k)| \leq k + |\phi(h_k)| < 2^{n_k}.$$

But $\phi(t_k)$ is a multiple of 2^{n_k}, since t_k is. So $\phi(t_k) = 0$ and therefore $2^{n_{k+1}}\phi(e_{k+1}) = \phi(t_{k+1}) - \phi(t_k) = 0$, a contradiction. □

Theorem 2.5.2 (cf. Corollary III.3.4 in [7])**.** *If H is slender and I is not ω-measurable cardinal, then for any $\phi : R^I \to H$, if $\phi_{|\bigoplus_I R} = 0$, then $\phi = 0$.*

We recall that a cardinal is *ω-measurable* if it supports non-principal ultrafilter which is closed under countable intersections. The existence of ω-measurable cardinals cannot be proved in ZFC, and it is unknown if the existence is consistent with ZFC. Moreover, if exists, the first ω-measurable cardinal κ must be strongly inaccessible, ie if $\lambda < \kappa$, then $2^\lambda < \kappa$ and κ cannot be obtained as λ-union of λ cardinals.

Let $\mathcal{S}$ be a family of subsets of I which is closed under taking subsets and finite unions. These conditions ensure that the set of $\mathcal{S}$-supported functions $M_\mathcal{S} = \{a : I \to R \mid |a| \in \mathcal{S}\}$ is a submodule of R^I, so we will call such $\mathcal{S}$ a *supportive* family. The dual of $\mathcal{S}$, given by $\mathcal{S}^* = \{T \subset I \mid \forall S \in \mathcal{S} \quad \#(S \cap T) < \infty\}$ is also supportive. Note that there is a natural bilinear pairing $\langle \, , \rangle : M_\mathcal{S} \times M_{\mathcal{S}^*} \to R$ given by $\langle a, b\rangle = \sum_{i \in I} a(i)b(i)$.

Lemma 2.5.3. *Suppose R is slender, the cardinality I is less than the first ω-measurable cardinal, and $\mathcal{S}$ is a supportive family which covers I. Then the evaluation map $e : M_{\mathcal{S}^*} \to \operatorname{Hom}(M_\mathcal{S}, R)$ given by $e(b)(a) = \langle a, b\rangle$ is an isomorphism.*

Proof. The injectivity of e follows easily from the hypothesis that $\mathcal{S}$ covers I.

To prove surjectivity, let $f \in \operatorname{Hom}(M_\mathcal{S}, R)$. We claim that $|f| \in \mathcal{S}^*$. Indeed, if the intersection $|f| \cap S$ is infinite for some $S \in \mathcal{S}$, then it contains an infinite countable subset C, and thus $R^C \subset M_\mathcal{S}$ and $|f_{|R^C}| = C$, contradicting slenderness of R.

We also claim that $|f| = \emptyset$ implies $f = 0$. If $f(a) \neq 0$ for some $a \in M_\mathcal{S}$, then since $M_\mathcal{S}$ contains $R^{|a|}$ we get a contradiction to Theorem 2.5.2 for $f|_{R^{|a|}}$.

The first claim shows that the characteristic function $1_{|f|} \in M_{\mathcal{S}^*}$, and the second claim implies that $f - e(1_{|f|}) = 0$. □

Let X be a metric space with two subspaces A and B. The families

$$\mathcal{H} = \{S \subset X \mid S \overset{c}{\subset} B \text{ and } \forall r \quad \#(S \cap N_r(A)) < \infty\}$$

$$\mathcal{C} = \{T \subset X \mid T \overset{c}{\cap} B \overset{c}{\subset} A\}$$

are supportive and cover X.

Lemma 2.5.4. *The families $\mathcal{H}$ and $\mathcal{C}$ are dual to each other: $\mathcal{H}^* = \mathcal{C}$ and $\mathcal{C}^* = \mathcal{H}$.*

Proof. If Y belongs to both families, then $Y \overset{c}{\subset} B$, by the first condition of $\mathcal{H}$. Then $Y \overset{c}{=} Y \overset{c}{\cap} B$, so $Y \overset{c}{\subset} A$, by definition of $\mathcal{C}$. So Y has to be finite by the second condition of $\mathcal{H}$. This implies that if $S \in \mathcal{H}$ and $T \in \mathcal{C}$, then $S \cap T$ is finite, which means $\mathcal{H}^* \supset \mathcal{C}$ and $\mathcal{C}^* \supset \mathcal{H}$.

Now let $Y \in \mathcal{C}^*$. Since $\mathcal{C}$ contains all subsets which are coarsely disjoint from B, Y cannot contain an infinite subset which is coarsely disjoint from B. This implies that $Y \overset{c}{\subset} B$. Since $\mathcal{C}$ also contains all metric neighborhoods of A, Y must also satisfy the second condition of $\mathcal{H}$, so $Y \in \mathcal{H}$. Thus $\mathcal{C}^* \subset \mathcal{H}$.

Finally, if $Y \notin \mathcal{C}$, then Y contains an infinite locally finite subset which is coarsely disjoint form A and coarsely contained in B. This subset is in $\mathcal{H}$, so $Y \notin \mathcal{H}^*$. Thus $\mathcal{H}^* \subset \mathcal{C}$. □

From now on we will always assume that our space X is not ω-measurable.

Lemma 2.5.5. *The evaluation map e of cochains on chains induces isomorphisms of (co)chain complexes:*

$$\mathrm{C\,X}_*(B - A \subset X) = \mathrm{Hom}(\mathrm{C\,X}^*(B - A \subset X), \mathbb{Z}),$$

$$\mathrm{C\,X}^*(B - A \subset X) = \mathrm{Hom}(\mathrm{C\,X}_*(B - A \subset X), \mathbb{Z}),$$

Proof. The chain and cochain groups correspond to the modules $M_{\mathcal{H}}$ and $M_{\mathcal{C}}$ for the choice of $R = \mathbb{Z}$, $X = X^{n+1}$, and $B = \Delta_B$ and $A = \Delta_A$.

So, Lemmas 2.5.3 and 2.5.4 give isomorphisms in each degree and it is clear that the evaluation map commutes with the differentials. ☐

2.6 Alexander Duality

Definition. A metric space X is a *coarse Poincaré duality space of formal dimension* n (*coarse PD(n) space* for short), if there exist chain maps $p : \mathrm{C}^*(X) \to \mathrm{C\,X}_{n-*}(X)$ and $q : \mathrm{C\,X}_{n-*}(X) \to \mathrm{C}^*(X)$, so that pq and qp are chain homotopic to identities via chain homotopies $G : \mathrm{C\,X}_*(X) \to \mathrm{C\,X}_{*+1}(X)$ and $F : \mathrm{C}^*(X) \to \mathrm{C}^{*-1}(X)$ which are controlled:

$$\begin{aligned} \forall \phi \in \mathrm{C}^*(X) &\quad |p(\phi)| \overset{c}{\subset} |\phi| \\ \forall \phi \in \mathrm{C}^*(X) &\quad |F(\phi)| \overset{c}{\cap} \Delta \overset{c}{\subset} |\phi| \\ \forall c \in \mathrm{C\,X}_*(X) &\quad |q(c)| \overset{c}{\cap} \Delta \overset{c}{\subset} |c| \\ \forall c \in \mathrm{C\,X}_*(X) &\quad |G(c)| \overset{c}{\subset} |c| \end{aligned}$$

The next Lemma shows that the maps in the definition of a coarse PD(n) space restrict to maps between $\mathrm{C\,X}^*(X - A)$ and $\mathrm{C\,X}_*(A \subset X)$,

Lemma 2.6.1.

$$p(\mathrm{C\,X}^*(X - A)) \subset \mathrm{C\,X}_{n-*}(A \subset X) \qquad p(\mathrm{C\,X}^*(A \subset X)) \subset \mathrm{C\,X}_{n-*}(X - A) \tag{2.6.1}$$

$$F(\mathrm{C\,X}^*(X - A)) \subset \mathrm{C\,X}^{*-1}(X - A) \qquad F(\mathrm{C\,X}^*(A \subset X)) \subset \mathrm{C\,X}^{*-1}(A \subset X) \tag{2.6.2}$$

$$q(\mathrm{C\,X}_*(A \subset X)) \subset \mathrm{C\,X}^{n-*}(X - A) \qquad q(\mathrm{C\,X}_*(X - A)) \subset \mathrm{C\,X}^{n-*}(A \subset X) \tag{2.6.3}$$

$$G(\mathrm{C\,X}_*(A \subset X)) \subset \mathrm{C\,X}_{*+1}(A \subset X) \qquad G(\mathrm{C\,X}_*(X - A)) \subset \mathrm{C\,X}_{*+1}(X - A) \tag{2.6.4}$$

Proof. (2.6.1) By assumption on p, $|p(\phi)| \overset{c}{\subset} |\phi|$. Since p maps into $\mathrm{C\,X}_{n-*}(X)$, $|p(\phi)| \overset{c}{\subset} \Delta$. If $\phi \in \mathrm{C\,X}^*(X - A)$, then $|\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A$. Combining these using Lemma 1.2.2 we get $|p(\phi)| \overset{c}{\subset} \Delta_A$ and thus $p(\phi) \in \mathrm{C\,X}_{n-*}(A \subset X)$.

If $\phi \in \mathrm{C\,X}^*(A \subset X)$, then $|\phi| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$. So $|p(\phi)| \overset{c}{\subset} \Delta_X \overset{c}{-} \Delta_A$ and thus $p(\phi) \in \mathrm{C\,X}_{n-*}(X - A)$.

(2.6.2) By assumption on F, $|F(\phi)| \overset{c}{\cap} \Delta \overset{c}{\subset} |\phi|$. If $\phi \in \mathrm{C\,X}^*(X - A)$ then $|\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A$. Combining these we get $|F(\phi)| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A$. So $F(\phi) \in \mathrm{C\,X}^{*-1}(X - A)$.

If $\phi \in \mathrm{C\,X}^*(A \subset X)$, then $|\phi| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$. It follows that $|F(\phi)| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$ and $F(\phi) \in \mathrm{C\,X}^{*-1}(X - A)$.

(2.6.3) Let $c \in \mathrm{C\,X}_*(A \subset X)$, so $|c| \overset{c}{\subset} \Delta_A$. By assumption on q, $|q(c)| \overset{c}{\cap} \Delta \overset{c}{\subset} |c|$. Thus $|q(c)| \overset{c}{\subset} \Delta_A$. Hence $q(c) \in \mathrm{C\,X}^{n-*}(X - A)$.

Let $c \in \mathrm{C\,X}^*(X - A)$ then $|c| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$ Hence by the property of q, $|q(c)| \overset{c}{\cap} \Delta_A \overset{c}{\subset} |c|$. By Lemma 1.2.2, we have $q(c) \in \mathrm{C\,X}^{n-*}(A \subset X)$.

(2.6.4) Let $c \in \mathrm{C\,X}_*(A \subset X)$, so $|c| \overset{c}{\subset} \Delta_A$. By assumption on G, $|G(c)| \overset{c}{\subset} |c|$. Thus $|G(c)| \overset{c}{\subset} \Delta_A$. Hence $G(c) \in \mathrm{C\,X}_{*+1}(A_X)$.

Let $c \in \mathrm{C\,X}^*(X - A)$, so $|c| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$. By assumption on G, $|G(c)| \overset{c}{\subset} |c|$. Combining this we get $|G(c)| \overset{c}{\cap} \Delta_A \overset{c}{\subset} *$. □

Theorem 2.6.2. *If X is a coarse PD(n) space, then for any $A \subset X$*

$$\mathrm{H\,X}^k(X - A) \cong \mathrm{H\,X}_{n-k}(A),$$

$$\mathrm{H\,X}_k(X - A) \cong \mathrm{H\,X}^{n-k}(A).$$

Proof. By Lemmas 2.6.1 and 2.2.2, p and q induce maps between $\mathrm{H\,X}^*(X-A)$ and $\mathrm{H\,X}_{n-*}(A)$, and $\mathrm{H\,X}^*(A)$ and $\mathrm{H\,X}_{n-*}(X - A)$, which are inverses of each other. □

2.7 Products

Let δ denote the diagonal map from X to $X \times X$, i.e $\delta(x) = (x, x)$ for all $x \in X$.

We follow closely [18, Chapter 6].

To have control over homotopies we need to fix a particular choice of chain homotopy equivalence between $\mathrm{C}_*(X \times X)$ and $\mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$.

Denote $\mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ by $\mathrm{C}_*(X \otimes X)$ and $\mathrm{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \mathbb{Z})$ by $\mathrm{C}^*(X \otimes X)$. Define the support of $\sigma^k \otimes \tau^l$ to be $|\sigma \times \tau| \subset (X \times Y)^{(k+1)(l+1)}$.

For a simplex $\sigma = (x_0, \ldots, x_i, \ldots, x_k)$ let ${}_i\sigma = (x_0, \ldots, x_i)$ and $\sigma_{k-i} = (x_i, \ldots, x_k)$ denote its front and back faces. The Alexander–Whitney map is given by:

$$A : \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y),$$
$$\sigma^k \mapsto \sum_i {}_i p_X(\sigma) \otimes p_Y(\sigma)_{k-i}.$$

Let $\sigma = (x_0, \ldots, x_k)$ and $\tau = (y_0, \ldots, y_l)$ be simplices. Arrange pairs (x_i, y_j) as the vertices of an $k \times l$ rectangular grid in $\mathbb{R}^2$. To a path γ in this grid starting at (x_0, y_0) and ending at (x_k, y_l), always moving either north or east, associate a simplex ρ_γ in $X \times Y$, obtained by reading the labels along the path. The collection of all such simplices is a simplicial subdivision of $\sigma \times \tau$. Define a simplicial cross product

$$S : \mathrm{C}_*(X \otimes Y) \to \mathrm{C}_*(X \times Y),$$
$$\sigma^k \otimes \tau^l \mapsto \sigma \times \tau = \sum_\gamma (-1)^{|\gamma|} \rho_\gamma.$$

where $|\gamma|$ is the number of squares in the grid lying below the path γ.

Lemma 2.7.1. *The maps A and S are coarsely support preserving chain homotopy equivalences between $\mathrm{C\,X}_*(X \times Y)$ and $\mathrm{C\,X}_*(X \otimes Y)$.*

Proof. Note that $|S(\sigma \otimes \tau)| \subset |\sigma \times \tau|$ and $|A(\sigma)| \subset |p_X(\sigma) \times p_Y(\sigma)|$. The acyclic models produce chain homotopies $H : \mathrm{C}_*(X \otimes Y) \to \mathrm{C}_{*+1}(X \otimes Y)$ and $H' : \mathrm{C}_*(X \times Y) \to \mathrm{C}_{*+1}(X \times Y)$ between their compositions and identities, which satisfy the same support conditions. (The point is that the acyclic models use the same vertices.) Since $|p_X(\sigma) \times p_Y(\sigma)| \overset{\operatorname{diam}\sigma}{\subset} |\sigma|$ all these maps extend to coarsely support preserving maps between coarse chain complexes. □

Remark. A similar statement, in a slightly different language, is proved by Hair in [8], where an explicit formulas for the homotopies are given.

The cohomological cross product $\mathrm{C}^*(X)\otimes\mathrm{C}^*(Y) \to C(X\otimes Y)$ is defined by $(\phi\times\psi)(\sigma\otimes\tau) = \phi(\sigma)\times\psi(\tau)$.

We now consider the case $X = Y$. The Alexander–Whitney diagonal approximation is the composition:

$$\mathrm{C}_*(X) \xrightarrow{\delta_*} \mathrm{C}_*(X \times X) \xrightarrow{A} \mathrm{C}_*(X \otimes X).$$

Its tensor product with itself is used for the cap and cup products in $X \otimes X$:

$$\begin{aligned}\delta_*^A : \mathrm{C}_*(X \otimes X) &\to \mathrm{C}_*(X \otimes X \otimes X \otimes X)\\ \sigma^k \otimes \tau^l &\mapsto \sum_{i,j}(-1)^{j(k-i)}({}_i\sigma \otimes {}_j\tau) \otimes (\sigma_{k-i} \otimes \tau_{l-j}).\end{aligned}$$

Lemma 2.7.2. *The map* δ_A^* *satisfies* $|\delta_A^*(\phi)| \overset{\mathrm{c}}{\subset} \delta^{-1}|\phi|$.

Let $T : X \times X \to X \times X$ be the involution map switching factors. It induces the chain involutions T_* on $\mathrm{C\,X}_*(X\times X)$ and $\mathrm{C\,X}_*(X\otimes X)$, the latter one is given by $\sigma^k\otimes\tau^l \mapsto \varepsilon_{kl}\tau^l\otimes\sigma^k$.

Define a chain homotopy D_* between T_* and id in $\mathrm{C}_*(X \times X)$ by

$$D_*(x_0, \dots, x_m) = \sum(-1)^k(x_0, \dots, x_k, T(x_k), \dots, T(x_m))$$

Since metric neighborhoods of $\delta(X)$ in $X \times X$ are invariant under T, D extend to coarsely support preserving maps between coarse chain complexes $\mathrm{C\,X}_*(\delta(X) \subset X \times X)$.

Note that S commutes with T_*: $ST_* = T_*S$. So we can concatenate the homotopies from Lemma 2.7.1 with D to obtain a chain homotopy D^T between T_* and id in $\mathrm{C\,X}_*(\delta(X) \subset X \otimes X)$:

$$T_* \overset{H_*T_*}{\simeq} AST_* = AT_*S \overset{AD_*S}{\simeq} AS \overset{H'_*}{\simeq} id.$$

In other words, $D^T = H' + ADS - HT_*$ and $T_* - id = \partial D^T + D^T\partial$.

Lemma 2.7.3.

1. *The homotopy* $D_*^T : \mathrm{C}_*(X \otimes X) \to \mathrm{C}_*(X \otimes X)$ *satisfies* $|D_*^T(c)| \overset{c}{\cap} \Delta_{\delta(X)} \overset{c}{=} |c| \overset{c}{\cap} \Delta_{\delta(X)}$.

2. *The homotopy* $D_T^* : \mathrm{C}^*(X \otimes X) \to \mathrm{C}^*(X \otimes X)$ *satisfies* $|D_T^*(\phi)| \overset{c}{\cap} \Delta_{\delta(X)} \overset{c}{=} |\phi| \overset{c}{\cap} \Delta_{\delta(X)}$.

3. *The map* $T_* : \mathrm{C\,X}_*(\delta(X) \subset X \otimes X) \to \mathrm{C\,X}_*(\delta(X) \subset X \otimes X)$ *is controlled chain homotopic to the identity.*

The slant product is the composition

$$\mathrm{C}^*(X \times Y) \otimes \mathrm{C}_*(Y) \xrightarrow{S^* \otimes id} \mathrm{C}^*(X \otimes Y) \otimes \mathrm{C}_*(Y) \cong \mathrm{C}^*(X; \mathrm{C}^*(Y) \otimes \mathrm{C}_*(Y)) \xrightarrow{ev} \mathrm{C}^*(X).$$

Let $\varepsilon'_n = (-1)^{n(n+1)/2}$ and $\varepsilon_n = (-1)^n$.

Lemma 2.7.4. *Let* $\phi \in \mathrm{C\,X}^n(X \otimes X - \delta(X))$ *be a cocycle and let* $c \in \mathrm{C\,X}_n(X)$ *be a cycle. Then:*

1. *The slant product map,* $\varepsilon'_{n-k-1}\phi/\cdot : \mathrm{C\,X}_k(X) \to \mathrm{C}^{n-k}(X)$ *is a chain map and* $|\phi/a| \overset{c}{\cap} \Delta_X \overset{c}{\subset} |a|$ *for all* $a \in \mathrm{C\,X}_*(X)$.

2. *The cap product map* $\varepsilon'_{n-k} \cdot \frown c : \mathrm{C}^k(X) \to \mathrm{C\,X}_{n-k}(X)$ *is a chain map and* $|\psi \frown c| \overset{c}{\subset} |\psi|$ *for all* $\psi \in \mathrm{C}^k(X)$.

3. *The chain maps* $\mathrm{C\,X}_*(X) \to \mathrm{C\,X}_*(\delta(X) \subset X \otimes X)$ *given by* $a \mapsto \varepsilon_{nk}\phi \frown (c \otimes a)$ *and* $a \mapsto T^*\phi \frown (a \otimes c)$ *are controlled chain homotopic: the chain homotopy* D *satisfies* $|D(a)| \overset{c}{\subset} \delta(|a|)$.

4. *The chain maps* $\mathrm{C}^*(X) \to \mathrm{C\,X}^*(X \otimes X - \delta(X))$ *given by* $\psi \mapsto \varepsilon_{nk}\phi \smile p_1^*(\psi)$ *and* $\psi \mapsto p_2^*(\psi) \smile \phi$ *are controlled chain homotopic: the chain homotopy* D *satisfies* $|D(\psi)| \overset{c}{\cap} \Delta_{X \times X} \overset{c}{\subset} \delta(|\psi|) \overset{c}{\cap} \Delta_{\delta(X)}$.

Proof. (1) The slant product is defined by $(\phi/\tau)(\sigma) = \phi(\sigma \otimes \tau)$. Since $a \in \mathrm{C\,X}_*(X)$ and $\phi \in \mathrm{C\,X}^n(X \otimes X - \delta(X))$, it follows that $(\phi/a)(\sigma)$ is well-defined and $|\phi/a| \overset{c}{\cap} \Delta_X \overset{c}{\subset} |a|$. The choice of signs together with $d\phi = 0$ shows that $\phi/\cdot$ is indeed a chain map:

$$d(\varepsilon'_{n-k-1}\phi/a) = (-1)^{n-k}\varepsilon'_{n-k-1}\phi/\partial a = \varepsilon'_{n-k}\phi/\partial a.$$

(2) The cap product is defined by $\psi \frown \sigma = \psi(\sigma_k)\,{}_{n-k}\sigma$. Since $c \in \mathrm{C\,X}_n(X)$, $\psi \frown c$ is a well-defined element of $\mathrm{C\,X}_{n-k}(X)$ and $|\psi \frown c| \overset{c}{\subset} |\psi|$. Again the choice of signs together with $\partial c = 0$ shows that $\varepsilon'_{n-k} \cdot \frown c$ is a chain map:

$$\partial(\varepsilon'_{n-k}\psi \frown c) = (-1)^{n-k}\varepsilon'_{n-k}d\psi \frown c = \varepsilon'_{n-k-1}d\psi \frown c.$$

(3) Using the diagonal approximation δ_A^*, we arrive to the following definition of the cap product on $X \otimes X$: $\phi \frown (\sigma^k \otimes \tau^l) = \sum_{i+j=n}(-1)^{i(l-j)}(\phi(\sigma_i \otimes \tau_j) \otimes {}_{k-i}\sigma \otimes {}_{l-j}\tau)$. It follows as before that both maps are well-defined chain maps. For $a \in \mathrm{C\,X}_k(X)$ we have

$$\begin{aligned} T^*\phi \frown (a \otimes c) &= \varepsilon_{nk}T^*\phi \frown T_*(c \otimes a) \\ &= \varepsilon_{nk}T_*(T^{*2}\phi \frown (c \otimes a)) = \varepsilon_{nk}T_*(\phi \frown (c \otimes a)). \end{aligned}$$

By assumptions on c and a we have $|c| \times |a| \overset{c}{\subset} \Delta_{X\times X}$ and $|c| \times |a| \overset{c}{\cap} \Delta_{\delta(X)} \overset{c}{\subset} \delta(|a|)$. Therefore, by assumptions on ϕ,

$$|\phi \frown (c \otimes a)| \overset{c}{\subset} |\phi| \overset{c}{\cap} (|c| \times |a|) \overset{c}{\subset} |\phi| \overset{c}{\cap} \Delta_{X\times X} \overset{c}{\cap} (|c| \times |a|) \overset{c}{\subset} \delta(|a|).$$

The claim follows, since the homotopy between T_* and id coarsely preserves supports by Lemma 2.7.3.

(4) For $\psi \in \mathrm{C}^*(X)$ we have

$$p_2^*(\psi) \smile \phi = \delta_A^*(p_2^*(\psi) \times \phi) = \delta_A^* T_{X\times X}^*(\varepsilon_{nk}\phi \times T^* p_1^*(\psi)).$$

Since $\cdot \otimes 1$ and $\varepsilon_{nk}\phi \times \cdot$ are chain maps, the concatenation of the homotopies of T^* and $T_{X\times X}^*$

to identities gives a chain homotopy between the maps in the claim:

$$D(\psi) := \delta_A^* T_{X\times X}^*(\varepsilon_{k-1}\phi \times D_T^* p_1^*(\psi)) + \delta_A^* D_{T_{X\times X}}^*(\varepsilon_{nk}\phi \times p_1^*(\psi))$$

For the support of the first term we have:

$$\begin{aligned}
&\left|\delta_A^* T_{X\times X}^*(\phi \times D_T^* p_1^*(\psi))\right| \overset{c}{\cap} \Delta_{X\times X} \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|T_{X\times X}^*(\phi \times D_T^* p_1^*(\psi))| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|D_T^* p_1^*(\psi) \times \phi| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) && \text{by Lemma 2.7.1} \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|D_T^* p_1^*(\psi)| \times |\phi| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) \\
&\qquad \overset{c}{\subset} |D_T^* p_1^*(\psi)| \overset{c}{\cap} |\phi| \overset{c}{\cap} \Delta_{X\times X} \\
&\qquad \overset{c}{\subset} |D_T^* p_1^*(\psi)| \overset{c}{\cap} \Delta_{\delta(X)} && \text{by assumption on } \phi \\
&\qquad \overset{c}{\subset} |p_1^*(\psi)| \overset{c}{\cap} \Delta_{\delta(X)} && \text{by Lemma 2.7.3 (2)} \\
&\qquad \overset{c}{=} \delta(|\psi| \overset{c}{\cap} \Delta_X).
\end{aligned}$$

Similarly, for the support of the second term we have:

$$\begin{aligned}
&\left|\delta_A^* D_{T_{X\times X}}^*\big(\phi \times p_1^*(\psi)\big)\right| \overset{c}{\cap} \Delta_{X\times X} \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|D_{T_{X\times X}}^*(\phi \times p_1^*(\psi))| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|\phi \times p_1^*(\psi)| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) && \text{by Lemma 2.7.3 (2)} \\
&\qquad \overset{c}{\subset} \delta^{-1}\big(|\phi| \times |p_1^*(\psi)| \overset{c}{\cap} \Delta_{\delta(X\times X)}\big) \\
&\qquad \overset{c}{\subset} |\phi| \overset{c}{\cap} |p_1^*(\psi)| \overset{c}{\cap} \Delta_{X\times X} \\
&\qquad \overset{c}{\subset} |p_1^*(\psi)| \overset{c}{\cap} \Delta_{\delta(X)} && \text{by assumption on } \phi \\
&\qquad \overset{c}{=} \delta(|\psi| \overset{c}{\cap} \Delta_X).
\end{aligned}$$

□

The next three formulas tell how cup product, cap product and slant product interact with each other on the chain level.

Lemma 2.7.5. [18, 6.1.4-6] *Let σ and τ be simplices in X and let $\phi \in \mathrm{C}^*(X \otimes X)$, and $\psi \in \mathrm{C}^*(X)$. Then :*

1. $\psi \smile (\phi/\tau) = (p_1^*(\psi) \smile \phi]/\tau$.

2. $\phi/(\psi \frown \tau) = (\phi \smile p_2^*(\psi))/\tau$.

3. $(\phi/\sigma) \frown \tau = p_{1*}(\phi \frown (\tau \otimes \sigma))$.

2.8 Algebraic Poincaré Duality

In this section we find a criterion for a space to be a coarse PD(n) space in terms of existence of certain pair of (co)homology classes.

Definition. A pair of classes $U \in \mathrm{H\,X}^n(X \times X - \delta(X))$ and $[X] \in \mathrm{H\,X}_n(X)$ is an *orientation pair* if $U/[X] = 1 \in \mathrm{H\,X}^0(X - X)$. We will call U a *diagonal class* and $[X]$ a *fundamental class*.

Remark. This condition is stronger than it looks. Since the only representative of $1 \in \mathrm{H\,X}^0(X - X)$ is the constant function, the above equation is satisfied on the (co)chain level, independent of the choice of representatives for U and $[X]$. It also follows that the support of any representative of $[X]$ is coarsely equal to X.

Lemma 2.8.1. *Let $U \in \mathrm{H\,X}^n(X \times X - \delta(X))$ and $[X] \in \mathrm{H\,X}_n(X)$ be a pair of classes. The following conditions are equivalent.*

1. *$(U, [X])$ is an orientation pair.*

2. *$[X] \neq 0$ and $p_{1*}(U \frown ([X] \times [X])) = [X]$.*

3. *For any $x \in X$, $U(x \times [X]) = 1$.*

Proof. (1) ⇔ (2). By Lemma 2.7.5(3), $p_{1*}(U \frown ([X] \times [X])) = (U/[X]) \frown [X]$. For nonzero $[X]$ this equals to $[X]$ if and only if $U/[X] = 1$.

(1) ⇔ (3) follows immediately from the formula $(U/[X])(x) = U(x \times [X])$. □

Theorem 2.8.2. *X is a coarse $PD(n)$ space if and only if it has an orientation pair.*

Proof. Let X be a coarse $PD(n)$ space and let p and q be the associated homotopy equivalences. Set $c = p(1) \in \mathrm{C\,X}_n(X)$ where $1 \in \mathrm{C\,X}^0(X - X)$, and define $\phi \in \mathrm{C}^*(X \otimes X)$ by the rule $\phi(\sigma^k \otimes \tau^{n-k}) = -\varepsilon'_{k+1} q(\tau)(\sigma)$. Note that, if $\sigma \otimes \tau \overset{r}{\subset} \Delta_{X\times X}$ and $q(\tau)(\sigma) \neq 0$ then it follows from the property of q that σ and τ will be r' close to each other where r' depends only on r. Hence we have $\phi \overset{c}{\cap} \Delta_{X\times X} \overset{c}{\subset} \Delta_{\delta(X)}$ i.e $\phi \in \mathrm{C\,X}^n(X \otimes X - \delta(X))$. Moreover, ϕ is a cocycle, since

$$d\phi(\sigma^k \otimes \tau^{n-k+1}) = \phi(\partial\sigma \otimes \tau + \varepsilon_k \sigma \otimes \partial\tau) = -\varepsilon'_k q(\tau)(\partial\sigma) - \varepsilon_k \varepsilon'_{k+1} q(\partial\tau)(\sigma)$$
$$= -(\varepsilon'_k + \varepsilon_k \varepsilon'_{k+1}) dq(\tau)(\sigma) = 0.$$

Then for $x \in X$ we have $\phi(x \otimes c) = -\varepsilon'_1 q(c)(x) = qp(1)(x) = 1(x) = 1$. Thus by Lemma 2.8.1(3) $([\phi], [c])$ is an orientation pair.

Conversely, suppose X has a fundamental class $[X] = [c]$ and an orientation class $U = [\phi]$. We claim that the chain maps $\varepsilon'_{n-k-1}\phi/\cdot : \mathrm{C\,X}_k(X) \to \mathrm{C}^{n-k}(X)$ and $\varepsilon'_n \varepsilon'_{n-k} \cdot \frown c : \mathrm{C}^k(X) \to \mathrm{C\,X}_{n-k}(X)$ are controlled chain homotopy equivalences as in the definition of a coarse $PD(n)$ space.

By Lemma 2.7.4(3) $\varepsilon_n T^* U \frown ([X] \times [X]) = U \frown ([X] \times [X])$, thus by Lemma 2.8.1(2), $\varepsilon_n T^* U$ is also a diagonal class for $[X]$ and $T^* U/[X] = \varepsilon_n$.

Note that $\varepsilon'_n\varepsilon'_{n-k}\varepsilon'_{n-k-1} = \varepsilon_{nk}\varepsilon_n$. Let $a \in \mathrm{C\,X}_k(X)$. Then we have

$$\begin{aligned}
\varepsilon_{nk}\varepsilon_n(\phi/a) \frown c &= \varepsilon_n p_{1*}(\varepsilon_{nk}\phi \frown (c \otimes a)) && \text{by Lemma 2.7.5(3)}\\
&\simeq \varepsilon_n p_{1*}(T^*\phi \frown (a \otimes c)) && \text{by Lemma 2.7.4(3)}\\
&= \varepsilon_n(T^*\phi/c) \frown a && \text{by Lemma 2.7.5(3)}\\
&= a.
\end{aligned}$$

Note that the only homotopy we used here is the composition of p_{1*} with the homotopy from Lemma 2.7.4(3), so it has the property $|D(a)| \overset{c}{\subset} |a|$.

Similarly, $\varepsilon'_n\varepsilon'_{k-1}\varepsilon'_{n-k} = \varepsilon_{nk}$, and for $\phi \in \mathrm{C}^k(X)$ we have

$$\begin{aligned}
\varepsilon_{nk}\phi/(\psi \frown c) &= \varepsilon_{nk}(\phi \smile p_2^*(\psi))/c && \text{by Lemma 2.7.5(2)}\\
&\simeq (p_1^*(\psi) \smile \phi)/c && \text{by Lemma 2.7.4(4)}\\
&= \psi \smile (\phi/c) && \text{by Lemma 2.7.5(1)}\\
&= \psi
\end{aligned}$$

Lemma 2.7.4(4) and (1) imply that the homotopy above satisfies $|D(\phi)| \overset{c}{\cap} \Delta_X \overset{c}{\subset} |\phi|$. ☐

Corollary 2.8.3. *If X is a proper, uniformly acyclic n-manifold, then X is a coarse PD(n) space.*

Proof. Since X is an acyclic manifold, it has an orientation class $U \in \mathrm{H}^n(X \times X, X \times X - \delta(X))$ such that $U|_{x\times(X,X-x)}$ is a generator of $\mathrm{H}^n(x \times (X, X - x))$ for all $x \in X$. It also has a fundamental class $[X] \in \mathrm{H}^{\mathrm{lf}}_n(X)$ which similarly restricts to generators of $\mathrm{H}_n(X, X - x)$. Choosing them in compatible way we obtain $U(x \times [X]) = 1$ for all $x \in X$. Let ϕ' and c' be singular (co)cycles representing these classes. We now construct their coarse versions.

By subdividing as necessary we can arrange that the singular simplices in the fundamental class c' have uniformly bounded diameter. Since X is proper and c' is locally finite, any bounded set intersects $|c'|$ in finitely many simplices. Hence taking the vertices of c' determines a coarse cycle $c \in \mathrm{CX}_n(X)$.

Using uniform contractibility of $X \times X$ we can inductively fill coarse simplices by singular chains in a controlled manner. Indeed, 0-simplices are naturally identified, and if the faces of a coarse simplex are already filled by singular chains, then their signed sum is a cycle, which by uniform contractibility bounds a singular chain contained in a controlled neighborhood of the simplex.

This gives a chain map $S : \mathrm{C}_*(X \times X) \to \mathrm{C}^{\mathrm{s}}_*(X \times X)$ with the property $|S(\sigma^k)| \overset{\rho_k(\mathrm{diam}(\sigma^k))}{\subset} |\sigma^k|$, where ρ_k is a suitable sequence of control functions.

Set $\phi(\sigma^n) = \phi'(S(\sigma^n))$. Since ϕ' vanishes on singular simplices in $X \times X - \delta(X)$, $\phi(\sigma) = 0$ if $\rho_n(\mathrm{diam}(\sigma)) < d(\sigma, \delta(X))$. Thus $\phi \in \mathrm{CX}^n(X \times X - \delta(X))$.

By construction, $\phi(x \times c) = 1$ for all $x \in X$. Therefore, by Lemma 2.8.1 $([\phi], [c])$ is an orientation pair, and the claim follows. $\square$

Chapter 3

On the computation of coarse cohomology

3.1 Introduction

Coarse cohomology is hard to compute in general. One way to deal with this problem is to relate coarse cohomology with a more computable cohomology. For nice spaces, Roe [15] proved the following.

Theorem 3.1.1 (Roe [15])**.** *If X is uniformly contractible and proper, then the coarse cohomology $\mathrm{H\,X}^*(X)$ is isomorphic to the compactly supported Alexander–Spanier cohomology $\mathrm{H}^*_{\mathrm{c}}(X)$.*

Examples of uniformly contractible and proper spaces include universal cover of compact aspherical spaces. For example, Euclidean space of any dimension is uniformly contractible and proper. By Roe's Theorem it follows that $\mathrm{H\,X}^*(\mathbb{R}^n) = \mathrm{H}^*_{\mathrm{c}}(\mathbb{R}^n)$.

In this chapter, our goal is to generalize the above theorem to compute coarse cohomology of more general spaces. Our main motivation for doing that is to compute coarse cohomology of the complement $\mathrm{H\,X}^*(X - A)$ which we discussed in the previous chapter. By Proposition 2.3.3, to get the coarse cohomology of the complement of A, we need to understand the coarse cohomology of $(X/\bar{A}, d_A)$. Unfortunately Theorem 3.1.1 does not apply to the computation of $\mathrm{H\,X}^*(X/\bar{A})$ because $(X/\bar{A}, d_A)$ rarely satisfies its hypothesis even if X does. On the positive side, if X is uniformly contractible and proper, then in some sense, $(X/\bar{A}, d_A)$ retains those properties at infinity. More precisely, $(X/\bar{A}, d_A)$ turns out to be uniformly contractible at infinity and proper at infinity in this case.

A space X is called *uniformly contractible at infinity* if there exist two non-decreasing control function $\rho, R : \mathbb{R}_+ \to \mathbb{R}_+$ and a basepoint $b \in X$ such that any set B of diameter r is contractible inside a set of diameter $\rho(r)$ if $d(b, B) \geq R(r)$. X is said to be *proper at infinity* if there exist a non decreasing control function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and a base point $b \in X$ such that closure of any set B with diameter r is compact if $d(b, B) \geq \rho(r)$. Our main theorem in this chapter is the following.

Theorem 3.1.2. *If X is uniformly contractible at infinity and is proper at infinity, then its coarse cohomology $\mathrm{H\,X}^*(X)$ is isomorphic to its boundedly supported cohomology $\mathrm{H\,B}^*(X)$.*

$\mathrm{H\,B}^*(X)$ will be defined later (Definition 3.2) in the paper. In particular, $\mathrm{H\,B}^*(X)$ coincides with compactly supported Alexander–Spanier cohomology when X is proper and contractible (Example 3.2.1). Hence, Theorem 3.1.2 generalizes Theorem 3.1.1. In general, $\mathrm{H\,B}^*(X)$ is isomorphic to the reduced Alexander–Spanier cohomology of X at infinity with degree shifted down by one (Proposition 3.2.2). As a consequence of Theorem 3.1.2, we prove the following.

Corollary 3.1.3. *Suppose X is uniformly contractible at infinity and proper at infinity. Let $A \subset X$ so that $X \neq N_r(A)$ for any r. Then $\mathrm{H\,X}^*(X - A) = \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(A))$ for $* \geq 1$.*

Remark. Let us contrast the hypothesis of Roe's Theorem 3.1.1 with the hypothesis of Theorem 3.1.2. Uniform contractibility at infinity is more flexible compared to uniform contractibility. For example, deleting a bounded set does not affect the property of being uniformly contractible at infinity. This is not true for uniformly contractible spaces because such spaces are necessarily contractible and complement of bounded set in a contractible set might not be contractible anymore. In general, spaces satisfying uniform contractibility at infinity can be very far from being contractible. Figure 3.1 illustrates an example of a space

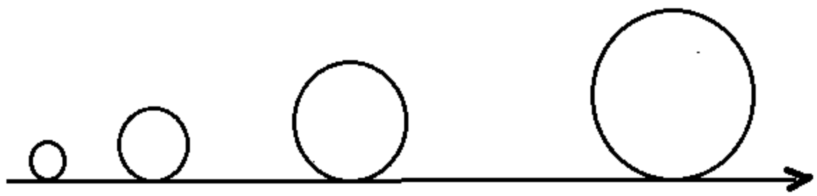

Figure 3.1: A subspace of $\mathbb{R}^2$ that consists of countable union of circles $\{C_i\}_{i\in\mathbb{N}}$ and the ray $r := [0,\infty) \times \{0\}$ such that the i^{th} circle has radius i and distance between two consecutive circle grows to infinity. This is an example of a space that is uniformly contractible at infinity.

that is uniformly contractible at infinity but complement of any bounded set has infinitely generated fundamental group.

Remark. Our approach to prove Theorem 3.1.2 is different from the proof of Roe's Theorem 3.1.1 in [15]. The proof of Theorem 3.1.1 in [15] relies heavily on tools from homological algebra. Whereas our approach follows an idea from [16] which is more geometric in the sense that we explicitly construct (co)chain homotopy to establish the isomorphism between the concerned cohomology. Another interesting aspect is that Theorem 3.1.2 does not hold if we replace uniform contractibility by uniform acyclicity (with respect to singular homology). That means that any proof of this theorem needs to be able to distinguish between uniform acyclicity and uniform contractibility. This is one of the subtleties involved in the proof.

Overview. In section 3.2, we set some terminologies, define boundedly supported cohomology $\mathrm{H\,B}^*(X)$ and recall coarse cohomology $\mathrm{H\,X}^*(X)$. In section 3.3, we define uniformly acyclicity at infinity and local acyclicity at infinity and establish isomorphism between $\mathrm{H\,X}^*(X)$ and $\mathrm{H\,B}^*(X)$ when X is uniformly acyclic at infinity and locally acyclic at infinity. This proof contains bulk of the main ideas that will go into the proof of the Theorem 3.1.2. Section 3.4 contains the proof of the Theorem 3.1.2. We will note at the end of the section that Theorem 3.1.2 does not hold if we replace uniform contractibility at infinity by uniform acyclicity at infinity. One of the main subtleties in the proof of Theorem 3.1.2 involves finding and using a feature that uniform contractibility can provide but uniform acyclicity cannot. Finally in section 3.5, we prove Corollary 3.1.3.

3.2 Boundedly supported cohomology

Definition (Boundedly supported cohomology)**.** Boundedly supported cohomology of a metric space X, denoted by $\mathrm{H\,B}^*(X;R)$, is the cohomology of the following cochain complex

$$\mathrm{C\,B}^*(X;R) := \{\phi \in \mathrm{C}^*(X;R) \mid ||\phi|| \stackrel{\mathrm{c}}{=} *\}$$

Example 3.2.1. If X is proper and contractible, then $\mathrm{H\,B}^*(X;R) = \mathrm{H}^*_{\mathrm{c}}(X;R)$. Indeed, if X is contractible, then $\mathrm{C}^*_0(X;R)$ is an acyclic complex. In this case, the sequence

$$0 \to \mathrm{C}^*_0 \to \mathrm{C}^* \to \mathrm{C}^*_{cas} \to 0$$

gives $\mathrm{H}^*_{\mathrm{c}}(X;R) = \mathrm{H}^*(\mathrm{C}^*_{\mathrm{c}}(X;R))$. Moreover, if X is proper, then $\mathrm{C}^*_{\mathrm{c}}(X;R) = \mathrm{C\,B}^*(X;R)$. Hence we get $\mathrm{H}^*_{\mathrm{c}}(X;R) = \mathrm{H}^*(\mathrm{C\,B}^*(X;R)) = \mathrm{H\,B}^*(X;R)$. For example,

$$\mathrm{H\,B}^*(\mathbb{R}^n;\mathbb{Z}) = \mathrm{H}^*_{\mathrm{c}}(\mathbb{R}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}$$

In general, the boundedly supported cohomology is the same as the (reduced) Alexander–Spanier cohomology at the infinity with degree shifted down by one. This is the content of the next proposition.

Proposition 3.2.2. *1. If X is bounded, then*

$$\mathrm{H\,B}^*(X;R) = \begin{cases} R & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. If X is unbounded and $b \in X$ then

$$\mathrm{H\,B}^*(X;R) = \begin{cases} 0 & \text{if } * = 0 \\ \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(b);R) & \text{otherwise} \end{cases}$$

Proof. (1) If X is bounded then $\mathrm{C\,B}^*(X;R) = \mathrm{C}^*(X;R)$. The cohomology of the latter complex is trivial except in degree 0. Hence, $\mathrm{H\,B}^*(X;R) = 0$ for $* \geq 1$. $\mathrm{H\,B}^0(X;R) = \{\text{constant functions on } X\} \cong R$.

(2) Elements in $\mathrm{H\,B}^0(X;R)$ consists of constant functions defined on X with support contained in a neighborhood of b. This means, if X is not bounded, then $\mathrm{H\,B}^0(X;R)$ is trivial. Hence in this case $\mathrm{H\,B}^0(X;R) = 0$.

Consider the following maps between the cochain complexes where j is the inclusion map and i is induced by canonical restriction maps (followed by quotient maps) $i_r : \mathrm{C}^*(X;R) \to \mathrm{C}^*(X - N_r(b);R) \to \mathrm{C}^*_{as}(X - N_r(b);R)$.

$$0 \to \mathrm{C\,B}^*(X;R) \xrightarrow{j} \mathrm{C}^*(X;R) \xrightarrow{i} \varinjlim \mathrm{C}^*_{as}(X - N_r(b);R) \to 0$$

The above is a short exact sequence because of the following

$$\begin{aligned} ker(i) &= \{\phi \in \mathrm{C}^*(X;R) \mid \phi \in \mathrm{C}^*_0(X - N_r(A);R) \text{ for some r}\} \\ &= \{\phi \in \mathrm{C}^*(X;R) \mid ||\phi|| \text{ is bounded}\} \\ &= Im(j) \end{aligned}$$

The above short exact sequence of cochain complexes induces a long exact sequence of the corresponding reduced cohomology. The reduced cohomology of the middle complex is trivial in all degrees. Hence, the long exact sequence implies that

$$\mathrm{H\,B}^*(X;R) \cong \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(b);R) \text{ for } * \geq 1$$

□

Example 3.2.3. Suppose X is the space appearing in figure 3.1. Since X is unbounded, $\mathrm{H\,B}^0(X)$ is trivial. Suppose $b \in X$. For $* \geq 1$, we have the following

$$\begin{aligned}\mathrm{H\,B}^*(X,\mathbb{Z}) &= \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(b);\mathbb{Z}) \\ &= \begin{cases} \Pi_{i=0}^{\infty}\mathbb{Z}/\oplus_{i=0}^{\infty}\mathbb{Z} & *=2 \\ 0 & *=1, *>2 \end{cases}\end{aligned}$$

3.3 Proof of a preliminary theorem

A space is called *locally acyclic at infinity* if it is locally acyclic outside a bounded set. X is *uniformly acyclic at infinity* if there exist two non-decreasing control functions $\rho, R : \mathbb{R}^+ \to \mathbb{R}^+$ and a basepoint $b \in X$ such that any set B of diameter r and $d(b, B) \geq R(r)$, the inclusion $B \hookrightarrow N_{\rho(r)}(B)$ induce trivial map between the singular homology.

In this section, our goal is to prove the following Theorem.

Theorem 3.3.1. *If X is uniformly acyclic at infinity and locally acyclic at infinity, then the inclusion $\mathrm{C\,X}^*(X) \hookrightarrow \mathrm{C\,B}^*(X)$ induces an isomorphism on cohomology:*

$$\mathrm{H\,X}^*(X) \cong \mathrm{H\,B}^*(X).$$

In a uniformly acyclic at infinity space, we can perform a version of the standard "connect the dots" construction. Every 1-simplex σ of diameter r outside of the ball of radius $R(r)$ is fillable, so we can pick a singular 1-chain c, such that $|c| \subset N_{\rho(r)}(\sigma)$ and $\partial c = \partial\sigma$. Proceeding by induction on the dimension, if a simplex is sufficiently far from the base point, its boundary is already filled by a singular cycle that bounds a singular chain contained in a controlled neighborhood of the simplex. Moreover, if the space is locally acyclic at infinity, we can fill small simplices outside a bounded set by chains of small diameter. More precisely, suppose $\mathcal{U}$ is a cover of X and for each $x \in X$ fix a set $U_x \in \mathcal{U}$ that contains x. We can choose the filling so that for any point x outside a bounded set, there is a neighborhood V_x of x, such that the filling of any simplex in V_x stays inside U_x. Moreover, by subdividing, we can arrange so that the filling singular chains of all simplices are supported by $\mathcal{U}$.

Note that not every simplex is fillable, and that the diameter of fillings grows with dimension, as well as the size of the balls that we have to avoid. To formalize the notion of sufficiently far, we make the following definition. Given an increasing sequence of increasing control functions $R_n : [0,\infty) \to [0,\infty)$, denote

$$\mathrm{C}_n^F(X) = \langle \sigma^n \mid d(\sigma^n, b) \geq R_n(\operatorname{diam}\sigma^n)\rangle \subset \mathrm{C}_n(X)$$

Since R_n is increasing, this defines a subcomplex of the chain complex of finitely supported chains.

Let $V : \mathrm{C}_*^{\mathrm{s}}(X) \to \mathrm{C}_*(X)$ denote the forgetful map, which maps a singular simplex to its vertices. Let $\mathrm{C}_*^{\mathcal{U}}(X)$ denote the complex of singular chains supported by $\mathcal{U}$. The discussion above gives the following lemma.

Lemma 3.3.2. *Suppose X is uniformly acyclic at infinity and locally acyclic at infinity. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, fix a set $U_x \in \mathcal{U}$ that contains x. Then there exist two increasing sequences of control functions R_n and ρ_n and a chain map $S : \mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathcal{U}}(X)$ where*

$$\mathrm{C}_n^F(X) = \langle \sigma^n \mid d(\sigma^n, b) \geq R_n(\mathit{diam}\,\sigma^n)\rangle \subset \mathrm{C}_n(X)$$

and a chain homotopy $G : \mathrm{C}_*^F(X) \to \mathrm{C}_{*+1}(X)$ between VS and the inclusion map so that

1. *$S : \mathrm{C}_0^F(X) \to \mathrm{C}_0^{\mathrm{s}}(X)$ is the identity.*

2. *$|S(\sigma^n)| \subset N_{\rho_n}(\sigma^n)$.*

3. *$|G(\sigma^n)| \subset N_{\rho_n}(\sigma^n)$.*

4. *There exist a bounded set B such that for any point $x \notin B$, there is a neighborhood V_x of x so that $|G(\sigma^n)| \subset U_x^{n+2}$ for any $\sigma^n \in V_x^{n+1}$.*

We now present the proof of Theorem 3.3.1.

Proof of 3.3.1. By the long exact sequence, we need to show $\mathrm{H}^*(\mathrm{C\,B}^*(X)/\mathrm{C\,X}^*(X)) = 0$. That means for $\phi \in \mathrm{C\,B}^n(X)$ with $d\phi \in \mathrm{C\,X}^{n+1}(X)$, we need to find $\psi \in \mathrm{C\,B}^{n-1}(X)$ so that $\phi - d\psi \in \mathrm{C\,X}^n$. Let U be a bounded neighborhood of $\|\phi\|$ in X and for each $x \in X - \|\phi\|$ choose a neighborhood U_x such that $U_x^{n+1} \cap |\phi| = \emptyset$. Let $\mathcal{U}$ denote the cover of X that consist of the collection of U_x together with U.

Lemma 3.3.2 produces the chain homotopy $G : \mathrm{C}_*^F(X) \to \mathrm{C}_{*+1}(X)$, which we use to define a linear map $D : \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$ by setting

$$D(\sigma^n) = \begin{cases} G(\sigma^n) & \text{if } \sigma^n \in \mathrm{C}_n^F(X), \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$T = id - \partial D - D\partial,$$

In particular $T(\sigma) = VS(\sigma)$ by Lemma 3.3.2. Dually we have

$$T^* = id - D^*d - dD^*.$$

We claim that $T^*(\phi)$ is coarse. Let σ^n be a simplex with $\mathrm{diam}(\sigma^n) \leq r$ for some $r \geq 0$. If $d(\sigma^n, b) \geq R_n(r)$ then $\sigma^n \in \mathrm{C}_n^F(X)$ and the standard argument shows that $T(\sigma)$ is supported by $\mathcal{U}$. If all vertices of σ^n are outside of $\rho_n(r)$-neighborhood of U, then $|T(\sigma)|$ does not meet

U^{n+1} and therefore $T^*(\phi)(\sigma) = \phi(T(\sigma)) = 0$, since $\phi|_{U_x^{n+1}} = 0$ for all U_x. Since U is bounded, the claim follows.

Next we claim that $D^*(\phi) \in \mathrm{C\,B}^{n-1}(X)$. By lemma 3.3.2, we can choose a bounded set B containing U such that for any point $x \notin B$, there is a neighborhood V_x of x so that $|G(\sigma^n)| \subset U_x^{n+2}$ for any $\sigma^n \in V_x^{n+1}$. Hence for $x \notin B \cup ||\phi||$, we have $|D_*(\sigma)| \notin |\phi|$ for all $\sigma \in V_x^{n+1}$. Therefore, $||D^*(\phi)|| \subset ||\phi|| \cup B$. The claim follows since $\phi \in \mathrm{C\,B}^*(X)$.

Finally we claim that, $D^*d(\phi) \in \mathrm{C\,X}^*(X)$. By construction of D, if $d(\sigma^n, b) \geq R_n(r)$ then $|D\sigma| \subset N_{\rho_n(r)}(\sigma)$ where r is the diameter of σ. Since $d\phi \in \mathrm{C\,X}^*(X)$, the claim follows.

Since $\phi - dD^*(\phi) = T^*(\phi) + D^*d\phi$, our desired cochain is $\psi = D^*(\phi)$. □

Combining Theorem 3.1.2 and proposition 3.2.2 we get the following

Corollary 3.3.3. *If X is unbounded, uniformly acyclic at infinity and locally acyclic at infinity, then for any $b \in X$*

$$\mathrm{H\,X}^*(X;R) = \begin{cases} 0 & \textit{if } * = 0 \\ \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(b); R) & \textit{otherwise} \end{cases}$$

Example 3.3.4. Let X be the space appearing in figure 3.1. X is unbounded, uniformly acyclic at infinity and locally acyclic. By Theorem 3.3.1, we have $\mathrm{H\,X}^*(X) = \mathrm{H\,B}^*(X)$. We computed $\mathrm{H\,B}^*(X)$ in the Example 3.2.3. That gives us the following

$$\mathrm{H\,X}^*(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ \Pi_{i=0}^{\infty}\mathbb{Z}/\oplus_{i=0}^{\infty}\mathbb{Z} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

Remark. Theorem 3.3.1 does not hold if we drop the local acyclicity at infinity condition from the assumption. Consider the space $X = \bigsqcup_{i=1}^{\infty} S_i$ of disjoint union of countably infinite Warsaw circles. We can embed X into $\mathbb{R}^2$ in a way so that $\mathrm{diam}(S_i) = 1$ for all i and X is coarsely equivalent to a ray $[0,\infty)$ with the subspace metric. This space is uniformly acyclic, since Warsaw circles are acyclic. Suppose $A \subset X$ is some bounded subset. $\mathrm{H\,X}^2(X;\mathbb{Z}) = \mathrm{H\,X}^2(X : \mathbb{Z}) = \mathrm{H\,X}^2([0,\infty);\mathbb{Z}) = 0$. On the other hand, each Warsaw circle has nontrivial Alexander–Spanier cohomology in degree 1. It follows that, $\varinjlim \tilde{\mathrm{H}}^1(X - N_r(b);\mathbb{Z}) \neq 0$. So, the conclusion of the above corollary fails in this case. That implies that Theorem 3.3.1 fails for this space.

3.4 Proof of the main theorem

In this section we prove the following.

Theorem 3.4.1. *If X is uniformly contractible at infinity and is proper at infinity, then the inclusion $\mathrm{C\,X}^*(X) \hookrightarrow \mathrm{C\,B}^*(X)$ induces an isomorphism on cohomology:*

$$\mathrm{H\,X}^*(X) \cong \mathrm{H\,B}^*(X).$$

The underlying idea of the proof is the same as the proof of Theorem 3.3.1. However there are two main differences that makes the proof more technical. The first difference is the absence of local acyclicity of the space. To make up for that, we will embed X into a locally acyclic space. This can be achieved by Kuratowski embedding theorem which says that any metric space X admits an isometric embedding into the infinite dimensional Banach space $\ell^\infty(X)$. Perhaps more subtle difference is the fact that the above theorem does not hold if we drop uniform contractibility at infinity by uniform acyclicity at infinity. This follows from the example in the remark 3.3. That means, the proof of the Theorem 3.4.1 needs to be able to distinguish between uniform acyclicity and uniform contractibility.

Our first goal is to prove the following modification of the Lemma 3.3.2 which will be one of the main ingredients in the proof of Theorem 3.4.1. By abuse of notation, we will denote $\ell^\infty(X)$ by ℓ^∞ throughout the paper.

Proposition 3.4.2. *Suppose $X \subset \ell^\infty$ is uniformly contractible at infinity and proper at infinity. Let $N(X)$ be an open neighborhood of X in ℓ^∞. Let $\mathcal{U}$ be an open cover of $N(X)$. Then there exists an increasing sequence of control functions $R_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a subcomplex $\mathrm{C}^F_*(X)$ of $\mathrm{C}_*(X)$ of the following form*

$$\mathrm{C}^F_n(X) = \langle \sigma^n \mid d(\sigma^n; b) > R_n(diam(\sigma^n)) \rangle \subset \mathrm{C}_n(X),$$

a chain map $S : \mathrm{C}^F_*(X) \to \mathrm{C}^{\mathcal{U}}_*(N(X))$ and a map $G : \mathrm{C}^F_*(X) \to \mathrm{C}_{*+1}(N(X))$ such that the following hold.

1. *$S|_{\mathrm{C}^F_0(X)}$ is the inclusion map $\mathrm{C}^F_0(X) \hookrightarrow \mathrm{C}_0(N(X))$.*

2. *G is a chain homotopy between VS and the inclusion map $i : \mathrm{C}^F_*(X) \hookrightarrow \mathrm{C}_*(N(X))$.*

$$\partial G(\sigma) = VS(\sigma) - i(\sigma) - G\partial(\sigma)$$

3. *There exists an increasing sequence of function $\rho_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|VS(\sigma^n)| \subset N_{\rho_n(diam(\sigma^n))}(\sigma^n)$ and $|G(\sigma^n)| \subset N_{\rho_n(diam(\sigma^n))}(\sigma^n)$.*

4. *For any $k \in \mathbb{N}$, there exists a bounded set $B \subset X$ such that for any $x \in X - B$ and a neighborhood V_x of x in ℓ^∞, there is a neighborhood W_x of x in X such that $|G(\sigma^n)| \subset V^{n+2}_x$ and $|VS(\sigma^n)| \subset V^{n+1}_x$ for all $\sigma^n \in W^{n+1}_x$ if $n \leq k$.*

Sketch of the proof: The main idea of the proof of Proposition 3.4.2 is similar to the Lemma 3.3.2. To get S, we first fill simplices in X by singular chains and then use barycentric subdivision on these chains until they fall inside $\mathrm{C}^{\mathcal{U}}_*(X)$. Most of this filling process can be done in X with the necessary control on the support as required by property (3) by using uniform contractibility of X at infinity. For property (4), we need to fill small simplices by small singular chains and unless X is locally acyclic this cannot be achieved by staying inside X. However, since $N(X)$ is an open subset of ℓ^∞, we can fill small enough simplices of X

in $N(X)$ by taking the convex hull of its vertices. In summary, we fill some simplices (big ones) in X and some simplices (small ones) in $N(X)$. The main difficulty is to choose these fillings in a compatible way so that it gives a chain map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(N(X))$. This would be immediate if we knew $N(X)$ is uniformly contractible but this might not be the case even if X is uniformly contractible.

This problem is solved in two steps. In the first step, we construct a chain map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(X)$ that sends small simplices to singular chains which can be homotoped to the convex filling by staying inside $N(X)$ (Lemma 3.4.3). Convex filling of a simplex is the image of that simplex under the following chain map

$$
\begin{aligned}
&c : \mathrm{C}_*(X) \to \mathrm{C}_*^{\mathrm{s}}(\ell^\infty) \\
&\quad (x_0, \ldots, x_n) \mapsto c(\sigma) : (s_0, \ldots, s_n) \mapsto \sum_{i=0}^{n} s_i x_i
\end{aligned}
$$

It is in the construction of this chain map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(X)$ where we crucially use the fact that our space is uniformly contractible at infinity not just uniformly acyclic at infinity. In figure 3.2, we illustrated that this construction cannot be performed when X is the Warsaw circle which is a uniform acyclic space.

In the second step, we modify the chain map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(X)$ produced in Lemma 3.4.3 to get a chain map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(N(X))$ that sends small enough simplices to its convex filling. The idea here is to glue the filling of small simplices obtained in Lemma 3.4.3 with the associated homotopy of this filling with the convex filling. This is done in Lemma 3.4.5.

Finally, to prove Proposition 3.4.2, we post compose the map $\mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(N(X))$ from Lemma 3.4.5 with a suitable subdivision operator to get the map S into $\mathrm{C}_*^{\mathcal{U}}(X)$ and the corresponding subdivision homotopy gives us G.

Lemma 3.4.3. *Suppose $X \subset \ell^\infty$ and X is uniformly contractible at infinity and proper at infinity. Let $N(X)$ be an open neighborhood of X in ℓ^∞. Then there exists an increasing sequence of control functions $R_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a subcomplex $\mathrm{C}_*^F(X)$ of $\mathrm{C}_*(X)$ of the*

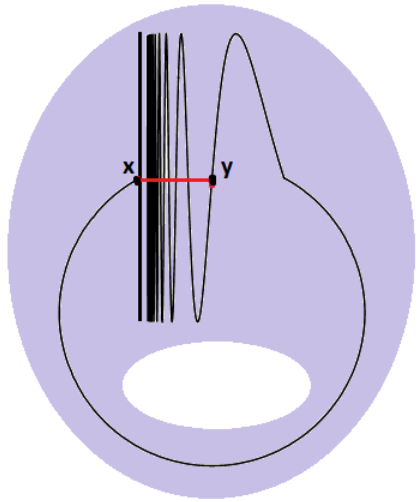

Figure 3.2: In the above figure, the space $X \subset \mathbb{R}^2 \subset \ell^\infty$ is the Warsaw circle and $N(X)$ is a tubular neighborhood of the grey region in ℓ^∞ that deforms retract to the grey region. Red line is the convex filling of the simplex (x, y). Any filling of (x, y) in X has to be around the circular part. Hence, there is no homotopy in $N(X)$ between the red convex filling and any filling of (x, y) in X. Furthermore, any neighborhood of x contains such a y where this happens.

following form

$$\mathrm{C}_n^F(X) = \langle \sigma^n \mid d(\sigma^n; b) > R_n(diam(\sigma^n)) \rangle \subset \mathrm{C}_n(X),$$

a chain map $f : \mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(X)$ *and* $D : \mathrm{C}_*^F(X) \to \mathrm{C}_{*+1}^{\mathrm{s}}(\ell^\infty)$ *so that the following hold.*

1. f *is a chain map where* $f|_{\mathrm{C}_0^F(X)}$ *is the inclusion map* $\mathrm{C}_0^F(X) \hookrightarrow \mathrm{C}_0(X)$.

2. D *is a chain homotopy between* f *and the convex filling* $c(\sigma)$

$$\partial D(\sigma) = f(\sigma) - c(\sigma) - D\partial(\sigma)$$

3. *There exist an increasing sequence of control functions* ρ_n *such that* $|f(\sigma^n)| \subset N_{\rho_n(diam(\sigma^n))}(\sigma^n)$ *and* $|D(\sigma^n)| \subset N_{\rho_n(diam(\sigma^n))}(\sigma^n)$ *where the tubular neighborhoods are taken in* ℓ^∞.

4. *For any* $k \in \mathbb{N}$, *there exists a bounded set* $Z \subset X$ *such that for each point* $x \in X - Z$, *there is a neighborhood* W_x *of* x *such that for all* $\sigma^n \in W_x^{n+1}$ *with* $n \leq k$, $D(\sigma^n) \in$

$\mathrm{C}^{\mathrm{s}}_{n+1}(N(X))$.

Before going into the proof of the above lemma, let us describe certain cone constructions that will be used to construct f and D in the proof. A standard n-simplex Δ^n is the following subset $\Delta^n = \{(s_0, \dots, s_n) \mid \sum_{i=0}^{n} s_i = 1, s_i \geq 0\} \subset \mathbb{R}^{n+1}$ whose vertices are the unit vectors along the coordinate axes. A singular n-simplex in X is a continuous function $\alpha : \Delta^n \to X$ and the image of α is called the support of the singular simplex.

Let H_t be a homotopy that contracts some set $B \subset \ell^\infty$ to a point in ℓ^∞. We call such homotopy to be a contracting homotopy. To H_t, we can associate a cone operator $H : \mathrm{C}^{\mathrm{s}}_*(B) \to \mathrm{C}^{\mathrm{s}}_{*+1}(\ell^\infty)$. The construction goes as follows. Let α be a singular n-simplex supported in $B \subset \ell^\infty$. Consider the following map $I \times \Delta^n \to \ell^\infty$ where $I = [0, 1]$.

$$(t, (s_0, \dots, s_n)) \mapsto H_t(\alpha(s_0, \dots, s_n))$$

Since H_1 is a constant map, the above map induces a map from $(I \times \Delta^n)/(\{1\} \times \Delta^n)$ to ℓ^∞. $(I \times \Delta^n)/(\{1\} \times \Delta^n)$ can be identified with a singular $(n+1)$-simplex. Hence the above map gives a singular $(n + 1)$-simplex and we define $H(\alpha)$ to be this simplex. The i^{th} face of $H(\alpha)$ corresponds to the above map restricted to the set of points that have zero in their i^{th} coordinates. When $\dim(\alpha) \geq 1$, one can check that that $\partial(H(\alpha)) = \alpha - H(\partial\alpha)$.

Now, we will define a different cone operator $\bar{H}$ associated to H_t. The inputs of $\bar{H}$ will be generated by convex fillings of simplices whose vertices lie in B (although we will later extend the domain of $\bar{H}$). If σ is a 0-simplex, we define $\bar{H}(\sigma) := H(\sigma)$. If $\sigma = (x_0, \dots, x_n)$ with $n \geq 1$, then we consider the following map $I \times \Delta^n \to \ell^\infty$.

$$(t, (s_0, \dots, s_n)) \mapsto \sum_{i=0}^{n} s_i H_t(x_i)$$

The above map restricted to $\{1\} \times \Delta^n$ is a constant map and hence induces a map from $(I \times \Delta^n)/(\{1\} \times \Delta^n)$. That means we can realize the above map as a singular $(n+1)$-simplex in ℓ^∞, which we define to be $\bar{H}(c(\sigma))$ (see figure 3.3). The i^{th} face of this simplex corresponds to the above map restricted to the set of points whose i^{th} coordinates are zero. In particular, the 0^{th} face is $c(\sigma)$ and hence $\bar{H}(c(\sigma))$ can be viewed as a cone on $c(\sigma)$.

We can extend the domain of $\bar{H}$ to simplices of the form $\bar{G}(c(\sigma))$ where $\bar{G}$ is the associated cone operator to some contracting homotopy G_s such that the composition $H_t \circ G_s$ is defined on the vertices of an σ for all $[t, s] \in [0, 1]^2$. If $n = 0$, then we define $\bar{H}(\bar{G}(c(\sigma))) := H(G(\sigma))$. If $n \geq 1$, we consider the following map from $I^2 \times \Delta^n$ to ℓ^∞.

$$(t, s, (s_0, \dots, s_n)) \mapsto \sum_{i=0}^{n} s_i H_t(G_s(x_i))$$

Since H_1 and G_1 are constant maps, the above map induce a map from two fold cone on Δ^n and hence gives a singular $(n + 2)$-simplex. We define $\bar{H}(\bar{G}(c(\sigma)))$ to be this simplex.

We can iterate the above process. Suppose $\sigma = (x_0, \dots, x_n)$ is an n-simplex for some $n \in \mathbb{N}$. Let $\{H^k_{t_k}, \dots, H^1_{t_1}\}$ be a set of k contracting homotopies such that the domain of the composition $H^k_{t_k} \circ \dots H^1_{t_1}$ contains the vertices of σ. When $n = 0$, we define

$$\bar{H^k}(\dots(\bar{H^1}(\sigma)))\dots) := H^k(\dots(H^1(\sigma)))\dots)$$

For $n \geq 1$, we consider the following map $I^k \times \Delta^n \to \ell^\infty$

$$(t_k, \dots, t_1, (s_0, \dots, s_n)) \mapsto \sum_{i=0}^{n} s_i H^k_{t_k}(\dots(H^1_{t_1}(x_i)))$$

Since H^i_1 is a constant map for each $i \in \{1, 2, \dots, k\}$, the above map induces a map from k-fold cone on Δ^n. So the above map gives a singular $(n + k)$-simplex and we define this simplex to be $\bar{H^k}(\dots(\bar{H^1}((c(\sigma)))\dots)$. Again note that, the above map restricted to the set of points whose first coordinates are zero give the simplex $H^{\bar{k}-1}(\dots(\bar{H^1}(c(\sigma)))\dots)$.

The above discussion tells us that, $\bar{H}$ can be defined to any singular simplices of the form $\bar{H^k}(\dots(\bar{H^1}(c(\sigma)))\dots)$ where the domain of the composition $H_t \circ H^k_{t_k} \circ \dots \circ H^1_{t_1}$ contains the vertices of σ for all $(t, t_k. \dots, t_1) \in [0, 1]^{1+k}$. Suppose $\beta = \bar{H^k}(\dots(\bar{H^1}(c(\sigma)))\dots)$ is such a singular simplex in the domain of $\bar{H}$. Then $\bar{H}(\beta)$ is the simplex defined by the following map $I^{k+1} \times \Delta^n \to \ell^\infty$

$$(t, t_k, \ldots, t_1, (s_0, \ldots, s_n)) \mapsto \sum_{i=0}^{n} s_i H_t(H^k_{t_k}(\ldots(H^1_{t_1}(x_i))\ldots))$$

In what follows, we will blur the distinction between the above map and the corresponding singular simplex for convenience. We will denote both of them by $\bar{H}(\beta)$.

Lemma 3.4.4. *If* $\dim(\beta) \geq 1$, *then* $\partial \bar{H}(\beta) = \beta - \bar{H}(\partial\beta)$.

Proof. Suppose $\beta = \bar{H^k}(\ldots(\bar{H^1}(c(\sigma)))\ldots)$. The i^{th} face of $\bar{H}(\beta)$ corresponds to $\bar{H}(\beta)|_{S(i,n,k+1)}$, where $S(i, n, k+1) \subset I^{k+1} \times \Delta^n$ is the set of points that have 0 in their i^{th} coordinates.

Case 1 ($n \geq 1$): We observe that

$$\bar{H}(\beta)|_{S(i,n,k+1)} = \begin{cases} \beta & \text{when } i = 0 \\ \bar{H}(\beta|_{S(i-1,n,k)}) & \text{when } 1 \leq i \leq n+k+2 \end{cases}$$

We can now check the following boundary formula where β is in the domain of $\bar{H}$ and $n \geq 1$.

$$\begin{aligned}
\partial \bar{H}(\beta) &= \sum_{i=0}^{n+k+2} (-1)^i \bar{H}(\beta)|_{S(i,n,k+1)} \\
&= \bar{H}(\beta)|_{S(0,n,k+1)} + \sum_{i=1}^{n+k+2} (-1)^i \bar{H}(\beta)|_{S(i,n,k+1)} \\
&= \beta + \sum_{i=1}^{n+k+2} (-1)^i \bar{H}(\beta|_{S(i-1,n,k)}) \\
&= \beta - \sum_{i=0}^{n+k+1} (-1)^i \bar{H}(\beta|_{S(i,n,k)}) \\
&= \beta - \bar{H}(\sum_{i=0}^{n+k+1} (-1)^i \beta|_{S(i,n,k)}) \\
&= \beta - \bar{H}(\partial\beta)
\end{aligned}$$

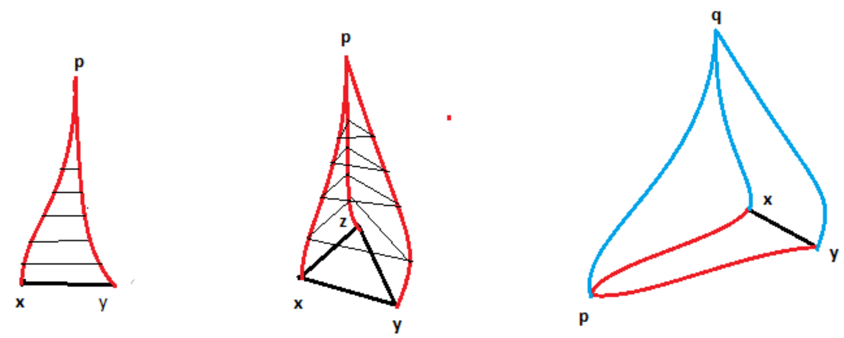

Figure 3.3: H_t and G_s are two contracting homotopy with p and q being their contracting point respectively. The first two pictures are of $\bar{H}(c(x,y))$ and $\bar{H}(c(x,y,z))$. The third picture is of $\bar{G}(\bar{H}(c(x,y)))$.

Case 2 ($n = 0$): In this case, by construction $\bar{H}(\beta) = H(\beta)$. We know that $\partial H(\beta) = \beta - H(\partial\beta)$ if $\dim(\beta) \geq 1$. Hence, $\partial\bar{H}(\beta) = \beta - \bar{H}(\partial\beta)$ in this case.

□

Proof of 3.4.3. Since X is *uniformly contractible at infinity*, there exist two non-decreasing control functions $\rho, R : \mathbb{R}^+ \to \mathbb{R}^+$ and a basepoint $b \in X$ such that any set B of diameter r is contractible inside a set of diameter $\rho(r)$ if $d(b, B) \geq R(r)$. For each bounded set $B \subset X$ with diameter r and $d(b, B) \geq R(r)$, we fix a homotopy H_t^B that contracts B inside $N_{\rho(r)}(B)$. Using paracompactness of X, we pick a locally finite open cover $\mathcal{U} = \{U_\alpha\}$ so that diameters of all U_α are uniformly bounded. In particular, for each 1-simplex $(x, y) \in \cup U_\alpha^2$, the set $S(x, y) = \{\alpha \mid (x, y) \in U_\alpha^2\}$ is finite. This locally finiteness of the cover will be used at the end of the proof.

Construction of the complex $\mathrm{C}_*^F(X)$: We can assume $\mathrm{C}_0^F(X)$ to be $\mathrm{C}_0(X)$ by letting

$R_0 = 0$. For each 1-simplex $\sigma = (x, y)$ in $\cup U_\alpha^2$, we fix an $\alpha \in S(x, y)$ and we let $B(\sigma) := U_\alpha$. For other 1-simplices, we let $B(\sigma)$ to be $\{x, y\}$. Proceeding inductively, for an n-simplex σ, we let

$$B(\sigma) = \bigcup_{\tau \in |\partial\sigma|} N_{\rho(\mathrm{diam}(B(\tau)))} B(\tau)$$

We observe that $\mathrm{diam}(B(\sigma))$ depends only on $\mathrm{diam}(\sigma)$ and $\dim(\sigma)$. Therefore by uniform contractibility at infinity, we can choose an increasing sequence of functions $R_n : [0, \infty) \to [0, \infty)$ indexed by natural numbers so that $R_n \geq R$ for $n \geq 1$ and that $B(\sigma)$ is contractible inside $N_{\rho(\mathrm{diam}(B(\sigma))}(B(\sigma))$ if $d(\sigma^n, b) \geq R_n(\mathrm{diam}(\sigma^n))$. For $n \geq 1$, we let

$$\mathrm{C}_n^F(X) = \langle \sigma^n \mid d(\sigma^n; b) > R_n(\mathrm{diam}(\sigma^n)) \rangle \subset \mathrm{C}_n(X)$$

By construction, we have an associated homotopy $H_t^{B(\sigma)}$ that contracts $B(\sigma)$ inside $N_{\rho(\mathrm{diam}(B(\sigma))}(B(\sigma))$ if $\sigma \in \mathrm{C}_*^F(X)$.

Construction of the map f: We let f to be the identity map on $\mathrm{C}_0^F(X) = \mathrm{C}_0(X)$ and in higher degrees we define f inductively to be the following (see figure 3.4).

$$f(\sigma) := \bar{H}^{B(\sigma)}(f(\partial\sigma))$$

Note that, in order for the above definition to make sense, $f(\partial\sigma)$ has to be in the domain of $\bar{H}^{B(\sigma)}$. We can show that by induction on dimension of σ. By induction hypothesis, suppose $f(\partial\sigma)$ is well defined and hence $|f(\partial\sigma)|$ consists of singular simplices of the form $\bar{H}^{B(\tau)}(\beta)$ where τ is a codimension one subsimplex of σ. According to the construction of $B(\sigma)$, for any subsimplex τ of σ, $H_t^{B(\tau)}(B(\tau)) \subset B(\sigma)$. That implies any simplex of the form $\bar{H}^{B(\tau)}(\beta)$ is in the domain of $\bar{H}^{B(\sigma)}$ and hence f is well defined.

Construction of the chain homotopy D: We first let D_0 to be trivial on $\mathrm{C}_0^F(X)$. We

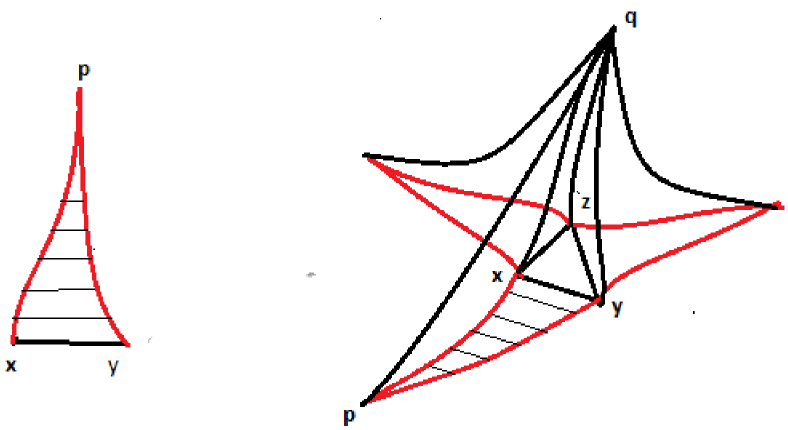

Figure 3.4: In the left picture, the straight line between x and y is $c((x,y))$ and the red singular 1-simplices give the filling $f((x,y)) = \bar{H}^{B((x,y))}(\partial(x,y))$ where $H_t^{B((x,y)}$ contracts the set $B((x,y))$ to the point p. The striped region is $D((x,y)) = \bar{H}^{B(x,y)}(c((x,y)))$. The picture on the right is the support of $\bar{H}^{B((x,y,z))}(D(\partial((x,y,z)))$ which is made of four singular 3-simplices. The one in the center is $\bar{H}^{B(x,y,z)}((x,y,z))$. Other three belong to the support of $\bar{H}^{B((x,y,z))}(D(\partial((x,y,z))))$.

then define $D_*(\sigma)$ inductively to be the following.

$$D_*(\sigma) := -\bar{H}^{B(\sigma)}(D_*(\partial\sigma)) - \bar{H}^{B(\sigma)}(c(\sigma))$$

To prove D_* is well defined, we need to check that $D_*(\partial\sigma)$ does belong to the domain of the $\bar{H}^{B(\sigma)}$. The proof of this is same as well definedness of f, so we will skip it. We illustrate f and D applied to 1 and 2-dimensional simplices in figure 3.4.

Proof of (1): First we notice that $f|_{C_0^F(X)}$ is the inclusion map $C_0^F(X) \hookrightarrow C_0(X)$ by construction. Next we show that f is a chain map by induction. For a 1-simplex (x,y), we have

$$\begin{aligned}\partial f((x,y)) &= \partial(\bar{H}^{B((x,y))}(\partial(x,y))) \\ &= \partial(H^{B((x,y))}(y-x)) \\ &= \partial(x,y) = f(\partial((x,y)))\end{aligned}$$

Inducting on the dimension of σ, let us assume that $f(\partial(\sigma)) = \partial(f(\sigma))$ if $\dim(\sigma) < n$ where $n \geq 2$. We know that, $\partial(\bar{H}^{B(\sigma)}(\alpha)) = \alpha - \bar{H}^{B(\sigma)}(\partial\alpha)$ if $\dim(\alpha) \geq 1$. Suppose $\dim(\sigma) = n \geq 2$, then $\dim(f(\partial\sigma)) \geq 1$. Then we have the following.

$$\begin{aligned}\partial f(\sigma) &= \partial\bar{H}^{B(\sigma)}(f(\partial(\sigma))) \\ &= f(\partial(\sigma)) - \bar{H}^{B(\sigma)}(\partial f(\partial(\sigma))) \quad \text{[by Lemma 3.4.4]} \\ &= f(\partial(\sigma))\end{aligned}$$

Proof of (2): This can be proved by induction on the dimension of σ. Since D_0 is trivial, property (3) is true for the base case $n = 1$. For a 1-simplex (x, y), we have

$$\begin{aligned}\partial D((x,y)) = -\partial\bar{H}^{B((x,y))}(c((x,y))) &= -c((x,y)) + \bar{H}^{B((x,y))}(\partial(c(x,y))) \\ &= -c((x,y)) + \bar{H}^{B((x,y))}(\partial(x,y)) \\ &= -c((x,y)) + f((x,y))\end{aligned}$$

Assume that $\partial D_*(\sigma) = f(\sigma) - c(\sigma) - D_*(\partial\sigma)$ if $\dim(\sigma) < n$ where $n \geq 2$. Suppose σ is an n-simplex with $n \geq 2$. Then we have

$$
\begin{aligned}
&\partial D_*(\sigma) \\
&= -\partial \bar{H}^{B(\sigma)}(D_*(\partial(\sigma))) - \partial \bar{H}^{B(\sigma)}(c(\sigma)) \\
&= \bar{H}^{B(\sigma)}(\partial(D_*(\partial\sigma))) - D_*(\partial(\sigma)) - \partial \bar{H}^{B(\sigma)}(c(\sigma)) && \text{[by Lemma 3.4.4]} \\
&= \bar{H}^{B(\sigma)}(f(\partial(\sigma))) - \bar{H}^{B(\sigma)}(c(\partial\sigma)) - \bar{H}^{B(\sigma)}(D_*(\partial^2(\sigma))) \\
&\qquad\qquad - D_*\partial(\sigma) + \bar{H}^{B(\sigma)}(\partial(c(\sigma))) - c(\sigma) && \text{[by induction hypothesis]} \\
&= \bar{H}^{B(\sigma)}(f(\partial(\sigma))) - c(\sigma) - D_*(\partial(\sigma)) \\
&= f(\sigma) - c(\sigma) - D_*\partial(\sigma)
\end{aligned}
$$

Proof of (3): By construction, both $|f(\sigma)|$ and $|D(\sigma)|$ are subsets of $N_{\rho(\mathrm{diam}(B(\sigma)))}(B(\sigma))$. As observed earlier, $\mathrm{diam}(B(\sigma))$ depends only on $\dim(\sigma)$ and $\mathrm{diam}(\sigma)$. It follows that $\mathrm{diam}(|f(\sigma)|$ and $\mathrm{diam}(|D(\sigma)|)$ depends only on $\dim(\sigma)$ and $\mathrm{diam}(\sigma)$. That means we can construct a non decreasing sequence of function $\rho_n : [0, \infty) \to [0, \infty)$ such that $|f(\sigma^n)| \leq \rho_n(r)$ and $|D(\sigma^n)| \leq \rho_n(r)$ where $\mathrm{diam}(\sigma^n) = r$. The claim follows.

Proof of (4): Our main claim towards proving (4) is the following.

Claim: Suppose $||d||$ denotes the union of support of all singular simplices in the support of a singular chain d. We claim that there exists a bounded set $Z \subset X$ such that for any $\epsilon > 0$, $k \in \mathbb{N}$ and $x \notin Z$ there is a neighborhood W_x of x in X such that $||D(\sigma)|| \subset N_\epsilon(||f(\sigma)||)$ for all $\sigma^n \in W_x^{n+1}$ with $n \leq k$.

We first explain how this claim implies property (4). Since X is proper at infinity, there exist a bounded set Z', such that for each $x \notin Z'$, there is a bounded neighborhood V_x of x such that $\cup_{\sigma \in V_x^{n+1}} |f(\sigma)|$ is contained in a compact set for any n. This is true because $\cup_{\sigma \in V_x^{n+1}} |f(\sigma)|$ is bounded as V_x is bounded and so closure of $\cup_{\sigma \in V_x^{n+1}} |f(\sigma)|$ is compact if x is chosen outside some bounded sets. Fix $k \in \mathbb{N}$. Hence for each $x \notin Z'$, there is an $\epsilon_x > 0$ such that $N_{\epsilon_x}(||f(\sigma)||) \subset N(X)$ for all $\sigma \in W_x^{n+1}$ and for all $n \leq k$. By the claim, for each

$x \notin Z \cup Z'$ and ϵ_x, we get a neighborhood $W_x \subset V_x$ of x so that $||D(\sigma)|| \subset N_{\epsilon_x}(||f(\sigma)||)$ for all $\sigma \in W_x^{n+1}$ with $n \leq k$. Since $N_{\epsilon_x}(||f(\sigma)||) \subset N(X)$ for such σ, we get property (4).

So it is now enough to prove the above claim. For that we first take a closer look at $|f(\sigma)|$ and $|D(\sigma)|$.

Support of $f(\sigma)$ and $D(\sigma)$: Consider the following set

$$\mathcal{P}(\sigma) := \{(\sigma^n, \sigma^{n-1}, \dots, \sigma^j) \mid \sigma^j \subset \sigma^{j+1} \subset \cdots \sigma^{n-1} \subset \sigma^n = \sigma, 1 \leq j \leq n\}$$

Where σ^k denotes a k-dimensional subsimplex of σ. Let $v(\tau)$ denotes the set of vertices of a simplex τ. We claim the following.

$$|f(\sigma)| = \{\bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\dots(\bar{H}^{B(\sigma^1)}(x)))\dots)) \mid (\sigma^n, \dots, \sigma^1) \in \mathcal{P}(\sigma), x \in v(\sigma^1)\} \quad (3.4.1)$$

It can be checked by induction on dimension of σ. The claim is true for $n = 1$. Suppose it is true for $(n-1)$ dimensional simplices. Let σ be an n-simplex and $\tau = f(\sigma^{n-1}) \in |f(\partial\sigma)|$ for some $(n-1)$-dimensional subsimplex σ^{n-1} of σ. By induction hypothesis, $|\tau|$ consists of the following simplices

$$\{\bar{H}^{B(\sigma^{n-1})}(\bar{H}^{B(\sigma^{n-2})}(\dots(\bar{H}^{B(\sigma^1)}(x)))\dots)) \mid (\sigma^{n-1}, \dots, \sigma^1) \in \mathcal{P}(\sigma^{n-1}), x \in v(\sigma^1)\}$$

Now the claim follows from the fact that $|f(\sigma)| = \cup_{\tau \in |f(\partial\sigma)|} |\bar{H}^{B(\sigma)}(\tau)|$.

Similarly one can check the following

$$|D(\sigma)| = \{\bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\dots(\bar{H}^{B(\sigma^j)}(c(\sigma^j)))\dots)) \mid (\sigma^n, \sigma^{n-1}, \dots, \sigma^j) \in \mathcal{P}(\sigma), 1 \leq j \leq n\} \quad (3.4.2)$$

Proof of the claim: Recall that the singular simplices in $|D(\sigma)|$ can be indexed by the set $\mathcal{P}(\sigma) := \{(\sigma^n, \sigma^{n-1}, \dots, \sigma^j) \mid \sigma^j \subset \sigma^{j+1} \subset \cdots \subset \sigma^{n-1} \subset \sigma^n = \sigma, 1 \leq j \leq n\}$, where σ^i is a i-dimensional subsimplex of σ. For each $s = (\sigma^n, \dots, \sigma^j) \in \mathcal{P}(\sigma)$, let

$$D_s(\sigma) := \bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\dots(\bar{H}^{B(\sigma^j)}(c(\sigma^j)))\dots))$$

Then, $|D(\sigma)| = \cup_{s\in\mathcal{P}(\sigma)} D_s(\sigma)$ by (3.4.2).

If $s = (\sigma^n, \dots, \sigma^j) \in \mathcal{P}(\sigma)$ and $\sigma^j = (x_0, \dots, x_j)$, then $D_s(\sigma)$ is supported in the following set

$$\{\sum_{i=0}^{j} s_i \cdot H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}(..(H_{t_j}^{B(\sigma^j)}(x_i)))..)\} \mid \sum_{i=0}^{j} s_i = 1, (t_n, \dots, t_j) \in [0,1]^{n-j+1}\}$$

Suppose

$$r_s(\sigma) = \max_{(t_n,\dots,t_j)\in[0,1]^{n-j+1}} \{\mathrm{diam}\, H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}(..(H_{t_j}^{B(\sigma^j)}\{v(\sigma^j)\})\}\}.$$

Where $v(\sigma^j)$ denotes the set of vertices of σ^j. We now claim that $||D_s(\sigma)|| \subset N_{r_s(\sigma)}(||f(\sigma)||)$. For that, it is enough to show that

$$\{H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}(..(H_{t_j}^{B(\sigma^j)}\{v(\sigma^j)\})\} \subset ||f(\sigma)|| \quad \text{for all } (t_n, \dots, t_j) \in [0,1]^{n-j+1} \qquad (*)$$

To show the above, let us take a look at the support of $f(\sigma)$. Recall from (3.4.1) that $|f(\sigma)|$ consists of the following set of simplices

$$\{\bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\dots(\bar{H}^{B(\sigma^1)}(x)))\dots)) \mid (\sigma^n, \dots, \sigma^1) \in \mathcal{P}(\sigma), x \in v(\sigma^1)\}$$

Each singular simplex of the form $\bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\dots(\bar{H}^{B(\sigma^1)}(x)))\dots))$ is supported on the following set

$$\{H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}(..(H_{t_1}^{B(\sigma^1)}(x)))..) \mid (t_n, \dots, t_1) \in [0,1]^n\}$$

Let $x \in v(\sigma^j)$. Choose $(\sigma^n, \dots, \sigma^j, \dots, \sigma^1) \in \mathcal{P}(\sigma)$ such that $s = (\sigma^n, \dots, \sigma^j)$ and $x \in v(\sigma^1)$.

Then for all $(t_n, \ldots, t_j) \in [0,1]^{n-j+1}$

$$H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}(\ldots(H_{t_j}^{B(\sigma^j)}(x))\ldots)) \in ||\bar{H}^{B(\sigma^n)}(\bar{H}^{B(\sigma^{n-1})}(\ldots(\bar{H}^{B(\sigma^1)}(x))\ldots))|| \subset ||f(\sigma)||$$

This proves $(*)$ and consequently we get $||D_s(\sigma)|| \subset N_{r_s(\sigma)}(||f(\sigma)||)$.

To prove our claim, we need to show that there is a bounded set Z so that for any $\epsilon > 0$ and $k \in \mathbb{N}$, we can choose small neighborhood W_x of $x \notin Z$ so that for any simplex $\sigma \in W_x^{n+1}$, $n \leq k$, $s \in \mathcal{P}(\sigma)$, we have $r_s(\sigma) < \epsilon$.

Recall that we chose the cover $\mathcal{U}$ to be locally finite. Hence there are only finitely many $U_\alpha \in \mathcal{U}$ containing any given $x \in X$. Let V_x be the intersection of those finitely many U_α. Then for any 1-simplex $\sigma \in V_x^2$, $B(\sigma)$ is one of $B(U_\alpha)$ by construction. In particular $\#\{B(\sigma) \mid \sigma \in V_x^2\}$ is finite. Note that for any simplex σ, $B(\sigma)$ is determined by $\cup_{\tau \in S} B(\tau)$ and $\dim(\sigma)$, where S is the set of 1-subsimplices of σ. This implies that, for any given $k \in \mathbb{N}$, the set $\{B(\sigma^n) \mid \sigma^n \in V_x^{n+1}, n \leq k\}$ is finite. Since X is proper at infinity, there is a bounded set Z so that these homotopies are uniformly continuous when restricted to some neighborhood $W'_x \subset V_x$ of $x \notin Z$. This implies that for any $\epsilon > 0$, there exists a neighborhood $W_x \subset W'_x$ of every $x \notin Z$, such that $\operatorname{diam}\{H_{t_n}^{B(\sigma^n)}(H_{t_{n-1}}^{B(\sigma^{n-1})}..(H_{t_j}^{B(\sigma^j)}(W_x))..)\} < \epsilon$ for all $\sigma \in W_x^{n+1}$, for all $(\sigma^n, \ldots, \sigma^j) \in \mathcal{P}(\sigma)$, $(t_n, \ldots, t_j) \in [0,1]^{n-j+1}$ and $n \leq k$. That implies $r_s(\sigma) < \epsilon$ for all $s \in \mathcal{P}(\sigma)$, $\sigma \in W_x^{n+1}$, $n \leq k$, and $x \notin Z$. That finishes the proof. □

Lemma 3.4.5. *Let X, $N(X)$ and $\mathrm{C}_*^F(X)$ be as in the Lemma 3.4.3 and $k \in \mathbb{N}$. Then there exist a chain map $g : \mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathrm{s}}(N(X))$ and a bounded set $Z \subset X$ such that for each $x \in X - Z$, there is an open neighborhood W_x of x in X such that $g(\sigma) = c(\sigma)$ for all $\sigma \in W_x^{n+1}$ with $n \leq k$.*

Proof. Let f, D, Z and W_x be as in the Lemma 3.4.3. We define $g(\sigma) := c(\sigma)$ if σ is supported in W_x for all $x \in X - Z$. If a i-simplex σ^i is not in W_x^{i+1} for any x but there is a $(i-1)$-subsimplex σ' of σ that lives in W_x^i for some x, then we define $g(\sigma) := f(\sigma) - [\sigma : \sigma']D(\sigma')$

(see figure 3.5), where $[\alpha : \beta] = \pm 1$ tells us whether orientations of α and β agree or not. For such σ, we will now check that $g(\partial\sigma) = \partial g(\sigma)$. For that we need to use the fact that D is a chain homotopy between f and c. We first note the following

$$\begin{aligned}\partial g(\sigma) &= \partial f(\sigma) - [\sigma : \sigma']\partial D(\sigma') \\ &= f(\partial\sigma) + [\sigma : \sigma']D(\partial\sigma') - [\sigma : \sigma']f(\sigma') + [\sigma : \sigma']c(\sigma')\end{aligned}$$

Now we will show that the above is same as $g\partial(\sigma)$. First we observe that any $(i-1)$-subsimplex $\tau \neq \sigma'$ of σ has a maximal $(i-2)$-subsimplex τ' supported in V. Hence for such τ, we have $g(\tau) = f(\tau) - [\tau : \tau']D(\tau')$. We use that to have the following

$$\begin{aligned}g(\partial(\sigma)) &= \sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau]g(\tau) + [\sigma : \sigma']g(\sigma') \\ &= \sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau]g(\tau) + [\sigma : \sigma']c(\sigma') \\ &= \sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau]f(\tau) - \sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau][\tau : \tau']D(\tau') + [\sigma : \sigma']c(\sigma') \\ &= f(\partial\sigma) - [\sigma : \sigma']f(\sigma') - \sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau][\tau : \tau']D(\tau') + [\sigma : \sigma']c(\sigma')\end{aligned}$$

To get $\partial g(\sigma) = g(\partial\sigma)$, it is now enough to show that

$$[\sigma : \sigma']D(\partial\sigma') = -\sum_{\substack{\tau\in\partial\sigma \\ \tau\neq\sigma'}} [\sigma : \tau][\tau : \tau']D(\tau')$$

For that, we need to show that for a common $(i-2)$-subsimplex β of σ' and τ', $[\sigma : \sigma'][\sigma' : \beta] = -[\sigma : \tau][\tau : \beta]$. This is true because coefficient of β in $\partial(\partial\sigma)$ is $[\sigma : \sigma'][\sigma' : \beta] + [\sigma : \tau][\tau : \beta]$ and so this has to be zero.

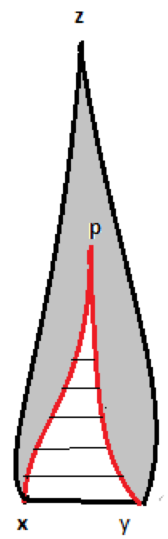

Figure 3.5: This is a picture of $g(\sigma)$ where $\sigma = (x, y, z)$. Here p is the contracting point of the homotopy $H^{B((x,y))}$. The grey part is $f(\sigma)$ and striped part is $D(\sigma')$ where $\sigma' = (x, y)$

In other cases, an i-simplex σ has some maximal subsimplices $\{\sigma'\}$ in W_x for some x with dimension less than $i-1$. In this case, by induction on dimension one can show that $g(\partial\sigma)$ deforms retract to $f(\partial\sigma) = \partial(f(\sigma))$ by deforming along $D(\sigma')$ for each maximal subsimplex σ' supported in V. Then we use $f(\sigma)$ to fill this deformed $g(\partial\sigma)$.

□

Proof of 3.4.2. To define the map $S : \mathrm{C}^F_*(X) \to \mathrm{C}^s_*(N(X))$, we first take the filling g from the Lemma 3.4.5. We then apply barycentric subdivision on g to get the support in $\mathrm{C}^{\mathcal{U}}_*(N(X))$. Let $G : \mathrm{C}^F_*(X) \to \mathrm{C}_{*+1}(N(X))$ be the corresponding subdivision operator. The only thing that remains to be checked is the property (4) in Proposition 3.4.2. By Lemma 3.4.5, simplices in W_x are filled by convex fillings and consequently, the support of the subdivision operator stays in the convex hull of that simplex. Since the diameter of convex hull of a small simplex is small, property (4) is satisfied. □

Our next goal is to prove Proposition 3.4.7 which shows that $\mathrm{HB}^*(X)$ is taut in the sense that any class in $\mathrm{HB}^*(X)$ is a restriction of a class from $\mathrm{HB}^*(N(X))$ for some neighborhood $N(X)$ of X inside ℓ^∞. Let $\mathcal{U}$ be a collection of open sets in X and A be a subset of X. The *star of A with respect to $\mathcal{U}$*, denoted by $st(A,\mathcal{U})$, is defined to be the union of those elements of $\mathcal{U}$ whose intersection with A is nonempty. An open covering of A in X is a collection $\mathcal{U}$ of open sets of X such that $A \subset st(A,\mathcal{U})$. We now state the following lemma from [19] which roughly says that there is a projection map from $st(X,\mathcal{V})$ to V that does not move close points too far apart.

Lemma 3.4.6 ([19]). *If $X \subset \ell^\infty$, then for every open covering $\mathcal{U}$ of X in ℓ^∞, there is an open covering $\mathcal{V}$ of X in ℓ^∞ and a function $f : st(X,\mathcal{V}) \to X$ such that*

1. *$f(x) = x$ for all $x \in X$*

2. *For each $V \in \mathcal{V}$ with $V \cap X \neq \emptyset$ there is a $U \in \mathcal{U}$ such that $V \cup f(V) \subset U$.*

Definition. A map $f : X \to Y$ between metric spaces is called *coarse* map, if inverse image of a bounded set is bounded and there exist a non decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \rho(d(x,y))$$

Proposition 3.4.7. *Let X be a subspace of ℓ^∞. Then for any $\phi \in \mathrm{CB}^*(X)$, there exists a neighborhood $N(X)$ of X in ℓ^∞ and a coarse map $f : N(X) \to X$ such that $f^*(\phi) \in \mathrm{CB}^*(N(X))$ and $i^* f^*(\phi) = \phi$ where $i : X \to N(X)$ is the inclusion.*

Proof. The proof is similar to the proof of tautness of Alexander-Spanier cohomology [19]. Suppose, $\phi \in \mathrm{CB}^*(X)$. That means there is a bounded set B such that for any $a \in X - B$, we can choose a neighborhood U_a so that $U_a^{*+1} \cap |\phi| = \emptyset$. Cover B with a bounded open set U_B in X. Let $\mathcal{U} = U_B \cup \{U_a\}_{a \in X-B}$ be the open cover of X. We can choose these open sets so that their diameter is less than r for some $r \geq 0$. Then Lemma 3.4.6 yields a refinement of $\mathcal{V}$ of $\mathcal{U}$, and a map $f : st(X, \mathcal{V}) \to X$ such that $f(a) = a$ for all $a \in X$ and for each $V \in \mathcal{V}$ with $V \cap X \neq \emptyset$, there is a $U \in \mathcal{U}$ such that $V \cup f(V) \subset U$.

By the second property of f in Lemma 3.4.6, it follows that $d(x, f(x)) \leq r$ for all $x \in st(X, \mathcal{V})$. Hence, $d(f(x), f(y)) \leq 2r + d(x, y)$ by triangle inequality which means f has an upper control function. Furthermore f sends unbounded set to unbounded set because $d(y, f(x)) \geq d(y, x) - r$ by triangle inequality. Hence f is a coarse map.

We let $N(X) := st(X, \mathcal{V})$. To show that $f^*(\phi) \in \mathrm{C\,B}^*(N(X))$, we notice that if an element $V \in \mathcal{V}$ is far from U_B, then $V \cup f(V)$ does not touch U_B, and hence $f^*(\phi)|_V \subset U_a$ for some $a \in X - B$. Finally $i^* f^*(\phi) = \phi$ because $f|_X = id_X$. □

Now we are ready to present the proof of our main theorem.

Proof of 3.4.1. By the long exact sequence, we need to show $\mathrm{H}^*(\mathrm{C\,B}^*(X)/\mathrm{C\,X}^*(X)) = 0$, in other words for $\phi \in \mathrm{C\,B}^n(X)$ with $d\phi \in \mathrm{C\,X}^{n+1}(X)$ we need to find $\psi \in \mathrm{C\,B}^{n-1}(X)$ so that $\phi - d\psi \in \mathrm{C\,X}^n$. By Kuratowski's embedding theorem, X can be embedded inside ℓ^∞. Proposition 3.4.7 produces a neighborhood of $N(X)$ and a coarse map $f : N(X) \to X$ such that $\phi' = f^*(\phi) \in \mathrm{C\,B}^*(N(X))$. Let U be a bounded neighborhood of $\|\phi'\|$ in X and for each $x \in X - \|\phi'\|$ choose a neighborhood U_x such that $U_x^{n+1} \cap |\phi| = \emptyset$. Let $\mathcal{U}$ denote the collection of U_x together with U. Moreover, we have $d(\phi') = f^*(d(\phi))$ is coarse since $d(\phi')$ is coarse and f is a coarse map.

Lemma 3.4.2 produces the chain homotopy $G : \mathrm{C}^F_*(X) \to \mathrm{C}_{*+1}(N(X))$, which we use to define a linear map $D : \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(N(X))$ by setting

$$D(\sigma^n) = \begin{cases} G(\sigma^n) & \text{if } \sigma^n \in \mathrm{C}^F_n(X) \\ 0 & \text{otherwise} \end{cases}$$

Let $i : \mathrm{C}_*(X) \to \mathrm{C}_*(N(X))$ is the inclusion map. We now define

$$T = i - \partial D - D\partial.$$

And dually

$$T^* = i^* - D^*d - dD^*$$

We claim that $T^*(\phi')$ is coarse. Let σ^n be of some fixed diameter r. If $d(\sigma^n, b) > R_n(r)$, then $\sigma^n \in \mathrm{C}_n^F(X)$ and it follows that $T(\sigma)$ is supported by $\mathcal{U}$. If all vertices of σ^n are outside of $\rho_n(r)$-neighborhood of U, then $|T(\sigma)|$ does not meet U^{n+1} and therefore $T^*(\phi')(\sigma) = \phi'(T(\sigma)) = 0$, since $\phi|_{U_x^{n+1}} = 0$ for all U_x. Since U is bounded, the claim follows.

We claim that $D^*d(\phi)$ is coarse. Indeed, by construction if σ has diameter r, then $|D\sigma| \overset{\rho_n(r)}{\subset} \sigma$. So, D^* preserves coarseness, and the claim follows since $d(\phi')$ is coarse.

Finally, we claim that $D^*(\phi') \in \mathrm{C\,B}^{*+1}(X)$. Because of the property (4) of G in lemma 3.4.2, we can choose a neighborhood V of x for all $x \notin ||\phi'||$ except points in some bounded set B, such that $D_*(V^{*+1})$ does not intersect $|\phi'|$. That implies $x \notin ||D^*\phi'||$ for all $x \notin ||\phi'|| \cup B$. Since $\phi' \in \mathrm{C\,B}^*(X)$, the claim follows.

Thus $\psi = D^*(\phi')$ is the desired cochain. □

3.5 Computation of coarse cohomology of the complement

In this section we will exploit Theorem 3.4.1 to express coarse cohomology of the complement in terms of Alexander–Spanier cohomology for nice spaces.

Theorem 3.5.1. *If $X/\bar{A}$ is uniformly contractible at infinity, proper at infinity and $X \overset{c}{\neq} A$, then*

$$\mathrm{H\,X}^*(X - A) = \begin{cases} 0 & \textit{if } * = 0 \\ \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(A)) & \textit{otherwise} \end{cases}$$

Proof. If $X/\bar{A}$ is uniformly contractible at infinity, then Theorem 3.4.1 yields $\mathrm{H\,X}^*(X/\bar{A}) = \mathrm{H\,B}^*(X/\bar{A})$. Combining this with Proposition 2.3.3 gives us $\mathrm{H\,X}^*(X - A) = \mathrm{H\,B}^*(X/\bar{A})$.

Note that $X \overset{c}{\neq} A$ is equivalent to saying $X/\bar{A}$ is unbounded. In this case, for any $b \in X/\bar{A}$, Proposition 3.2.2 gives us the following

$$\mathrm{H\,B}^*(X/\bar{A}) = \begin{cases} 0 & \text{if } * = 0 \\ \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(A)) & \text{otherwise.} \end{cases}$$

Finally we observe that $\varinjlim \tilde{\mathrm{H}}^*(X/\bar{A} - N_r(b)) = \varinjlim \tilde{\mathrm{H}}^*(X - N_r(A))$. That finishes the proof. □

Let $q : X \to X/\bar{A}$ be the quotient map. We observe that the ball $B_r(x)$ in (X, d) with center at x is isometric to the ball $B_r(q(x))$ in $(X/\bar{A}, d_A)$ if $d(x, A) > 2r$. That means, if X is uniformly contractible at infinity and proper at infinity, then so is $X/\bar{A}$. Hence, as a consequence of Theorem 3.5.1, we get the following.

Corollary 3.5.2. *If X is uniformly contractible at infinity, proper at infinity and $X \overset{c}{\neq} A$, then*

$$\mathrm{H\,X}^*(X - A) = \begin{cases} 0 & \textit{if } * = 0 \\ \varinjlim \tilde{\mathrm{H}}^{*-1}(X - N_r(A)) & \textit{otherwise.} \end{cases}$$

Chapter 4

Obstruction to coarse embedding

4.1 Introduction

Van Kampen [17] developed an obstruction theory for embeddings of n-dimensional simplicial complexes into $\mathbb{R}^{2n}$. A modern approach to his theory uses (co)homology of the configuration space. In this final chapter of this thesis we develop an analogous theory for coarse embedding by using coarse cohomology theory.

Let us first briefly describe the classical van Kampen obstruction for a topological space X. Let $\delta(X)$ be the diagonal set $\{(x, x) \mid x \in X\} \subset X \times X$. Consider the following deleted product

$$\tilde{X} := X \times X - \delta(X) = \{(x, y) \in X \times X \mid x \neq y\}$$

$\tilde{X}$ has a natural free action by $\mathbb{Z}_2$ by switching the coordinates. Consider the corresponding $\mathbb{Z}_2$ covering map $q : \tilde{X} \to \tilde{X}/\mathbb{Z}_2$. The space $\tilde{X}/\mathbb{Z}_2$ is called the (2^{nd} ordered) configuration space of X. There exists a classifying map from the $\mathbb{Z}_2$-bundle $q : \tilde{X} \to \tilde{X}/\mathbb{Z}_2$ to the universal

$\mathbb{Z}_2$-bundle $S^\infty \to \mathbb{R}P^\infty$ as follows.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\phi}} & S^\infty \\ \downarrow & & \downarrow \\ \tilde{X}/\mathbb{Z}_2 & \xrightarrow{\phi} & \mathbb{R}P^\infty \end{array}$$

If there is an embedding $g : X \hookrightarrow \mathbb{R}^m$, then we can choose $\tilde{\phi}$ so that it factors through S^{m-1}. More precisely, we can choose $\tilde{\phi}$ to be the following map $\tilde{X} \to S^{m-1} \subset S^\infty$

$$(x, y) \mapsto \frac{g(x) - g(y)}{|g(x) - g(y)|}$$

In this case, ϕ maps $\tilde{X}/\mathbb{Z}_2$ to $\mathbb{R}P^{m-1} \subset \mathbb{R}P^\infty$. So the induced map $\phi^* : \mathrm{H}^m(\mathbb{R}P^\infty) \to \mathrm{H}^m(\tilde{X}/\mathbb{Z}_2)$ is trivial as it factors through $\mathrm{H}^m(\mathbb{R}P^{m-1})$. In particular, if $\eta^m \in \mathrm{H}^m(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is the nonzero class, then $\phi^*(\eta^m)$ will be trivial. Hence the cohomology class $\phi^*(\eta^m)$, gives an obstruction for the embedding of X into $\mathbb{R}^m$. The class $\phi^*(\eta^m)$ is called the van Kampen obstruction class of degree m and is denoted by $vk^m(X)$.

Let us now turn our attention to the coarse world. A map $f : X \to Y$ between two metric spaces is said to be a *coarse embedding* if there exist two proper non decreasing maps $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that $\rho_-(d(x, y)) \leq d(f(x), f(y)) \leq \rho_+(d(x, y))$ for all $x, y \in X$. Our main goal is to find obstruction for the coarse embedding of a metric space X into a given space Y.

Roe [15] defined coarse cohomology of a metric space that roughly measures how uniformly large bounded sets fit together. Building on Roe's theory, a notion of coarse cohomology of complement of a subspace A in X was introduced in [1]. In this paper, we use an equivariant version of that theory to define $\mathbb{Z}_2$-equivariant coarse cohomology of the complement of $\delta(X)$ in $X \times X$. If X is a separable metric space, we then find a class in the n^{th} degree of that cohomology, denoted by $cvk^n(X)$, that obstructs coarse embedding of X into $\mathbb{R}^{n-1}$. In fact, the class $cvk^n(X)$ can be used to get obstruction to coarse embedding into any other metric space. We define coarse obstruction dimension $\operatorname{cobdim}(X)$ of a separable metric space X which is roughly the largest n such that $cvk^n(X)$ is nonzero. Our main theorem is the following which gives a necessary condition for the existence of coarse embedding maps.

Theorem 4.1.1. *If X admits a coarse embedding into Y, then* $\operatorname{cobdim}(X) \leq \operatorname{cobdim}(Y)$.

Hence, cobdim of a space gives a measure of how large that space is coarsely. In general cobdim is hard to compute. We give estimates of cobdim for certain spaces.

Below we list some examples and their implications due to Theorem 4.1.1.

- We show that $\operatorname{cobdim}(\mathbb{R}^n) = n$. Hence Theorem 4.1.1 implies that X does not admit coarse embedding into $\mathbb{R}^{n-1}$ if $\operatorname{cobdim}(X) \geq n$.

- Let K be a finite simplicial complex and $Cone(K) = K \times [0,1]/K \times \{0\}$ be the finite cone on K. If $vk^{n-1}(Cone(K)) \neq 0$, then any infinite Euclidean cone on K, denoted by $Cone^e_\infty(K)$, have cobdim $\geq n$ (Theorem 4.5.2). For example, $vk^3(Cone(K_{3,3})) \neq 0$ where $K_{3,3}$ is the complete bipartite graph on three points. This implies that $\operatorname{cobdim}(Cone^e_\infty(K_{3,3})) \geq 4$. Since $\operatorname{cobdim}(\mathbb{R}^3) = 3$, we obtain that $Cone^e_\infty(K_{3,3})$ does not admit coarse embedding into $\mathbb{R}^3$ by Theorem 4.1.1 (Example 4.5.2). This was observed earlier by Bestvina–Kapovich–Kleiner [4].

- If X is a proper, uniformly contractible n-manifold with uniformly locally contractible boundary, then $\operatorname{cobdim}(X) \leq n$ (Theorem 4.7.7). Example of such spaces include universal cover of compact aspherical n-manifolds. Hence Theorem 4.1.1 tells us, if $\operatorname{cobdim}(X) \geq n$, then X does not admit coarse embedding into the universal cover of any compact aspherical $(n-1)$-manifold.

- If X is a uniformly contractible, locally contractible space and $\mathrm{HX}^*(X^2 - \delta(X)) = 0$ for $* \leq n-1$, then $\operatorname{cobdim}(X) \geq n$ (Theorem 4.8.1). In particular, any proper, uniformly contractible n-manifold has cobdim $\geq n$ and hence it does not admit coarse embedding into any uniformly contractible manifold with dimension $\leq n-1$ by Theorem 4.1.1. This was obtained earlier by Yoon [20].

- If a finitely generated group G acts properly on X by isometries, then there exists a coarse embedding of G, equipped with a word metric, into X by mapping G into an orbit of the action. That means, from coarse point of view, any space X with a proper G-action has to be at least as large as G. Using Theorem 4.1.1, we show that , if G acts properly, cocompactly on a contractible manifold M (possibly with boundary), then $\dim(M) \geq \operatorname{cobdim}(G)$ (Corollary 4.7.9).

Remark. The study of obstruction to coarse embedding of finitely generated groups into proper contractible n-manifold in terms of certain homology classes was initiated by Bestvina, Kapovich and Kleiner [4] and was later generalized by Yoon [20]. One of the main motivations behind the present work was to formulate their obstruction in terms of coarse cohomology.

Overview. In section 4.2, we define the equivariant coarse cohomology of the complement. In section 4.3, we produce a computational tool for this cohomology by relating it to the equivariant Alexander–Spanier cohomology for many spaces. We define the coarse obstruction dimension of a space and prove Theorem 4.4.8 which is a slightly stronger version of Theorem 4.1.1. In section 4.5, we give a relation between classical van Kampen obstruction and coarse van Kampen obstruction. We use this relation to compute coarse van Kampen obstruction for certain Euclidean cones on simplicial complexes. In section 4.6, we produce a coarse version of Gysin sequence for $\mathbb{Z}_2$-bundle to compute coarse cohomology of configuration space. In section 4.7 and 4.8, we use the coarse Gysin sequence to estimate cobdim of certain spaces.

4.2 Equivariant coarse cohomology of the complement

Let G be a group acting on X by isometries and R be a G-module. Let G acts on X^{n+1} by the diagonal action, $g(x_0, \ldots, x_n) = (gx_0, \ldots, gx_n)$. G-equivariant coarse cohomology of the complement of A in X, denoted by $\mathrm{HX}^*_{\mathrm{G}}(X - A; R)$, is defined to be the cohomology of the following cochain complex.

$$\mathrm{C\,X}^*_{\mathrm{G}}(X - A; R) := \{\phi \in \mathrm{C\,X}^*(X - A; R) \mid \phi \text{ is } G\text{-equivariant}\}$$

Suppose X and Y both admit a G-action and $A \subset X$ and $B \subset Y$. Let us now address the following question: What kind of maps between X and Y induce a map between $\mathrm{H\,X}^*_{\mathrm{G}}(Y - B)$ and $\mathrm{H\,X}^*_{\mathrm{G}}(X - A)$.

A map $f : X \to Y$ is called a *proper map* if $f^{-1}(B)$ is bounded in X whenever B is bounded in Y. A map $f : X \to Y$ is said to have an *upper control* if there exist a proper function $\rho_+ : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $d(f(x), f(y)) \leq \rho_+(d(x, y))$ for all $x, y \in X$. A map $f : X \to Y$ is called a *coarse map* if f is proper and it has a upper control.

We now recall that the space that models coarse complement of A in X. It is the space X equipped with the following pseudo-metric.

$$d_A(x, y) = \min\{d(x, A) + d(y, A), d(x, y)\}.$$

Note that the definition of a proper map and coarse map makes sense with respect to pseudo metric.

Recall that A map $f : X \to Y$ is called a *coarse map between the coarse complements of $A \subset X$ and $B \subset Y$*, if the induced map $f : (X, d_A) \to (Y, d_B)$ is a coarse map.

We also recall the proposition 2.3.5 that tells us that any coarse map $f : X \to Y$ between the coarse complements of A and B induces a chain map $f^* : \mathrm{C\,X}^*(Y - B) \to \mathrm{C\,X}^*(X - A)$. In the equivariant setting, this immediately implies the following.

Proposition 4.2.1. *If $f : X \to Y$ is a G-equivariant coarse map between the coarse complements of $A \subset X$ and $B \subset Y$, then f induces a chain map $f^* : \mathrm{C\,X}^*_{\mathrm{G}}(Y - B) \to \mathrm{C\,X}^*_{\mathrm{G}}(X - A)$ and hence induce a map $f^* : \mathrm{H\,X}^*_{\mathrm{G}}(Y - B) \to \mathrm{H\,X}^*_{\mathrm{G}}(X - A)$.*

4.3 Computation of equivariant coarse cohomology

In this section, we will show that $\mathrm{H\,X}^*_{\mathrm{G}}(X-A)$ is equal to equivariant boundedly supported cohomology $\mathrm{H\,B}^*_{\mathrm{G}}(X-A)$ if X is reasonably nice (Theorem 4.3.1). This section is essentially an equivariant version of section 3.3 where we related coarse cohomology with boundedly supported cohomology.

Let us begin by defining boundedly supported cohomology of the complement. It is the cohomology of the following cochain complex:

$$\mathrm{C\,B}^*(X-A;R) := \{\phi \in \mathrm{C}^*(X;R) \mid \|\phi\| \overset{c}{\subset} \Delta_A\}$$

Suppose R is a G-module. We define equivariant boundedly supported cohomology of the complement with coefficients in R to be the cohomology of the following cochain complex.

$$\mathrm{C\,B}^*_{\mathrm{G}}(X-A;R) := \{\phi \in \mathrm{C\,B}^*(X-A;R) \mid \phi \text{ is } G\text{-equivariant}\}$$

We denote the cohomology of the above complex by $\mathrm{H\,B}^*_{\mathrm{G}}(X-A;R)$. For the rest of the discussion, we will omit R from the notation unless it is crucial.

We will use the notation $G \curvearrowright (X,A)$ to mean that G is acting on X by isometries where the action stabilizes the subset $A \subset X$. We now state our main theorem of this section which is essentially an equivariant version of Theorem 3.3.1.

Theorem 4.3.1. *Suppose X is uniformly contractible away from A, locally contractible away from A and $G \curvearrowright (X,A)$. Then the inclusion $\mathrm{C\,X}^*_{\mathrm{G}}(X-A) \hookrightarrow \mathrm{C\,B}^*_{\mathrm{G}}(X-A)$ induces an isomorphism on the cohomology:*

$$\mathrm{H\,X}^*_{\mathrm{G}}(X-A) \cong \mathrm{H\,B}^*_{\mathrm{G}}(X-A).$$

If X is uniformly contractible away from A, then we can perform a version of the standard "connect the dots" construction. Every 1-simplex σ of diameter r outside of $N_{\mu(r)}(A)$ is fillable, so we can pick a singular 1-chain c, such that $|c| \subset N_{\rho(r)}(\sigma)$ and $\partial c = \partial \sigma$. Proceeding by induction on the dimension, if a simplex is sufficiently far from A, its boundary is already filled by a singular cycle that bounds a singular chain contained in a controlled neighborhood of the simplex. If the space is locally contractible away from A, we can fill small simplices outside a metric neighborhood of A by chains of small diameter. Moreover, by subdividing, we can arrange so that filling singular chains of all simplices are supported by $\mathcal{U}$.

Note that not every simplex is fillable, and that the diameter of fillings grows with dimension, as well as the size of the neighborhood of A that we have to avoid. To formalize the notion of sufficiently far, we make the following definition. Given an increasing sequence of control functions $\mu_n(r)$, denote

$$\mathrm{C}_n^F(X) = \langle \sigma^n \mid d(\sigma^n, A) \geq \mu_n(\operatorname{diam} \sigma^n) \rangle \subset \mathrm{C}_n(X)$$

Since μ_n is increasing, this defines a subcomplex of the chain complex of finitely supported chains.

If $G \curvearrowright (X, A)$, then $\sigma \in \mathrm{C}_n^F(X)$ implies $g\sigma \in \mathrm{C}_n^F(X)$ for all $g \in G$. In this setting, we can choose the filling of the simplices in $\mathrm{C}_n^F(X)$ G-equivariantly. Suppose, $\mathcal{U}$ be a G-invariant open over of X. First, we choose the filling of simplices from a set P that contains one simplex from each orbit of the action of G on the simplices in $\mathrm{C}_n^F(X)$ as discussed earlier and then extend this G-equivariantly. If the space is locally contractible away from A, there exist neighborhoods V_x of x for every $x \notin N_r(A)$ for some $r > 0$ such that filling of simplices in V_x^{n+1} can be chosen to be supported in some set $U \in \mathcal{U}$. We choose these V_x so that $gV_x = V_{gx}$ and fill simplices in $P \cap V_x^{n+1}$ by singular chains supported in some set in $U \in \mathcal{U}$. When we extend these fillings G-equivariantly, every simplex in $\cup_{x \notin N_r(A)} V_x^{n+1}$ gets filled by singular simplices in some $U \in \mathcal{U}$. Similarly, we can choose the subdivision of the filling to be G-equivariant. We first subdivide the filling of the simplices in P so that they are supported in $\mathcal{U}$ and then extend G-equivariantly to all simplices in $\mathrm{C}^F_n(X)$. Since $\mathcal{U}$ is G-invariant, filling of all simplices in $\mathrm{C}^F_n(X)$ stay inside $\mathcal{U}$ in this equivariant extension.

Let $V : \mathrm{C}^{\mathrm{s}}_*(X) \to \mathrm{C}_*(X)$ be the forgetful map, which maps a singular simplex to its vertices. Let $\mathrm{C}^{\mathcal{U}}_*(X)$ be the complex of singular chains supported by $\mathcal{U}$. The discussion above gives the following lemma.

Lemma 4.3.2. *Suppose X is uniformly contractible away from A and locally contractible away from A and $G \curvearrowright (X, A)$. Let $\mathcal{U}$ be a G-invariant open cover of X. Then there exist two non-decreasing sequence of control functions $\rho_n, \mu_n : [0, \infty) \to [0, \infty)$, a G-equivariant chain map $S : \mathrm{C}^F_*(X) \to \mathrm{C}^{\mathcal{U}}_*(X)$ where*

$$\mathrm{C}^F_n(X) = \langle \sigma^n \mid d(\sigma^n, A) \geq \mu_n(diam\, \sigma^n)\rangle \subset \mathrm{C}_n(X)$$

and a G-equivariant chain homotopy $H : \mathrm{C}^F_*(X) \to \mathrm{C}_{*+1}(X)$ between VS and the inclusion map so that

1. *$S : \mathrm{C}^F_0(X) \to \mathrm{C}^{\mathrm{s}}_0(X)$ is the identity.*
2. *$|S(\sigma^n)| \subset N_{\rho_n}(\sigma^n)$.*
3. *$|H(\sigma^n)| \subset N_{\rho_n}(\sigma^n)$.*
4. *There exist $r > 0$ so that for every $x \notin N_r(A)$, there is a neighborhood V_x of x such that for all $\sigma^n \in V_x^{n+1}$, $H(\sigma^n) \in \mathrm{C}^{\mathcal{U}}_{n+1}(X)$.*

Proof of Theorem 4.3.1. By the long exact sequence, we need to show that $\mathrm{H}^*(\mathrm{C\,B}^*_{\mathrm{G}}(X - A)/\mathrm{C\,X}^*_{\mathrm{G}}(X - A)) = 0$. In other words, for $\phi \in \mathrm{C\,B}^n_{\mathrm{G}}(X - A)$ with $d\phi \in \mathrm{C\,X}^{n+1}_{\mathrm{G}}(X - A)$ we need to find $\psi \in \mathrm{C\,B}^{n-1}_{\mathrm{G}}(X - A)$ so that $\phi - d\psi \in \mathrm{C\,X}^n_{\mathrm{G}}(X - A)$. Choose a metric neighborhood U of A such that $||\phi|| \subset U$. Since ϕ is G-equivariant, we have $g \cdot U = U$ for all $g \in G$. For each $x \in X - ||\phi||$, choose a metric neighborhood U_x with diameter ≤ 1 such that $U_x^{n+1} \cap |\phi| = \emptyset$. Since ϕ is G-equivariant, we can choose the association $x \mapsto U_x$ so that

$U_{gx} = gU_x$ for all $g \in G$. Let $\mathcal{U}$ denote the collection of U_x together with U. By construction, this is a G-invariant cover.

Lemma 4.3.2 produces the G-equivariant chain homotopy $H : \mathrm{C}^F_*(X) \to \mathrm{C}_{*+1}(X)$ corresponding to the cover $\mathcal{U}$. We now use H to define a linear map $D : \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$ by setting

$$D(\sigma^n) = \begin{cases} H(\sigma^n) & \text{if } \sigma^n \in \mathrm{C}^F_n(X) \\ 0 & \text{otherwise.} \end{cases}$$

We define the following map

$$\tau = id + \partial D + D\partial$$

In particular, $\tau(\sigma) = VS(\sigma)$ when $\sigma \in \mathrm{C}^F_*(X)$. After dualizing we can apply the above on ϕ to get the following

$$\tau^*\phi = \phi + dD^*\phi + D^*d\phi$$

We claim that $\tau^*\phi \in \mathrm{C\,X}^n_{\mathrm{G}}(X - A)$. If $\mathrm{diam}(\sigma^n) \leq k$ and $d(\sigma^n, A) > \mu_n(k)$ for some $k \geq 0$, then $\sigma^n \in \mathrm{C}^F_*(X)$ and hence $\tau(\sigma^n) = VS(\sigma^n)$. Moreover, if σ^n is outside of $\rho_n(k)$-neighborhood of U, then $|\tau\sigma^n|$ does not touch U^{n+1} because $|\tau\sigma^n| = |VS\sigma^n| \subset N_{\rho_n(k)}(\sigma^n)$. Since $U \overset{c}{\subset} A$, we obtain $(\tau^*\phi)(\sigma^n) = 0$. Hence, $|\tau^*\phi| \cap N_k(\Delta) \overset{c}{\subset} \Delta_A$. This proves the claim.

Next we claim that $D^*(\phi) \in \mathrm{C\,B}^{n-1}_{\mathrm{G}}(X - A)$. By Lemma 4.3.2, there exist $r > 0$ so that for every $x \notin N_r(A)$, there is a neighborhood V_x of x with diameter ≤ 1 such that for all $\sigma \in V^n_x$, $H(\sigma) \in \mathrm{C}^{\mathcal{U}}_{n+1}(X)$. Let $s = r + \rho_{n-1}(1)$. We will show that $||D^*(\phi)|| \subset N_s(U)$. Let $x \notin N_s(U)$ and $\sigma \in V^n_x$. It is enough to show that $\phi(D(\sigma)) = 0$. By construction of D, it is enough to show that $\phi(H(\sigma)) = 0$. By property (3) of Lemma 4.3.2, we have $H(\sigma) \subset N_{\rho_{n-1}(1)}(\sigma)$. This implies $H(\sigma)$ does not touch U^{n+1} since $s \geq \rho_{n-1}(1)$. This means $|H(\sigma)| \subset U^n_{x'}$ for some $x' \in X$ by property (4) of Lemma 4.3.2. This implies $\phi(H(\sigma)) = 0$. Hence, $||D^*(\phi)|| \subset N_s(U)$. Since $U \overset{c}{\subset} A$, the claim follows.

Finally we claim that $D^*d(\phi) \in \mathrm{C\,X}^n_{\mathrm{G}}(X - A)$. By construction of D, if $d(\sigma^n, A) \geq \mu_n(r)$

then $|D\sigma^n| \subset N_{\rho_n(r)}(\sigma^n)$ where r is the diameter of σ^n. Now the claim follows from the assumption that $d(\phi) \in \mathrm{C\,X}^*_{\mathrm{G}}(X - A)$.

Then setting $\psi = -D^*(\phi)$ we get what we want. □

Honkasalo defined a notion of equivariant Alexander–Spanier cohomology in [11]. To define this in general, one needs a contravariant coefficient system—a contravariant functor from the category of G-spaces G/H $(H \leq G)$ and G-maps between them to the category of R-modules. If R is a G-module, it defines a contravariant coefficient system $G/H \mapsto R$, each G-map $G/H \to G/K$ inducing identity $M \to M$. With this coefficient system, the equivariant Alexander–Spanier cochain complex takes the following form

$$\mathrm{C}^*_{\mathrm{G}}(X; R) := \{\phi \in \mathrm{C}^*_{as}(X; R) \mid \phi \text{ is } G\text{-equivariant}\}$$

We will denote the cohomology of this complex by $\mathrm{H}^*_{\mathrm{G}}(X; R)$.

The next proposition relates the equivariant boundedly supported cohomology with the equivariant Alexander–Spanier cohomology. Roughly we show that under certain condition $\mathrm{H\,B}^*_{\mathrm{G}}(X - A)$ is the same as the equivariant Alexander–Spanier cohomology of the space far away from A. In the proof, we need to use the following complex

$$\mathcal{C}^*_G(X; R) := \{\phi \in \mathrm{C}^*(X; R) \mid \phi \text{ is } G \text{ equivariant}\}$$

We denote the corresponding homology by $\mathcal{H}^*_G(X; R)$.

Proposition 4.3.3. *Suppose* $G \curvearrowright (X, A)$ *and* $Fix_G(X) := \{x \in X \mid g \cdot x = x \quad \forall g \in G\} \neq \emptyset$.

1. *If* $X - A \overset{\mathrm{c}}{=} *$, *then*

$$\mathrm{H\,B}^*_{\mathrm{G}}(X - A; R) = \begin{cases} R & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. If $X - A \overset{c}{\neq} *$, *then*

$$\mathrm{H\,B}^*_{\mathrm{G}}(X - A; R) = \begin{cases} 0 & \textit{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}_G(X - N_r(A); R) & \textit{otherwise.} \end{cases}$$

Proof. 1. If $X - A \overset{c}{=} *$, then $\mathrm{C\,B}^*_{\mathrm{G}}(X - A; R) = \mathcal{C}^*_G(X; R)$. In particular, we observe that $\mathrm{H\,B}^0_{\mathrm{G}}(X - A; R) = \{\text{constant functions on } X\} \cong R$. We claim that $\mathcal{H}^*_G(X) = 0$ for $* \geq 1$. To see that, choose $a \in Fix_G(X)$. Now consider the following G-equivariant cone operator

$$D : (x_0, x_1, \ldots, x_n) \mapsto (a, x_0, \ldots, x_n)$$

Since $D\partial - \partial D = id$, we get $\mathcal{H}^*_G(X) = 0$ for $* \geq 1$. Hence, $\mathrm{H\,B}^*_{\mathrm{G}}(X - A; R) = 0$ for $* \geq 1$.

2. Elements in $\mathrm{H\,B}^0_{\mathrm{G}}(X - A)$ are constant functions on X with support contained in a neighborhood of A. This means, if $X - A \overset{c}{\neq} *$, then $\mathrm{H\,B}^0_{\mathrm{G}}(X - A) = 0$.

   To calculate $\mathrm{H\,B}^*_{\mathrm{G}}(X - A)$ for $* \geq 1$, consider the following short exact sequence of G-equivariant cochain complexes,

$$0 \to \mathrm{C\,B}^*_{\mathrm{G}}(X - A) \xrightarrow{j} \mathcal{C}^*_G(X) \xrightarrow{i} \varinjlim \mathrm{C}^*_{\mathrm{G}}(X - N_r(A)) \to 0 \qquad (4.3.1)$$

   Indeed

$$\begin{aligned} ker(i) &= \{\phi \in \mathcal{C}^*_G(X) \mid \phi \in \mathrm{C}^*_0(X - N_r(A)) \text{ for some r}\} \\ &= \{\phi \in \mathcal{C}^*_G(X)) \mid |\phi| \text{ is bounded}\} = Im(j) \end{aligned}$$

   The short exact sequence (4.3.1) induce a long exact sequence of corresponding reduced cohomologies. The reduced cohomology of the middle cochain complex in 4.3.1 is trivial in all degrees. Hence, the long exact sequence implies that

$$\mathrm{H\,B}^*_{\mathrm{G}}(X - A) \cong \varinjlim \tilde{\mathrm{H}}^{*-1}_{\mathrm{G}}(X - N_r(A)) \quad \text{for } * \geq 1$$

□

The following theorem of Honkasalo relates the G-equivariant Alexander–Spanier cohomology with the Alexander–Spanier cohomology of the quotient by the G-action.

Theorem 4.3.4 (cf. Corollary 6.8 [11])**.** *Let R be an abelian group with trivial G action. There is a natural isomorphism $\mathrm{H}^*_{\mathrm{G}}(X; R) \cong \mathrm{H}^*(X/G; R)$, where the right hand side is the ordinary Alexander–Spanier cohomology of X/G.*

We can now use the above theorem to get an analogous version in the coarse settings for certain spaces.

Theorem 4.3.5. *Suppose X is uniformly contractible away from A, locally contractible away from A, $X - A \overset{c}{\neq} *$ and $G \curvearrowright (X, A)$. Then*

$$\mathrm{H\,X}^*_{\mathrm{G}}(X - A; R) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}((X - N_r(A))/G; R) & \text{otherwise} \end{cases}$$

where R is an abelian group with trivial G-action.

Proof. By Theorem 4.3.1, we get $\mathrm{H\,X}^*_{\mathrm{G}}(X - A; R) \cong \mathrm{H\,B}^*_{\mathrm{G}}(X - A; R)$. The rest follows from the following where the first equality comes from Proposition 4.3.3 and the second equality comes from Theorem 4.3.4.

$$\mathrm{H\,B}^*_{\mathrm{G}}(X-A;R) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}_G(X-N_r(A);R) & \text{otherwise.} \end{cases}$$
$$= \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}((X-N_r(A))/G;R) & \text{otherwise.} \end{cases}$$

□

Example 4.3.6. Consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ by antipodal map. Let $\mathbb{R}^m$ be a vector subspace. Then for any $\mathbb{Z}_2$-module R with trivial action and $i \geq 1$

$$\begin{aligned} \mathrm{H\,X}^i_{\mathrm{G}}(\mathbb{R}^n - \mathbb{R}^m;R) &= \varinjlim \tilde{\mathrm{H}}^{i-1}((\mathbb{R}^n - N_r(\mathbb{R}^m))/\mathbb{Z}_2;R) \quad \text{[by Theorem 4.3.5]} \\ &= \tilde{\mathrm{H}}^{i-1}((\mathbb{R}^n-\mathbb{R}^m)/\mathbb{Z}_2;R) \\ &= \tilde{\mathrm{H}}^{i-1}(\mathbb{R}P^{n-m-1};R) \end{aligned}$$

The second equality holds because $\mathbb{R}^n - N_r(\mathbb{R}^m)$ is $\mathbb{Z}_2$-equivariantly homotopic to $\mathbb{R}^n - \mathbb{R}^m$ and the last equality follows because $\mathbb{R}^n - \mathbb{R}^m$ is $\mathbb{Z}_2$-equivariantly homotopic to S^{n-m-1} with the antipodal $\mathbb{Z}_2$-action.

Example 4.3.7. Consider $\mathbb{R}^\infty$ with $\mathbb{Z}_2$-action by antipodal map. For any vector subspace M of codimension n, $\mathbb{R}^\infty - N_r(M)$ is $\mathbb{Z}_2$-equivariantly homotopic to S^{n-1} with antipodal action. If R is a $\mathbb{Z}_2$-module with trivial $\mathbb{Z}_2$-action then, by similar argument as before, $\mathrm{H\,X}^i_{\mathbb{Z}_2}(\mathbb{R}^\infty - M;R) = \tilde{\mathrm{H}}^{i-1}(\mathbb{R}P^{n-1};R)$ for all $i \geq 1$.

4.4 Coarse van Kampen obstruction

This section develops an obstruction theory for coarse expanding maps which include coarse embedding maps. We begin by defining the coarse cohomology of the configuration space.

Coarse cohomology of the configuration space. Let (X, d) be a metric space. Equip X^2 with the sup metric. Consider the $\mathbb{Z}_2$-action on X^2 that flips the coordinates. The fixed point set for this action is the diagonal subspace $\delta(X) = \{(x, x) \mid x \in X\} \subset X^2$. In particular, we have $\mathbb{Z}_2 \curvearrowright (X^2, \delta(X))$. Let R be a $\mathbb{Z}_2$-module. We define *coarse cohomology of the configuration space of* X with coefficients in R to be the cohomology of the complex $\mathrm{C\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X); R)$.

Notation 4.4.1. For the rest of this chapter, the coefficients for ordinary cohomology will be $\mathbb{Z}_2$ and the coefficients for equivariant cohomology will be $\mathbb{Z}_2$ with trivial $\mathbb{Z}_2$-action. For convenience, we will omit the coefficients from the notation.

Proposition 4.4.2. *If X is uniformly contractible, locally contractible and unbounded then*

$$\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}((X^2 - N_r(\delta(X)))/\mathbb{Z}_2) & \text{otherwise.} \end{cases}$$

.

Proof. This follows immediately from Theorem 4.3.5. □

Example 4.4.3. Euclidean space $\mathbb{R}^n$ satisfies the hypothesis of Proposition 4.4.2. Hence, $\mathrm{H\,X}^*_{\mathbb{Z}_2}((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n)) = \varinjlim \tilde{\mathrm{H}}^{*-1}(((\mathbb{R}^n)^2 - N_r(\delta(\mathbb{R}^n)))/\mathbb{Z}_2)$ for $* \geq 1$. For any r, there is a $\mathbb{Z}_2$-equivariant deformation retraction of $(\mathbb{R}^n)^2 - \delta(\mathbb{R}^n)$ to $(\mathbb{R}^n)^2 - N_r(\delta(\mathbb{R}^n))$. Hence using

Proposition 4.4.2, we obtain

$$\mathrm{H\,X}^*_{\mathbb{Z}_2}((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n)) = \begin{cases} 0 & \text{if } * = 0 \\ \tilde{\mathrm{H}}^{*-1}(((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n))/\mathbb{Z}_2) & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0 & \text{if } * = 0 \\ \tilde{\mathrm{H}}^{*-1}(\mathbb{R}P^{n-1}) & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathbb{Z}_2 & \text{if } 2 \leq * \leq n \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we have the following

$$\mathrm{H\,X}^*_{\mathbb{Z}_2}((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)) = \begin{cases} 0 & \text{if } * = 0 \\ \tilde{\mathrm{H}}^{*-1}(\mathbb{R}P^{\infty}) & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathbb{Z}_2 & \text{for } \quad * \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.4.4. Recall that $\ell^\infty = L^\infty(\mathbb{N})$ is the space of bounded sequences with the sup-norm metric. $(\ell^\infty)^2 - N_r(\delta(\ell^\infty))$ deformation retracts $\mathbb{Z}_2$-equivariantly to $(\ell^\infty)^2 - \delta(\ell^\infty)$. Since $(\ell^\infty)^2 - \delta(\ell^\infty)$ is contractible, $((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2$ is a classifying space for $\mathbb{Z}_2$ and hence is homotopy equivalent to $\mathbb{R}P^\infty$. Hence arguing as before we obtain the following.

$$\mathrm{H\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) = \begin{cases} \mathbb{Z}_2 & \text{for } \quad * \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Definition. A map $f : X \to Y$ is said to be a proper map with respect to $A \subset X$ and $B \subset Y$ if for any r there exists an r' such that $f^{-1}(N_r(B)) \subset N_{r'}(A)$.

Equivalently, f is proper with respect to A and B if the map $f : (X, d_A) \to (Y, d_B)$ is proper.

Definition. A map $f : X \to Y$ is a *coarse expanding map* if it has an upper control and for any sequence of pair of points (x_i, x_i') in X, we have $d(f(x_i), f(x_i')) \to \infty$ whenever $d(x_i, x_i') \to \infty$.

Equivalently, f is coarse expanding if it has an upper control and the induced map $(x, y) \mapsto (f(x), f(y))$ is proper with respect to $\delta(X)$ and $\delta(Y)$.

Remark. Any coarse expanding map is proper and has upper control and hence is a coarse map. But the converse is not true. For instance, the map $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$, is a coarse map but not a coarse expanding map.

Lemma 4.4.5. *If $f : X \to Y$ is a coarse expanding map, then the induced map $(x, y) \mapsto (f(x), f(y))$ between $X \times X$ and $Y \times Y$ is a coarse map between the coarse complements of $\delta(X)$ and $\delta(Y)$.*

Proof. We will denote by d the sup metric on $X \times X$ and $Y \times Y$ induced from the metric of X and Y respectively. Let $g : X \times X \to Y \times Y$ be the map $(x, y) \mapsto (f(x), f(y))$. Our goal is to prove that g is a coarse map with respect to the metrics $d_{\delta(X)}$ and $d_{\delta(Y)}$.

Since f is a coarse expanding map, by definition, g is proper with respect to $\delta(X)$ and $\delta(Y)$. Or equivalently, g is proper with respect to $d_{\delta(X)}$ and $d_{\delta(Y)}$. The only thing that remains to be checked is that g has an upper control with respect to $d_{\delta(X)}$ and $d_{\delta(Y)}$. Since f has an upper control, g has an upper control ρ with respect to d. Suppose, $d_{\delta(X)}(x, y) \le r$ for some $r > 0$.

Case 1: If $d(x, y) \le r$, then $d(g(x), g(y)) \le \rho(r)$. This implies $d_{\delta(Y)}(g(x), g(y)) \le \rho(r)$.

Case 2: If $d(x, \delta(X)) + d(y, \delta(X)) \le r$, then $d(x, \delta(X)) \le r$ and $d(y, \delta(X)) \le r$. This implies $d(g(x), \delta(Y)) = d(g(x), g(\delta(X))) \le \rho(r)$. Similarly, we have $d(g(y), \delta(Y))) \le \rho(r)$. These imply $d(g(x), \delta(Y)) + d(g(y), \delta(Y)) \le 2\rho(r)$. That means $d_{\delta(Y)}(g(x), g(y)) \le 2\rho(r)$.

Combining case 1 and case 2, we get that 2ρ is an upper control for g with respect to $d_{\delta(X)}$ and $d_{\delta(Y)}$. Hence g is a coarse map with respect to the metrics $d_{\delta(X)}$ and $d_{\delta(Y)}$. □

The following Lemma now follows from the Proposition 4.2.1 and Lemma 4.4.5.

Lemma 4.4.6. *If $f : X \to Y$ is a coarse expanding map, then the map $(x, y) \mapsto (f(x), f(y))$ induces a map $f^* : \mathrm{H\,X}^*_{\mathbb{Z}_2}(Y^2 - \delta(Y)) \to \mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$.*

The next proposition tells us that for any coarse expanding map $f : X \to \ell^\infty$, the induced map $f^* : \mathrm{H\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$ depends only on X, not on f.

Proposition 4.4.7. *Any two coarse expanding maps from X to ℓ^∞ induce the same maps from $\mathrm{H\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$.*

Proof. Suppose $f, g : Y = X^2 \to (\ell^\infty)^2$ are two maps induced by two coarse expanding maps from X to ℓ^∞. We will also denote the induced map between $\mathrm{C}_*(X^2)$ and $\mathrm{C}_*((\ell^\infty)^2)$ by f and g respectively. Let f^* and g^* be the corresponding map from $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$. We will show that there is a chain homotopy $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \to \mathrm{C\,X}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X))$ between f^* and g^*.

Let $V : \mathrm{C}^{\mathrm{s}}_* \to \mathrm{C}_*$ denotes the map that sends a singular simplex to its vertices. Using uniform contractibility of $(\ell^\infty)^2$ one can construct a chain map $s : \mathrm{C}_*((\ell^\infty)^2) \to \mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$ such that $|s(\sigma)| \subset N_{\rho_n(\mathrm{diam}(\sigma))}(\sigma)$ for some sequence of functions $\rho_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that $V \circ s = id$. We first construct a chain homotopy $D : \mathrm{C}_*(X^2) \to \mathrm{C}^{\mathrm{s}}_{*+1}((\ell^\infty)^2)$ between the two maps $s \circ f, s \circ g : \mathrm{C}_*(X^2) \to \mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$ with certain properties.

We define $D_0 : \mathrm{C}_0(X^2) \to \mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$ as follows: For all $y \in Y$, join $f(y)$ and $g(y)$ by an arc which stays inside the following cylinder

$$\{z \mid \min\{d(f(y), \delta(\ell^\infty)), d(g(y), \delta(\ell^\infty))\} \le d(z, \delta(\ell^\infty)) \le \max\{d(f(y), \delta(\ell^\infty)), d(g(y), \delta(\ell^\infty))\}$$

with subdivision of length one. We define $D_0(y)$ to be the singular chain supported by this subdivided arc. For a singular simplex α define size(α):=diam($|\alpha|$) and for a singular chain $c \in \mathrm{C}^{\mathrm{s}}_*(X)$, define $\mathrm{size}(c) := \sup_{\tau \in |c|}\{\mathrm{size}(\tau)\}$. For a simplex $\sigma \in (X^2)^*$, let $r_\sigma := \min\{d|s(f(\sigma))|, \delta(\ell^\infty)), d(|s(g(\sigma))|, \delta(\ell^\infty))\}$ and $R_\sigma := \max\{d(|s(f(\sigma))|, \delta(\ell^\infty)), d(|s(g(\sigma))|, \delta(\ell^\infty))\}$.

Let C_σ be the following cylinder

$$\{z \mid r_\sigma \le d(z, \delta(\ell^\infty)) \le R_\sigma\}$$

Inductively, let us assume that $D_i : \mathrm{C}_i(X^2) \to \mathrm{C}^{\mathrm{s}}_{i+1}((\ell^\infty)^2)$ has been defined for all $i \le n$ with the following properties.

1. $s \circ f - s \circ g = \partial D_i + D_{i-1}\partial$

2. $|D_i(\sigma)| \subset C_\sigma$.

3. $\mathrm{size}(D_i(\sigma)) \le \mathrm{size}(s(f(\sigma)) - s(g(\sigma)) - D_{i-1}\partial(\sigma))$

4. D_i is $\mathbb{Z}_2$-equivariant.

To define $D_{n+1}(\sigma)$, let $K = s(f(\sigma)) - s(g(\sigma)) - D_n(\partial(\sigma))$. By induction hypothesis (1), $\partial K = D_{n-1}(\partial^2(\sigma)) = 0$ and hence K is a cycle. By (2), $|K| \subset C_\sigma$. Since C_σ is contractible, we can get a singular chain c which is supported in C_σ such that $\partial c = K$. After applying appropriate subdivision, we can make c to satisfy $\mathrm{size}(c) \le \mathrm{size}(K)$ without changing its boundary. Define $D_{n+1}(\sigma)$ to be that c. By construction, D_{n+1} satisfies condition (1), (2), and (3). To make D_* satisfy (4), we can first define it on an element from each orbit of $(*+1)$-tuples in X under the $\mathbb{Z}_2$-action and then extend equivariantly.

Since $V \circ s = id$ and $V\partial = \partial V$, applying V on both side of (1), we have the following

$$f - g = \partial V D_i + V D_{i-1}\partial$$

Dualizing the above we get f^* and g^* are chain homotopic via the chain homotopy $(VD)^*$.

It is now enough to prove that $(VD)^*$ maps $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X))$. Let $\phi \in \mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ and $\sigma \in |(VD)^*(\phi)| \cap N_r(\Delta)$ for some $r \ge 0$. Since $\sigma \in N_r(\Delta)$, property (3) implies that $|VD(\sigma)| \subset N_s(\Delta)$ for some s that depends only on r. Since $\phi \in \mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$and $\phi(VD(\sigma)) \ne 0$, it then follows that $|VD(\sigma))| \subset N_t(\Delta_{\delta(\ell^\infty)})$

where t depends only on s and hence depends only on r. It now follows from Property (2) that $\sigma \in N_p(\delta_X)$ for some p that depends only on t and hence depends only on r. Hence, we proved that for each $r \geq 0$ there exist $p \geq 0$ such that $|(VD)^*(\phi)| \cap N_r(\Delta) \subset N_p(\delta_X)$. Finally, $(VD)^*(\phi)$ is $\mathbb{Z}_2$-equivariant by property (4). Hence, $(VD)^*(\phi) \in \mathrm{CX}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X))$. This finishes the proof.

□

Coarse van Kampen obstruction class. Let X be a separable metric space and $g : X \to \ell^\infty$ be a coarse expanding map. Such embedding exists due to the Kuratowski embedding theorem. We consider the induced map $g^* : \mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \to \mathrm{HX}^n_{\mathbb{Z}_2}(X^2 - \delta(X))$. In Example 4.4.4, we saw that $\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) = \mathbb{Z}_2$ if $n \geq 2$. Let e^n be the nontrivial element in $\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ for $n \geq 2$. Proposition 4.4.7 implies that $g^*(e^n)$ depends only on the space X, not on g. We call the class $g^*(e^n)$ to be the n^{th} *degree coarse van Kampen obstruction class* of X and denote it by $cvk^n(X)$ where $n \geq 2$.

Assumption. From now on all our spaces will be separable metric spaces.

Definition (Coarse obstruction dimension)**.** *Coarse obstruction dimension* of a space Y, denoted by $\mathrm{cobdim}(Y)$, is 0 if Y is bounded. For unbounded Y, we say $\mathrm{cobdim}(Y) := \max\{n, 1\}$, where n is the largest such that $cvk^n(Y) \neq 0$.

Now we present the main theorem of this paper.

Theorem 4.4.8. *If there exists a coarse expanding map $X \to Y$, then* $\mathrm{cobdim}(X) \leq \mathrm{cobdim}(Y)$.

Proof. If $\mathrm{cobdim}(X) = 0$, then there is nothing to prove. If $\mathrm{cobdim}(X) = 1$, then X is unbounded by definition. This means Y is also unbounded and hence $\mathrm{cobdim}(X) \geq 1$. Suppose $\mathrm{cobdim}(X) = n \geq 2$. Let $g : Y \to \mathbb{R}^\infty$ be a coarse expanding map. Consider the following composition.

$$X \times X \xrightarrow{f} Y \times Y \xrightarrow{g} \ell^\infty \times \ell^\infty$$

By Proposition 4.4.6, the above maps induce the following maps between coarse cohomology of the configuration spaces

$$\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \xrightarrow{g^*} \mathrm{HX}^n_{\mathbb{Z}_2}(Y^2 - \delta(Y)) \xrightarrow{f^*} \mathrm{HX}^n_{\mathbb{Z}_2}(X^2 - \delta(X))$$

Let $e^n \in \mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ be the generator. Then $cvk^n(Y) = g^*(e^n)$ and $cvk^n(X) = f^*g^*(e^n)) = f^*(cvk^n(Y))$. By assumption $cvk^n(X) \neq 0$ and hence $cvk^n(Y) \neq 0$. So, we get $\mathrm{cobdim}(Y) \geq n$. □

One immediate corollary of the above theorem is the following.

Corollary 4.4.9. *If X and Y are coarsely equivalent, then* $\mathrm{cobdim}(X) = \mathrm{cobdim}(Y)$.

In Example 4.3.6, we saw that $\mathrm{HX}^*_{\mathbb{Z}_2}(Conf(\mathbb{R}^n)) = 0$ for all $* > n$. Hence $\mathrm{cobdim}(\mathbb{R}^n) \leq n$. Using Theorem 4.4.8, we get the following.

Corollary 4.4.10. *If* $\mathrm{cobdim}(X) \geq n$, *then X does not admit a coarse expanding map into* $\mathbb{R}^{n-1}$.

4.5 Relation with the classical van Kampen obstruction

Let us recall the classical van Kampen obstruction class. Let X be a topological space. Any continuous embedding $f : X \hookrightarrow \mathbb{R}^\infty$ induces a map $f : \mathrm{H}^*((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)/\mathbb{Z}_2) \to \mathrm{H}^*((X^2 - \delta(X))/\mathbb{Z}_2)$. This map depends only on X because the quotient map $((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)) \to ((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty))/\mathbb{Z}_2$ is a universal $\mathbb{Z}_2$-bundle. Let $\eta^* \in \mathrm{H}^*((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)/\mathbb{Z}_2)$ be the generator. Then $vk^*(X) := f(\eta^*)$ is called the van Kampen obstruction class in degree $*$. Note that $(\ell^\infty)^2 - \delta(\ell^\infty) \to ((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2$ can also be considered as a universal $\mathbb{Z}_2$-bundle because $(\ell^\infty)^2 - \delta(\ell^\infty)$ is contractible. Hence we can use ℓ^∞ instead of $\mathbb{R}^\infty$ to define $vk^*(X)$. We use this viewpoint in the next Proposition.

Proposition 4.5.1. *Let X be a uniformly contractible, locally contractible and unbounded space. Suppose $i : \tilde{\mathrm{H}}^*((X^2 - \delta(X))/\mathbb{Z}_2) \to \varinjlim \tilde{\mathrm{H}}^*((X^2 - N_r(\delta(X)))/\mathbb{Z}_2)$ is the map induced by inclusions $X^2 - N_r(\delta(X)) \hookrightarrow X^2 - \delta(X)$ for each $r > 0$. Then given $n \geq 2$, $cvk^n(X)$ is nontrivial if and only if $i(vk^{n-1}(X))$ is nontrivial.*

Proof. Let $f : X \to \ell^\infty$ be an isometry. We then have the following commutative diagram where the top vertical isomorphism maps are due to the Proposition 4.4.2.

$$\begin{array}{ccc}
\mathrm{HX}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) & \xrightarrow{\quad f \quad} & \mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \\
\downarrow \cong & & \downarrow \cong \\
\varinjlim \tilde{\mathrm{H}}^{*-1}(((\ell^\infty)^2 - N_r(\delta(\ell^\infty)))/\mathbb{Z}_2) & \xrightarrow{f} & \varinjlim \tilde{\mathrm{H}}^{*-1}((X^2 - N_r(\delta(X)))/\mathbb{Z}_2) \\
\cong \uparrow & & i \uparrow \\
\tilde{\mathrm{H}}^{*-1}(((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2) & \xrightarrow{\quad f \quad} & \tilde{\mathrm{H}}^{*-1}((X^2 - \delta(X))/\mathbb{Z}_2)
\end{array}$$

We note that the image of the generator of $\tilde{\mathrm{H}}^{*-1}(((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2)$ under the bottom horizontal map is $vk^{*-1}(X)$ if $* \geq 2$. By the commutativity of the above diagram, it follows that $cvk^*(X)$ is nontrivial if and only if $i(vk^{*-1}(X))$ is nontrivial when $* \geq 2$. □

Let K be a finite complex. There is a natural way to put a metric on the open cone $Cone_\infty(K) := K \times [0, \infty)/K \times \{0\}$. Embed K piecewise linearly into a high-dimensional sphere S^{N-1} and identify $Cone_\infty(K)$ with the union of all the rays coming from the origin in $\mathbb{R}^N$ that meet the embedded copy of K. Equip $Cone_\infty(K)$ with the subspace metric induced from $\mathbb{R}^N$. We call this space a Euclidean cone on K and denote it by $Cone^e_\infty(K)$.

Theorem 4.5.2. *Let K be a finite simplicial complex and let $Cone(K) = K \times [0, 1]/K \times 0$ be the finite cone on K. Then $vk^{*-1}(Cone(K)) \neq 0$ iff $cvk^*(Cone^e_\infty(K)) \neq 0$, when $* \geq 2$. In particular, $vk^{n-1}(Cone(K)) \neq 0$ and $n \geq 2$ implies* $\operatorname{cobdim}(Cone^e_\infty(K)) \geq n$.

Proof. Since K is a finite complex, $Cone^e_\infty(K)$ satisfies the hypothesis of Proposition 4.4.2. In this case, the map i from the Proposition 4.4.2 is in fact isomorphism. Hence we get $cvk^*(Cone^e_\infty(K)) \neq 0$ if and only if $vk^{*-1}(Cone^e_\infty(K)) \neq 0$ when $* \geq 2$. One can check that the inclusion map $Cone(K) \hookrightarrow Cone^e_\infty(K)$ induces isomorphism between the cohomology of their configuration spaces. It follows that $vk^{*-1}(Cone(K)) \neq 0$ iff $vk^{*-1}(Cone^e_\infty(K)) \neq 0$. Hence we obtain that $vk^{*-1}(Cone(K)) \neq 0$ iff $cvk^*(Cone^\infty_e(K)) \neq 0$, when $* \geq 2$. □

4.6 A coarse Gysin sequence

Let $p : X^2 \to X^2/\mathbb{Z}_2$ denotes the quotient map where $\mathbb{Z}_2$ acts by switching the co-ordinates. Suppose, $B_X := p(X^2)$ and $B'_X := p(\delta(X))$. Equip B_X with the metric $d_p(x, y) = \min\{d(x', y') \mid p(x') = x, p(y') = y\}$. For any $r \geq 0$, we have the following Gysin sequence (cf [9], page 438) for $\mathbb{Z}_2$-bundles.

$$\to \mathrm{H}^{*-1}(B_X - N_r(B'_X)) \to \mathrm{H}^*(B_X - N_r(B'_X)) \to \mathrm{H}^*(X^2 - N_r(\delta(X))) \to$$

Since direct limit is an exact functor, letting $r \to \infty$ in the above sequence, we obtain the following long exact sequence

$$\to \varinjlim \mathrm{H}^{*-1}(B_X - N_r(B'_X)) \to \varinjlim \mathrm{H}^*(B_X - N_r(B'_X)) \to \varinjlim \mathrm{H}^*(X^2 - N_r(\delta(X))) \to \quad (4.6.1)$$

If X is uniformly contractible, locally contractible and unbounded, then Theorem 4.3.5 implies that

$$\varinjlim \mathrm{H}^*(B_X - N_r(B'_X)) = \begin{cases} \mathrm{H\,X}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \oplus \mathbb{Z}_2 & \text{if } * = 0, \\ \mathrm{H\,X}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\varinjlim \mathrm{H}^*(X^2 - N_r(\delta(X))) = \begin{cases} \mathrm{H\,X}^1(X^2 - \delta(X)) \oplus \mathbb{Z}_2 & \text{if } * = 0, \\ \mathrm{H\,X}^{*+1}(X^2 - \delta(X)) & \text{otherwise.} \end{cases}$$

Hence for such X, we can rewrite the long exact sequence (4.6.1) in terms of coarse cohomology. We record this in the following lemma.

Lemma 4.6.1 (Coarse Gysin sequence)**.** *If X is uniformly contractible, locally contractible and unbounded, then there is a long exact sequence as follows*

$$\begin{aligned} 0 \to \mathrm{H\,X}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \oplus \mathbb{Z}_2 \to \mathrm{H\,X}^1(X^2 - \delta(X)) \oplus \mathbb{Z}_2 \to \mathrm{H\,X}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \oplus \mathbb{Z}_2 \to \\ \cdots \to \mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{H\,X}^*(X^2 - \delta(X)) \to \mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \cdots \end{aligned} \tag{4.6.2}$$

where $* \geq 2$.

Observation 4.6.2. Suppose X is uniformly contractible, locally contractible and unbounded. Additionally assume that $\mathrm{H\,X}^1(X^2 - \delta(X)) = 0$. It follows from Lemma 4.6.1 that $\mathrm{H\,X}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$. As a consequence, the second map in (4.6.2) is an isomorphism. That means, in this case, we can write the beginning part of the coarse Gysin sequence (4.6.2) as follows

$$0 \to \mathbb{Z}_2 \to \mathrm{H\,X}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{H\,X}^2(X^2 - \delta(X)) \to \mathrm{H\,X}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \cdots \tag{4.6.3}$$

This observation will be useful for us in several occasion.

We will use Lemma 4.6.1 to estimate cobdim for certain spaces in the next two sections.

4.7 On the upper bound of cobdim

Our first goal in this section is to prove that $\mathrm{cobdim}(X) \leq n$ where X is a proper, uniformly contractible n-manifold (corollary 4.7.3). We begin by calculating $\mathrm{H\,X}^*(X^2 - \delta(X))$ for such spaces.

Lemma 4.7.1. *If X is a proper, uniformly contractible n-manifold, then*

$$\operatorname{HX}^*(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & * = n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If X is a proper, uniformly contractible n-manifold, then $X \times X$ is a proper, uniformly contractible $2n$-manifold. By corollary 2.8.3, we get

$$\operatorname{HX}^*(X^2 - \delta(X)) = \operatorname{HX}_{2n-*}(\delta(X)) = \begin{cases} \mathbb{Z}_2 & * = n, \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 4.7.2. *If X is a proper, uniformly contractible n-manifold, then*

$$\operatorname{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & 2 \leq * \leq n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Observe that X is unbounded because it is proper and it does not have a boundary. Elements of $\operatorname{HX}^0_{\mathbb{Z}_2}(X^2 - \delta(X))$ are constant functions on $X \times X$ with support contained in a neighborhood of $\delta(X)$. Since X is unbounded, we have $\operatorname{HX}^0_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$.

By Lemma 4.7.1, we have

$$\operatorname{HX}^*(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Next we will show that $\operatorname{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ if $* > n$. For $* \geq 2$, consider the following part of the coarse Gysin sequence.

$$\cdots \to \operatorname{HX}^*(X^2 - \delta(X)) \to \operatorname{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \operatorname{HX}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \cdots \qquad (4.7.1)$$

Since $\operatorname{HX}^*(X^2 - \delta(X)) = 0$ for $* > n$, the middle map is an isomorphism for $* > n$.

Hence, $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X)) = \mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2-\delta(X))$ if $* \geq n+1$. That means, if $\mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2-\delta(X)) \neq 0$ then $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X)) \neq 0$ for all $* \geq n+1$. But $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X)) = \varinjlim \tilde{\mathrm{H}}^{*-1}(B_X - N_r(B'_X)) = 0$ for $* \geq 2n+2$ since $B_X - N_r(B'_X)$ is a $2n$-manifold. That means, $\mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$ and consequently, $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$ for all $* \geq n+1$. We divide rest of the calculations in three cases.

Case 1 ($n=1$). We only have to show that $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$. Consider the following part of the coarse Gysin sequence

$$\begin{aligned} 0 \to \mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) \oplus \mathbb{Z}_2 \to \mathrm{HX}^1(X^2-\delta(X)) \oplus \mathbb{Z}_2 \to \mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) \oplus \mathbb{Z}_2 \\ \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) \to \mathrm{HX}^2(X^2-\delta(X)) \to \cdots \quad (4.7.2) \end{aligned}$$

Since $n=1$, we have $\mathrm{HX}^1(X^2-\delta(X)) = \mathbb{Z}_2$. Since the second map in the above exact sequence is injective, $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X))$ is either $\mathbb{Z}_2$ or it is trivial. We claim that it cannot be $\mathbb{Z}_2$. On the contrary, suppose $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) = \mathbb{Z}_2$, then the second map in the above sequence is an isomorphism. This implies that the fourth map is injective. Hence $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) \neq 0$ which is a contradiction because $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$ for $* \geq n+1 = 2$. Hence, $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X) = 0$.

Case 2 ($n=2$). Since $\mathrm{HX}^1(X^2-\delta(X)) = 0$, we observed in 4.6.2 that $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$ Furthermore, we have $\mathrm{HX}^2(X^2-\delta(X)) = \mathbb{Z}_2$ by hypothesis and we already showed $\mathrm{HX}^3_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$. So the sequence (4.6.3) takes the following form.

$$0 \to \mathbb{Z}_2 \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) \to \mathbb{Z}_2 \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) \to 0$$

So there is an injective map and a surjective map from $\mathbb{Z}_2$ into $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))$. It follows that $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X)) = \mathbb{Z}_2$.

Case 3 ($n>2$). In this case, $\mathrm{HX}^1(X^2-\delta(X)) = 0$. Hence, observation 4.6.2 tells us $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X)) = 0$.

Since $\mathrm{H\,X}^2(X^2 - \delta(X)) = 0$, the sequence (4.6.3), in this case, takes the following form.

$$0 \to \mathbb{Z}_2 \to \mathrm{H\,X}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) \to 0 \to \dots$$

Hence $\mathrm{H\,X}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) = \mathbb{Z}_2$.

Since $\mathrm{H\,X}^*(X^2 - \delta(X)) = 0$ for $* \leq n-1$, it follows from (4.7.1) that $\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \cong \mathrm{H\,X}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X))$ for $2 \leq * \leq n-2$. That implies

$$\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \mathrm{H\,X}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) = \mathbb{Z}_2 \text{ when } 2 \leq * \leq n-1.$$

Finally to compute $\mathrm{H\,X}^n_{\mathbb{Z}_2}(X^2 - \delta(X))$, consider the following part of the coarse Gysin sequence.

$$\cdots \to \mathrm{H\,X}^n_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{H\,X}^n(X^2 - \delta(X)) \to \mathrm{H\,X}^n_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{H\,X}^{n+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \cdots$$

Since $\mathrm{H\,X}^n(X^2 - \delta(X)) \neq 0$ by assumption, it follows from the above sequence that $\mathrm{H\,X}^n_{\mathbb{Z}_2}(X^2 - \delta(X)) \neq 0$. We already know that the fourth term is trivial in the above sequence. That means the third map is surjective. Since $\mathrm{H\,X}^n(X^2 - \delta(X)) = \mathbb{Z}_2$, we can conclude that $\mathrm{H\,X}^n_{\mathbb{Z}_2}(X^2 - \delta(X)) = \mathbb{Z}_2$.

□

As a consequence of the above theorem, we obtain the following.

Corollary 4.7.3. *If X is a proper, uniformly contractible n-manifold, then* $\mathrm{cobdim}(X) \leq n$.

Our next goal is to improve Corollary 4.7.3 to include manifolds with boundaries. For that, we need to impose a condition on the metric of the boundary. The following definition is inspired by the locally k-connected space defined in [6].

Definition. A metric space (X, d) is uniformly locally contractible if for every $\epsilon > 0$, there is a $\delta > 0$ such that any ball of radius δ is contractible inside a ball of radius ϵ.

Lemma 4.7.4. *Let (X, d) be a uniformly locally contractible metric space. Then the space $X \times [1, \infty)$ has a metric that makes the space uniformly contractible away from $X \times \{1\}$ and the map $x \mapsto (x, 1)$ is an isometric embedding of X into $X \times [1, \infty)$.*

Proof. The construction of the metric is the same as the one appearing in Lemma 2.2 of [6]. Choose a continuous strictly increasing function $\phi : [1, \infty) \to [1, \infty)$ with $\phi(1) = 1$. Let d be the original metric on X and define a function ρ' by

1. $\rho'((x, t), (x', t)) = \phi(t)d(x, x')$.

2. $\rho'((x, t), (x, t')) = |t - t'|$.

We then define $\rho : (X \times [1, \infty))^2 \to [0, \infty)$ to be

$$\rho((x, t), (x', t')) = \inf \sum_{i=1}^{l} \rho'((x_i, t_i), (x_{i-1}, t_{i-1}))$$

where the sum is over all chains

$$(x, t) = (x_0, t_0), (x_1, t_1), \ldots, (x_l, t_l) = (x', t')$$

and each segment is either horizontal or vertical. It pays to move towards 1 before moving in the X-direction. Also $\phi(1) = 1$ implies that $X \times \{1\}$ with the subspace metric is isometric to X via the map $(x, 1) \mapsto x$. Now we will describe a ϕ, so that corresponding metric ρ makes $X \times [1, \infty)$ uniformly contractible away from $X \times \{1\}$. Since X is uniformly locally contractible, we have an infinite positive decreasing sequence $\{r_i\}$ with $r_1 = 1$ such that for every $x \in X$, the inclusions $\ldots \subset B^d_{r_i}(x) \subset B^d_{r_{i-1}}(x) \subset$ are nullhomotopic maps. Set $\phi(t) = \frac{1}{r_t}$ for $t \in \mathbb{N}$. For nonintegral values of t, we set

$$\phi(t) = \phi([t]) + (t - [t])\phi([t] + 1)$$

Suppose, $N_i = \frac{\phi(i)}{\phi(i-1)}$. Now we consider the ball $B^\rho_k(x, i) \subset cX$. Note that $B^\rho_k(x, i) \subset$

$B^d_{\frac{k}{N_{i-k}}}(x) \times [i-k, i+k]$ and that $B^\rho_k(x,i)$ contracts in itself to $B^\rho_k(x,i) \cap (X \times [i-k,i]) \subset B^d_{\frac{k}{N_{i-k}}}(x) \times [i-k,i]$. Also, $B^d_{\frac{k}{N_{i-k-1}}}(x) \times \{i-k-1\} \subset B^\rho_{k+2}(x,i)$. So, $B^\rho_k(x,i)$ can be contracted inside $B^\rho_{k+2}(x,i)$ by pushing it down to $(i-k-1)$-level and contracting it there.

□

The following gluing lemma can be found in [5].

Lemma 4.7.5 ([5], Lemma I.5.24)**.** *Let X_1 and X_2 be two proper metric spaces. Let $A_i \subset X_i$ be closed subset and $f : A_1 \to A_2$ be an isometry. Let Y be the space obtained by gluing (X_i, A_i) along A_i via the map f. Define $d : Y \times Y \to \mathbb{R}$ as follows*

$$d(x,y) = \begin{cases} d_i(x,y) & \text{if } x, y \in X_i \\ \inf_{a \in A_1}\{d_1(x,a) + d_2(f(a),y)\} & \text{if } x \in X_1, y \in X_2 \end{cases}$$

Then

1. *d is a proper metric on Y.*
2. *The canonical inclusions $X_i \hookrightarrow Y$ are isometric embedding.*

Proposition 4.7.6. *Any uniformly contractible, proper n-manifold with uniformly locally contractible boundary admits an isometric embedding into a uniformly contractible, proper n-manifold.*

Proof. Since ∂M is uniformly locally contractible, Lemma 4.7.4 allows us to equip $\partial M \times [1,\infty)$ with a metric so that it is uniformly contractible away from $\partial M \times \{1\}$. Let $\rho, j : \mathbb{R}_+ \to \mathbb{R}_+$ be two functions such that any ball $B(x,r)$ in $\partial M \times [1,\infty)$ is contractible inside $B(x,\rho(r))$ whenever $d(x, \partial M \times \{1\}) \geq j(r)$. We glue $\partial M \times [1,\infty)$ to M along ∂M by the attaching map $(x,1) \mapsto x$. We call this space Y. By Lemma 4.7.5, there is a proper metric on Y such that the canonical maps $M \hookrightarrow Y$ and $\partial M \times [1,\infty) \hookrightarrow Y$ are isometric embedding.

We claim that Y is uniformly contractible. Since M is uniformly contractible, there exists a function $\tau : [0,\infty) \to [0,\infty)$ such that, for any $r \geq 0$, any ball of radius r in M is contractible inside a concentric ball of radius $\tau(r)$. Take a ball $B_r(x)$ of radius r in Y. If $x \in M$ and $d(x, \partial M) \geq r$, then $B_r(x)$ is contained inside M because points in $\partial M \times [1, \infty)$ are at least as far from x as points in ∂M by the construction of the metric on Y. Hence, $B_r(x)$ is contractible inside $B_{\tau(r)}(x)$. If $x \in \partial M \times [1, \infty)$ and $d(x, \partial M) \geq j(r) + r$, then $B(x, r) \subset \partial M \times [1, \infty)$ and hence is contractible in $B(x, \rho(r))$. If $d(x, \partial M) \leq j(r) + r$, we can deformation retract $B_r(x) \cap (\partial M \times [1, \infty))$ by sliding it along $[1, \infty)$ until it lands on $\partial M \times \{1\}$. This homotopy takes place in a set of diameter at most $2j(r) + 2r$ because the diameter shrinks as we approach towards $X \times \{1\}$. So the deformed ball is now contained in M and has diameter at most $2r$. This set is contractible inside a set of diameter at most $\tau(2r)$ by uniform contractibility of M. Hence $B_r(x)$ is contractible inside a set of diameter at most $2j(r) + 2r + \tau(2r)$. Hence Y is uniformly contractible. □

As a consequence of the above proposition, we get the following.

Theorem 4.7.7. *If X is a proper, uniformly contractible n-manifold with uniformly locally contractible boundary, then* $\mathrm{cobdim}(X) \leq n$.

Proof. By Proposition 4.7.6, we have a uniformly contractible proper n-manifold Y such that X embeds isometrically in Y. By Theorem 4.4.8, it follows that $\mathrm{cobdim}(X) \leq \mathrm{cobdim}(Y)$. By Corollary 4.7.3, we know $\mathrm{cobdim}(Y) \leq n$ and hence $\mathrm{cobdim}(X) \leq n$. □

Now we can prove the following improvement of Corollary 4.4.10. The proof is immediate from Theorem 4.4.8 and Theorem 4.7.7.

Corollary 4.7.8. *If* $\mathrm{cobdim}(X) \geq n$*, then X cannot be coarsely embedded into a proper, uniformly contractible* $(n-1)$*-manifold with uniformly locally contractible boundary.*

Definition (Cocompact action dimension)**.** The *cocompact action dimension* $\mathrm{cadim}(G)$ of a group G is the least dimension of a contractible manifold (possibly with boundary) that admits a proper cocompact G-action.

Corollary 4.7.9. $\mathrm{cadim}(G) \geq \mathrm{cobdim}(G)$.

Proof. Suppose $\mathrm{cadim}(G) = n$. Then G admits a proper, cocompact action on a contractible n-manifold M. By Milnor–Schwarz Lemma, there exists a coarse equivalence $f : G \to M$. Since M is contractible and it admits a cocompact action, M is uniformly contractible. Similarly, since ∂M is locally contractible and it admits a cocompact action, it is uniformly locally contractible. Since M is proper, by Theorem 4.7.8, we get $\mathrm{cobdim}(G) \leq n = \mathrm{cadim}(G)$.

□

4.8 On the lower bound of cobdim

In this section our main theorem is the following.

Theorem 4.8.1. *If X is uniformly contractible, locally contractible and* $\mathrm{HX}^*(X^2 - \delta(X)) = 0$ *for* $* \leq n-1$*, then* $\mathrm{cobdim}(X) \geq n$.

Proof. For $n = 0$, the claim is trivial. If $n = 1$, the assumption says $\mathrm{HX}^0(X^2 - \delta(X)) = 0$. This means X is unbounded, otherwise non zero constant functions from X^2 to the $\mathbb{Z}_2$ give nontrivial elements in $\mathrm{HX}^0(X^2 - \delta(X))$. Hence, $\mathrm{cobdim}(X) \geq 1$ in this case.

Suppose $n \geq 2$ and $f : X \to \ell^\infty$ is an isometry. We first show that $f^* : \mathrm{HX}^2_{\mathbb{Z}_2}(\ell^\infty)^2 - \delta(\ell^\infty)) \xrightarrow{f^*} \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X))$ is a nontrivial map. To see that, consider the following part of the maps between the concerned coarse Gysin sequences. Our goal is to show the second vertical map is nontrivial.

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathrm{HX}^1((\ell^\infty)^2 - \delta(\ell^\infty)) \oplus \mathbb{Z}_2 & \longrightarrow & \mathrm{HX}^2_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) & \longrightarrow & \ldots \\ & & \downarrow f^* & & \downarrow f^* & & \\ \ldots & \longrightarrow & \mathrm{HX}^1(X^2 - \delta(X)) \oplus \mathbb{Z}_2 & \xrightarrow{j} & \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) & \longrightarrow & \ldots \end{array}$$

By commutativity of the diagram, Our claim follows if we can show that j is injective and the first vertical map is non trivial. Since $\mathrm{HX}^1(X^2 - \delta(X)) = 0$, it follows from (4.6.3) that j is injective. Next we show that the first vertical map in the above commutative diagram is non trivial. It is equivalent to showing that the following map is not trivial.

$$f^* : \varinjlim \mathrm{H}^0(((\ell^\infty)^2 - N_r(\delta(\ell^\infty)))/\mathbb{Z}_2) \to \varinjlim \mathrm{H}^0((X^2 - N_r(\delta(X)))/\mathbb{Z}_2)$$

This follows from the fact that the domain of those maps are isomorphic to $\mathbb{Z}_2$ where the nontrivial elements are coming from the nontrivial constant maps $(\ell^\infty)^2 \to \mathbb{Z}_2$ and $X^2 \to \mathbb{Z}_2$. Hence we can conclude that the map $f^* : \mathrm{HX}^2_{\mathbb{Z}_2}(\ell^\infty)^2 - \delta(\ell^\infty)) \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X))$ is non trivial.

Let us now consider the maps between the following parts of the coarse Gysin sequences where $* \geq 2$.

$$\begin{array}{ccccccc} \to \mathrm{HX}^*((\ell^\infty)^2 - \delta(\ell^\infty)) & \to & \mathrm{HX}^*_{\mathbb{Z}_2}(\ell^\infty)^2 - \delta(\ell^\infty)) & \to & \mathrm{HX}^{*+1}_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \to \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ \longrightarrow \mathrm{HX}^*(X^2 - \delta(X)) & \longrightarrow & \mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) & \longrightarrow & \mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \longrightarrow \end{array}$$

Note that the first terms of both sequences above are trivial for $* \leq n-1$. That implies that the third horizontal maps in the above diagram are injective. Hence by commutativity of the diagram, if the second vertical map is nontrivial then so is the third vertical map whenever $* \leq n-1$. We saw previously that, when $* = 2$ the second vertical map is injective. It now follows by induction that the third vertical map is injective when $* \leq n-1$. In particular, when $* = n-1$, injectivity of the third vertical map means $cvk^n(X) \neq 0$. Hence, $\mathrm{cobdim}(X) \geq n$.

□

As a consequence of the above theorem we get the following.

Corollary 4.8.2. *If X is a proper, uniformly contractible and locally contractible coarse $PD(n)$ space, then X cannot be coarsely embedded into a proper, uniformly contractible $(n-1)$-manifold with uniformly locally contractible boundary.*

Proof. If X is a proper, uniformly contractible and locally contractible coarse $PD(n)$ space, then it satisfies the hypothesis of Theorem 4.8.1. Hence, $\operatorname{cobdim}(X) \geq n$. The claim now follows from Corollary 4.7.8. □

Example 4.8.3. If X is a proper, uniformly contractible n-manifold, then it satisfies all the hypothesis of the above corollary. Hence $\operatorname{cobdim}(X) \geq n$. Theorem 4.7.7 implies that $\operatorname{cobdim}(X) \leq n$. Hence $\operatorname{cobdim}(X) = n$ whenever X is a proper, uniformly contractible n-manifold.

BIBLIOGRAPHY

[1] A. Banerjee and B. Okun. Coarse cohomology of the complement, in preparation.

[2] A. Banerjee. On computation of coarse cohomology, in preparation.

[3] A. Banerjee. Coarse embedding obstruction, in preparation.

[4] M. Bestvina, M. Kapovich, and B. Kleiner. Van Kampen's embedding obstruction for discrete groups. *Invent. math. 150, 219–235 (2002).*

[5] M. Bridson and A. Haefliger. *Metric Spaces of Non-Positive Curvature*, Vol. 319, 2009.

[6] A. N. Dranishnikov, Steven C. Ferry, and Shmuel Weinberger. Large Riemannian Manifolds Which Are Flexible. *Annals of Mathematics* 157 (3):919–938, 2003.

[7] P. C. Eklof and A. H. Mekler. *Almost free modules.* North-Holland Mathematical Library, vol. 65. North-Holland Publishing Co., Amsterdam, Revised, 2002.

[8] S. Hair, *Homological methods in coarse geometry.* Ph.D. Thesis, 2010.

[9] A. Hatcher. *Algebraic topology.* Cambridge University Press, 2000.

[10] H. Honkasalo and E. Laitinen. Equivariant Lefschetz classes in Alexander-Spanier cohomology. *Osaka J. Math.* 33 (3):793–804, 1996.

[11] H. Honkasalo. Equivariant Alexander-Spanier cohomology. *Mathematica Scandinavica* 63:179-195, 1988.

[12] M. Kapovich and B. Kleiner. Coarse Alexander duality and duality groups. *J. Differential Geom.* 69 (2):279–352, 2005.

[13] A. J. Margolis. Quasi-isometry invariance of group splittings over coarse Poincaré duality groups. *Proc. Lond. Math. Soc. (3)* 116 (6):1406–1456, 2018.

[14] L. Mosher, M. Sageev, and K. Whyte. Quasi-actions on trees II: Finite depth Bass-Serre trees. *Mem. Amer. Math. Soc.* 214 (1008), 2011.

[15] J. Roe. Coarse cohomology and index theory on complete Riemannian manifolds. *Mem. Amer. Math. Soc.* 104 (497), 1993.

[16] J. Roe. *Lectures on coarse geometry.* University Lecture Series, vol. 31. American Mathematical Society, Providence, RI, 2003.

[17] E. R. van Kampen. Komplexe in euklidischen Raumen. *Abh. Math. Sem. Univ. Hamburg 9*, 1933.

[18] E. H. Spanier. *Algebraic topology.* Springer-Verlag, New York-Berlin, 1981.

[19] E. H. Spanier. Tautness for Alexander-Spanier cohomology. *Pacific Journal of Mathematics* 75 (2):561 – 563, 1978.

[20] S. Y. Yoon. A lower bound to the action dimension of a group. *Algebr. Geom. Topol.* 4 (1):273–296, 2004.