August 2023

An Optimal Decay Estimation of the Solution to the Airy Equation

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by

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A Thesis Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Master of Science

in Mathematics

at

University of Wisconsin-Milwaukee

August 2023

ABSTRACT

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by
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The University of Wisconsin-Milwaukee, 2023
Under the Supervision of Dr. Lijing Sun of Mathematics

In this thesis, we investigate the initial value problem to the Airy equation

$$\partial_t u + \partial_x^3 u = 0 \tag{0.1}$$

$$u(0, x) = f(x). \tag{0.2}$$

We found the optimal decay estimate of the solution to the initial value problem $(1) - (2)$. The main tool we used is Fourier analysis and oscillatory integrals.

ACKNOWLEDGEMENTS

I'd like to recognize the University of Wisconsin - Milwaukee for providing a space for me to succeed and grow academically and personally.

This endeavor would not have been possible without the tremendous support of Dr. Lijing Sun. She provided the space for me to succeed, grow, and pushed me to challenge myself more than I ever have.

Lastly, I could not have undertaken this journey without the support of my family. A special thank you to Emily for keeping me sane, Edgar for keeping me positive, my mom for always believing me, and my grandparents who will never understand their impact on pushing me to achieve my dreams. To my grandma who won't see the completion of my degree, thank you for being my biggest cheerleader, the best shoulder to lean on, and the greatest role model I could have asked for.

1 Introduction

Dispersive partial differential equations have been the topic of intensive studies for centuries, starting from the classical physics and going widely beyond. These equations serve to model various vibrating media such as waves on string, in liquids or in air. The most familiar problems in which dispersive effects appear are the classical problems of water wave descriptions. One of the most famous equations is the Korteweg–de Vries (KdV) equation derived to model the propagation of low amplitude long water waves in a shallow canal. Nowadays many versions and higher order generalizations of the KdV equation are of use in different areas including hydrodynamics, plasma physics, electrodynamics, in studies of electromagnetic and acoustic waves, waves in elastic media, turbulence, traffic flows, mass transport and others. Another common dispersive partial differential equation (PDE) is the Airy equation. The Airy equation is the solution to the time-dependent Schrodinger's equation and has applications in many fields such as physics, quantum mechanics and probability. In the field of probability the Airy equation is connected to Chernoff's distribution. Some everyday applications of the Airy equation are electromagnetism, the propagation of light, and calculating the amplitude of polarized rainbows. One of the most common applications is in fluid dynamics but more specifically the study of waves or the movement of water. The behavior of waves and the amplitude of rainbows both require finding the asymptotic approximation of the Airy equation. [1] An approximation is needed because the function cannot be evaluated using traditional techniques. We wish to find the optimal approximation solution of the Airy equation.

In the present paper we consider the Airy equation

$$\partial_t u + \partial_x^3 u = 0 \quad (t,x) \in (0,\infty) \times \mathbb{R} \tag{1.1}$$

$$u(0,x) = f(x) \quad x \in \mathbb{R}. \tag{1.2}$$

Initial value problem (1.1)-(1.2) was treated in [2], namely, the asymptotic behavior was analyzed for the rescaled solutions under weakly dependent stationary initial data.

The purpose of the thesis is to find the best decay rate of the solution to the initial value problem $(1)-(2)$. Our result is

Theorem 1.1. *Assume that $u(t,x)$ is a solution to*

$$\partial_t u + \partial_x^3 u = 0 \quad (t,x) \in (0,\infty) \times \mathbb{R} \tag{1.3}$$

$$u(0,x) = f(x) \quad x \in \mathbb{R}, \tag{1.4}$$

and $f \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ we have

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\alpha(p)} \|f(x)\|_{L^p},$$

where $\alpha(p) = |\frac{1}{3} - \frac{2}{3p}|$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Notation: We will use $A \lesssim B$ or $A = O(B)$ to denote the estimate $A \le CB$ where $C > 0$ is some absolute constant. If the constant C depends on a list of parameters L we will write $A \lesssim_L B$ or $A = O_L(B)$.
We will also define the differential operator D by $Df(t) = \frac{1}{i\phi'(t)} \frac{df}{dt}$, such that $D^N f(t) = \frac{1}{i\phi'(t)} \frac{d^N f}{dt^N}$.

The thesis is organized as follows. In section 2, we provide some preliminaries. In section 3 we give the proof of the main results. In section 4, we give my future research plan.

2 Preliminaries

Definition 2.1. Let $f(x) \in L^1$. The Fourier Transform of a function $f(x)$ is defined by

$$F\{f(\omega)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$

The result is a function of ω or a frequency. $\hat{f}(\omega)$ gives how much power $f(x)$ contains at a frequency ω.[4]

Definition 2.2. The Inverse Fourier Transform is as follows.
$F^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega = f(x)$.

The properties of the Fourier transform.

1. $F\left[\frac{d^n}{dx^n}f(x)\right] = (i\omega)^n\hat{f}(\omega)$.

2. A convolution of two functions is defined by

$$f(x) * h(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau.$$

Then, the Fourier Transform of the convolution is $F\{f(x) * h(x)\} = \hat{f}(\omega)\hat{h}(\omega)$.

3. Plancherel Theorem. If $f \in L^1 \cap L^2$; then $\hat{f} \in L^2$; and $\mathscr{F}|\left(L^1 \cap L^2\right)$ extends uniquely to a unitary isomorphism on L^2. [3]

We also need the properties of the oscillatory integral defined by

$$I(\lambda) = \int_a^b e^{i\lambda\phi(\omega)}\psi(\omega)d\omega.$$

Theorem 2.3. *Principle of non-stationary phase. Let $\psi \in C^\infty(\mathbb{R})$ have compact support in (a,b) (so that in particular $\psi(a) = \psi(b) = 0$) and suppose that the phase $\phi \in C^\infty$ satisfies $\phi'(t) \neq 0$ for all $t \in [a,b]$. We claim that in this case the integral $I_\psi(\lambda)$ decreases very fast in λ, in particular*

Proposition 2.4. *(Principle of non-stationary phase). Let ψ, ϕ be as above, that is $\psi \in C_c^\infty((a,b))$ and $\phi \in C^\infty$ is such that $\phi' = 0$ on all of $[a,b]$. Then for every, $N > 0$ we have $|I_\psi(\lambda)| \lesssim_{N,\psi,\phi} |\lambda|^{-N}$.*

Corollary 2.5. *Let $\psi \in C^1$ and let the phase ϕ satisfy $|\phi^{(k)}| > 1$ in (a,b) for some $k \geq 1$ (if $k = 1$, assume additionally that ϕ' is monotonic). Then the inequality*
$|I_\psi(\lambda)| \leq C'\left[|\psi(b)| + \int_a^b |\psi'(t)|dt\right]|\lambda|^{-\frac{1}{k}}$ holds, with $C'_k > 0$ an absolute constant depending only on k.

Theorem 2.6. *The Riesz - Thorin Interpolation Theorem. Suppose that* (X, M, μ) *and* (Y, N, ν) *are measure spaces and* $p_0, p_1, q_0, q_1 \in [1, \infty]$. *If* $q_0 = q_1 = \infty$, *suppose also that* ν *is semifinite. For* $0 < n < 1$, *define* p_n *and* q_n *by* $\frac{1}{p_n} = \frac{1-n}{p_0} + \frac{n}{p_1}$ *and* $\frac{1}{q_n} = \frac{1-n}{q_0} + \frac{n}{q_1}$.
If T *is a linear map from* $L^{p_0}(\mu) + L^{p_1}(\mu)$ *into* $L^{q_0}(\nu) + L^{q_1}(\nu)$ *such that* $||Tf||_{q_0} \leq M_0||f||_{p_0}$ *for* $f \in L^{p_0}(\mu)$ *and* $||Tf||_{q_1} \leq M_1||f||_{p_1}$ *for* $f \in L^{p_1}(\mu)$, *then* $||Tf||_{q_n} \leq M_0^{1-n}M_1^n||f||_{p_n}$ *for* $f \in L^{p_n}(\mu), 0 < n < 1$. *[3]*

3 Proof of Main Theorem

The Airy equation is the dispersive partial differential equation,

$$\partial_t u + \partial_x^3 u = 0 \quad (t, x) \in (0, \infty) \times \mathbb{R} \tag{3.1}$$
$$u(0, x) = f(x) \quad x \in \mathbb{R}. \tag{3.2}$$

Solving the initial value problem we have the following,

$$\hat{f}(u_t) = -\hat{f}(u_{xxx}) \tag{3.3}$$
$$\frac{d}{dt}\hat{u} = i\omega^3\hat{u} \tag{3.4}$$
$$\frac{d}{dt}\hat{u}(t, \omega) = i\omega^3\hat{u}(t, \omega) \tag{3.5}$$
$$\frac{d\hat{u}(t, \omega)}{\hat{u}(t, \omega)} = i\omega^3 dt \tag{3.6}$$
$$\int_0^t \frac{1}{\hat{u}(t, \omega)} d\hat{u}(t, \omega) = \int_0^t i\omega^3 dt \tag{3.7}$$
$$ln(\hat{u}(t, \omega))|_0^t = i\omega^3 t|_0^t \tag{3.8}$$
$$ln(\hat{u}(t, \omega)) - ln(\hat{u}(0, \omega)) = i\omega^3 t \tag{3.9}$$
$$e^{ln(\hat{u}(t,\omega))} = e^{ln(\hat{u}(0,\omega))}e^{i\omega^3 t} \tag{3.10}$$
$$\hat{u}(t, \omega) = e^{i\omega^3 t}(\hat{u}(0, \omega)) \tag{3.11}$$
$$\hat{u}(t, \omega) = e^{i\omega^3 t}\hat{f}(\omega) \tag{3.12}$$

Now apply the inverse Fourier transform,
$$u(t, x) = \int e^{i\omega^3 t}\hat{f}(\omega)e^{i\omega x} d\omega \tag{3.13}$$
$$= \int e^{i\omega^3 t + i\omega x}\hat{f}(\omega) d\omega \tag{3.14}$$
$$= \int e^{i(\omega^3 t + \omega x)}\hat{f}(\omega) d\omega \tag{3.15}$$

By taking a Fourier transform in the spatial variable x, it is not hard to see that the solution to the Airy equation can be written formally as the convolution

$$u(t, x) = \frac{1}{2\pi} \int f(y) \frac{1}{t^{\frac{1}{3}}} Ai\left(\frac{x-y}{t^{\frac{1}{3}}}\right) dy,$$

where $Ai(x)$ denotes the Airy function

$$Ai(x) := \int_{\mathbb{R}} e^{i\left(\omega^3 + x\omega\right)} d\omega,$$

which is clearly an oscillatory integral (provided the integral exists!). Hence, integration by parts will not be sufficient to estimate the solution.

Proposition 3.1. *For $x > 1$ we have superpolynomial decay, that is for every $N > 0$ we have $|Ai(x)| \lesssim |x|^{-N}$.*

Proof. We wish to evaluate $Ai(x) := \int_{\mathbb{R}} e^{i\left(\omega^3 + x\omega\right)} d\omega$, since we are unable to evaluate this integral on its own, we will make a substitution to help.
Let

$$\varphi(\omega) + \sum_{j\in\mathbb{N}} \psi_j(\omega) = 1.$$

Where φ is a C^∞ bump function supported in [-2,2], with $\varphi \equiv 1$ on [-1,1].
Define $\psi(\omega) = \varphi(\frac{\omega}{2}) - \varphi(\omega)$ and $\psi_j(\omega) = \psi(2^{-j}\omega)$. Now to evaluate the integral we will take the limit as $n \to \infty$ of

$$\int e^{i(\omega^3 + x\omega)} \varphi(\omega) d\omega + \sum_{j=0}^{n} \int e^{i(\omega^3 + x\omega)} \psi_j(\omega) d\omega$$

.
First, we will focus on $\int e^{i(\omega^3 + x\omega)} \varphi(\omega) d\omega$. Let $\lambda = x^{\frac{3}{2}}$ and $\phi(\omega) = \omega^3 + \omega$.

$$\int e^{i(\omega^3 + x\omega)} \varphi(\omega) d\omega = \int x^{\frac{1}{2}} e^{i(x^{\frac{3}{2}} \phi(\omega))} \varphi(\omega) d\omega \tag{3.16}$$

$$= x^{\frac{1}{2}} \left[\frac{-ie^{i\lambda\phi(\omega)} \varphi(\omega)}{\lambda \phi'(\omega)} - \int \frac{1}{\lambda} e^{i\lambda\phi(\omega)} \left(\frac{\varphi'(\omega)}{\phi'(\omega)} \right) + C d\omega \right] \tag{3.17}$$

$$= x^{\frac{1}{2}} \left(\frac{1}{x^{\frac{3}{2}}} \right) \int \left| e^{i\lambda\phi(\omega)} \frac{\varphi'(\omega)}{\phi'(\omega)} + C d\omega \right| \tag{3.18}$$

$$= \frac{1}{x} \int \left| e^{i\lambda\phi(\omega)} \frac{\varphi'(\omega)}{\phi'(\omega)} + C d\omega \right| \tag{3.19}$$

$$= \frac{1}{x} \left[\frac{\varphi'(\omega)}{\phi'(\omega)} \left(\frac{-ie^{i\lambda\phi(\omega)}}{\lambda\phi'(\omega)} \right) - \int \left| \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} D^1 \left(\frac{\varphi'(\omega)}{\phi'(\omega)} \right) + C d\omega \right| \right] \tag{3.20}$$

$$= \frac{1}{x} \left(\frac{1}{x^{\frac{3}{2}}} \right) \int \left| \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} D^1 \left(\frac{\varphi'(\omega)}{\phi'(\omega)} \right) d\omega \right| \tag{3.21}$$

$$\vdots \tag{3.22}$$

$$\text{WLOG, apply this N times} \tag{3.23}$$

$$= \frac{1}{x} \left(\frac{1}{x^{\frac{3}{2}}} \right)^N \int_a^b \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} D^N \left(\frac{\varphi'(\omega)}{\phi'(\omega)} \right) d\omega \tag{3.24}$$

$$\lesssim x^{-\frac{3}{2}N} \tag{3.25}$$

We see that $\phi^{(N)}(\omega)$ is always nonzero and it is a continuous function. Since $\int_a^b \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} D^N \left(\frac{\psi'(\omega)}{\phi'(\omega)} \right)$

is continuous on a compact domain we know that it is bounded.
Next we will look at the second part, $\sum_{j=0}^{n}\int e^{i(\omega^3+x\omega)}\psi_j(\omega)d\omega$. First let's take a look at the domains of $\psi_0(\omega)$ and $\psi_1(\omega)$ so that we can extend it to $\psi_j(\omega)$.

$$\psi_0(\omega) = \begin{cases} 0 & |\omega| \leq 1 \\ \varphi(\frac{\omega}{2}) - \varphi(\omega) & 1 \leq |\omega| \leq 4 \\ 0 & |\omega| \geq 4 \end{cases}$$

$$\psi_1(\omega) = \begin{cases} 0 & |\omega| \leq 2 \\ \varphi(\frac{\omega}{2}) - \varphi(\omega) & 2 \leq |\omega| \leq 8 \\ 0 & |\omega| \geq 8 \end{cases}$$

$$\vdots$$

$$\psi_j(\omega) = \begin{cases} 0 & |\omega| \leq 2^j \\ \varphi(\frac{\omega}{2^{j+1}}) - \varphi(\frac{\omega}{2^j}) & 2^j \leq |\omega| \leq 2^{j+2} \\ 0 & |\omega| \geq 2^{j+2} \end{cases}$$

We will start by evaluating from $\psi_0(\omega)$ and extend to $\psi_j(\omega)$.

$$\int_1^4 x^{\frac{1}{2}} e^{i\lambda\phi(\omega)}\psi_0(\omega)d\omega = x^{\frac{1}{2}}\int_1^4 e^{i\lambda\phi(\omega)}\psi_0(\omega)d\omega \tag{3.26}$$

$$= x^{\frac{1}{2}}\left[\frac{-ie^{i\lambda\phi(\omega)}\psi_0(\omega)}{\lambda\phi'(\omega)}|_1^4 - \int_1^4 \frac{1}{\lambda} ie^{i\lambda\phi(\omega)}\left(\frac{\psi_0'(\omega)}{\phi'(\omega)}\right)d\omega\right] \tag{3.27}$$

$$= x^{\frac{1}{2}}\left(\frac{1}{x^{\frac{3}{2}}}\right)\int_1^4 ie^{i\lambda\phi(\omega)}\frac{\psi_0'(\omega)}{\phi'(\omega)}d\omega \tag{3.28}$$

$$= \frac{1}{x}\int_1^4 \left|e^{i\lambda\phi(\omega)}\frac{\psi_0'(\omega)}{\phi'(\omega)}d\omega\right| \tag{3.29}$$

$$= \frac{1}{x}\int_1^4 \left|\frac{1}{\lambda}e^{i\lambda\phi(\omega)}D^1\left(\frac{\psi_0'(\omega)}{\phi'(\omega)}\right)d\omega\right| \tag{3.30}$$

$$= \frac{1}{x}\left(\frac{1}{x^{\frac{3}{2}}}\right)\int_1^4 \left|e^{i\lambda\phi(\omega)}D^1\left(\frac{\psi_0'(\omega)}{\phi'(\omega)}\right)d\omega\right| \tag{3.31}$$

$$\vdots \tag{3.32}$$

WLOG, apply this N times (3.33)

$$= \frac{1}{x}\left(\frac{1}{x^{\frac{3}{2}}}\right)^N\int_1^4 \left|e^{i\lambda\phi(\omega)}D^N\left(\frac{\psi_0'(\omega)}{\phi'(\omega)}\right)d\omega\right| \tag{3.34}$$

$$\lesssim x^{-\frac{3}{2}N} \tag{3.35}$$

$$\int_2^8 x^{\frac{1}{2}} e^{i\lambda\phi(\omega)}\psi_1(\omega)d\omega = x^{\frac{1}{2}}\int_2^8 e^{i\lambda\phi(\omega)}\psi_1(\omega)d\omega \tag{3.36}$$

$$= x^{\frac{1}{2}} \left[\frac{-ie^{i\lambda\phi(\omega)}\psi_1(\omega)}{\lambda\phi'(\omega)}|_2^8 - \int_2^8 \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} \left(\frac{\psi_1'(\omega)}{\phi'(\omega)}\right) d\omega \right] \tag{3.37}$$

$$= x^{\frac{1}{2}} \left(\frac{1}{x^{\frac{3}{2}}}\right) \int_2^8 ie^{i\lambda\phi(\omega)} \frac{\psi_1'(\omega)}{\phi'(\omega)} d\omega \tag{3.38}$$

$$= \frac{1}{x} \int_2^8 \left| e^{i\lambda\phi(\omega)} \frac{\psi_1'(\omega)}{\phi'(\omega)} d\omega \right| \tag{3.39}$$

$$= \frac{1}{x} \int_2^8 \left| \frac{1}{\lambda} e^{i\lambda\phi(\omega)} D^1 \left(\frac{\psi_1'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.40}$$

$$= \frac{1}{x} \left(\frac{1}{x^{\frac{3}{2}}}\right) \int_2^8 \left| e^{i\lambda\phi(\omega)} D^1 \left(\frac{\psi_1'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.41}$$

$$\vdots \tag{3.42}$$

WLOG, apply this N times (3.43)

$$= \frac{1}{x} \left(\frac{1}{x^{\frac{3}{2}}}\right)^N \int_2^8 \left| e^{i\lambda\phi(\omega)} D^N \left(\frac{\psi_1'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.44}$$

$$\lesssim x^{-\frac{3}{2}N} \tag{3.45}$$

$$\int_{2^j}^{2^{j+2}} x^{\frac{1}{2}} e^{i\lambda\phi(\omega)} \psi_j(\omega) d\omega = x^{\frac{1}{2}} \int_{2^j}^{2^{j+2}} x^{\frac{1}{2}} e^{i\lambda\phi(\omega)} \psi_j(\omega) d\omega \tag{3.46}$$

$$= x^{\frac{1}{2}} \left[\frac{-ie^{i\lambda\phi(\omega)}\psi_j(\omega)}{\lambda\phi'(\omega)}|_{2^j}^{2^{j+2}} + \int_{2^j}^{2^{j+2}} \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right] \tag{3.47}$$

$$= x^{\frac{1}{2}} \left(\frac{1}{x^{\frac{3}{2}}}\right) \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.48}$$

$$= \frac{1}{x} \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.49}$$

$$= \frac{1}{x} \int_{2^j}^{2^{j+2}} \left| \frac{1}{\lambda} ie^{i\lambda\phi(\omega)} D^1 \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right|, \psi'(\omega) = 2^{-j}\psi(2^{-j}\omega) \tag{3.50}$$

$$= \frac{1}{x} \left(\frac{1}{x^{\frac{3}{2}}}\right)^N \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} D^N \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.51}$$

$$\vdots \tag{3.52}$$

WLOG, apply this N times (3.53)

$$= x^{-\frac{3}{2}N} \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} D^N \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \right| \tag{3.54}$$

$$= x^{-\frac{3}{2}N} \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} \left(\frac{D^N(\psi_j'(\omega))}{D^N(\phi'(\omega))}\right) d\omega \right| \tag{3.55}$$

$$= x^{-\frac{3}{2}N} \int_{2^j}^{2^{j+2}} \left| ie^{i\lambda\phi(\omega)} \left(\frac{(2^{-jk})\psi_j^{(k)}\left(2^{-j}\omega\right)\omega}{D^N(\phi'(\omega))} \right) \right| d\omega \tag{3.56}$$

$$= x^{-\frac{3}{2}N} \int_{2^j}^{2^{j+2}} \left| \left(\frac{e^{i\lambda\phi(\omega)}\omega}{D^N(\phi'(\omega))} \right) \sum_{k=0}^{N} (2^{-jk})\psi_j^{(k)}(2^{-j}\omega) d\omega \right| \tag{3.57}$$

$$\lesssim x^{-\frac{3}{2}N} \int_{2^j}^{2^{j+2}} \left| \frac{e^{i\lambda\phi(\omega)}}{D^N(\phi'(\omega))}(1) d\omega \right| \tag{3.58}$$

We know $\sum_{k=0}^{N}(2^{-jk})\psi_j^{(k)}(2^{-j}\omega) \lesssim 1$ since 2^{-j} is a geometric series that converges to 1 and $\psi_j^{(k)}(2^{-j}\omega)$ is bounded because it is continuous on a compact domain.
Now apply Proposition 2.4 and we have

$$|I_\psi(\lambda)| \lesssim_{N,\psi,\phi} |\lambda|^{-N} \lesssim_{N,\psi,\phi} |x|^{-\frac{3}{2}N}.$$

Then for $x > 1$, we have $|I_\psi(\omega)| \lesssim \min(1,|x|^{-\frac{3}{2}N}) \implies \int e^{i\lambda\phi(\omega)}\varphi(\omega)d\omega \lesssim \min(1,|x|^{-\frac{3}{2}N})$. □

Proposition 3.2. *For $-1 \leq x \leq 1$ we have $|Ai(x)| \lesssim 1$.*

Proof. We wish to show that for $-1 \leq x \leq 1$ we have

$$|Ai(x)| \lesssim 1.$$

To do this we will follow a similar argument to the previous part. We need to evaluate

$$Ai(x) := \int_{\mathbb{R}} e^{i(\omega^3+\omega x)} d\omega.$$

Since we are unable to evaluate this integral on it's own, we will make a substitution to help.
Let

$$\varphi(\omega) + \sum_{j\in\mathbb{N}} \psi_j(\omega) = 1.$$

Where φ is a C^∞ bump function supported in [-2,2], with $\varphi \equiv 1$ on [-1,1]. Define

$$\psi(\omega) = \varphi(\frac{\omega}{2}) - \varphi(\omega)$$

and

$$\psi_j(\omega) = \psi(2^{-j}\omega).$$

Now to evaluate the integral we will take the limit as $n \to \infty$ of

$$\int e^{i(\omega^3+x\omega)}\varphi(\omega)d\omega + \sum_{j=0}^{n} \int e^{i(\omega^3+x\omega)}\psi_j(\omega)d\omega.$$

First, we will focus on

$$\int e^{i(\omega^3+x\omega)}\varphi(\omega)d\omega.$$

Note: we will not be making a substitution for λ but

$$\phi(\omega)=\omega^3+x\omega$$

which is different from the last part.

Since, $\phi(\omega)=\omega^3+x\omega$ we have $\phi'(\omega)=3\omega^2+x$ hence a critical point exists at $\omega=-\sqrt{\frac{|x|}{3}}$ for $-1\le x\le 1$. Since we have a critical point, when we evaluate the integral we cannot use integration by parts, however we know that $e^{i(\omega^3+x\omega)}$ is bounded and so is $\varphi(\omega)$. Hence, we have the following,

$$\int_{-2}^{2}\left|e^{i(\omega^3+x\omega)}\varphi(\omega)\right|d\omega=\int_{-2}^{2}\left|e^{i(\omega^3+x\omega)}\right||\varphi(\omega)|\,d\omega \tag{3.59}$$

$$\lesssim 1 \tag{3.60}$$

Therefore, $\left|e^{i(\omega^3+x\omega)}\right|<1$ and $|\varphi(\omega)|$ is bounded by some constant since it is continuous on a compact set. Differentiating N times we have,

$$D^N\left(\frac{\varphi''(\omega)}{\phi'(\omega)}\right)=D^N\left(\frac{\varphi''(\omega)}{3\omega^2+x}\right)$$

where $\left|D^N\left(\frac{\varphi''(\omega)}{3\omega^2+x}\right)\right|$ is continuous on [-2,2] when $-1\le x\le 1$ hence it is bounded by some constant. Therefore we have,

$$\int_{-2}^{2}\left|e^{i\phi(\omega)}D^N\left(\frac{\varphi''(\omega)}{\phi'(\omega)}\right)\right|d\omega\lesssim 1.$$

Next we will look at the second part,$\sum_{j=0}^{n}\int e^{i(\omega^3+x\omega)}\psi_j(\omega)d\omega$. By the previous part, we can evaluate directly from the $j^{th}\psi$ function. We have the following,

$$\sum\int_{2^j}^{2^{j+2}}e^{i\phi(\omega)}\psi_j(\omega)d\omega=\frac{-ie^{i\phi(\omega)}\psi_j(\omega)}{\phi'(\omega)}|_{2^j}^{2^{j+2}}-\int_{2^j}^{2^{j+2}}ie^{i\phi(\omega)}\frac{\psi_j'(\omega)}{\phi'(\omega)}d\omega \tag{3.61}$$

$$=0-\int_{2^j}^{2^{j+2}}ie^{i\phi(\omega)}\frac{\psi_j'(\omega)}{\phi'(\omega)}d\omega \tag{3.62}$$

$$\vdots \tag{3.63}$$

$$\text{WLOG, applying this N times}=\int_{2^j}^{2^{j+2}}e^{i\phi(\omega)}D^N\left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right)d\omega \tag{3.64}$$

$$=\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}\left(\left(2^{-j}\right)^k\psi_j^{(k)}(2^{-j}\omega)\omega\right)d\omega \tag{3.65}$$

$$=\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}\sum_{k=0}^{N}\left(2^{-j}\right)^k\left|\psi_j^{(k)}(2^{-j}\omega)\right|d\omega \tag{3.66}$$

Since, $\sum_{k=0}^{N}\left(2^{-j}\right)^k\left|\psi_j^{(k)}(2^{-j}\omega)\right|\lesssim 1$, then
$\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}\sum_{k=0}^{N}\left(2^{-j}\right)^k\left|\psi_j^{(k)}(2^{-j}\omega)\right|d\omega\lesssim\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}(1)d\omega.$
Above we noted that $\left|e^{i(\omega^3+x\omega)}\right|<1$. We also have $\frac{1}{D^N(\phi'(\omega))}$ which is continuous on $[2^j,2^{j+2}]$ when $-1\leq x\leq 1$ hence it is bounded by some constant.
Thus, we have

$$|Ai(x)|=\int e^{i(\omega^3+x\omega)}\varphi(\omega)d\omega+\sum_{j=0}^{n}\int e^{i(\omega^3+x\omega)}\psi_j(\omega)d\omega \tag{3.67}$$

$$=\int_{-2}^{2}\left|e^{i\phi(\omega)}D^N\left(\frac{\varphi''(\omega)}{\phi'(\omega)}\right)\right|d\omega \tag{3.68}$$

$$+\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}\sum_{k=0}^{N}\left(2^{-j}\right)^k\left|\psi_j^{(k)}(2^{-j}\omega)\right|d\omega \tag{3.69}$$

$$\lesssim\int_{-2}^{2}\left|e^{i\phi(\omega)}D^N\left(\frac{\varphi''(\omega)}{\phi'(\omega)}\right)\right|d\omega+\int_{2^j}^{2^{j+2}}\frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))}d\omega \tag{3.70}$$

Let C $\in\mathbb{R}$ be some constant. $\lesssim 1*C+1*C$ (3.71)

$$\lesssim 1 \tag{3.72}$$

□

Proposition 3.3. *For $x<-1$ we have $|Ai(x)|\lesssim|x|^{-\frac{1}{4}}$.*

Proof. We wish to show that for $x<-1$ we have

$$|Ai(x)|\lesssim|x|^{-\frac{1}{4}}.$$

To do this we will follow a similar argument to the previous part. We need to evaluate

$$Ai(x):=\int_{\mathbb{R}}e^{i(\omega^3+\omega x)}d\omega.$$

Since we are unable to evaluate this integral on it's own, we will make a substitution to help.
Let

$$\varphi(\omega)+\sum_{j\in\mathbb{N}}\psi_j(\omega)=1.$$

Where φ is a C^∞ bump function supported in [-2,2], with $\varphi\equiv 1$ on [-1,1]. Define

$$\psi(\omega)=\varphi(\frac{\omega}{2})-\varphi(\omega)$$

and

$$\psi_j(\omega)=\psi(2^{-j}\omega).$$

Now to evaluate the integral we will take the limit as $n \to \infty$ of

$$\int e^{i(\omega^3+x\omega)}\varphi(\omega)d\omega + \sum_{j=0}^{n} \int e^{i(\omega^3+x\omega)}\psi_j(\omega)d\omega.$$

First we will take a look at $\int e^{i(\omega^3+x\omega)}\varphi(\omega)d\omega$ but since $x < -1$ we need to take the absolute value of x so we have $\int e^{i(\omega^3+-|x|\omega)}\varphi(\omega)d\omega$. Then let $\lambda = |x|^{\frac{3}{2}}$ such that $\phi(\omega) = \omega^3 - \omega$. We see that $\phi'(\omega) = 3\omega - 1$ so then critical points exists when $\omega = \pm\frac{1}{\sqrt{3}}$. Therefore we need to evaluate the integral on an interval that does not include $\omega = \pm\frac{1}{\sqrt{3}}$. We will also split $\varphi(\omega)$ such that

$$\varphi(\omega) = \varphi(100\omega) + \varphi(\omega) - \varphi(100\omega).$$

$$\begin{aligned}
\int e^{i(\omega^3+-|x|\omega)}\varphi(\omega)d\omega &= \int_{-2}^{2} |x|^{\frac{1}{2}} e^{i\lambda\phi(\omega)}\varphi(\omega)d\omega && (3.73)\\
&= \int_{-2}^{2} |x|^{\frac{1}{2}} e^{i\lambda\phi(\omega)}(\varphi(100\omega) + \varphi(\omega) - \varphi(100\omega))d\omega && (3.74)\\
&= \int_{-2}^{2} |x|^{\frac{1}{2}} e^{i\lambda\phi(\omega)}\varphi(100\omega)d\omega + \int_{-2}^{2} |x|^{\frac{1}{2}} e^{i\lambda\phi(\omega)}\varphi(\omega) - \varphi(100\omega)d\omega && (3.75)\\
&= |x|^{\frac{1}{2}} \left(\int_{-2}^{2} e^{i\lambda\phi(\omega)}\varphi(100\omega)d\omega + \int_{-2}^{-\frac{1}{50}} e^{i\lambda\phi(\omega)}\varphi(\omega) - \varphi(100\omega)d\omega \right) && (3.76)\\
&+ |x|^{\frac{1}{2}} \left(\int_{\frac{1}{50}}^{2} e^{i\lambda\phi(\omega)}\varphi(\omega) - \varphi(100\omega)d\omega \right) && (3.77)
\end{aligned}$$

Since, no critical points lie in the following integral, $|x|^{\frac{1}{2}} \int_{-2}^{2} e^{i\lambda\phi(\omega)}\varphi(100\omega)d\omega$ we can use the argument shown in the proof of Proposition 4.2 and conclude that

$$|x|^{\frac{1}{2}} \int_{-2}^{2} e^{i\lambda\phi(\omega)}\varphi(100\omega)d\omega \lesssim 1.$$

However, both $\int_{-2}^{-\frac{1}{50}} e^{i\lambda\phi(\omega)}\varphi(\omega) - \varphi(100\omega)d\omega$ and $\int_{\frac{1}{50}}^{2} e^{i\lambda\phi(\omega)}\varphi(\omega) - \varphi(100\omega)d\omega$ have critical points so we cannot preform integration by parts and must use a corollary.
We apply Corollary 3.5 to each integral, we have

$$|I_\varphi(\lambda)| \leq C_k' \left[|\varphi\left(-\frac{1}{50}\right)| + \int_{-2}^{-\frac{1}{50}} |\varphi'(\omega)|d\omega \right] |\lambda|^{-\frac{1}{k}}.$$

We have $\phi''(\omega) = 3 > 1$, such that $k = 2$. Since $\varphi'(\omega)$ is bounded by some constant we have

$$|I_\varphi(\lambda)| \leq C_k' \left[|\varphi\left(-\frac{1}{50}\right)| + \int_{-2}^{-\frac{1}{50}} |\varphi'(\omega)|d\omega \right] |\lambda|^{-\frac{1}{k}} \qquad (3.78)$$

$$\leq C|\lambda|^{-\frac{1}{2}} \tag{3.79}$$

$$\leq C\left(|x|^{\frac{3}{2}}\right)^{-\frac{1}{2}} \tag{3.80}$$

$$\leq C|x|^{-\frac{3}{4}} \tag{3.81}$$

The for the other integral we have

$$|I_\varphi(\lambda)| \leq C_k' \left[|\varphi(2)| + \int_{\frac{1}{50}}^{2} |\varphi'(\omega)| d\omega\right] |\lambda|^{-\frac{1}{k}}.$$

We have $\phi''(\omega) = 3 > 1$, such that $k = 2$. Since $\varphi'(\omega)$ is bounded by some constant we have

$$|I_\varphi(\lambda)| \leq C_k' \left[|\varphi(2)| + \int_{-\frac{1}{50}}^{2} |\varphi'(\omega)| d\omega\right] |\lambda|^{-\frac{1}{k}} \tag{3.82}$$

$$\leq C|\lambda|^{-\frac{1}{2}} \tag{3.83}$$

$$\leq C\left(|x|^{\frac{3}{2}}\right)^{-\frac{1}{2}} \tag{3.84}$$

$$\leq C|x|^{-\frac{3}{4}} \tag{3.85}$$

Then we have the following,

$$\int e^{i(\omega^3 + -|x|\omega)} \varphi(\omega) d\omega = \int |x|^{\frac{1}{2}} e^{i\lambda\phi(\omega)} \varphi(\omega) d\omega \tag{3.86}$$

$$\leq |x|^{\frac{1}{2}} \left(C|x|^{-\frac{3}{4}} + C|x|^{-\frac{3}{4}}\right) \tag{3.87}$$

$$\lesssim |x|^{-\frac{1}{4}} \tag{3.88}$$

Now we will take a look at

$$\sum_{j=0}^{n} \int e^{i(\omega^3 - |x|\omega)} \psi_j(\omega) d\omega.$$

The argument will follow as in the proof of proposition 4.2, since we the two critical points do not lie in the domain $[2^j, 2^{j+2}]$.

$$\sum \int_{2^j}^{2^{j+2}} e^{i\phi(\omega)} \psi_j(\omega) d\omega = \frac{-ie^{i\phi(\omega)} \psi_j(\omega)}{\phi'(\omega)} |_{2^j}^{2^{j+2}} - \int_{2^j}^{2^{j+2}} ie^{i\phi(\omega)} \frac{\psi_j'(\omega)}{\phi'(\omega)} d\omega \tag{3.89}$$

$$= 0 - \int_{2^j}^{2^{j+2}} ie^{i\phi(\omega)} \frac{\psi_j'(\omega)}{\phi'(\omega)} d\omega \tag{3.90}$$

$$\vdots \tag{3.91}$$

$$\text{WLOG, applying this N times} = \int_{2^j}^{2^{j+2}} e^{i\phi(\omega)} D^N \left(\frac{\psi_j'(\omega)}{\phi'(\omega)}\right) d\omega \tag{3.92}$$

$$= \int_{2^j)}^{2^{j+2}} \frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))} \left(\left(2^{-j}\right)^k \psi_j^{(k)}(2^{-j}\omega)\omega \right) d\omega \tag{3.93}$$

$$= \int_{2^j}^{2^{j+2}} \frac{e^{i\phi(\omega)}}{D^N(\phi'(\omega))} \sum_{k=0}^{N} \left(2^{-j}\right)^k \left|\psi_j^{(k)}(2^{-j}\omega)\right| d\omega \tag{3.94}$$

Since, $\sum_{k=0}^N \left(2^{-j}\right)^k \left|\psi_j^{(k)}(2^{-j}\omega)\right| \lesssim 1$, then for $x < -1$ we have $\left|e^{i(\omega^3+x\omega)}\right| < 1$. We also have $\frac{1}{D^N(\phi'(\omega))}$ which is continuous on $[2^j, 2^{j+2}]$ when $x < -1$ hence it is bounded by some constant and contributes to $O_N\left(|x|^{-\frac{3}{2}N}\right)$ so it does not affect our estimate.

Therefore we have $|Ai(x)| \lesssim |x|^{-\frac{1}{4}}$, as desired. □

Combine the proposition 3.1, proposition 3.2 and proposition 3.3, we can get the following lemma:

Lemma 3.4. $||u(\cdot,t)||_{L^\infty(\mathbb{R},dx)} \lesssim |t|^{-\frac{1}{3}}||f||_{L^1(\mathbb{R})}$

Proof.

$$||u(t,x)||_{L^\infty} \leq \text{esssup}\frac{1}{2\pi}\int f(y)\frac{1}{t^{\frac{1}{3}}}Ai\left(\frac{x-y}{t^{\frac{1}{3}}}\right)dy \tag{3.95}$$

$$\leq \frac{1}{t^{\frac{1}{3}}}\left(\frac{1}{2\pi}\right)\int \left|f(y)Ai\left(\frac{x-y}{t^{\frac{1}{3}}}\right)\right|dy \tag{3.96}$$

$$\lesssim t^{-\frac{1}{3}}||f||_{L^1(\mathbb{R})} \tag{3.97}$$

□

When we apply Plancherel's theorem, we also have the following lemma:

Lemma 3.5.

$$||u(t,x)||_{L^2} \lesssim ||f(x)||_{L^2}.$$

Now to find the L^p estimate which is the optimal estimate, we will apply the Reisz-Thorin Theorem. $p_0 = q_0 = 2, p_1 = 1, q_1 = \infty, M_0 = 1$ and $M_1 = t^{-\frac{1}{3}}$. $n = -1 + \frac{2}{p}$ for $1 \leq p \leq 2$.
We have the following,

$$||u(\cdot,t)||_{L^q} \leq ||e^{t\partial_x^3}u||_{L_p} \tag{3.98}$$

$$\leq 1^{1-n}\left(t^{-\frac{1}{3}}\right)^n ||f||_{L_p} \tag{3.99}$$

$$\leq \left(t^{-\frac{1}{3}}\right)^{\left(-1+\frac{2}{p}\right)} ||f||_{L^p} \tag{3.100}$$

$$\lesssim \left(t^{-\left(-\frac{1}{3}+\frac{2}{3p}\right)}\right) ||f||_{L^p}, \text{for } 1 \leq p \leq 2 \tag{3.101}$$

By the dual properties of the L^p spaces, the main theorem is proved.

4 Future Research Plan

In the future, I am going to investigate the nonlinear Airy equation, namely

$$\partial_t u + \partial_x^3 u = F(u) \quad (t,x) \in (0,\infty) \times \mathbb{R} \tag{4.1}$$

$$u(0,x) = f(x) \quad x \in \mathbb{R}, \tag{4.2}$$

where F is a nonlinear function of u.

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