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Random Quotients of Hyperbolic Groups and Property (T)

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ABSTRACT

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What does a typical quotient of a group look like? Gromov looked at the density model of quotients of free groups. The density parameter d measures the rate of exponential growth of the number of relators compared to the size of the Cayley ball. Using this model, he proved that for $d < 1/2$, the typical quotient of a free group is non-elementary hyperbolic. Ollivier extended Gromov's result to show that for $d < 1/2$, the typical quotient of many hyperbolic groups is also non-elementary hyperbolic.

Żuk and Kotowski–Kotowski proved that for $d > 1/3$, a typical quotient of a free group has Property (T). We show that (in a closely related density model) for $1/3 < d < 1/2$, the typical quotient of a large class of hyperbolic groups is non-elementary hyperbolic and has Property (T). This provides an answer to a question of Gromov (and Ollivier).

Dedicated to the city of Milwaukee and its people

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1 INTRODUCTION

1.1 Background and Motivation

A major goal of group theory is to understand the quotients of a group. A fundamental question is what does a "random" quotient of a group look like? What properties does it have?

A widely studied, large class of groups in geometric group theory are *hyperbolic groups*. These are groups whose Cayley graphs are (in an appropriate sense) negatively curved. A hyperbolic group is *non-elementary* if it doesn't have finite index cyclic subgroups. Free groups are the simplest example of such groups.

A natural question to ask is whether hyperbolicity is robust:

Question 1. *Is a random quotient of a non-elementary hyperbolic group, non-elementary hyperbolic?*

In [Gro93] Gromov introduced the density model of quotients of free groups. Let $\mathcal{B}_\ell$ be the Cayley ball of length ℓ of a free group F_n. Let R be a random choice of $|\mathcal{B}_\ell|^d$ elements from $\mathcal{B}_\ell$. The parameter d is called *density*. The resulting quotients $F_n/\langle\langle R\rangle\rangle$ are called random groups. More precisely, this is a *random quotient of a free group at density d, and length ℓ*. A random quotient of F_n is said to have a certain property *with overwhelming probability (w.o.p)* at density d if as $\ell \to \infty$ the proportion of quotients having the property $\to 1$. Gromov showed that for any $d < 1/2$ a random quotient of F_n is non-elementary hyperbolic.

The success of the density model owes itself to the fact that different properties show up at different densities. One such property is Kazhdan's Property (T), a rigidity property for groups that plays an important role in many different areas of mathematics and computer science. For a long time, it was surprisingly difficult to find examples of groups having

Property (T). Keeping in mind that hyperbolic groups are in abundance, Gromov asked in [Gro87]:

Question 2. *Does a random quotient of a non-elementary hyperbolic group have Property (T)?*

Żuk and Kotowski–Kotowski showed that a random quotient of a free group has Property (T) for density $d > 1/3$. Combining this result with Gromov's non-elementarity result for $d < 1/2$ we have:

Theorem 1.1 ([KK13],[Żuk03],[Gro93])**.** *For density* $1/3 < d < 1/2$*, a random quotient of* F_n *has Property (T) and is non-elementary hyperbolic w.o.p.*

The goal of this dissertation is to generalize this theorem to a large class of non-elementary hyperbolic groups.

Ollivier in [Oll04] introduces a framework for proving analogous results for quotients of any hyperbolic group and posed the following problem:

Problem 1 ([Oll05])**.** *Does there exist a model for taking quotients of a non-elementary hyperbolic group* G *such that a random quotient of* G *has Property (T) and is non-elementary hyperbolic for* $1/3 < d < 1/2$ *with overwhelming probability?*

1.2 Statement of Main results

For a non-elementary hyperbolic group G with finite symmetric generating set A, we introduce a new model of taking its random quotients called the frayed geodesic model.

We start with $\mathcal{L}_{\ell-2}$, a set of geodesic words uniquely representing each element of the Cayley Ball $\mathcal{B}_{\ell-2}$. A *frayed geodesic* is a concatenation of a generator from A, a geodesic from $\mathcal{L}_{\ell-2}$ followed by another generator from A. X_ℓ is the set of all frayed geodesics (these words will induce a non-uniform measure on the Cayley Ball $\mathcal{B}_\ell$). To get a member of the *frayed geodesic model* at density d, we quotient G by a random choice of $|X_\ell|^d$ words from X_ℓ.

Informally, we say a non-elementary hyperbolic group is of large type if there are lots of ways of extending any given geodesic.

Our main results are the following:

Theorem 1.2. *A random quotient of a non-elementary hyperbolic group G in the frayed-geodesic model has Property (T) for density $d > 1/3$ with overwhelming probability.*

Theorem 1.3. *A random quotient of a torsion-free non-elementary hyperbolic group G of large type in the frayed-geodesic model is non-elementary hyperbolic for $d < 1/2$ for G with overwhelming probability.*

Actually, Theorem 1.3 is true for non-elementary hyperbolic groups of large type with harmless torsion (See [Oll04] for the definition of harmless torsion).

Combining these two results give a partial solution to Problem 1.

Chapter 2 is devoted to proving Theorem 1.2 and Chapter 3 is devoted to proving Theorem 1.3.

2 RANDOM QUOTIENTS IN THE FRAYED-GEODESIC MODEL HAVE PROPERTY (T)

2.1 Introduction

Kazhdan's Property (T) was first introduced by Kazhdan to study lattices in Lie groups. This property plays an important role in many other areas of mathematics and also computer science. For example, groups with Property (T) were used by Margulis to give the first explicit construction of families of expanders, graphs with good spectral properties which are important in theoretical computer science. It was shown by Zuk ([Żuk03]) and Kotowski–Kotowski ([KK13]) that for density $d \in (1/3, 1)$, a random quotient of a free group has Property (T) almost surely.

The goal of this chapter is to prove

Theorem. *1.2 A random quotient of a non-elementary hyperbolic group G in the frayed-geodesic model has Property (T) for density $d > 1/3$ with overwhelming probability.*

A similar result appears in [Ash22] independently of the results in this dissertation using a different strategy and model of randomness. The proof of Theorem 1.2 builds off of the proof of Property (T) in the case of free groups. The proof in [KK13] doesn't directly generalize from free groups to hyperbolic groups. We need to find a set of words in G that play the role of positive words as in [KK13]. To attain this goal, we use the automatic structure of hyperbolic groups and elements of Perron–Frobenius theory.

2.2 Preliminaries

We provide some preliminary definitions in this section.

Let Γ be a finitely generated group.

Definition 2.1 (Property (T))**.** Γ has a Property (T) if every affine isometric action of Γ on any real Hilbert space $\mathcal{H}$ has a fixed point.

We observe that Property (T) is preserved by quotients.

Remark. Let $\phi : \Gamma \to H$ be an epimorphism. Any representation of H yields a representation of Γ, and fixed points for Γ must be fixed points for H.Then if Γ has Property (T), so does H.

Definition 2.2 (Hyperbolic group)**.** A geodesic space X is said to be δ-hyperbolic if it satisfies the thin triangle condition, namely, if there is some δ such that any geodesic triangle is contained within the δ–neighborhood of any two of its sides. A group G with generating set $\mathcal{A}$ is said to be δ–hyperbolic if its Cayley graph is δ–hyperbolic.

We note that hyperbolic groups have exponential growth.

Lemma 2.3 ([Coo93]Cooernart's theorem)**.** *Let G be a non-elementary hyperbolic group with a finite generating set, $\mathcal{A}$. Let $B(\ell) = \left\{ x \in G \;\middle|\; \|x\| \leq \ell \right\}$ be the Cayley ball of radius ℓ. Then, there exists a constant $\lambda > 0$ and constants $c_1, c_2 > 0$ such that for all ℓ*

$$c_1 \lambda^\ell \leq B(\ell) \leq c_2 \lambda^\ell \tag{2.1}$$

Definition 2.4 (Growth rate of a group)**.** For a non-elementary hyperbolic group G, the number quantity λ in the above lemma is known as its *growth rate.*

2.3 Probability Lemmas

Lemma 2.5 (Intersection of high probability events is high probability)**.** *Let Ω_n be a sequence of sample spaces. Let $A_n, B_n \subset \Omega_n$ be sequences of events such that $\lim_{n\to\infty} P_n(A_n) = 1$ and $\lim_{n\to\infty} P_n(B_n) = 1$. Then $\lim_{n\to\infty} P_n(A_n \cap B_n) = 1$.*

Proof. We have:

$$1 \geq P_n(A_n \cup B_n) = P_n(A_n) + P_n(B_n) - P_n(A_n \cap B_n)$$

which implies $P_n(A_n \cap B_n) \geq P_n(A_n) + P_n(B_n) - 1$

As $n \to \infty$, $P_n(A_n \cap B_n)$ limits to 1 by squeeze theorem. □

Lemma 2.6 (induced uniformity)**.** *Let A and B be finite sets such that $A \subset B$. Let $X_1, X_2, ...X_N$ be N elements chosen uniformly and independently from B.Let F be the event that at least k of them were from A. Let $Y_1, Y_2, \cdots Y_k$ be the first k of the X_i's that were from A conditioned on the event F. Then, the random variables $Y_1, Y_2, \cdots Y_k$ are uniform and independent choices on A.*

Proof. Let $a_1, a_2, ...a_k$ be an arbitrary k-tuple from A^k. There are $\binom{N}{k}$ choices for the positioning i of the first k slots from A.

Let T_i be the number of N-tuples having the sequence $a_1, a_2, ...a_k$ as the first k slots from A in the specific positioning i. The total number of N-tuple sequences having $a_1, a_2, ...a_k$ as the first k elements from A is $\sum_i T_i$ where i runs from 1 to $\binom{N}{k}$.

We observe that the calculation is the same for any k-tuple from A. □

Lemma 2.7 (Picking from a subset)**.** *Let $A \subset B$, where A and B are finite sets. Further, let there exists constants $C_1, C_2 > 0$ such that $C_1 \leq |A|/|B| \leq C_2$. Suppose $X_1, X_2, X_3, ...X_N$ be N pickings uniformly and independently from $B(N < \sqrt{n})$.*
Then,

1. *P(at least $C_1N/2$ elements will be from A) $\geq 1 - \frac{K}{N}$ for some constant K (independent of N).*

2. *Let $Y_1, Y_2, Y_3, ...Y_{C_1N/2}$ be the random variables representing the first $C_1N/2$ elements from A. Then, they are uniformly and independently distributed.*

Proof. Let 1_A be the indicator function for A. We have for the expectation $E(1_A)$ and variance σ_{1_A} :

$$C_1 \leq E(1_A) \leq C_2 \text{ which implies,}$$

$$C_1^2 - C_2^2 \leq \sigma_{1_A} = E[1_A^2] - (E[1_A])^2 \leq C_2^2 - C_1^2.$$

Let S_N denote the number of items from A. Thus, $S_N = 1_A.X_1 + 1_A.X_2 + \cdots 1_A.X_N$. By Chebyshev's inequality, we have

$$1 - \frac{\sigma_{1_A^2}}{N\epsilon^2} \leq P(|\frac{S_N}{N} - E(1_A)| \leq \epsilon|)$$

Taking $\epsilon = \frac{C_1}{2}$ we get,

$$\begin{aligned} 1 - \frac{\sigma_{1_A^2}}{N\epsilon^2} &\leq P(-\frac{C_1}{2} \leq \frac{S_N}{N} - E(1_A) \leq \frac{C_1}{2}) \\ &\leq P(-\frac{C_1}{2} + E(1_A) \leq \frac{S_N}{N}). \end{aligned}$$

Which implies,

$$\begin{aligned} 1 - \frac{\sigma_{1_A^2}}{N\epsilon^2} &\leq P(C_1 - \frac{C_1}{2} \leq -\frac{C_1}{2} + E(1_A) \leq \frac{S_N}{N}) \\ &\leq P(\frac{C_1 N}{2} \leq S_N) \leq 1. \end{aligned}$$

As, $N \to \infty, P \to 1$. (Since, $\epsilon, \sigma_{1_A^2}$ are bounded quantities).

By, (1) we know that at least $C_1N/2$ elements are known to be from A. Applying Lemma 3.2, for N and $k = C_1N/2$ we see that every k tuple is equally likely, and hence the random variables $Y_1, Y_2, Y_3, ...Y_{C_1N/2}$ representing the k tuple are uniformly and independently distributed. □

2.4 Triangular model and modifications

In this section, we will review the positive triangular model studied in [KK13] and show that a presentation in a slightly modified triangular model has Property (T) for $d > \frac{1}{3}$ as well.

Definition 2.8 (positive word)**.** Let F_m be a free group with a basis $S \bigcup S^{-1}$.

A positive word uses elements only from S.

Definition 2.9 (positive triangular model)**.** Fix $d \in (0, 1)$. A presentation in the positive triangular model $\mathcal{M}^+(m, d)$ is given by $F_m / \langle\langle R \rangle\rangle$, where R is a m^{3d}-tuple of relators, chosen independently and uniformly from the set of positive words of length 3.

Definition 2.10 (with overwhelming probability)**.** Fix $d \in (0, 1)$. We say that a random presentation in the positive triangular model has a property with overwhelming probability if and only if we have

$$\lim_{m \to \infty} \frac{|\text{presentations in } \mathcal{M}^+(m, d) \text{ having the property}|}{|\mathcal{M}^+(m, d)|} = 1$$

Theorem 2.11 ([KK13], Theorem 3.14)**.** *A random presentation in the $\mathcal{M}^+(m, d)$ for $d > \frac{1}{3}$ has Property (T) with overwhelming probability.*

First, we observe that all positive words except the ones with the same letter repeated thrice have three equivalent cyclic permutations. We show below that choosing from these relators gives us a similar theorem as Theorem 2.11, as the presentations do not change.

Definition 2.12 ($3to1$ positive words)**.** A positive word of length 3 is $3to1$ if it has 3 cyclic permutations.

Remark. All positive words of length 3 except the ones with the same letter have 3 cyclic permutations. So, the number of length 3, $3to1$ positive words is $m^3 - m$.

Definition 2.13 ($3to1$ triangular model)**.** Fix $d \in (0, 1)$. A presentation in the $3to1$ triangular model $M^+_{3to1}(m, d)$ is given by $F_m / \langle\langle R \rangle\rangle$, where R is a tuple of m^{3d} relators, chosen uniformly and independently from the set of $3to1$ positive words.

Definition 2.14 (with overwhelming probability)**.** Fix $d \in (0, 1)$. We say that a random presentation in the $3to1$ triangular model has a property with overwhelming probability if and only if we have

$$\lim_{m \to \infty} \frac{\left|\text{presentations in } \mathcal{M}^+_{3to1}(m, d) \text{ having the property}\right|}{\left|\mathcal{M}^+_{3to1}(m, d)\right|} = 1$$

Lemma 2.15. *A random presentation in $\mathcal{M}^+(m,d)$ is from $\mathcal{M}^+_{3to1}(m,d)$ for $\frac{1}{3} < d < \frac{2}{3}$ with overwhelming probability.*

Proof. The number of presentations in the $3to1$ triangular model is $(m^3 - m)^{m^{3d}}$.The number of presentations in the positive triangular model was $(m^3)^{m^{3d}}$.

Now, let

$$L_m = \frac{(m^3 - m)^{m^{3d}}}{(m^3)^{m^{3d}}}$$

Taking the log of both sides gives,

$$\log L_m = m^{3d} \log(1 - m^{-2}) = \frac{\log(1 - m^{-2})}{m^{-3d}}$$

Applying limits and simplifying we get,

$$\lim_{m\to\infty} \log L_m = \lim_{m\to\infty} \frac{-2}{3d} \frac{1}{m^{2-3d} - \frac{1}{m^{3d}}}.$$

This goes to 0 for $\frac{1}{3} < d < \frac{2}{3}$. Thus, $\lim_{m\to\infty} L_m = 1$. □

Lemma 2.16. *A random presentation in $M^+_{3to1}(m,d)$ for $\frac{1}{3} < d < \frac{2}{3}$ has Property (T) with overwhelming probability.*

Proof. Let A_m be the set of presentations in $M^+(m,d)$ having Property (T). By Theorem 2.11, $|A_m|/|M^+(m,d)| \to 1$. Let B_m be the set of presentations in $M^+(m,d)$ that are from $M^+_{3to1}(m,d)$. By Lemma 2.15, $\frac{|B_m|}{|M^+(m,d)|} \to 1$. By Lemma 2.5, $\frac{|A_m \cap B_m|}{|M^+(m,d)|} \to 1$.

Now, the probability that a random presentation in $M^+_{3to1}(m,d)$ has Property (T) is greater than $\frac{|A_m \cap B_m|}{|M^+_{3to1}(m,d)|}$. We have,

$$\begin{aligned}
\lim_{m\to\infty} \frac{|A_m \cap B_m|}{|M^+_{3to1}(m,d)|} &= \lim_{m\to\infty} \frac{|A_m \cap B_m|}{|M^+(m,d)|} \frac{|M^+(m,d)|}{|M^+_{3to1}(m,d)|} \\
&= \lim_{m\to\infty} \frac{|A_m \cap B_m|}{|M^+(m,d)|} \cdot \lim_{m\to\infty} \frac{|M^+(m,d)|}{|M^+_{3to1}(m,d)|} \\
&= 1.
\end{aligned}$$

□

Now, we will define a preferred order on the positive relators of length 3 to pick only one of the three permutations.

Let F_m be a free group with a basis $S \bigcup S^{-1}$. Let $s_1 < s_2 < s_3 < \cdots < s_m$ be an order on the set S.

Definition 2.17 (preferred positive word)**.** A positive word of length 3 is *preferred* if it has the following properties.

1. It is a $3to1$ positive word.
2. If there are 3 distinct letters in the word, then the greatest index is at the end.
3. If there are 2 distinct letters in the word, then the following holds:
   (a) The greatest index is at the end if the lower index is repeated.
   (b) The greatest index is at the beginning and the end if the greatest index is repeated.

We will denote the set of preferred positive words by S^3_{pref}.

Remark. For example, $a_2a_1a_2, a_1a_1a_2$ are preferred instead of $a_1a_2a_2, a_1a_2a_1$. Further, note that $|S^3_{pref}| = (m^3 - m)/3$.

Now, we are ready to define the preferred positive triangular model which is the model we will use later.

Definition 2.18 (preferred positive triangular model)**.** Fix $d \in (0,1)$, A presentation in the preferred positive triangular model $\mathcal{M}^+_{pref}(m,d)$ is given by $F_m/\langle\langle R\rangle\rangle$, where R is a tuple of m^{3d} relators, chosen uniformly and independently from S^3_{pref}.

Theorem 2.19. *A random presentation in* $M^+_{pref}(m,d)$ *for* $\frac{1}{3} < d < \frac{1}{2}$ *has Property (T) with overwhelming probability.*

Proof. Using Tietze transformations from the presentations in $\mathcal{M}^+_{3to1}(m,d)$ to change every relator to the preferred permutation gives us isomorphic maps from presentations in $\mathcal{M}^+_{3to1}(m,d)$ to $\mathcal{M}^+_{pref}(m,d)$, so having/not having Property (T) is preserved.

A presentation in $\mathcal{M}^+_{pref}(m,d)$ has exactly $3^{m^{3d}}$ preimages in$\mathcal{M}^+_{3to1}(m,d)$. In $\mathcal{M}^+_{3to1}(m,d)$, the number of total presentations is the sum of the number of Property (T) presentations and the number of non-Property (T) presentations. Let $N = m^3 - m$ and let u_m be the share of presentations having Property (T) in $\mathcal{M}^+_{3to1}(m,d)$.

Dividing throughout by $3^{m^{3d}}$ we get in $\mathcal{M}^+_{pref}(m,d)$

$$\frac{\mu_m N^{m^{3d}}}{3^{m^{3d}}} + \frac{(1-\mu_m)N^{m^{3d}}}{3^{m^{3d}}} = \frac{N^{m^{3d}}}{3^{m^{3d}}}.$$

Hence, share of presentations in $\mathcal{M}^+_{pref}(m,d)$ having Property (T) equals

$$\frac{\mu_m N^{m^{3d}}}{3^{m^{3d}}} \frac{3^{m^{3d}}}{N^{m^{3d}}} = \mu_m.$$

And, we know μ_m goes to 1 as $m \to \infty$ by Lemma 2.16. This concludes the proof. □

2.5 General Property (T) theorem

We will map the preferred triangular model to a group G to get Property (T) quotients. First, as a warm-up, we map the positive triangular model:

Theorem 2.20. *Fix $d > \frac{1}{3}$. Let G be an infinite group with finite generating set A such that $e \notin A$ and $|A| = n$. Let $\{W_{m'}\}_{m'=1}^{\infty}$ be a sequence of sets of words of G with*

- *$|W_{m'}| = m'$.*
- *$W_{m'} \cap A = \emptyset$ for m' large enough.*

Let $m = m' + n$. If R' is a m^{3d}-tuple of relators chosen uniformly and independently from $(W_{m'} \cup A)^3$, then $G/\langle\langle R'\rangle\rangle$ has Property (T) with overwhelming probability.

Proof. Let $A = \{a_1, a_2, \cdots a_n\}$. Choose an enumeration $w_i \in W_{m'}$.

We define a $\phi : F_m \to G$ by

$$\phi(s_i) = \begin{cases} w_i, & \text{for } i = 1 \text{ to } m' \\ a_{i-m'}, & \text{for } i = m'+1 \text{ to } m \end{cases}.$$

We see that $\phi|_S : S \to W_{m'} \cup A$ is a bijection and hence $\phi|_{S^3} \to (W_{m'} \cup A)^3$ is a bijection. Hence, $\phi|_{S^3}$ induces a uniform measure on the words in $(W_{m'} \cup A)^3$. Thus, for every uniform and independent choice R from S^3, $R' := \phi(R)$ is also a uniform and independent choice from $(W_{m'} \cup A)^3$ and vice versa. We have :

$$
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\langle\langle R \rangle\rangle & \xrightarrow{f} & \langle\langle R' \rangle\rangle \\
\downarrow & & \downarrow \\
F_m & \xrightarrow{\phi} & G \\
\downarrow & & \downarrow \\
F_m/\langle\langle R \rangle\rangle & \xrightarrow{\bar{\phi}} & G/\langle\langle R' \rangle\rangle \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
$$

Further, ϕ is a surjection and as above induces a surjection $\bar{\phi}$ from $F_m/\langle\langle R \rangle\rangle$ to $G/\langle\langle R' \rangle\rangle$. Since, Property (T) is preserved by surjective homomorphism and as $F_m/\langle\langle R \rangle\rangle$ has Property (T) with overwhelming probability, so does $G/\langle\langle R' \rangle\rangle$. □

Definition 2.21 (preferred images)**.** Let ϕ be as in the above proof with domain the set of preferred positive words S^3_{pref}. The set of *preferred images*, denoted by $(W_{m'} \cup A)^3_{pref}$ is the image of S^3_{pref} under ϕ.

Theorem 2.22. *Fix $d > \frac{1}{3}$. Let G be an infinite group with finite generating set A such that $e \notin A$ and $|A| = n$. Let $\{W_{m'}\}_{m'=1}^{\infty}$ be a sequence of sets of words of G such that*

- *$|W_{m'}| = m'$.*
- *$W_{m'} \cap A = \phi$, for m' large enough.*

Let $m = m' + n$. If R' is a m^{3d}-tuple of relators chosen uniformly and independently from $(W_{m'} \cup A)^3_{pref}$, then $G/\langle\langle R' \rangle\rangle$ has Property (T) with overwhelming probability.

Proof. The same as above with the observation that $\phi|_{S^3_{pref}} \to (W_{m'} \cup A)^3_{pref}$ is a bijection. □

2.6 Good words

In the proof that a random quotient of a free group has Property (T) in [KK13], the set of positive words of F_n plays a crucial role. Now, we will describe a sequence of sets $\{W_{m'}\}_{m'=1}^{\infty}$ for any non-elementary hyperbolic group G that behaves like the set of positive words, as in [KK13].

Proposition 2.23. *Let G be a non-elementary hyperbolic group with a finite (symmetric) generating set A. Let λ be the growth rate of G in the generating set A. Then there exists a set of words $\mathcal{W} \subset A^*$ with the following properties:*

1. *$\mathcal{W}$ is closed under concatenation.*

2. *The evaluation map: $\mathcal{W} \to G$ mapping a word w to an element $\bar{w}$ is injective, and the sequence obtained by evaluating a word letter by letter is a geodesic.*

3. *There exist constants $C_1'', C_2'' > 0$ and $h \in \mathbb{N}$ such that,*

$$C_1''\lambda^{rh} \leq |\mathcal{W}_{rh}| \leq C_2''\lambda^{rh} \text{ for all } r \in \mathbb{N}$$

   where, $\mathcal{W}_{rh} = \{w \in \mathcal{W} \mid |w| = rh\}$.

Proof. Given G as above. Fix a total order on A.

A geodesic word w representing a group element G is *shortlex geodesic* if it is lexicographically first (in the ordering induced by the total order on A). We will denote the set of all shortlex geodesic words by $\mathcal{L}$. Note, the evaluation map $\mathcal{L} \to G$ is bijective.

Canon showed that $\mathcal{L}$ is a regular language. See [Cal13] for details.

Choose Γ a finite state automaton accepting $\mathcal{L}$ without redundant vertices. Perron–Frobenius theory implies that there exists a strongly connected subgraph Γ' of Γ whose Perron-Frobenius eigenvalue is equal to that of Γ. (See, for instance, Theorem 3.3, [DFW19]). This eigenvalue is equal to the growth rate λ of G (see, for instance, Corollary 3.7, [DFW19]).

Take any vertex v_* in Γ'. Let $\mathcal{W}$ be the set of all loops of Γ' based at v_*.

Then, let ℓ_1, ℓ_2 be loops based at v_*, then $\ell_1\ell_2$ is also a loop based at v_*. This shows that $\mathcal{W}$ is closed under concatenation.

Consider $\mathcal{L}' = \{\gamma l \delta \mid l \in \mathcal{W}\} \subset \mathcal{L}$, where γ is a fixed path from the start to v_* and δ is a fixed path from v_* to an accept state. It follows that $\mathcal{W}$ is a subset of $\mathcal{L}$ as subwords of shortlex words are also shortlex. Since $\mathcal{L} \to G$ is bijective, $\mathcal{W} \to G$ is injective.

Now, let $\mathcal{W}_{rh} = \{$all loops based at v_* of length exactly $rh\}$. We have $C_1'' > 0$ and a $h \in \mathbb{N}$ such that, $|\mathcal{W}_{rh}| > C_1''\lambda^{rh}$. For a proof, see step 2 in proof of Proposition 3.5, [DFW19].

□

Concatenating words from $\mathcal{W}$ (along with the generators) in the preferred order gives us the set of good words. The preferred order ensures that the generators show up only in the end.

Definition 2.24 (set of good words)**.** Fix ℓ and let h be as in Proposition 2.23. Dividing ℓ by $3h$ we get $\ell = q(3h) + r$ for a q and a r such that $0 \leq r < 3h$. We will call the set $(\mathcal{W}_{qh} \cup A)^3_{pref}$ *good words of parameter* ℓ. We will denote this set by X^Δ_ℓ.

We note below that the set of good words of parameter ℓ has the same growth rate λ as the Cayley ball of radius ℓ. This fact will be used in the next section.

Lemma 2.25. *There exist constants* $C_1', C_2' > 0$ *such that*

$$C_1'\lambda^\ell \leq |X^\Delta_\ell| \leq C_2'\lambda^\ell \text{ for all } \ell \in \mathbb{N}.$$

Proof. Let $|A| = n$. Now, for big enough ℓ, we have $\mathcal{W}_{qh} \cap A = \emptyset$. Hence, for such ℓ,

$$C_1''\lambda^{qh} + n \leq |\mathcal{W}_{qh} \cup A| \leq C_2''\lambda^{qh} + n \text{ with } C_1'', C_2'' \text{ as in Proposition 2.23.}$$

We can find constants $a, b > 0$ such that

$$a\lambda^{q(3h)} \leq |(\mathcal{W}_{qh} \cup A)^3_{pref}| \leq b\lambda^{q(3h)}$$

Indeed, if we let $m = |\mathcal{W}_{qh} \cup A|$, then $|(\mathcal{W}_{qh} \cup A)^3_{pref}| = (m^3 - m)/3$.

Further, observe that

$$\frac{m^3}{3} \geq \frac{m^3 - m}{3} = \frac{m(m-1)(m+1)}{3} \geq \frac{(m-1)^3}{3}.$$

Now from the definition, it follows that $q(3h) = l - r$, where $r < 3h$. □

Definition 2.26 (Intermediary model)**.** Fix a parameter $d \in (0, 1)$. Let X_ℓ^Δ be the set of good words as above. A presentation in the *intermediary model* $\mathscr{G}^\Delta(G, X_\ell^\Delta, d)$ is given by $G/\langle\langle R' \rangle\rangle$ where R' is a $\left|X_\ell^\Delta\right|^d$-tuple chosen uniformly and independently from X_ℓ^Δ.

Applying the general Property (T) theorem to the set of good words gives us Property (T) in the Intermediary model.

Theorem 2.27 (Property (T) in the Intermediary model)**.** *Fix $d > \frac{1}{3}$. A presentation in the intermediary model $\mathscr{G}^\Delta(G, X_\ell^\Delta, d)$ has Property (T) with overwhelming probability.*

Proof. Apply Theorem 2.22 to the sequence of sets $\{\mathcal{W}_{qh}\}$, where $\mathcal{W}_{qh}$ are as described above. □

2.7 Proof of Theorem 1.2

Let G be a non-elementary hyperbolic group with the generating set A. Let $\mathcal{B}_\ell$ be a set of geodesic words in A of length less than or equal to ℓ of G, which uniquely represent the group elements in the Cayley ball of length ℓ. We define a large set of words, of which the good words defined in the previous section are a large subset.

Definition 2.28 (frayed-geodesics)**.** A frayed geodesic of word length ℓ is a word of the form aba', where b is a geodesic from $\mathcal{B}_\ell$ of length exactly $l - 2$, a, a' are generators. The set X_ℓ of frayed-geodesics of word length less than or equal to $\ell = \{aba' \mid a, a' \in A, b \in \mathcal{B}_{\ell-2}\}$.

We observe that the set of frayed geodesics contains the set of good words as a density-one subset and that it grows at the same rate as the Cayley ball.

Lemma 2.29. *1. There exist constants $K_1, K_2 > 0$ such that*

$$K_1\lambda^l \leq |X_\ell| \leq K_2\lambda^l \text{ for all } l \in \mathbb{N},$$

where λ is the growth rate of the group G.

2. $X_\ell^\Delta \subset X_\ell$ and there exist constants $C_1, C_2 > 0$ such that $C_1 \leq |X_\ell^\Delta|/|X_\ell| \leq C_2$, $\forall l \in \mathbb{N}$.

Proof. By [Coo93] , there exists $c_1, c_2 > 0$ such that $c_1\lambda^l \leq |\mathcal{B}_{\ell-2}| \leq c_2\lambda^l$. Let $|A| = n$, we have $|X_l| = |A||\mathcal{B}_{\ell-2}||A|$. We also note that $\mathcal{B}_\ell \subset X_\ell$.

Fix ℓ large enough. Let $x \in X_\ell^\Delta$. If x is a concatenation of three (shortlex) geodesics, then $x \in X_\ell$ as $\mathcal{B}_{3qh} \subset X_\ell$. Else, if x is a word of the form b_1b_2a' where b_1, b_2 are (shortlex) words of length qh, then we observe that b_1b_2 can be read as $a(b_1'b_2)$ for some generator a where $b_1'b_2$ is also a shortlex word (as subwords of shortlex words are shortlex). Therefore, we get constants C_1, C_2 by Lemma 2.25 and (1) above. □

Definition 2.30 (frayed-geodesic model)**.** Fix a parameter $d \in (0, 1)$.Choose a length ℓ.Let X_ℓ be as above. A presentation in the frayed-density model $\mathscr{G}(G, X_\ell, d)$ is given by $G/\langle\langle R\rangle\rangle$, where R is a tuple of $\lfloor |X_\ell|^d \rfloor$ relators chosen uniformly and independently from X_ℓ.

Definition 2.31 (with overwhelming probability)**.** Fix G and d. We say that a random presentation in the frayed density model has a property with overwhelming probability if and only if we have

$$\lim_{\ell\to\infty} \frac{|\text{presentations in } \mathscr{G}(G, X_\ell, d) \text{ having the property}|}{|\mathscr{G}(G, X_\ell, d)|} = 1$$

Theorem 2.32. *Fix density $d > \frac{1}{3}$. A random presentation in the frayed geodesic model $\mathscr{G}(G, X_\ell, d)$ has Property (T) with overwhelming probability.*

Proof. Fix G, ℓ and $1/3 < d < 1/2$. Let R be a tuple of $\lfloor |X_\ell|^d \rfloor$ relators chosen uniformly and independently from X_ℓ. Let A_ℓ be the event that R contains at least a $C_1\lfloor |X_\ell|^d \rfloor/2$ large subtuple of good words. We note that by Lemma 2.29 and Lemma 2.3, $P_\ell(A_\ell) \to 1$ as

$\ell \to \infty$. Now, let us pick a d' such that $1/3 < d' < d$. By Lemma 2.29 we have:

$$|X_\ell^\Delta|^{d'} \leq C_2 |X_\ell|^{d'} \leq \frac{C_1 \lfloor |X_\ell|^d \rfloor}{2}$$

for ℓ large enough. Thus, R will contain a $|X_\ell^\Delta|^{d'}$-long subtuple R' of good words with probability $P_\ell(A_\ell)$. Let B_ℓ be the event that $G/\langle\langle R' \rangle\rangle$ has Property (T). $P_\ell(B_\ell) \to 1$ by Theorem 2.27.

Hence,

$$P_\ell(G/\langle\langle R \rangle\rangle \text{ has Property (T)}) \geq P_\ell(B_\ell \mid A_\ell) = \frac{P_\ell(B_\ell \cap A_\ell)}{P_\ell(A_\ell)}$$

which goes to 1 by Lemma 2.5. □

3 RANDOM QUOTIENTS IN THE FRAYED-GEODESIC MODEL ARE NON-ELEMENTARY HYPERBOLIC

3.1 Introduction

The question of whether a random quotient of a hyperbolic group is non-elementary hyperbolic was studied by Ollivier in [Oll04]. He proves in [Oll04], a general theorem that deals with random quotients by any type of word. However, depending on the type of words used (reduced words, plain words, geodesic words etc.) , we are going to have a "phase shift" $\beta \geq 0$. For example, a random quotient of a hyperbolic group by reduced words (as in the case of free groups) will be trivial for $d \in (1/2 - \beta, 1)$ for a β strictly positive. In order to calculate β for various types of words, a set of axioms are laid out in [Oll04]. A proof is provided showing that $\beta = 0$ for the case of geodesic words representing uniquely elements of the Cayley sphere.

The goal of this chapter is to prove

Theorem. *1.3 A random quotient of a torsion-free non-elementary hyperbolic group G of large type in the frayed- geodesic model is non-elementary hyperbolic for $d < 1/2$ for G with overwhelming probability.*

The proof follows the proof sketch provided by Ollivier in [Oll04] for the case of the Cayley Sphere. In doing so we fill in the technical details gaps in the proof of the sphere model of Ollivier using new ideas.

3.2 Preliminaries

In this section, we collect some properties of non-elementary hyperbolic groups of large type.

Lemma 3.1 ([Coo93], Cooernaert's theorem)**.** *Let G be a non-elementary hyperbolic group. Let $B(s) = \{x \in G \mid ||x|| \leq s\}$ be the Cayley ball of radius s. We have, for all s ,constants $C_1, C_2 > 0$ and $\lambda > 0$ such that*

$$C_1 e^{\lambda s} \leq B(s) \leq C_2 e^{\lambda s}$$

Definition 3.2 (Growth rate)**.** We will call the constant λ in the above Lemma as the growth rate of the group G.

Let $\mathcal{L}$ be a set of unique geodesic representatives of the group elements of G with a generating set $\mathcal{A}$.

Definition 3.3 (Cone of a geodesic)**.** For $w \in \mathcal{L}$, the cone of w, $C(w)$ is defined as $C(w) = \{g \in \mathcal{L} \mid ||gw|| = ||g|| + ||w||\}$.

According to Lemma 2.3 in [GTT18] cones may be of large type or small type.

Lemma 3.4 (see Lemma 2.3, [GTT18])**.** *Let G be a hyperbolic group with growth rate λ. There exists $C_1, C_2 > 0$ such that for all r*

$$C_1 e^{\lambda r} \leq |C(w) \cap S(e, r)| \leq C_2 e^{\lambda r} \tag{3.1}$$

or there exists $c > 0$ and $\lambda_1 < \lambda$ such that

$$|C(w) \cap S(e, r)| \leq c e^{\lambda_1 r} \tag{3.2}$$

where $S(e, r)$ be the set of all elements in G at distance r from e.

Definition 3.5 (Groups of large type)**.** A hyperbolic group G with a generating set $\mathcal{A}$ is said to be of large type if every cone is of the first type as in the above lemma.

Example 3.6. Free groups, closed surface groups with the standard generating set, groups with infinitely many ends. See [HMM18] for a discussion.

Remark. It is not known if every hyperbolic group is of large type with respect to some generating set.

In the next lemma, we note that the number of geodesics with a given subword in a fixed position is large.

Lemma 3.7. *Let G be a non-elementary hyperbolic group of large type. Let $w \in \mathcal{L}$ and a p_1 positive integer be given. Let $B = \{g \in \mathcal{L}$ of norm $r \mid$ there exists $g_1, g_1' \in \mathcal{L}$ such that $g = g_1 w g_1'$ where $||g_1 w g_1'|| = ||g_1|| + ||w|| + ||g_1'||$ and $||g_1|| = p_1\}$. We have,*

$$C_1^2 e^{\lambda(r-||w||)} < |B|.$$

Proof. We want to estimate the number of geodesics of length r such that w arises as a subword at position p_1. By Lemma 3.4, we get the following.

$$\begin{aligned} |B| &\geq |\{g_1 \mid g_1 w \text{ is a geodesic}\} \cap \{g_1' \mid (g_1 w) g_1' \text{ is a geodesic }\}| \\ &\geq |\{g_1 \mid g_1 \in C(w^{-1}) \cap S(e, p_1)\}| \times |\{g_1' \mid g_1' \in C(g_1 w) \cap S(e, r - p_1 - ||w||) \\ &\geq C_1 e^{\lambda p_1} . C_1 e^{\lambda(r - p_1 - ||w||)} \end{aligned}$$

□

3.3 Probability Lemmas

We state some probability lemmas that will be used in later sections. The reader may skip ahead to later sections and come back if these need to be recalled.

Lemma 3.8. *Let Ω_1, Ω_2 be two discrete probability spaces. Let C be an event in the product space, $\Omega_1 \times \Omega_2$. Then there exists an event $A \subset \Omega_1$ and for every $a \in A$, events $B_a \subset \Omega_2$ such that $C = \cup_{a \in A}(B_a \times \{a\})$. We have $Pr_{\Omega_1 \times \Omega_2}(C) = \sum_{a \in A} Pr_{\Omega_2}(B_a)$. Further, if there exists K such that $Pr_{\Omega_2}(B_a) \leq K$ for all B_a, then:*

$$Pr_{\Omega_1 \times \Omega_2}(C) \leq K Pr_{\Omega_1}(A)$$

.

Proof. The proof is left to the reader. □

Lemma 3.9 ([Ros76], Markov's Inequality). *If Z is a random variable that takes only non-negative values, then*

$$P(Z \geq 1) \leq E[Z]$$

3.4 Reduction of the frayed ball to the frayed annulus

In this section, we show that with overwhelming probability the relators we chose for quotients in the frayed geodesic model are from a frayed annulus that we define below.

Definition 3.10 (frayed annulus model). Fix a parameter $d \in (0,1)$.Choose a length ℓ. Let X_{ann_ℓ} be the set of frayed geodesics of word length $\in (\frac{\ell}{2}, \ell)$. A presentation in the frayed-annulus model $\mathscr{G}(G, X_{ann_\ell}, d)$ is given by $G/\langle\langle R\rangle\rangle$, where R is a tuple of $\lfloor |X_\ell|^d \rfloor$ relators chosen uniformly and independently from X_{ann_ℓ}.

Lemma 3.11. *For $\frac{1}{3} < d < \frac{1}{2}$ a random presentation in the frayed geodesic model is from the frayed annulus model with overwhelming probability.*

Proof. Fix, $d \in (\frac{1}{3}, \frac{1}{2})$. Let $G/\langle\langle R\rangle\rangle$ be a presentation in the frayed-annulus model $\mathscr{G}(G, X_{ann_\ell}, d)$. Let, $Z=$ Number of relators of length k in R.

Then,

$$\begin{aligned} E[Z] &= \lfloor |X_\ell|^d \rfloor \cdot P(\text{ one word is of length } k) \\ &\leq K_2^d \lambda^{\ell d} (\frac{K_2 \lambda^k}{K_1 \lambda^l}) \quad \text{for some } K_1, K_2 \\ &= K \lambda^{k-\ell(1-d)} \quad \text{for some } K(K_1, K_2, d). \end{aligned}$$

Here K_1, K_2 are as in Lemma 2.29.

The exponent will be positive if , $k > \ell(1-d) > \frac{\ell}{2}$. By Markov's inequality, $P(Z > 1) \leq E[Z]$. So, with overwhelming probability, R does not contain frayed geodesics of length less than or equal to $\frac{l}{2}$.

□

Definition 3.12 (with overwhelming probability)**.** Fix G and d. We say that a random presentation in the frayed density model has a property with overwhelming probability if and only if we have

$$\lim_{\ell\to\infty} \frac{|\text{presentations in } \mathscr{G}(G, X_{ann_\ell}, d) \text{ having the property}|}{|\mathscr{G}(G, X_{ann_\ell}, d)|} = 1$$

We will prove in section 3.6 that a random presentation in the frayed annulus model is non- elementary hyperbolic with overwhelming probability, which is enough to prove that a random presentation in the frayed ball model is non-elementary hyperbolic with overwhelming probability.

3.5 Analysis of random geodesic segments

In this section, we will study the properties of random geodesic segments. We will use the results in this section to prove our main theorems in the later sections.

Let G be a non-elementary hyperbolic group. We note the following geometric lemma based on hyperbolicity.

Lemma 3.13. *There exists a constant K such that for all positive integers s, for all positive real L, and for all $x \in G$*

$$P((x^{-1}|X_{\leq s}) \geq L) \leq Ke^{-\lambda L} \tag{3.3}$$

where $X_{\leq s}$ is a random variable on $\Omega_{\leq s} = \{x \in G|\ ||x|| \leq s\}$ and $(x^{-1}|X_{\leq s})_e$ is the Gromov product based at identity.

Proof. Let $K = \frac{C_2}{C_1}e^{\lambda\delta}$, we claim that this is the required constant. Indeed, choose s, L, x.

If $s < L$ or if $||x|| < L$, then there is no $X_{\leq s}$ such that $(x^{-1}|X_{\leq s}) \geq L$ and the upper bound holds.

If $s > L$ and $||x|| \geq L$, there exists a y at a distance L from e on the geodesic joining e to x. If $(x^{-1}|X_{\leq s}) \geq L$, then let y' be another element on the geodesic joining e to $X_{\leq s}$ such that $d(y, y') < \delta$. Such a y' exists as G is hyperbolic.

Hence, we have $d(y, X_{\leq s}) \leq d(y, y') + d(y', X_{\leq s}) \leq \delta + ||X_{\leq s}|| - L$. So,

$$\begin{aligned} P((x^{-1}|X_{\leq l}) \geq L) \leq P(d(y, X_{\leq l}) &\leq ||X_{\leq l}|| - L + \delta) \\ &= \frac{|B(||X_{\leq s}|| - L + \delta)|}{|B(s)|} \\ &\leq \frac{|B(s - L + \delta)|}{|B(s)|} \\ &\leq \frac{C_2 e^{\lambda(s-L+\delta)}}{C_1 e^{\lambda s}} \\ &= e^{-\lambda L} \frac{C_2 e^{\lambda\delta}}{C_1} \end{aligned}$$

Here C_1, C_2, λ are as in Lemma 3.2.

□

Next, we use the above lemma to study the multiplication of a fixed group element by a random geodesic word. We conclude that the probability to get at most L amount of cancellation is low.

Lemma 3.14. *There exists a constant K such that for all s, for any positive real L, and for any $x \in G$*

$$P(||xX_{\leq s}|| \leq ||x^{-1}|| + ||X_{\leq s}|| - L) \leq Ke^{\frac{-\lambda}{2}L} \tag{3.4}$$

$$P(||X_{\leq s}x|| \leq ||x^{-1}|| + ||X_{\leq s}|| - L) \leq Ke^{\frac{-\lambda}{2}L} \tag{3.5}$$

where $X_{\leq s}$ is a random variable on $\Omega_{\leq s} = \{x \in G| \ ||x|| \leq s\}$.

Proof. Let K be as in Lemma 3.13. Choose $s \geq K$, L and $x \in G$.

Applying Lemma 3.13 to L, we get

$$\begin{aligned} &P(x^{-1}|X_{\leq s}) \geq L) \leq Ke^{-\lambda L} \\ \Longrightarrow\ &P(\frac{1}{2}(||x^{-1}|| + ||X_{\leq s}|| - ||xX_{\leq s}|| \geq L) \leq Ke^{-\lambda L} \\ \Longrightarrow\ &P(||x^{-1}|| + ||X_{\leq s}|| - ||xX_{\leq s}|| \geq 2L) \leq Ke^{-\lambda L} \\ \Longrightarrow\ &P(||xX_{\leq s}|| \leq ||x^{-1}|| + ||X_{\leq s}|| - 2L) \leq Ke^{-\lambda L} \end{aligned}$$

This finishes the proof of the first inequality.

Notice that $||x^{-1}X_{\leq s}^{-1}|| = ||(X_{\leq s}x)^{-1}|| = ||X_{\leq s}x||$. To get the second inequality, we replace x by x^{-1} and $X_{\leq s}$ by $X_{\leq s}^{-1}$ in the first inequality.

□

We now prove an analogous result involving multiplication by random geodesic segment

Lemma 3.15 (variable length segments). *For any positive integer k, there exists a constant $K(k)$ such that for all k-tuples $S = (s_1, s_2, \dots, s_k)$, for any L, and for any $x \in G$, we have*

$$P(||X_S x|| \leq ||x|| + \sum_{i=1}^{k} ||Y_i|| - L) \leq K(k) \cdot L^{k-1} \cdot e^{\frac{-\lambda}{2}L} \tag{3.6}$$

where $X_S = \{(Y_1, Y_2, ...Y_k)\}$ is a random variable on $\Omega_{\leq S} = \{(x_1.x_2, ..x_k) \in G|\ ||x_i|| \leq s_i\}$.

Also, for any positive integer k', there exists a constant $K(k')$ such that for all k'-tuples $S' = (s'_1, s'_2, ..s'_{k'})$, for any L and for any $x \in G$ we have

$$P(||xX_{S'}|| \leq ||x|| + \sum_{i=1}^{k'} ||Y'_i|| - L) \leq K(k') \cdot L^{k'-1} \cdot e^{\frac{-\lambda}{2}L} \tag{3.7}$$

where $X_{S'} = \{(Y'_1, Y'_2, ...Y'_k)\}$ is a random variable on $\Omega_{\leq S'} = \{(x'_1, x'_2, ..x'_k) \in G|\ ||x'_i|| \leq s'_i\}$.

Further, for positive integers k, k', there exists a constant $K(k, k')$ such that for all k-tuples S and k'-tuples S', we have for any L and for any $x \in G$

$$P_{X_S \times X_{S'}}(||X_S x X_{S'}|| \leq ||x|| + (\sum_{i=1}^{k} ||Y_i||) + (\sum_{i=1}^{k'} ||Y'_i||) - L) \leq K(k, k') \cdot L^{k+k'-1} \cdot e^{\frac{-\lambda}{2}L} \tag{3.8}$$

where $X_S, X_{S'}$ are as before.

Proof. We proceed via induction on k. The base case is: there exists $K(1)$ such that, for all s, for all L, for all $x \in G$

$$P(||X_{\leq s}x|| \leq ||x^{-1}|| + ||X_{\leq s}|| - L) \leq K(1)e^{\frac{-\lambda}{2}L}.$$

Lemma 3.14 gives us such a $K(1)$ and hence the base case is true.

We assume the induction hypodissertation: for a positive integer $k-1$ there exists a constant $K(k-1)$ such that for all $(k-1)$-tuples $S^{k-1} = (s_1, s_2, ..s_{k-1})$, for any L , for any $x \in G$, we have the following.

$$P(||X_S^{k-1}x|| \leq ||x|| + \sum_{i=1}^{k-1} ||Y_i|| - L) \leq K(k-1) \cdot L^{k-2} \cdot e^{\frac{-\lambda}{2}L}.$$

We now prove the statement for k. We claim that $K(k) = K(k-1) \times K(1)$. Indeed, let $S^k = (s_1, s_2, ..s_k)$ be a k tuple. Choose a L and $x \in G$.

Let $C_L = \{(Y_1, Y_2, \ldots, Y_k) \mid ||x(Y_1 Y_2 \cdots Y_k)|| \leq ||x|| + ||Y_1 Y_2 \cdots Y_k|| - L\}$. For $i \leq L$, we define the event

$$C_i = \{(Y_1, Y_2, \ldots Y_{k-1}) \mid ||x(Y_1 Y_2 \cdots , Y_{k-1})|| \leq ||x|| + \sum_{j=1}^{k-1} ||Y_j|| - i, Y_k \in \mathcal{Y}\}$$

where

$$\mathcal{Y} := \{Y \mid \ ||(xY_1 Y_2 \cdots Y_{k-1})Y|| \leq ||x(Y_1 Y_2 \cdots Y_{k-1})|| + ||Y|| - (L-i)\}$$

Now, $C_L \subset \bigcup_{i \leq L} C_i$. Also, $C_i = \bigcup_{a \in A} B_a \times \{a\}$ by Lemma 3.8. We get by induction hypodissertation

$$P(A) \leq K(k-1) \cdot L^{k-2} \cdot e^{\frac{-\lambda}{2}i}$$

and by the base case

$$P(B_a) \leq K(1) \cdot e^{\frac{-\lambda}{2}(L-i)}.$$

Hence, we have

$$P(C_L) \leq L \cdot K(k-1) \cdot L^{k-2} \cdot e^{\frac{-\lambda}{2}i} \cdot K(1) \cdot e^{\frac{-\lambda}{2}(L-i)}$$

by Lemma 3.8.

The proof of inequality (3.7) is similar. To prove (3.8) we combine (3.6) and (3.7) using Lemma 3.8.

□

For our applications, we will have k and k' such that $\sum_{i=1}^{k} s_i \leq \ell$, $\sum_{i=1}^{k'} s'_i \leq \ell$, and $L \leq \ell$ for some given positive real number ℓ.

Lemma 3.16. *For any $\epsilon > 0$, and any positive integers k, k', there exists a $\bar{M}(\epsilon, k, k')$ such that for all $S = (s_1, \ldots, s_k)$ and $S' = (s'_1, \ldots, s'_{k'})$ with $(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i) > \bar{M}(\epsilon, k, k')$, we have for any L, for any $x \in G$*

$$P_{X_S \times X_{S'}}(||X_S x X_{S'}|| \leq ||x|| + (\sum_{i=1}^{k} ||Y_i||) + (\sum_{i=1}^{k'} ||Y'_i||) - L) \leq e^{\frac{-\lambda}{2} L} e^{\epsilon(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i)} \quad (3.9)$$

where $X_S = (Y_1, Y_2, \ldots, Y_k)$ is a random variable on $\Omega_{\leq S} = \{(x_1.x_2, ..x_k) \in G|\ ||x_i|| \leq s_i\}$ and $X_{S'} = (Y'_1, Y'_2, \ldots, Y'_k)$ is a random variable on $\Omega_{\leq S'} = \{(x'_1, x'_2, ..x'_k) \in G|\ ||x'_i|| \leq s'_i\}$.

Proof. Given positive integers k, k', by 3.15 we have

$$\begin{aligned} P_{X_S \times X_{S'}}(||X_S x X_{S'}|| \leq ||x|| + (\sum_{i=1}^{k} ||Y_i||) + (\sum_{i=1}^{k'} ||Y'_i||) - L) \\ \leq K(k, k') \cdot L^{k+k'-1} \cdot e^{\frac{-\lambda}{2} L} \\ \leq K(k, k') \cdot \ell^{k+k'-1} \cdot e^{\frac{-\lambda}{2} L}. \end{aligned}$$

Given $\epsilon > 0$, we let $\bar{M}(\epsilon, k, k')$ be such that for $(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i) > \bar{M}$

$$K(k, k') \ell^{k+k'-1} \leq e^{\epsilon((\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i))}.$$

This finishes the proof. ☐

Lemma 3.17. *There exists a γ_3 such that for any $\epsilon > 0$, and any two positive integers k, k', there exists a $\bar{M}(\epsilon, k, k')$, such that for any $n : \mathbb{R}^+ \to \mathbb{R}^+$, for all $S = (s_1, \ldots, s_k)$ and $S' = (s'_1, \ldots, s'_{k'})$ with $(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i) > \bar{M}(\epsilon, k, k')$*

$$\begin{aligned} P(\textit{there exist } u, |u| \leq n(\ell) \mid ||X_S u X'_S|| \leq n(\ell)) \\ \leq e^{\gamma_3 n(\ell)} e^{\frac{-\lambda}{2}(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k} s'_i)} e^{\epsilon((\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k} s'_i))} \quad \textit{for all } \ell \end{aligned} \quad (3.10)$$

Further, for any given $C > 0$

$$\begin{aligned} P(\textit{there exist } u, |u| \leq n(\ell) \mid ||X_S u X'_S|| \leq C \log(\ell)) \\ \leq e^{\gamma_{4'} n(\ell)} e^{\frac{-\lambda}{2} C \log \ell} e^{\frac{-\lambda}{2}(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k} s'_i)} e^{\epsilon((\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k} s'_i))} \quad \textit{for all } \ell \end{aligned} \quad (3.11)$$

where X_S is a random variable on $\Omega_S = \{(x_1, x_2, \ldots, x_k) \in G| \; ||x_i|| = s_i\}$, and $X_{S'}$ is a random variable on $\Omega_{S'} = \{(x'_1, x'_2, \ldots, x'_k) \in G| \; ||x'_i|| = s'_i\}$ respectively.

Proof. To prove the first inequality, fix $u \leq n(\ell)$. Applying Lemma 3.16 to the case $\sum_{i=1}^{k} ||Y_i|| = \sum_{i=1}^{k} s_i$ and $\sum_{i=1}^{k'} ||Y_i|| = \sum_{i=1}^{k'} s'_i$ and $L = ||u|| + \sum_{i=1}^{k} s_i + \sum_{i=1}^{k} s'_i - n(\ell)$, we get

$$P(\text{there exist } u, |u| \leq n(\ell) \mid ||X_S u X'_S|| n(\ell))$$
$$\leq \sum_{\text{choices of u}} e^{\frac{-\lambda}{2}(||u|| + \sum_{i=1}^{k} s_i + \sum_{i=1}^{k} s'_i - n(\ell))} e^{\epsilon((\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i))}$$
$$\leq [(2m)^{n(\ell)} e^{-\frac{\lambda}{2} n(\ell)}] e^{\frac{-\lambda}{2}((\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i))} e^{\epsilon(\sum_{i=1}^{k} s_i) + (\sum_{i=1}^{k'} s'_i)}$$

Choosing an appropriate γ_3 gives us the required upper bound.

The Proof of the 2nd inequality is similar with the only difference being we choose $L = ||u|| + \sum_{i=1}^{k} s_i + \sum_{i=1}^{k} s'_i - C \log(\ell)$. □

3.6 Olivier's axioms for the frayed-annulus model

In this section, we describe the axioms laid out by Ollivier in [Oll04] for the case of words in the frayed annulus X_{ann_ℓ}. We further prove that X_{ann_ℓ} satisfies these axioms with the correct exponents, and hence the quotients in the frayed-annulus model will be non-elementary hyperbolic with overwhelming probability.

3.6.1 Notation

By $|w|$ we mean the length of the word, i.e., the number of letters in w. By, $||w||$ we mean norm, i.e., the smallest length of a word equal to w.

By, X we will mean a random word chosen from a specified sample space. For a,$b \in [0, 1]$, $X_{[a;b]}$ will mean the projection of the original word to its subword starting from $(a|X|)$-th letter to the $(a + b)|X|$-th letter. Small cases of letters will mean specific instances of the random variables.

$\Omega_s = \{x \in G \mid ||x|| = s\}$ would mean the sample of space of geodesics of length exactly s for the specified s. We will use this notation inside the upcoming proofs.

3.6.2 Statement and proof of axiom 1 and axiom 2

Let G be a torsion-free non-elementary hyperbolic group of large type. Let X_{ann_ℓ} be the set of words in its frayed annulus. The first axiom says that X_{ann_ℓ} should only contain words of length roughly ℓ up to some constant factor.

Theorem 3.18 (Axiom 1)**.** *There is a constant κ_1 such that for every ℓ, there are only words of length between $\frac{\ell}{\kappa_1}$ and $\kappa_1 \ell$ in X_{ann_ℓ}.*

Proof. Recall from the Lemma 3.10 that the lengths of words in X_{ann_ℓ} are in $(\frac{\ell}{2}, \ell)$. We can take $\kappa_1 = 2$ □

The second axiom states that subwords of words in X_{ann_ℓ} probably do not represent short elements of the group G.

Theorem 3.19 (Axiom 2)**.** *Let X be a random word from the frayed annulus $\Omega_{ann_\ell} = \{x | \frac{\ell}{2} < |x| < \ell, x \text{ a frayed geodesic}\}$. . Then, there exist a constant κ_2 such that for any $\epsilon > 0$, $\xi > 0$, there exists a natural number $M(\epsilon, \xi)$ such that for all $a \in [0,1]$, $b \in [\xi, 1]$, for any $t \leq 1$, we have for all $\ell > M$ and for all $r \in (\frac{\ell}{2}, \ell)$*

1. *if $a + b < 1$, then for any w of length ar*

$$P(||X_{[a;b]}|| \leq \kappa_2(1-t)|X_{[a;b]}| \mid X_{[0;a]} = w, |X| = r) \leq e^{-\frac{\lambda}{2}t(br)}e^{\epsilon\ell}$$

2. *if $a + b > 1$, then for any w of length $(1-b)r$*

$$P(||X_{[a;b]}|| \leq \kappa_2(1-t)|X_{[a;b]}| \mid X_{[a+b-1;a]} = w, |X| = r) \leq e^{-\frac{\lambda}{2}t(br)}e^{\epsilon\ell}$$

Proof. We can take $\kappa_2 = \frac{1}{3}$ and given ϵ, ξ we will show that such an $M(\epsilon, \xi)$ exists. Choose $\ell > M$, a, b, t and a $\frac{\ell}{2} < r < \ell$.

Case 1 If $a+b<1$, for any w we have $\frac{br}{2} < br-2 \leq ||X_{[a;b]}|| \leq \frac{1-t}{3}br$. This inequality is always false, and hence the probability is 0. So $M=1$ would work.

Case 2 If $a+b>1$, pick a word w of length $(1-b)r$, then the the probability listed looks like

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|}.$$

Here, $A=\{x\in\Omega_r \mid ||x_{[a;b]}|| \leq \frac{1}{3}br(1-t)\}$ and $B=\{x\in\Omega_r \mid x_{[a+b-1;a]}=w\}$.

We have that, w arises as a middle subword of the geodesic and

$$C_1^2 e^{\lambda(br)} \leq |B|$$

by 3.7.

Now let b_1, b_2 be such that $a+b_1=1, b_2=b-b_1$. We observe that,

$$|A\cap B| \leq |\bar{A}|$$

Where $\bar{A}=\{(x_1,x_2)\in\Omega_{b_1r}\times\Omega_{b_2r} \mid ||x_1x_2|| \leq \frac{1}{3}(br)(1-t)\}$.

We get by combining the above two inequalities,

$$P(A|B) \leq \frac{1}{C_1^2} e^{-\lambda(br)}|\bar{A}|.$$

Now, by piecewise geodesic lemma, for $\epsilon' := \frac{\epsilon}{2}$ there exists a $\bar{M}(\epsilon',\xi)$ such that

$$|\bar{A}| \leq e^{-\frac{\lambda}{2}t(br)}e^{\epsilon'(br)}|\Omega_{b_1r}\times\Omega_{b_2r}| \leq C_2^2 e^{-\frac{\lambda}{2}t(br)}e^{\epsilon'(br)}e^{\lambda(br)} \qquad \text{for } r>\bar{M}.$$

By combining the last two inequalities, we obtain

$$P(A|B) \leq \frac{C_2^2}{C_1^2}e^{-\frac{\lambda}{2}t(br)}e^{\epsilon'(br)} \leq \frac{C_2^2}{C_1^2}e^{-\frac{\lambda}{2}t(br)}e^{\epsilon' l} \qquad \text{for } r>\bar{M}.$$

The last inequality hold because $br \leq l$. Since $\epsilon'=\frac{\epsilon}{2}$, there exists $M_1=M_1(C_1,C_2,\epsilon)$ such that

$$\frac{C_2^2}{C_1^2}e^{\epsilon'\ell} \leq e^{\epsilon\ell} \qquad \text{for } \ell > M_1.$$

Let $M := \max\{2\bar{M}, M_1\}$. Since $M > 2\bar{M}$, it follows that $r > \bar{M}$. Then combination of the last two inequalities yields

$$P(A|B) \leq e^{-\frac{\lambda}{2}t(br)}e^{\epsilon\ell} \qquad \text{for } l \geq M.$$

□

3.6.3 Statement and proof of axiom 3

Axiom 3 controls the probability that subwords of words in X_{ann_ℓ} are almost inverse in the group. The subwords can also come from the same word. The proofs we provide in this dissertation will be for geodesics, the proofs when they are frayed geodesics will be similar.

Theorem 3.20 (Axiom 3)**.** *Let X, X' be random words from the frayed annulus $\Omega_{ann_\ell} = \{x | \frac{\ell}{2} < |x| < \ell, x$ a frayed geodesic$\}$. Then, there exists a constant γ_3 such that for any $n : \mathbb{R}^+ \to \mathbb{R}^+$, for any $\epsilon > 0, \xi > 0$, there exists $M(\epsilon, \xi)$ such that for all $a, a' \in [0,1], b, b' \in [\xi, 1]$ we have for all $l > M$ and for all $\frac{l}{2} < r < l, \frac{l}{2} < r' < l$*

1. *Case 1 :If $a+b, a'+b' \leq 1$(No wrap around) for any w, w' of lengths $ar, a'r'$*

$$\begin{aligned} P(\textit{there exists } u, v, |u|, |v| \leq n(l) \textit{such that } X_{[a;b]}uX'_{[a';b']}v = 1 \mid \\ X_{[0;a]} = w, X'_{[0;a']} = w', |X| = r, |X'| = r') \\ \leq e^{\gamma_3 n(\ell)}.e^{-\lambda\frac{(br+b'r')}{2}}.e^{\epsilon\ell} \end{aligned}$$

2. *Case 2: If $a+b, a'+b' > 1$ (2 wrap arounds) for any w, w' of lengths $(1-b)r, (1-b')r'$*

$$\begin{aligned} P(\textit{there exists } u, v, |u|, |v| \leq n(l) \textit{such that } X_{[a;b]}uX'_{[a';b']}v = 1 \mid \\ X_{[a+b-1;a]} = w, X'_{[a'+b'-1;a']} = w', |X| = r, |X'| = r') \\ \leq e^{\gamma_3 n(l)}.e^{-\lambda\frac{(br+b'r')}{2}}.e^{\epsilon\ell} \end{aligned}$$

3. *Case 3: If $a+b \leq 1, a'+b' > 1$ (1 wrap around) for any w of length ar, for any w' of length $(1-b)r'$*

$$P(\textit{there exists } u, v, |u|, |v| \leq n(l) \textit{such that } X_{[a;b]}uX_{[a';b']}v = 1 \mid X_{[0;a]} = w, X'_{[a'+b'-1;a']} = w', |X| = r, |X'| = r') \leq e^{\gamma_3 n(l)}.e^{-\lambda\frac{(br+b'r')}{2}}.e^{\epsilon\ell}$$

Let X be a random word from the frayed annulus $\Omega_{ann_\ell} = \{x | \frac{\ell}{2} < |x| < \ell, x$ a frayed geodesic$\}$, for all $a, a' \in [0,1], b, b' \in [\xi, 1]$ such that

1. *Case 4: $a \leq a+b \leq a' \leq a'+b' \leq 1$ (No wrap around in the same word) for any w of length ar, for any w' of length $[a'-(a+b)]r$*

$$P(\textit{there exists } u, v, |u|, |v| \leq n(l) \textit{such that } X_{[a;b]}uX_{[a';b']}v = 1 \mid X_{[0;a]} = w, X_{[a+b;a']} = w', |X| = r) \leq e^{\gamma_3 n(l)}.e^{-\lambda\frac{(br+b'r)}{2}}.e^{\epsilon\ell}$$

2. *Case 5: $a \leq a+b \leq a' \leq 1 < a'+b'$ (1 wrap around) for any w of length $[a-(a'+b'-1)]r$, for any w' of length $[a'-(a+b)]r$*

$$P(\textit{there exists } u, v, |u|, |v| \leq n(l) \textit{such that } X_{[a;b]}uX_{[a';b']}v = 1 \mid X_{[0;a'+b'-1]} = w, X_{[a+b;a']} = w', |X| = r) \leq e^{\gamma_3 n(l)}.e^{-\lambda\frac{(br+b'r)}{2}}.e^{\epsilon\ell}$$

Proof. Given n, ϵ, ξ , we will show that such an γ_3 and $M(\epsilon, \xi)$ exist. Pick $l > M$. Then let $r, r' \in (\frac{l}{2}, l)$, pick a, $a' \in [0,1]$, b, $b' \in [\xi, 1]$.

Case 1 : If $a+b$, $a'+b' < 1$, pick two words w, w' of lengths ar, $a'r'$. Then, the probabilities listed look like $P(A|B)$. Now,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$$

Here,

$$A = \{(x,x') \in \Omega_r \times \Omega_{r'} \mid \text{ there exists } u,v \text{ such that } x_{[a;b]}ux'_{[a';b']}v = 1\}$$

and

$$B = \{(x,x') \in \Omega_r \times \Omega_{r'} \mid x_{[0;a]} = w, x_{[0;a']} = w'\}.$$

Since w, w' arise as initial subwords, we have

$$C_1 e^{\lambda(1-a)r} \cdot C_1 e^{\lambda(1-a')r'} < |B|$$

by Lemma 3.7. Further, we claim the following.

$$|A \cap B| \leq (C_2 e^{\lambda(1-(a+b)r)})(C_2 e^{\lambda(1-(a'+b')r')})|\bar{A}|$$

where $\bar{A} = \{(x_1,x_2) \in \Omega_{br} \times \Omega_{b'r'} \mid$ there exists u,v such that $x_1ux_2 = v^{-1}\}$.

Indeed, define $f : A \cap B \to \bar{A}$ as $f(x,x') := (x_{[a;b]}, x_{[a',b']})$. Now, given $(x_1,x_2) \in \bar{A}$, consider $|f^{-1}((x_1,x_2))|$, there is only one choice of attaching w, w' but at most $|S((1-(a+b)r) \times S((1-(a'+b'))r')|$ ways of attaching the remaining subword to get back to an element in $A \cap B$.

Hence, by combining the above two inequalities we get,

$$P(A|B) \leq \frac{C_2^2}{C_1^2} e^{\lambda(-br)} e^{\lambda(-b'r')} |\bar{A}|$$

By the Lemma 3.17, we have:

$$\begin{aligned}
|\bar{A}| &\leq e^{\gamma_3 n(\ell)} \cdot e^{-\lambda(\frac{br+b'r'}{2})} \cdot e^{\epsilon(br+b'r')} \cdot |\Omega_{br} \times \Omega_{b'r'}| \\
&\leq e^{\gamma_3 n(\ell)} \cdot e^{-\lambda(\frac{br+b'r'}{2})} \cdot e^{\epsilon(br+b'r')} \cdot C_2 e^{\lambda(br)} \cdot C_2 e^{\lambda(b'r')}
\end{aligned}$$

Hence, we get

$$P(A|B) \leq e^{\gamma_3 n(\ell)} e^{-\lambda \frac{br+b'r'}{2}} e^{\epsilon \ell}$$

by combining the above two inequalities and choosing ℓ large enough to absorb the constants.

Case 2: If $a+b, a'+b' > 1$, pick two words w, w' of length $(1-b)r, (1-b')r'$. We have,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|}$$

where $A = \{(x,x') \in \Omega_r \times \Omega_{r'} \mid \text{ there exists } u, v \text{ such that } x_{[a;b]}ux'_{[a';b']}v = 1\}$ and $B = \{(x,x') \in \Omega_r \times \Omega_{r'} \mid x_{[a+b-1;a]} = w, x_{[a'+b'-1;a']} = w'\}$

Since w, w' arise as middle subwords of geodesics, we have

$$C_1^2 e^{\lambda(br)}.C_1^2 e^{\lambda(b'r')} < |B|$$

by 3.7.

Further, we claim,

$$|A\cap B| \leq |\bar{A}|$$

where, $\bar{A} = \{(x_1,x_2,x_3,x_4) \in \Omega_{(1-a)r} \times \Omega_{(a+b-1)r} \times \Omega_{(1-a')r'} \times \Omega_{(a'+b'-1)r'} \mid$ there exists u, v such that $x_1x_2ux_3x_4 = v^{-1}\}$

Indeed, define $f : A\cap B \to \bar{A}$ as follows

$$f(x,x') = (x_{[a;1-a]}, x_{[0;a+b-1]}, x_{[a';1-a']}, x_{[0;a'+b'-1]}).$$

We observe that f is injective: for any given $(x_1,x_2,x_3,x_4) \in \bar{A}$ there is at most one way of getting back an element of $A\cap B$ (which is by concatenating the pieces involved with w, w').

Combining the above two inequalities, we get

$$P(A|B) \leq \frac{|\bar{A}|}{C_1^2 e^{\lambda(br)}.C_1^2 e^{\lambda(b'r')}}$$

By piecewise geodesic lemma we have:

$$|\bar{A}| \leq e^{\gamma_3 n(\ell)} e^{-\lambda(\frac{br+b'r'}{2})} e^{\epsilon(br+b'r')} C_2^2 e^{\lambda(br)} e^{\lambda(b'r')}$$

Hence, we get

$$P(A|B) \leq e^{\gamma_3 n(\ell)} e^{-\lambda\frac{br+b'r'}{2}} e^{\epsilon\ell}$$

by combining the above two inequalities and choosing ℓ large enough to absorb the constants.

The proofs of *Case 3*, *Case 4* and *Case 5* will be similar.

□

3.6.4 Statement and proof of Axiom 4'

Theorem 3.21 (Axiom 4')**.** *There exists a constant $\gamma_{4'}$ such that, for any $n : \mathbb{R}^+ \to \mathbb{R}^+$, for any $C > 0$, for any $\epsilon > 0, \xi > 0$ there exists $M(\epsilon, \xi, C)$ such that for all $a \in [0;1], b \in [\xi, 1]$.*

We have for all $\ell > M$ and for all $\frac{\ell}{2} < r < \ell$

1. *Case 1 (No wrap around): If $a + b < 1$ for any w of length ar*

$$\begin{aligned} &P(\text{there exists } u, |u| \leq n(l) \text{ such that some cyclic permutation} \\ &x' \text{ of } X_{[a,b]}u \text{ satisfies } ||x'|| \leq C \log \ell \mid X_{[0;a]} = w, |X| = r) \\ &\leq e^{\gamma_{4'} n(l)}.e^{-\lambda \frac{(br)}{2}}.e^{\epsilon \ell} \end{aligned}$$

2. *Case 2 (wrap around) : If $a + b > 1$ for any w of length ar*

$$\begin{aligned} &P(\text{there exists } u, |u| \leq n(l) \text{ such that some cyclic permutation} \\ &x' \text{ of } X_{[a,b]}u \text{ satisfies } ||x'|| \leq C \log \ell \mid X_{[a+b-1;a]} = w, |X| = r) \\ &\leq e^{\gamma_{4'} n(l)}.e^{-\lambda \frac{(br)}{2}}.e^{\epsilon \ell} \end{aligned}$$

where X is a random variable on the frayed annulus $\Omega_{ann_\ell} = \{x | \frac{\ell}{2} < |x| < \ell, x \text{ is a frayed geodesic}\}$

Proof. Given C, n, ϵ, ξ. The claim is that $M(\epsilon, \xi, C)$ exists. Pick $l > M$. Then let $r \in (\frac{l}{2}, l)$ Pick $a \in [0, 1], b \in [\xi, 1]$

Case 1: If $a + b < 1$, pick a word w of length ar. We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$$

Here, $A = \{x \in \Omega_r \mid$ there exists u such that some cyclic permutation x' of $x_{[a,b]}u$ satisfies $||x'|| \leq C \log \ell\}$ and $B = \{x \in \Omega_r \mid x_{[0;a]} = w\}$.

Since w arises as an initial subword of a geodesic, we have

$$C_1 e^{\lambda(1-a)r} < |B|$$

by Lemma 3.7.

Further, we claim that

$$|A \cap B| \leq (\text{Number of ways to cut } x_{[a;b]})(C_2 e^{\lambda(1-a)r}))|\bar{A}|$$

where $\bar{A} = \{(x_1, x_2) \in \Omega_{b_1 r} \times \Omega_{b_2 r} \mid ||x_1 u x_2|| \leq C \log \ell\}$.

This is true since for every $x \in A \cap B$, there exists a piecewise geodesic formation such that $||x_{b_1 r} u x_{b_2 r}|| \leq C \log \ell$ with $x_{b_1 r} x_{b_2 r} = x_{[a;b]}$((with initial fixed piece w attached to $x_{b_1 r}$). Since there will be a free length of $(1-a)r$ there can be at max $C_2 e^{\lambda(1-a)r}$ length r geodesics with w as the initial subword that can result in such a piecewise geodesic formation.

Hence,

$$P(A|B) \leq \ell \frac{C_2}{C_1} e^{\lambda(-br)} |\bar{A}|$$

by combining the last two inequalities and noting that the maximum ways to cut is ℓ.

By the 3.17, we have:

$$\begin{aligned} |\bar{A}| &\leq e^{\gamma_{4'} n(\ell)} e^{-\frac{\lambda}{2} C \log \ell} e^{-\lambda(\frac{b_1 r + b_2 r}{2})} e^{\epsilon(b_1 r + b_2 r')} |\Omega_{b_1 r} \times \Omega_{b_1 r}| \\ &\leq e^{\gamma_{4'} n(\ell)} e^{-\frac{\lambda}{2} C \log \ell} e^{-\lambda(\frac{br}{2})} e^{\epsilon(br)} C_2^2 e^{\lambda(br)} \end{aligned}$$

Hence, we get

$$P(A|B) \leq e^{\gamma_{4'} n(\ell)} e^{-\lambda(\frac{br}{2})} e^{\epsilon \ell}$$

by combining the above two inequalities and choosing ℓ large enough to absorb the constants and polynomials.

Case 2 If $a + b > 1$ Pick a word w of length $(1-b)r$. We have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$$

Here, $A = \{x \in \Omega_r \mid$ there exists u such that some cyclic permutation x' of $x_{[a,b]} u$ satisfies $||x'|| \leq C \log \ell\}$ and $B = \{x \in \Omega_r \mid x_{[a+b-1;a]} = w\}$.

Since w arises as a middle subword we have

$$C_1^2 e^{\lambda(br)} < |B|$$

by Lemma 3.7. Further, we claim that

$$|A \cap B| \leq (\text{Number of ways to cut } x_{[a;b]})|\bar{A}|$$

where $\bar{A} = \{(x_1, x_2, \dots x_k) \in \Omega_{s_1} \times \Omega_{s_2} \cdots \Omega_{s_k} \mid ||x_1 u x_2|| \leq C \log \ell\}$ for some k and $\sum_{i=1}^{k} s_i = br$.

We proceed as before in Case 1 to get the desired inequality. □

Corollary 3.22. *By a theorem of ollivier the quotients are non-elementary*

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