DEVELOPING UNDERSTANDING OF FRACTION AND DECIMAL MAGNITUDE: A TEACHING EXPERIMENT

Elizabeth Cutter-Lin

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DEVELOPING UNDERSTANDING OF FRACTION AND DEcimal MAGNITUDE:

A TEACHING EXPERIMENT

by

Elizabeth Cutter-Lin

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in Urban Education

at

The University of Wisconsin-Milwaukee

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ABSTRACT

DEVELOPING UNDERSTANDING OF FRACTION AND DECIMAL MAGNITUDE:
A TEACHING EXPERIMENT

by

Elizabeth Cutter-Lin

The University of Wisconsin-Milwaukee, 2023
Under the Supervision of Professor DeAnn Huinker

This study investigated how three fifth-grade students’ understanding of fraction and decimal magnitude evolved over the course of a five-week teaching experiment. Students participated in teaching and learning sessions focused on developing concepts of fraction and decimal magnitude. The following questions guided this study: (1) How do fifth grade students reason about the magnitude of fractions and decimals? (2) What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude? (3) What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?

A teaching experiment methodology was used to investigate these questions. In keeping with the essential elements of a teaching experiment, the study included several iterations of teaching, student thinking, reflection, and analysis, and then teaching again. Data was analyzed both during the teaching experiment and retrospectively following the conclusion of the teaching and learning sessions. Primary data included recordings and transcripts from the teaching sessions as well as students’ written work.

Retrospective analysis generated three major themes as most prominent in the students’ reasoning. First, experiences physically partitioning fraction strips and number lines emerged as
critical to students’ developing reasoning. The students drew upon the foundations they established when engaging in their own partitioning work as they worked with questions of size, equivalency, and density. Second, critical relationships such as the relationship between the numerator and denominator and between fractions to other fractions and decimals were essential, but also complex and challenging for the students. In particular, students appeared to anchor much of their work in relationships of fractions and decimals to the benchmarks of 1/2 and one whole. Distance reasoning (i.e., considering the distance of a fraction to a benchmark) appeared to become a preferred reasoning strategy for the students. Lastly, the number line appeared more complicated and challenging than initially anticipated but powerful in illuminating misconceptions. Pushing the students to place both decimals and fractions on the same number line appeared especially critical in supporting the students in integrating their developing understanding of both of these notations and expanding and deepening their mental number lines.
Dedication

Elodie and Zoe
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CHAPTER ONE

INTRODUCTION

The teaching and learning of fractions and decimals is a consistent area of struggle for many students and teachers (Chan et al., 2007; Geary et al., 2008; Rittle-Johnson et al., 2001; Siegler et al., 2012). Difficulties with understanding fractions are especially problematic because fractions are a critical area of mathematics, often viewed as a “gatekeeper” to higher levels of mathematics such as algebra (Bailey et al., 2012; Booth & Newton, 2012; Booth et al., 2013; Torbeyns et al., 2015). Fractions not only serve as a foundation for many mathematical concepts, they are also important to everyday life (Booth & Newton, 2012; Chan et al., 2007; Siegler et al., 2013). All of these factors coalesce to form a pressing issue not only in mathematics education, but beyond.

National Assessment of Educational Progress (NAEP) results have shown repeatedly that students experience serious difficulties acquiring and applying critical rational number concepts (National Mathematics Advisory Panel [NMAP], 2008). Difficulties with fractions begin early in elementary school (Empson & Levi, 2011; Moss & Case, 1999), continue through middle school (Armstrong & Larson, 1995; Kamii & Clark, 1995), and then persist into secondary and even later education (Fazio et al., 2016; Orpwood et al., 2011). Wu (2009) asserted that fractions are a major source of “math phobia” (p. 7). Wu argued that this fear can be traced to two sources. First, students lose their natural reference point of their fingers as a representation. Second, the “inherently abstract nature of the concept of a fraction” produces discomfort and fear as students no longer ground their sense of numbers in their fingers and must find a mental substitute for their fingers and “by its very nature, this mental substitute has to be abstract because most fractions (e.g., 19/13 or 251/604) tend not to show up in the real world” (p. 8).
Student struggles in the area of fractions are particularly problematic given the connection between fraction understanding and later areas of mathematics. Fraction understanding in elementary school has been found to predict algebraic knowledge in high school, even after controlling for other factors such as socioeconomic status, family education, and intellectual capacity (Siegler et al., 2013). Mathematically speaking, fractions lay the foundation for understanding algebra (Behr et al., 1983; Wu, 2007). Thus, challenges understanding fractions have “serious long-term consequences” (Jordan et al., 2013, p. 46). These consequences extend from gaps in the understanding necessary for algebra to problems in daily life such as managing personal finances, medicine routines, and home repairs (Jordan et al., 2013). In short, understanding fractions matters.

In tandem with fractions, difficulties with students lacking a conceptual understanding of decimals have been well documented in the literature (Behr & Post, 1992; Durkin & Rittle-Johnson, 2015; Hiebert & Wearne, 1989; Rittle-Johnson et al., 2001). Students may feel that they have entered a whole new number system when they encounter decimals. This may be in part because the way we notate fractions and decimals appears to be quite different. Many students perceive these two numeration systems as completely separate with little to no connection (Hiebert & Behr, 1988). In reality, fractions and decimals are simply two different ways to write the same value. Fractions with denominators that are multiples of 10 can also be written as decimals and are referred to as decimal fractions. For example, 8/10 and 132/100 can be written as 0.8 and 1.32 respectively.

Though decimals are essentially a different notation system for fractions, they are nevertheless challenging in their own right for students. Behr and Post (1992) pointed out that decimals may present specific problems for students due to possessing characteristics in line with
both fractions and whole numbers. Hiebert et al. (1991) also noted this challenge and argued that decimals are “complex mathematical entities” that “represent a confluence of common fractions and whole numbers” (p. 322). Despite these challenges, deep conceptual understanding of decimals is mathematically valuable for students. Wearne and Hiebert (1988) positioned decimals as mathematically rich and as “a more powerful representation system” than any students would have encountered previously (p. 221). Alongside fractions, decimals require time and attention within the curriculum as they are a critical extension of the base-ten value system and rational numbers (Behr & Post, 1992).

It is also important that students understand the relationship between decimals and fractions. The National Council of Teachers of Mathematics (NCTM) (2000) drew attention to this connection when they called for students to explore the relationship between fractions and decimals in their seminal work, *Principles and Standards for School Mathematics*. This goal was upheld with the advent of the *Common Core Standards for Mathematics* (CCSSM) in 2010. The CCSSM specifically asked for students to be able to use decimal notation for fractions with denominators of 10 and 100, as well as locate such numbers on a number line. In another policy document, the Institute of Education Sciences’ (IES) (2010) research-based practice guide for developing effective fraction instruction called for teachers to ensure students understand that fractions can be represented as fractions, decimals, and percents and can translate among these forms (Seigler et al., 2010). This understanding of the relationship between fractions and decimals is an integral component of students’ overall rational number sense (Lamon, 2007). However, the different appearances of the two notations may lead students to believe that they are quite different and have little or no connection (Hiebert & Behr, 1988).
Statement of the Problem

Challenges with learning fractions and decimals are significant, persistent, widespread, and an obstacle to growth in mathematics and other mathematics-dependent domains (Bailey et al., 2012; Bonato, 2007; Booth & Newton, 2012; Siegler et al., 2012). Students and teachers struggle with many critical fraction and decimal concepts (Lortie-Forgues et al., 2015). These struggles with rational numbers may limit not only students’ success in school but may also hinder them in solving real world problems such as those related to medicinal doses and tax and loan rates (Booth & Newton, 2012).

Success with fractions has been tied to success in later mathematics and life opportunities (Lortie-Forgues et al., 2015; Torbeyns et al., 2015). Bailey et al. (2012) and Booth and Newton (2012) found success with fractions, particularly understanding of fraction magnitude, to be correlated to later success in algebra. Lortie-Forgues et al. (2015) posited that success with fraction and decimal computation extends beyond mathematics courses to many other subjects such as biology, physics, chemistry, engineering, economics, sociology, and psychology among others. They went on to argue that knowledge that builds from fractions is also central to many jobs even those where more advanced mathematics is not a prerequisite such as nursing and pharmaceuticals.

Understanding of fraction magnitude and specifically the ability to place fractions accurately on a number line appears to be of particular importance to students’ understanding of fractions (Fuchs et al., 2014; Jordan et al., 2013; Resnick et al., 2016; Siegler & Pike, 2013). Siegler et al.’s (2011) integrated theory of numerical development identified understanding of fraction size (magnitude) as critical to children’s overall numerical development. In 2016, Siegler went further, positioning understanding fraction and decimal magnitude as just as
important as understanding the magnitude of whole numbers. He noted that for both whole and rational numbers, understanding of magnitude is predictive and casually related to other critical areas of mathematics including arithmetic and general mathematics achievement. Though further exploration is needed, Siegler and Pyke (2013) and Fazio et al. (2014) have found evidence supporting theories asserting early mental magnitude representations of fractions as foundational to the development of procedures with fractions and mathematical knowledge as a whole. Additionally, Torbeyns et al. (2015) studied almost 200 students across three continents including the United States and found additional support for Siegler et al.’s (2011) integrated theory of numerical development. Torbeyns and colleagues concluded their results provided correlational evidence for the specific role fraction magnitude plays in supporting understanding of fractions and overall mathematics achievement even when controlling for fraction arithmetic skills.

Despite the clear importance of understanding fractions, students still appear to struggle significantly with them (Bailey et al., 2012; Boothe & Newton, 2012; Siegler et al., 2012). In particular, students appear to struggle with understanding fraction magnitude. For example, Kamii and Clark (1995) found two-thirds of 13-year-old students could not correctly place a simple fraction on a number line. The NAEP found only roughly 50 percent of eighth graders could correctly arrange three fractions in the order of their ascending magnitude on their 2004, 2005, and 2007 assessments (NCES). Such results are especially concerning given that this skill should be developed by the time students are in fourth grade (National Governors Association & Council of Chief State School Officers [NGA & CCSSO], 2010; National Assessment Governing Board [NAGB], 2017).
Students also struggle in understanding magnitude of decimals and translating between and connecting fractions and decimals (Rittle-Johnson et al., 2001). When asked to translate 0.029 to fraction form, only 29.2 percent of 17-year-olds were able to do so correctly on the 2004 NAEP (NCES 2004). On the 1992 NAEP, only 52 percent of eighth graders could correctly identify the fraction closest to 0.52 out of the following set: 1/50, 1/5, 1/4, 1/3, 1/2 (NCES, 1992).

Evidence has begun to accumulate that student understanding of fractions, and fraction magnitude in particular, is a critical foundation for understanding fractions and later mathematics (Booth & Newton, 2012; Fuchs et al., 2014; Siegler et al., 2012). What this research does not tell us yet is how students are thinking about the magnitude of these numbers or how they come to a place of understanding. I conjectured that experiences in comparing and placing fractions and decimals on the same number line, as well as critical discussions of their magnitudes, may help students make shifts in their mathematical understanding toward generalizations in their growing understanding of fractions and decimals. This study will explore students’ developing understanding of the magnitude of fractions and decimals and the instructional experiences that support or do not support this development.

**Purpose and Research Questions**

The purpose of this study was to document and explore how upper elementary students think mathematically when presented with fraction and decimal magnitude tasks and the means teachers might use to support their learning. The following questions guided this study:

1. How do fifth grade students reason about the magnitude of fractions and decimals?
2. What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude?

3. What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?

To investigate these questions, I used a teaching experiment methodology which is a form of design research. The teaching experiment methodology first emerged in the United States from a constructivist framework in the 1970s to fill a gap in the types of research used at that time (Molina et al., 2007; Steffe & Thompson, 2000). Contrary to traditional notions of a scientific experiment, the teaching experiment does not compare treatment effects between different groups. Rather, a teaching experiment is “primarily an exploratory tool” developed out of Piaget’s clinical interview and with an aim to explore students’ mathematical knowledge (Steffe & Thompson, 2000, p. 273).

A teaching experiment is made up of a series of teaching episodes that can span a period of a few weeks to a few years (Cobb & Steffe, 2010). Teaching episodes allow the researcher to explore how students are interacting with tasks and activities as they explore the mathematics and revise theories of learning (Steffe & Thompson, 2000). The teaching experiment goes beyond the clinical interview in that it involves “experimentation with the ways and means of influencing students’ mathematical knowledge” (Steffe & Thompson, 2000, p. 273). As Steffe and Thompson explained, a primary reason for the teaching experiment methodology is so researchers “experience, firsthand, students’ mathematical learning and reasoning” (p. 267). As such, this methodology provided a framework for this study to explore and seek to understand students’ conceptions of fraction and decimal magnitude.
This study used the teaching experiment methodology as a tool to explore one small group of students’ conceptions of decimal and fraction magnitude and the tasks and questions that supported their emerging understanding. I worked as the teacher-researcher using iterations of instructional tasks to develop models of students’ thinking. Tasks were conducted with a small group of three students from one classroom. I reviewed teaching and learning interactions following each teaching episode to make plans for the following teaching episode. Students’ actions, verbalizations, and written work were coded and analyzed throughout the study to work towards models of student learning of the magnitude of decimals and fractions.

Though challenges in conceptual understanding of fractions extend to all areas of fraction and decimal work, this study focused on understanding the magnitude of common and “unfriendly” fractions and decimals. Within the context of this study, I am defining “unfriendly” fractions and decimals to be those fractions and decimals that cannot easily be represented using a visual model by students (e.g., $\frac{27}{29}$, $\frac{3}{52}$, 0.6127). Exploring students’ ability to reason about “unfriendly” fractions and decimals provides important insight as to the level of students’ understanding of the magnitude of these numbers.

**Significance of the Study**

Much of the existing research on fraction magnitude and comparison is quantitative in nature and focused on particular intervention methods or representations use. Additionally, a significant body of research exists on students’ understanding of fractions and a growing body of research on students’ understanding of decimals, less research has been conducted in exploring students’ development of these concepts in connection with each other. Siegler has begun important work examining the role and significance of understanding fraction magnitude, but much of it remains theoretical in nature. This study seeks to dig more deeply into how students
reason about the magnitude of fractions and decimals in conjunction with each other and the
tasks and questions that support a growing understanding.

The Common Core State Standards in Mathematics (CCSSM) called for students to
understand fractions as numbers on the number line by the end of third grade. Recommendation
Two in the research-based instructional guide through the U.S. Department of Education pushed
for teachers to support students in expanding the number system beyond whole numbers and
recognizing that fractions are numbers (Siegler et al., 2010). Thus, the call for students to
understand fractions as numbers is clear. What is still missing is literature connecting
instructional practices in the classroom with students’ developing reasoning and models of
thinking for decimal and fraction magnitude.

One goal of this study is to begin to fill in part of the gap in the literature by exploring
what students are thinking as they navigate and develop models of thinking for fraction and
decimal magnitude. If we are to help students who are struggling with fractions, then we must
understand what and how students are thinking about the magnitude of fractions and decimals.
We cannot achieve this simply though assessments scored on a quantitative scale. Rather than
simply implementing interventions and testing children, I contend that we need to delve into
students’ actual reasoning to learn what they know (and do not know) about these numbers. We
want to build from students’ reasoning and knowledge and learn how to move them forward.
This teaching experiment sought to begin to address the gap in research and develop a model for
how students’ conceptions of fraction and decimal magnitude develop and the tasks and
instructional experiences that support their developing conceptions.
Definition of Terms

It is important to clarify distinctions and overlaps within the vocabulary I am using in this teaching experiment. The term fraction is prevalent throughout this study. I am most often simultaneously referring to the conceptual underlying notion of a fraction as well as the notation which may be written in fraction or decimal form when using the term fraction. There may be places where I say fraction for simplicity, but the ideas can be mathematically extended to include decimals. Thus, the reader may note the use of the term fraction more frequently than decimal. When I use the term decimal, I am still referring to the underlying conceptual idea, but am specifically indicating when that idea is represented using a decimal notation. If I am referring specifically to the notation of a fraction, I will specifically indicate it as such.

The term “rational number” is also prominent in much of the literature that informs this study. As Lamon (2012) and NCTM (20120) pointed out, fractions and rational numbers are not synonymous, as examples can be found in either set which do not fit with the other set; however, Lamon goes on to also explain that when we say fraction, “we are really referring to the underlying rational number” (p. 30). Ultimately, this study is looking to explore and build students’ understanding of positive rational numbers. Fractions and decimals are the specific types of rational numbers appropriate to work at this grade level and thus where I focus. Because much of the literature employs the term “rational number” in discussions of concepts which inform this study, the reader can assume that within this study, the use of the term “rational number” also aligns with the conceptual understanding of fractions upon which we are focused.

The following list presents brief clarifications on how certain terms will be used throughout this study.
Benchmark fractions: Fractions such as $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2. These fractions help students to “benchmark” or “anchor” the size of other fractions to them.

Fraction Strips: Medium-sized (often 8 inches x 2 inches) paper strips students fold to create physical representations of fractions.

Generalize: To be able to go beyond reliance on concrete models and visuals and apply reasoning to more challenging numbers and concepts. Generalization is a statement or idea that results from this process.

Iterate: The action of copying or repeating something. In this case, we might consider iterating a fraction such as $\frac{1}{4}$. If we iterate $\frac{1}{4}$ three times, then we have $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ or $\frac{3}{4}$.

Magnitude: The size or value of a number, how “big” a number is.

Manipulatives: Tactile or visual models (such as fraction strips) used to try and illustrate mathematical relationships for and by students.

Partition: To split apart or divide an area, region, or length into even subdivisions (may also be used as a noun). This rectangle is partitioned into 3 equal parts. I may also call each of these parts partitions.

Tape Diagrams (also known as strip diagrams and fraction bars): Rectangular drawings students use to represent fractions (among other concepts). This tape diagram represents the value $\frac{2}{3}$.

“Unfriendly” fractions and decimals: Within this study, this term refers to fractions or decimals that would not be easily drawn or regularly used by students such as $\frac{8}{26}$, $\frac{97}{99}$ or 0.137.
CHAPTER TWO

REVIEW OF THE LITERATURE

This chapter describes several important domains within the realm of developing understanding of fractions. First, I examine why fractions are valuable as an area of focus, including their role as a mathematical foundation and as a recurring challenge in instruction, as well as the current state of fraction learning in this country. I then unpack what it means to understand fractions, including core instructional foundations that will be used within this study. From there, I explore identified antecedents to success or struggle with fractions. Lastly, I attempt to frame student understanding within two key theories.

Why Focus on Fractions and Decimals?

The teaching and learning of fractions and decimals is worthy of our attention for several reasons. First, fractions are a critical foundation for other areas of mathematics (Bailey et al., 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2012). Second, the importance of fraction understanding extends beyond mathematics to impact everyday life (Booth & Newton, 2012; Chan et al., 2007; Geary et al., 2008; NMAP, 2008; Siegler et al., 2012). Lastly, despite their importance, many students and teachers still struggle to understand fractions and decimals (Chan et al., 2007; Geary et al., 2008; Rittle-Johnson et al., 2001; Siegler et al., 2012).

Fractions as a Foundation

It is widely accepted within the mathematics community that fraction knowledge is a critical foundation for later success in algebra (Empson & Levi, 2011; NMAP, 2008). This idea appears repeatedly in the literature on fractions and rational numbers.
Behr et al. (1983) explored students’ rational number understanding and learning processes through the Rational Number Project. They found that rational-number ideas are some of the most complex and important mathematical concepts children grapple with during elementary school. They further asserted that fractions establish the foundation for elementary algebraic operations upon which elementary algebraic operations can later be based. Chan et al. (2007) similarly emphasized the powerful role fractions play. They positioned fractions as “a key foundation of elementary mathematics” which “not only serve as a gate to real-world situations but also form a basis for many mathematics concepts introduced in primary school (e.g., ratios, proportions, decimals, percents, rational numbers)” (p. 27). Lamon (1993) went even further, asserting that fractions seem to occupy a pivotal moment in students’ mathematical lives. She argued that “proportional reasoning plays such a critical role in a student's mathematical development that it has been called a watershed concept, a capstone of elementary concepts” (p. 41).

In keeping with the general mathematics community, the NMAP asserted that “difficulty with the learning of fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra” (NMAP, 2008, p. 28). Since then, this statement has been regularly and consistently cited throughout the literature as one of the primary reasons fraction learning is critical. However, two sets of researchers questioned the level of empirical support for this assertion. Bailey et al. (2012) acknowledged that this determination logically follows from the structure of algebra but contended that no empirical link had been established between fraction competence and success in algebra. Booth and Newton (2012) further questioned the NMAP’s 2008 statement and subsequent citations arguing for
research to critically study the significance of fraction understanding and its impact on algebra learning.

Two studies, one by Siegler et al. (2012) and one by Brown and Quinn (2007), are regularly cited in support of the NMAP’s (2008) claim. In the first, Seigler et al. (2012) conducted a regression analysis of two large data sets and did ultimately find support for the link between fractions and algebra. In the second, Brown and Quinn (2007) also concluded empirical support existed for this link in their ex post facto study of 191 high school students.

Siegler et al. (2012) used regression analysis to examine long term predictors of high school students’ knowledge of algebra and overall math achievement using longitudinal, nationally representative data sets of students from fifth grade to high school from the United Kingdom and the United States. Siegler and colleagues’ goal was to determine what types of mathematical knowledge are most predictive of later mathematics achievement. The authors found that elementary school students’ knowledge of fractions and whole-number division uniquely predicted mathematics achievement in high school. Knowledge of fractions and whole-number division were found to be even more impactful than knowledge of whole-number addition, subtraction, and multiplication, verbal and nonverbal IQ, working memory, family education, and family income. Results were consistent across both data sets from the two countries.

Brown and Quinn (2007) conducted an ex post facto study of 191 high school students comparing students’ elementary and intermediate algebra performance with their understanding of and manipulation of fractions. In their study, they found a significant relationship between students’ ability to understand and manipulate fractions and their test scores in algebra. Brown
and Quinn concluded that a relationship did exist between fraction understanding and the successful study of algebra.

However, in critique of Siegler et al. (2012) and Brown and Quinn (2007), Booth and Newton (2012) posited that the two studies were too broad to determine if skill with fractions specifically was critical or if fraction knowledge itself was representative of a stronger understanding of number which also leads to success in algebra. They further criticized that algebra knowledge was measured with one performance score, which left it unclear as to whether fraction knowledge contributes to both conceptual and procedural facets of algebra knowledge. Bailey et al. (2012) further questioned Siegler et al.’s (2012) conclusion that their results indicated fraction understanding led to achievement in algebra because Siegler and colleagues were not able to control for general mathematics achievement at the end of elementary school and fractions competence in high school. Bailey and colleagues noted that it is quite possible Siegler et al.’s conclusion was correct but cautioned that an “alternative (not mutually exclusive) hypothesis…is that children with a firm grasp of basic mathematical concepts… will more easily understand and learn to solve fractions problems” (p. 448). Both sets of authors noted that Siegler et al.’s 2012 work was progress in the direction of supporting the link between fractions and algebra, but also concluded that additional investigations were needed.

To investigate the potential link further, Booth and Newton (2012) conducted a series of number line estimation tasks and measures of algebra readiness with 32 middle school students in a pre-algebra class. The authors found that knowledge of fraction magnitude, even more so than whole number magnitude, was related to students’ skill in early algebra. Booth and Newton went even further to examine which components of fraction understanding are most impactful to
skills in early algebra. They found that knowledge of unit fraction magnitude specifically appeared to impact algebra readiness.

In a follow-up study to their 2012 study, Booth et al. (2013) studied 72 eighth graders enrolled in non-honors Algebra 1. Results from the study extended their original findings to students’ actual algebra knowledge, going beyond algebraic readiness in pre-algebra students as demonstrated in the 2012 study. The authors found that fraction knowledge was in fact predictive of students’ improvement in algebra. They further found evidence that proportional reasoning skills, as evidenced by placement of non-unit fractions on a number line, are predictive of algebra learning. This finding that non-unit fractions were especially important, differed somewhat from Booth and Newton’s 2012 findings that unit fractions were particularly influential. Further research exploring the predictive and correlative role of unit and non-unit fractions in students’ algebra readiness and overall understanding and facility with fractions would be beneficial to the field.

Despite their important findings, Booth and Newton’s (2012) and Booth et al. (2013)’s studies do face several limitations. First, and perhaps most significant, the sample size of the 2012 study was small (32 students) and not necessarily representative of a typical population of incoming Algebra 1 students. Second, the studies did not include a measure of prior mathematics achievement. Since general mathematics achievement is related to fraction and whole number magnitude knowledge (Booth & Siegler, 2008), it may be that that relationship is mediating the relationship found here between fraction magnitude and algebra readiness. Additionally, it is important to note that the authors showed a strong relationship in both studies, but not necessarily causation. It is possible there is another unknown variable at play that assists students in understanding both the magnitude of fractions and being ready for algebra. Lastly, as the
authors themselves admit, they still have not addressed questions relating to whether or not targeted fraction interventions could improve students’ work in algebra. Studies are needed to determine *how* fraction knowledge impacts algebra readiness and learning.

In their own investigation of the fraction-algebra link, Bailey et al. (2012) analyzed results from a longitudinal study of mathematical development of 212 children from 1998-2010. The authors examined measures of intelligence, achievement, and working memory alongside mathematical tasks consisting of number sets, number line estimation, computational arithmetic, computational fractions, and fraction comparison. They found that achievement with fractions predicted subsequent gains in mathematics achievement, independent of general mathematics competence. Ultimately, Bailey et al. concluded their results did in fact add empirical support to NMAP’s (2008) and Siegler and colleagues’ (2012) assertions on the critical importance of fraction understanding for mathematics learning. However, as the authors acknowledge, if they had assessed fraction knowledge and mathematics achievement at different grades or for shorter or longer intervals, they may have obtained different results.

Not only have fractions been found to be a key foundation for higher levels of mathematics, but they may also be a key component of a deep understanding of number itself (Booth & Newton, 2012; Siegler et al., 2013). Based on their investigation, Booth and Newton (2012) went so far as to assert that it is possible that fraction magnitude knowledge is a better indication of “deep number knowledge” than whole number magnitude (p. 251). Because of their nature, fractions require a deeper understanding of number than is typically attained with solely whole number experiences. As Siegler et al. (2013) put forth, “Learning fractions requires a reorganization of numerical knowledge, one that allows a deeper understanding of numbers than is ordinarily gained through experience with whole numbers” (p. 13). Siegler et al. further
asserted that the one property which unites all real numbers is their possession of magnitude and their ability to be placed upon a number line. If this is the case, then understanding fraction magnitude may serve as a key foundation to a rich and full understanding of number itself.

**Current State of Fraction Learning**

Despite clear assertions on the mathematical importance of understanding fractions, fractions and decimals continue to be an area of struggle for many students (Chan et al., 2007; NMAP, 2008; Siegler et al., 2012; Tian & Siegler, 2018). Proficiency in fractions as a whole is low, and certain subareas have been found to be particularly problematic. All groups assessed in the United States, even adults, continue to struggle with operations and comparison tasks involving fractions and decimals (NMAP, 2008).

The National Assessment of Educational Progress [NAEP] found that 50 percent of eighth graders could not correctly order the magnitude of three fractions ($\frac{2}{7}$, $\frac{1}{12}$, and $\frac{5}{9}$) (NMAP, 2008). Issues with fraction understanding extend beyond ordering fractions by magnitude. When asked to estimate the sum of $\frac{12}{13}$ and $\frac{7}{8}$ and given the options of 1, 2, 19, and 21, 55 percent of 13-year-olds and 36 percent of 17-year-olds chose the answer 19 or 21 (Carpenter et al., 1981). Perhaps even more concerning, recently, 15 percent of college students at a major university also chose 19 or 21 as their estimate (Lewis & Hubbard, 2015). Though these may appear to be very specific skills, given the data on the impact of fraction understanding, such assessment results warn of serious issues.

Additional NAEP results show reason for continuing concern for student understanding of fractions and decimals even several years after the advent of the Common Core State Standards. On the 2017 NAEP, only 32 percent of fourth graders were correct when comparing all six fractions ($\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{6}$, $\frac{4}{6}$, $\frac{2}{8}$, $\frac{4}{8}$) to the benchmark of one half. Twenty percent of students’
answers were considered partially correct, and a troubling 47 percent of students’ answers were classified as incorrect (1 percent of students’ answers were omitted) (NCES, 2017). Students also struggled with comparing unit fractions in context. Given the scenario that four children each had a 10-foot-long string and each child cut their string into different size shares (fifths, fourths, sixths, and thirds, respectively), only 59 percent of fourth grade students were able to correctly identify that the piece of string cut into thirds will be the longest (NCES, 2017). Such results speak to a lack of understanding of fundamental concepts of partitioning and fraction size.

Results for questions concerning decimals are slightly better, but still give reason to pause. When asked to match three shaded hundred grids with their decimal symbolic notation from a possible group of five decimals (0.02, 0.20, 0.25, 2.0, 2.5) only 58 percent of fourth graders were able to successfully pair all three numbers (correct decimals included 0.20, 0.02, 0.25) (NCES, 2017). Twenty percent of students were partially correct, and 21 percent of students were considered fully incorrect. Students performed only slightly better when asked to identify the decimal number matching a particular tick mark on a pre-partitioned number line (NCES, 2017).

Students did not seem to fare better when asked about fractions and decimals in conjunction with each other. Eighth grade students were asked to order the following fractions and decimals from least to greatest on the 2022 NAEP: 0.75, \( \frac{1}{2} \), \( \frac{1}{10} \), 0.25 (NCES, 2022). Only 55 percent of eighth graders successfully placed all four numbers in the correct order. Given that all of the required numbers should be relatively familiar to students and all of the numbers lend themselves easily to visual representations, student performance on this assessment item is concerning.
Student struggles with fractions and decimals are also supported widely within the literature ((Behr & Post, 1992; Chan et al., 2007; Durkin & Rittle-Johnson, 2015; Geary et al., 2008; NMAP, 2008; Rittle-Johnson et al., 2001; Siegler et al., 2012; Wearne & Hiebert, 1989). Though the advent of the Common Core State Standards in 2010 brought new prominence to the importance of fractions and called for significant shifts in the way fractions are being taught, results show the learning of fractions continues to be an issue for many individuals.

Fractions are a critical building block for students as they move forward in elementary mathematics and into algebra in high school. Currently, fractions are weakly understood by many students and adults in this country. The importance of fractions and the difficulties experienced when working with fractions for many indicates we should feel an urgency in addressing how we teach fractions in our schools. First however, it may be helpful to identify concepts critical to understanding fractions. The following section explores several of critical fraction ideas as they relate to the teaching and learning that will be explored in this study.

What Does it Mean to Understand Fractions?

One of the goals of this teaching experiment was to support students in moving toward a stronger conceptual understanding of fractions and decimals through a series of instructional interactions. This section explores concepts and constructs that provided the mathematical foundation of learning in this study. I begin with a discussion of what fractions are, including the five fraction subconstructs. This is followed by a discussion of additional concepts and constructs surfaced within the literature as key to building a rich understanding of fractions.

The discussion of fraction concepts (as opposed to decimals) occupies the bulk of this section. This is in large part due to the way in which concepts of rational numbers are addressed. Though they are linguistically linked more explicitly with fractions, all of the concepts explored
in the following section are also building blocks to a rich conceptual understanding of decimals. If we recall that decimals and fractions are simply two different notational systems to represent the same numerical value, then it follows that both will share many of the same important mathematical foundations.

A Multi-Faceted Concept of Fractions

The Common Core State Standards in Mathematics [CCSSM] (2010) defined a fraction as “a number expressible in the form a/b where a is a whole number and b is a positive whole number (The word fraction in these standards always refers to a non-negative number)” (p. 85). Though this definition appears relatively simple, a great deal more is required to reach a full and rich understanding of fractions.

What are Fractions?

Students must have a concept of fractions that goes much deeper to reason flexibly and richly with fractions. Students must understand fractions as numbers (CCSSM, 2010; Lamon, 2012; National Research Council, 2001). Though this may seem obvious, it is critical, and not always automatic or easy. As the National Research Council [NRC] (2001) pointed out, this fact is “so fundamental it is often overlooked” (p. 235). This includes understanding that $\frac{3}{4}$ refers to the same relative amount in all of the pictures in Figure 2.1 regardless of how we name the number (e.g., $\frac{6}{8}, \frac{9}{12}$).

Figure 2.1

Images of $\frac{3}{4}$ or Equivalents

![Diagram of fraction images]
Understanding fractions as numbers also means students can see fractions as numbers with a specific location on the number line. The CCSSM (2010) specified that students should understand fractions as numbers on the number line and be able to represent them on a number line diagram by the end of third grade. The CCSSM placed understanding fractions as numbers front and center despite its historical absence in classrooms, heading an entire third grade cluster: “Develop understanding of fractions as numbers” (p. 24).

Though this idea of fractions as numbers may seem simple and straightforward, Carraher (1996) pointed out that it is "pedagogically naive as well as psychologically inaccurate to believe that it is easy for students to understand this concept" (p. 241). In their 2004 study of students’ understanding of the numerical value of fractions, Stafylidou and Vosniadou found students struggled in making the transition from seeing fractions as two independent quantities or as part of a whole. Lamon (1999) also addressed the qualitative leap that students must go through when moving from whole numbers to fractions. As Lamon pointed out, fractions are rejected as numbers in children’s initial number theories because they are not part of the counting sequence. Students resist accepting fractions as numbers and instead conceptualize fractions as two distinct whole numbers (Charalombos & Pitta-Pantazi, 2006).

**Fraction Sub-Constructs**

Fractions are in part so challenging because they are composed of several subconstructs. Kieren (1976) was the first to put forth the idea that fractions as a concept are not a single construct; rather they are made up of several interrelated subconstructs. Behr et al. (1983) and Lamon (2012) among others later built off Kieren’s work to develop five subconstructs. These subconstructs now appear regularly in the fraction literature. These subconstructs form a multifaceted construct for understanding fractions. Pantziara and Philippou (2012) among others,
attributed a portion of students’ difficulties in learning fractions to the complexity of this multifaceted model.

The *part-whole subconstruct* identifies that students need to understand that the parts into which the whole is partitioned have to be equal in size and also need to be able to partition a continuous area. Students must also understand several additional key ideas within this subconstruct. First, that when the parts are taken together, they should represent the entire whole. Second, that the greater the number of parts the whole is divided into, the smaller the resulting parts become. Third, that the relationship between the whole and the parts is maintained, regardless of the shape, size, or arrangement of the equivalent parts (Charalombos & Pitta-Pantazi, 2006).

Following the part-whole subconstruct, Charalombos and Pitta-Pantazi (2006) identified the *ratio subconstruct* as conveyance of the idea that a fraction is a comparison between the two quantities in the numerator and denominator. This recognizes fractions as a “comparative index” rather than a number, pushing students to grasp the idea of relative amounts and proportionality (Charalombos & Pitta-Pantazi, 2006, p. 297).

The *operator subconstruct* identifies rational numbers as *functions* applied to a number, object or set (Behr et al., 1983; Charalombos & Pitta-Pantazi, 2006; Marshall, 1993). For example, students need to be able to consider what the value of $\frac{2}{5}$ of $20$ would be. In this case $\frac{2}{5}$ is operating upon the quantity $20$ and we will see a resulting quantity that is less than $20$. Lamon (2012) expanded upon this idea when she wrote, “More simply put, the operator notion of rational numbers is about shrinking and enlarging, contracting and expanding, enlarging and reducing, or multiplying and dividing” (p. 191). In this case fractions (operators) *transform* a quantity.
Next, the *quotient subconstruct* conveys that any fraction can be seen as the result of a division situation (Charalombos & Pitta-Pantazi, 2006). This differs from the part-whole subconstruct in that two different measurement spaces are under consideration (e.g., four cookies are shared among five brothers) and no limitation occurs regarding the size of the fraction. The numerator can be larger, smaller, or equal to the denominator. NCTM (2010) clarified that within this interpretation of rational numbers, there are two related sub-interpretations. First, it can represent a division operation. In this case we might see $\frac{2}{3}$ as another way to write $2 ÷ 3$.

Alternatively, we could also see $\frac{2}{3}$ as “the single number that results from performing the operation” (NCTM, 2010, p. 24). Overall, this subconstruct is rooted in the idea of equal partitioning.

Finally, in the *measurement subconstruct*, a fraction is typically the measure assigned to an interval or region (Lamon, 2012). Charalombos and Pitta-Pantazi (2006) explained this further, identifying “two interrelated and interdependent notions” to help us understand this subconstruct of fraction understanding (p. 300). First, fractions are considered a number, which addresses the “quantitative personality” of fractions. Second, a fraction is “associated with the measure assigned to some interval” (Charalombos & Pitta-Pantazi, 2006, p. 300). More specifically, a unit fraction (such as $\frac{1}{4}$ or $\frac{1}{5}$) can be iterated to determine a distance from a predetermined starting point (Lamon, 2001; Marshall, 1993). For example, $\frac{3}{5}$ corresponds to the distance of three $\frac{1}{5}$ size units from any given point. This definition likely makes clear why the measurement subconstruct is most consistently connected with the number line and other measurement tools such as rulers to determine the distance between two points. This was also the
subconstruct most heavily explored and relied upon within this study as I examined students’
conceptions of fraction magnitude in relation to the number line.

Students do not need to be able to name and explain all five subconstructs, but it is
important that educators are aware of the subconstructs and understand their benefits and
potential limitations. The “multifaceted” nature of fractions and the five constructs may play a
significant role in students’ challenges in understanding fractions (Behr et al., 1992; Kieren,
1993; Lamon, 2012; Pantziara & Philippou, 2012). It is worth educators taking the time to notice
which subconstructs they are focusing on and which subconstructs they may be neglecting at the
expense of their students’ learning.

**Background on Fraction Learning in the United States**

Fraction instruction in the United States has traditionally focused on the part-whole
interpretation of fractions (Charalombos & Pitta-Pantazi, 2006; Fuchs et al., 2016; Tzur & Hunt,
2015). This is problematic because sole reliance on the part-whole subconstruct has several
limitations for students’ reasoning (Fuchs et al., 2017; Siegler et al., 2010). The part-whole
interpretation naturally lends itself to “out of” language such as five out of eight parts. In this
language, it is easy for us to interpret both the numerator and denominator as separate whole
numbers. Siebert and Gaskin (2006) argued that this is problematic because in such language,
“The actions that led to the creation of the parts (partitioning) or the creation of the whole
(iterating) are never acknowledged, thus obscuring the relationship of the parts to the whole and
the very actions that can be used to create and operate on fractions” (p. 397). Not only is this
critical relationship obscured, but the language itself may even lead to misconceptions.

When we overemphasize the part-whole interpretation we contribute to the idea that
fractions are composed of two distinct numbers and that fractions such as $\frac{5}{3}$ cannot exist because
you cannot have 5 parts of a whole that has been split into 3 parts (Fuchs et al., 2017). Siegler et al. (2010) also identified that a heavy focus in early fraction instruction focused on the part-whole interpretation can lead to students failing to develop the critical concept that fractions are numbers with magnitudes. Students become caught up in struggles understanding the relationship between the numerator and denominator and are not able to conceptualize the fraction as a number itself. The part-whole interpretation is not incorrect, nor am I arguing it should be removed from students’ fraction learning. Rather, we must balance this interpretation with others and be cautious of the language we use when describing part-whole relationships so as not to build misconceptions.

**A Shifting Instructional Focus**

The *Common Core State Standards in Mathematics* (CCSSM) shifted the focus of fraction instruction for US schools away from a part-whole focus, towards a measurement-based interpretation of fractions. A careful reading of the Number and Operations in Fractions Standards will show there is little discussion of the part-whole interpretation. Rather, the standards focus on developing an “understanding of fractions as numbers” (National Governors Association & Council of Chief State School Officers [NGA & CCSSO], 2010). Within the Standards’ cluster “Develop understanding of fractions as numbers,” the ensuing standards directly called for students to develop an understanding of fractions as built on unit fractions and as numbers on the number line with a focus on magnitude. In connection with these critical foundations, I have also found the actions of partitioning and iterating, work with benchmark fractions, and the density of fractions to be important for student understanding. All of these focus areas will form the foundation for my instructional interactions with students on fraction and decimal magnitude.
Unit Fractions

A solid understanding of fractions may be founded upon an understanding of unit fractions. Simply defined, the unit fraction is a fraction with a numerator of one which is formed by partitioning a whole into equal parts and taking one of those parts. For example, if a whole is partitioned into eight equal parts, then one part is one-eighth of the whole and eight copies of that part make the whole (NGA & CCSSO, 2010).

The Standards explained that “the goal is for students to see unit fractions as basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers” (NGA & CCSSO, 2010, p. 7). The standards encourage us to use language to describe fractions that connects back with this understanding. For example, using the language of the standards we might describe \( \frac{3}{4} \) as 3 parts of size \( \frac{1}{4} \). In this way, unit fractions are a fundamental element of all fractions and thus all concepts relating to fractions.

The importance of unit fractions is also supported in another significant policy document developed through the Institute of Education Sciences [IES]. The IES guide (Siegler et al., 2010) called for teachers to begin with sharing situations that result in unit fractions such as splitting one cookie between six people. From here, students can move on to more complex sharing situations with unit fractions providing a foundation to which they can return. Unit fractions help students visualize the size of fractions as they picture the iteration of the unit fraction to construct a non-unit fraction (i.e., \( \frac{3}{4} \) is 3 parts of size \( \frac{1}{4} \)). NCTM (2010) recommended the use of unit fractions as a foundation, noting unit fractions can help students make sense of improper fractions in particular. Using unit fractions, students can visualize a fraction such as \( \frac{4}{3} \) by seeing three copies of the unit \( \frac{1}{3} \). This image is more supportive than a part-whole reliant image which
may make conceptualizing four out of three parts difficult, if not even impossible, for some students.

Unit fractions first provide students with a foundation for understanding what fractions are. This understanding continues to be an important conceptual foundation throughout many areas of fraction instruction including comparisons, operations on fractions, and connections to decimal concepts. Understanding non-unit fractions as composed of unit fractions helps students to work with non-unit fractions. For example, if we are to conceptualize multiplication using a part-whole interpretation we might become befuddled at how we can create five groups of three out of four parts and how that results in $\frac{15}{4}$. Alternatively, when we think of $\frac{3}{4}$ as three parts of size $\frac{1}{4}$, we can picture taking five groups of three pieces of size $\frac{1}{4}$. We are simply iterating three groups of $\frac{1}{4}$ five times or $\frac{1}{4} \times 15$ times in total.

Unit fractions have also been noted to be particularly impactful on students’ algebra readiness (Booth & Newton, 2012). Though the literature on use of unit fractions is limited, Fazio et al. (2016) conducted two studies with interventions focused on developing an understanding of unit fractions and using this understanding to place fractions on the number line. They concluded that instruction focused on unit fractions and the number line helped students improve in their number line estimates, magnitude comparisons, and recall accuracy.

In my survey of the literature, unit fractions appeared less frequently than the other foundational concepts discussed within this section. It is possible that unit fractions remained unnoticed by many prior to 2010 when they appeared in the Standards. Despite the limited literature on unit fractions within the mathematics education world, I believe that they have mathematical relevance and value to the work of this study. In my work with the students, we
sought to build and use an understanding of fractions based on unit fractions. We used “unit fraction language” to describe and discuss fractions in both fraction and decimal notation.

**Partitioning and Iteration**

Unit fractions are implicitly linked to the actions of partitioning and iteration. Students must be able to partition in order to create unit fractions and must understand the process of iteration to create all other fractions out of unit fractions. For example, a student may first partition or split a whole into four equal pieces creating pieces of size one-fourth. In order to create the non-unit fraction 3-fourths the student would iterate or copy their one-fourth piece three times. The actions of partitioning and iterating thus form the core of fraction creation. As Lamon (2010) wrote, the act of partitioning “is the action through which fractions come into existence” (p. 172).

The Standards (2010) placed partitioning at the beginning of students’ interactions with fractions. Students begin their work by partitioning basic geometric shapes equally in grades one through three. In third grade, students partition even more shapes and begin to build non-unit fractions by iterating the unit fractions they have created through their partitions. Siebert and Gaskin (2006) presented the terms partition and iterate as two “images” and “tools” students can use to “create and act on fractions.” When students partition, they divide or “split” into equal sized shares. Students may envision iteration as repeating or making copies. These tools can provide means for justification as well as “enable children to develop robust meanings for fractions and fraction operations” because they facilitate reasoning and talking about fractions (Siebert & Gaskin, 2006, p. 394). Pothier and Sawada (1990) similarly found partitioning to be a powerful, if underused, tool for students in their work with students in grades kindergarten through six.
Tzur and Hunt (2015) found that work with iteration facilitates a deeper understanding of unit fractions themselves. In tasks asking students to determine the whole from various fraction portions such as $\frac{1}{4}$ or $\frac{3}{8}$, the researchers found that students needed to return to the unit fraction to be successful while also developing a stronger understanding of unit fractions. Tzur and Hunt argued that students’ work partitioning and iterating helped them begin to conceptualize unit fractions not only as shaded or folded pieces of the whole (e.g., one of four parts) but as a “multiplicative relationship between a unit and the whole into which it fits a given number of times” (p. 151). Partitioning and iteration can also help students address misconceptions as well as build a conceptual understanding of operations on fractions (Siebert & Gaskin 2006).

**The Number Line and Fraction Magnitude**

A number of policy documents call for the number line as a critical representation for building students’ understanding of fractions as numbers. The Standards (2010), in particular, brought new prominence to the number line as a critical representation for fractions. The NGA & CCSSO (2010) asked students to locate fractions and later decimals on the number line as they build their understanding of fractions as numbers (Standard 3.NF.a2, Standard 4.NF.c6). The number line is repeatedly called for as a representation for exploring and justifying thinking throughout the grades three through five fraction standards.

The idea that fractions are numbers is a critical component of understanding fractions and the number system (NGA & CCSSO, 2010; Lamon, 2012; Petit et al., 2022). The number line plays a valuable role in helping students achieve this understanding of fractions as numbers. The IES Practice Guide (Siegler et al., 2010) placed the importance of the number line as a tool for developing understanding of fractions as numbers at the center of its second recommendation which urged educators to “help students recognize that fractions are numbers and that they
expand the number system beyond whole numbers” and to “Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward” (p. 19). In their review of accumulated evidence, the authors of the guide found number lines to be an effective tool in helping students understand fractions as numbers with magnitude.

The importance of the number line as a representational tool for understanding and working with fractions appears repeatedly in both studies and texts geared towards helping teachers effectively teach fractions. In their research-based guide to fraction learning, Petit et al. (2022) supported the idea of using the number line as a tool to develop understanding of fractions as numbers as well for understanding equivalence and even operating on fractions. Wu (2008)’s frequently cited report on fractions, decimals, and rational numbers used the number line as the principal foundation for defining what a fraction is. Additionally, in their study surveying over 3,000 Finnish fifth and seventh graders, Hannula (2003) posited that the number line is a place where other constructs could come together: “Then, ability to locate a fraction on a number-line could be regarded as an indication (although not a guarantee) of confluence of several subconstructs” (Hannula, 2003, p. 1). Hamdan and Gunderson (2016) compared the effects of interventions using a number line and area model. They found the number line intervention played a “causal role” in magnitude understanding, leading to more transfer to other magnitude tasks and benefiting students more than the more widely used area model (p. 2). Booth and Newton (2012) found the ability to reason about fractions using a number line to both be an indicator of understanding of magnitude, as well as a facilitator of later learning of eventual learning of algebra.

Recent research has brought new prominence to the number line as a powerful tool for and window into students’ understanding of fractions. For example, in their work exploring
developmental predictors of fraction concepts and procedures, Jordan et al. (2013) found number line estimation to be the most significant factor contributing to variance in student fraction understanding. Their summary identified that they found fraction reasoning to be related to a “mix of cognitive, behavioral, and numerical abilities.” However, understanding of numerical magnitude, as evidenced by accuracy in estimating using a number line, was an especially strong early predictor of performance, even more predictive than proficiency with whole number computation or attention.

Prior to success with fractions, students first need to develop an understanding that whole numbers have magnitudes and are located on the number line. Students who “acquire this insight with whole numbers seem to have an advantage in learning fraction concepts as well” (Jordan et al., 2013, p. 55). Vukovic et al. (2014) extended and corroborated Jordan et al.’s (2013) findings on the importance of number line estimation. Vukovic and colleagues’ study examined second-graders’ estimation on number lines and found a unique association with fraction concepts two years following in fourth grade. In explanation of this, they posited that children’s “rudimentary proportional reasoning ability” may also provide “a foundation for learning fractions” (p. 1,473). In a separate longitudinal study examining growth of fraction understanding for students with identified mathematics difficulties, Jordan et al. (2017) further supported the idea that estimating magnitude on the number line serves as a key marker of success with fractions.

In 2011, Siegler and Lortie-Forgues put forth a now oft-cited integrated theory of numerical development specifically connected to the number line, asserting that “the unifying theme of numerical development is children’s growing understanding of numerical magnitudes” (Siegler, 2016, p. 341). Siegler (2016) summarized the main tenets of this theory with six major points:
1. The magnitudes of all rational numbers are represented on a mental number line, a dynamic structure that begins with small whole numbers and over the course of development expands rightward to include larger whole numbers, leftward to include negative numbers, and interstitially to include fractions and decimals.

2. Within specific ranges of whole numbers (e.g., 0–10, 0–100, 0–1000), magnitude representations progress from a compressive, approximately logarithmic distribution to an approximately linear one. Transitions occur earlier for smaller than for larger numerical ranges, corresponding to when children gain experience with the numbers in the range.

3. Development of rational numbers involves learning that many properties of whole numbers do not characterize other types of numbers but that all real numbers have magnitudes and can be represented on number lines.

4. Numerous processes influence development of numerical magnitude knowledge, but two that play especially large roles are association and analogy.

5. Because magnitude knowledge is central to numerical development, as posited by the theory, knowledge of the magnitudes of both whole and rational numbers should be both correlated with and causally related to other aspects of mathematics, including arithmetic and mathematics achievement test scores.

6. Because magnitude knowledge is central to numerical development, interventions designed to improve knowledge of both whole and rational number magnitudes should have substantial positive effects on a wide range of mathematical outcomes. (p. 342-343)

Siegler asserted that the mental number line and understanding of magnitude is critical to students’ mathematical development. One of the most important ideas that the integrated theory
of numerical development has put forth is that “rational and whole numbers are viewed as co-
equal parts of numerical development” (p. 350). Thus, this theory positioned fractions as just as
important as whole numbers within students’ development of mathematical proficiency. In
another study on developmental individual differences in understanding fractions, Siegler and
Pyke (2013) also found that “understanding magnitudes appears to be a particularly important
aspect of conceptual understanding of fraction.”

Fuchs et al. (2017) found that emphasis upon, and intentional teaching on, the magnitude
of fractions as opposed to a part-whole model had a statistically significant impact on at-risk
students’ learning of fractions. This corroborates Kerslake’s (1986) findings that a part-whole
model may inhibit students from developing an understanding of fractions as numbers. This
parallels NMAP’s (2008) hypothesis that improving student understanding of fraction magnitude
is more critical to strong understanding than the part-whole interpretation. As Fuchs et al. (2017)
argued, a part-whole interpretation encourages separate counting of the numerator and
denominator pieces and thus reinforces children’s predisposition to conceptualize view fractions
as composed of two separate whole numbers. Whereas, emphasizing an understanding of fraction
magnitude “encourages relational thinking about numerators and denominators as determinants
of a single number” (Fuchs et al., 2017, p.632). In a different 2013 study, Fuchs et al. (2013) also
found that focusing intervention on the magnitude of fractions using a magnitude model had a
significant impact on at-risk students’ fraction aptitude even when controlling for other
mediating effects. Magnitude teaching of fractions is typically connected with number lines as
called for by Seigler et al. (2011). Unfortunately, despite the 2008 NMAP assertion and a
decided emphasis on number lines as opposed to part-whole models in the Common Core
Standards, part-whole reasoning continues to overshadow the measurement interpretation in many American schools (Fuchs et al., 2017).

The number line played an important role in this study as I engaged with the students in instructional interactions to understand and build their conceptions of fraction and decimal magnitude. As explored in this section, the number line is a core component of the measurement interpretation of fractions which supports understanding of the magnitude of fractions. The number line is also an important representation for students as they make sense of fractions as numbers. Translating from experiences with concrete models to the number line supports students in conceptualizing fractions as numbers with magnitudes. I also utilized the number line as a model to help students relate fractions and decimals to each other as we worked to place fractions and decimals on the same number line.

**Benchmark Numbers**

One potentially valuable framework for students as they assess the magnitude of fractions and decimals is the comparison back to benchmark numbers such as one-half and one-whole. In using this strategy, students compare two fractions to a third fraction as a reference point. For example, a student might determine $\frac{4}{7}$ is larger than $\frac{2}{5}$ because $\frac{4}{7}$ is larger than one-half, while $\frac{2}{5}$ is smaller than one-half. This strategy has also been called “transitive” (Post et al., 1986) or “reference point comparison” (Behr et al., 1984). Behr et al. (1984) and Clarke and Roche (2009) found benchmarking strategies emerged naturally among certain children, typically those displaying a stronger conceptual understanding of fractions. Given correlations between success on rational number magnitude tasks and the use of this strategy, Clark and Roche recommended benchmark (or transitive thinking) strategies be explicitly brought forward for all students through class discussions and activities where students sort fractions as closest to benchmarks.
such as zero, one-half, and one. In keeping with this recommendation, I utilized benchmark sorts and connections back to benchmarks as I worked to support the students in developing their sense of fraction and decimal magnitude.

**The Density of Numbers**

One of the properties that separates fractions from whole numbers is the density property. The density property means that in between any two fractions there is an infinite number of other fractions. This differs from whole numbers which follow a set counting sequence. As one might imagine, understanding this property is a significant and complex shift for students. As Smith (1999) pointed out, “Because there is no ‘successor,’ no ‘next’ rational number, listing rational numbers in order is substantially more complicated” (p.5). Developing a robust understanding of the density property remains challenging for many students (McMullen et al., 2015; Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2010).

Hansen et al. (2017) named understanding of fraction density as one of the critical ideas students must understand to overcome misconceptions about fractions. Their research in this area links back to the importance of magnitude and the number line. The researchers found that the difficulty of understanding the density of fractions among other concepts was mitigated when students were taught fractions using an emphasis on the cardinal size of fractions.

One may note that several of the subconstructs explored earlier do *not* lend themselves to the idea of density. For example, it is unlikely one would explore or consider the density of fractions if working within a part-whole, ratio, quotient, or operator context. The measurement subconstruct thus emerges again as a critical dimension of fractions for students to explore and master. The literature supports the importance and value of the measurement construct and specifically the number line in developing concepts of the density of numbers. Lamon (2012),
Petit et al. (2022), and Siegler et al. (2010) all specifically mentioned the power of the number line for helping students build a sense of the density of rational numbers. This is not surprising if one considers how well the number line lends itself to successive partitioning and thus infinite numbers.

Understanding the density of numbers is one important subcomponent to true understanding of fractions as numbers. Thus, work building an understanding of the density of numbers informed many of the tasks used within this teaching experiment.

**Relational Thinking**

All of the previously explored areas are also critical foundations for decimal understanding. Given the nature of their notation, decimals also have one additional critical concept. Students need to understand decimals as fractions, but they also must understand the connection between decimals and the base ten system (Hiebert et al., 1991). Decimals represent a “confluence of common fractions and whole numbers” using the “base ten (whole-number-like) notation to stand for fractional quantities” (Hiebert et al., 1991, p. 322). Because decimal notation has its roots in the base-ten system, students must understand place value and the relationship between each place, recognizing that moving to the left, numbers become ten times greater and moving to the right, numbers become ten times smaller (Hiebert et al., 1991).

D’Ambrosio and Kastberg (2012) identified thinking relationally and understanding the relationship between the different subdivisions of the whole as a critical understanding for students. D’Ambrosio and Kastberg also contended that students need opportunities to build these understandings *for themselves* versus being given pre-partitioned models from the beginning such as the decimal grid.
Berryman (1972) also highlighted the importance of students understanding the place value system in their work with decimals. He argued that we in fact focus too much on the value of the place value column and its relationship to the decimal point. Instead, he suggested we emphasize the relationship between columns, focusing on deeply understanding that the magnitude as we move to the left is ten times greater and ten times less as we move to the right. He postulated that when we assign a particular base ten block as the one or the unit, we are forced to consider all the other pieces in relation to it and thus obscure the simpler structure and consistency of the changing magnitude.

Baturo (1998) found that grade 6 students struggled with the relationship between tenths and hundredths and that understanding the magnitude of hundredths was particularly challenging for students. Martini and Bay-Williams (2003) also studied sixth grade students’ understanding of decimals using four different representations: a number line, a 10 x 10 grid, money, and place value. They found only 14% of the students tested were able to represent the two decimals, 0.6 and 0.06 correctly across all four representations. The number line, in particular, emerged as a representation where students’ gaps or misconceptions surfaced. Students in their study struggled to represent the relative value of both decimals on the number line. Follow-up assessment found that students struggled most with placing decimals under one on the number line, especially those less than one-tenth.

**Connections Between Decimals and Fractions**

The NGO & CCSSO (2010) and NCTM (2000) both called for students to understand the relationship between decimals and fractions. Lamon (2007) similarly positioned this understanding of the relationship between fractions and decimals as a critical component of students’ overall rational number sense. Braithwaite et al. (2022) noted that multiple notations
may be challenging for learning but could provide an opportunity for students to develop a
stronger understanding of rational numbers than could be achieved from solely within-notation
knowledge. However, many students appear to struggle with this connection as evidenced in the
NAEP tasks addressed earlier.

In a recent study investigating students’ cross-notational understanding between
fractions, decimals, and percents, Schiller and Seigler (2023) found that even after several years
of experience with rational numbers that most students have not “integrated understanding of
rational number notations into a unified mental model” (p. 3). Schiller and Seigler further noted
that cross-notational knowledge does not automatically develop over time. They contend that
part of these challenges may be due to textbooks lacking cross-notational problems.

Overall, work on connections and conversions between fractions and decimals appears
somewhat scarce in the literature (Schiller & Seigler, 2023; Shaughnessy, 2009). Currently, the
connection between fractions and decimals appears to occupy much less of the research space
than other fraction topics. This study looked to explore this connection and the potential for it to
help students as they develop understanding of fraction and decimal magnitude.

Impediments to Fraction Understanding

The literature consistently identified several variables as contributing to difficulties in
understanding fractions. Number sense and skill with whole numbers, dominance and
interference of whole number reasoning, and transitions to multiplicative thinking all appear as
factors affecting development of fraction understanding. The section below explores possibilities
of what may be making fractions so difficult for students.
Number Sense and Whole Number Reasoning

Perhaps unsurprisingly, students’ previous experiences and success with whole numbers and number sense appear to impact students as they enter into the fraction world. Vukovic et al. (2014) found mastery of whole number concepts and skills to be a critical precursor to fraction understanding. They asserted that students’ fluency in identifying, processing, and manipulating small quantities seemed to have a specific impact on fraction learning. Interestingly, Vukovic et al. even found that procedural skill with whole numbers predicted conceptual understanding of fractions two years later. These findings are especially significant because the authors controlled for general mathematics and academic ability, as well as proportional reasoning.

Resnick et al. (2016) found that student success with estimation of whole number magnitude in third grade was a strong predictor of success with fractions in fourth grade. In fact, the authors found that students who struggled with accuracy in whole number line estimation skills were roughly twice as likely to appear in the low-growth category in grade four. This fits with Siegler and Lortie-Forgues (2014) integrated theory of numerical development. Students first grow in their understanding of the magnitude of whole numbers, which then expands interstitially to incorporate rational numbers. If students struggle with understanding the magnitude of whole numbers, then it translates that they will also struggle as they begin to work with fractions.

Whole Number Bias

Students need to begin with a strong understanding of whole number; however, inappropriate application of whole number reasoning strategies is also one of the most frequently cited complicating factors in students’ work with fractions. As Lamon (2012) wrote, “Children experience cognitive obstacles as they encounter fractions because they try to make connections
with the whole numbers and operations with which they are familiar” (p. 25). This makes sense given the amount of time children have spent living and working in the whole number world by the time they reach fractions.

The theory of whole number bias connects students’ struggles with fractions to the understandings and biases they have already built from their work in the whole number world (Mack, 1995; Ni & Zhou, 2005; Post et al., 1995). Ni and Zhou (2005) defined whole number bias as the “robust tendency to use the single unit counting scheme to interpret instructional data on fractions” (p. 28). In essence, the foundations and understandings children build in the whole number system lead to challenges as they move into an expanding number system where the same rules no longer apply (Kainulainen et al., 2017).

Whole number bias manifests itself in multiple ways. One example of an explanation impacted by misconceptions stemming from whole number bias could be when a student says one-eighth is larger than one-third because eight is larger than three. In a fourth-grade teaching experiment, Post et al. (1995) found this type of misconception rooted in whole-number based ordering strategies endured even after substantial instruction.

Recently, researchers have worked to identify two additional manifestations of whole number bias in fraction comparison problems (Gonzalez-Forte, 2022). DeWolf and Vosniadou (2015), Gomez and Dartnell (2011), Obersteiner et al. (2017), and Rinne et al. (2017) have found evidence of gap thinking and reverse bias which help expand our understanding of student misconceptions. In gap thinking, students focus on the difference or “gap” between the numerator and denominator to compare fractions (Behr et al., 1984; Pearn & Stephens, 2004). For example, a student might say the “gap” of one between the 2 and the 3 in 2/3 is smaller than the gap of two between the 8 and the 10 in 6/8 and thus 2/3 is larger. In reverse bias, students
may become stuck in the idea that a smaller denominator indicates fewer pieces, and a larger
denominator indicates more pieces and forget to consider the relationship between the numerator
and denominator (DeWolf & Vosniadou, 2014). This may look like a student saying 1/3 is larger
than 4/5 because thirds are larger than fifths. Gonzalez-Forte (2022) found that both of these
types of thinking were resistant to change even when students were presented with alternative
strategies.

Students also may see fractions as two separate and distinct numbers and struggle to
understand them as a single entity (NRC, 2005; Siegler et al., 2010). In fact, Siegler et al. (2010)
identified this failure to consider the “essential relation between each fraction’s numerator and its
denominator” as one of three facets comprising a general lack of conceptual understanding of
fractions (p. 7). Struggles with conceptualizing fractions as one number or entity leads to issues
with comparison such as the one identified above, as well as trouble operating on fractions (Behr
et al., 1984). Students may attempt to add, subtract, etc. each part of a fraction individually
(Carpenter et al., 1981).

Students may also have developed certain ideas regarding operations that no longer apply
when they reach fractions (Lamon, 2012). For example, a student may develop the idea that
multiplying always results in a larger product and dividing always results in a smaller dividend.
When that student reaches fractions and finds that multiplying four by \( \frac{1}{2} \) results in two or
dividing four by \( \frac{1}{2} \) results in eight, their assumptions no longer apply.

Though whole number reasoning can be applied inappropriately to fractions thus causing
problems, we do not want to be overly vigilant in our separation of fractions from whole
numbers. As Barnett-Clarke et al. (2010) and the NRC (2001) pointed out, students need to also
use their understanding of whole numbers and their properties to understand fractions fully. We
do not want students to abandon all of their understandings of whole numbers as they move into fractions and view fractions as “meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher” (Siegler et al., 2010, p. 6). Rather, we want to help students understand how “rational numbers build on and extend whole numbers” (Barnett-Clarke et al., 2010, p. 70). As the CCSWT (2011) pointed out, the meanings, uses, and properties of the operations stay true for rational numbers, though the algorithms we use for the operations may change.

Students may still struggle with whole number interference even with targeted instruction addressing the aforementioned issues. Working memory appears to be an issue for some students as they must remember and consider two information pieces (the numerator and denominator) which likely requires greater working memory resources than representing just whole numbers. Siegler et al. (2013) also found that fraction knowledge may require “inhibitory control, so that the numerator and denominator are not treated as independent whole numbers” (p. 16). Behr et al. (1984) noted that even after students have had success through instruction they may still struggle when asked to apply their knowledge in novel situations. This was based on evidence that “even late into instruction a substantial number of children ‘back slide’ into a whole-number-dominance strategy when confronted with problem-solving situations where they must apply their knowledge of the order and equivalence of fractions” (p. 333).

As Siegler (2016) asserted, “Understanding rational numbers requires a reorganization of the conceptual knowledge acquired in learning about natural numbers” (p. 350). Students must take what they have learned about whole numbers and apply understandings where appropriate as well as shift understandings for specific properties.
Transitions to Multiplicative Thinking

Multiplicative reasoning is different from additive thinking and develops slowly over time (Clark & Kamii, 1996). Piaget (1983/1987) identified that multiplication is not simply a faster way of adding, but rather is an operation that calls for students to practice higher order thinking. This type of thinking requires a higher number of levels of abstraction as well as a number of inclusion relationships to make simultaneously (Piaget, 1983/1987). Multiplicative reasoning also involves a shift from whole number to rational numbers and movement from discrete quantities to continuous quantities (Neagoy, 2017). Discrete quantities are things we can count, such as balls, bottles, or children; whereas continuous quantities are things we can measure such as length or depth. The act of measurement of these continuous quantities is inherently a multiplicative process (Kieren, 1976; Neagoy, 2017).

Multiplicative thinking is critical for students as they transition into the world of rational numbers (Fuchs et al., 2017; Neagoy, 2017). Students must move beyond whole number ideas where a number represents a fixed quantity to understand numbers that are expressed in relation to other numbers. This type of new proportional relationship is grounded in multiplicative reasoning which differs significantly from additive reasoning (NRC, 2005). When students “think proportionally,” they understand the “multiplicative relation between two quantities” (Siegler et al., 2010, p. 35).

Multiplicative thinking is also critical to work with fraction equivalency (Fuchs et al., 2017; Jordan et al., 2017; Resnick et al., 2016). In fact, the ability to visualize and understand multiplicative relationships between fractions underlies success with fractions (Boyer & Levine, 2012). Students need to be able to understand that when they are creating an equivalent fraction through multiplication, they are really multiplying by one, as the numerator and denominator
change at the same rate and maintain the same proportion to each other. For example, if their original fraction is $\frac{3}{4}$ and they are multiplying it by $\frac{2}{2}$ to result in their equivalent fraction $\frac{6}{8}$ they have not actually changed the size of the fraction.

However, as Neagoy (2017) pointed out, we must be cautious of focusing on multiplicative relationships solely relating to equivalent fractions. “Fraction instruction that focuses on the multiplicative relationship between equivalent fractions almost to the exclusion of the multiplicative relationship within each fraction” results in a limited understanding of the equivalent fraction algorithm (Neagoy, 2017, p. 122). Students must also see the multiplicative relationship between the numerator and denominator of the fraction (Neagoy, 2017). If a student only focuses on the additive relationship of the numerator to the denominator, they will not be able to fully reason about the size of the fraction. Additive reasoning in this case could create a misconception such as $\frac{6}{8}$ is the same as $\frac{8}{10}$ because they are both two away from the whole (Neagoy, 2017).

The importance of multiplicative thinking also extends to magnitude and the number line. Resnick et al. (2016) found that, “improving understanding of multiplicative relations and their connection to fraction magnitudes also is likely to increase learning” (p. 755). Vukovic et al. (2014) asserted that students need to be able to reason about proportions and multiplicative relations in order to be successful in number line activities. This becomes especially significant when we consider improper fractions. Resnick et al. (2016) found that fourth graders and “low-growth” students estimate both proper and improper fraction as between zero and one. The authors suggested that these students did not engage effectively with these numbers due to their limited understanding of the relationship between the numerator and denominator and limitations in thinking multiplicatively. In their instructional recommendations, Siegler and Pyke (2013)
noted that, “Placing greater instructional emphasis on the need to view each fraction as an integrated magnitude that expresses the relation between its numerator and its denominator might avoid subsequent difficulties not only in fraction arithmetic but in learning of mathematics more generally” (p. 17).

**Framing Student Understanding**

One of the primary goals of a teaching experiment is to explore and understand student thinking as they interact with instructional tasks (Steffe & Thompson, 2000). The following section explores how understanding and students’ mathematics will be conceptualized within this study. I begin with Steffe and Thompson’s (2000) concepts of mathematics *of* and *for* students. I then explore shifts in understanding.

**Mathematics of Students**

The teaching experiment is structured to help researchers experience and seek to understand students’ mathematical learning and reasoning (Steffe & Thompson, 2000). Essentially, the researcher is working to create a model of students’ “mathematical realities” (p. 267). As Cobb and Steffe (2010) noted, “The explanations we formulate consist of models—constellations of theoretical constructs—that represent our understanding of children’s mathematical realities” (p. 19) At the same time, we must recognize that, “the models must be distinguished from what might go on in children’s heads” (p. 19).

Steffe and Thompson (2000) differentiated between “students’ mathematics” and “mathematics of students” (p. 268). The authors used the phrase “students’ mathematics” to refer to mathematics that are independent of our interactions with them. This “mathematical reality” is independent of ours as students are distinct, autonomous individuals. Steffe and Thompson argued we can never truly know students’ mathematics because mathematics is a living subject,
and each individual may conceptualize it differently. A basic goal of the teaching experiment is to construct models of students’ mathematics based on what students say and do when engaged in mathematical work (Steffe & Thompson, 2000). “Mathematics of students” refers to these models that we create as we try to conceptualize students’ mathematical realities. Mathematics of students is our closest approximation as we attempt to understand students’ mathematical models and thus what we might do in order to influence them.

**Shifts in Understanding**

I conceptualize shifts in students’ understanding within this study as shifts or movement in student thinking towards understanding or generalizing a concept. I connect this back to Piagetian ideas of abstraction. Mason (1989) discussed the idea of abstraction as referring to a shared “root experience: an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression as an expression of generality, to seeing the expression as an object or property” (p. 2). He went on to explain that in this sense, abstracting “lies between the expression of generality” and manipulating that expression for the aim of building a conjecture (as one possibility) (p. 2). Though I am not sure that these shifts need to happen in the “twinkling of an eye,” I do think Mason provides a starting place for understanding abstraction as the result of a small shift of attention or movement towards conjecture.

Mason helps us frame a shift in terms of movement based on Floyd’s (1982) helix used to the experience of moving through the process of abstracting. Students move from “manipulating objects (physical, pictorial, symbolic, mental) to getting-a-sense-of some feature or property of those objects, to articulating that property as an expression of generality” and back to using that generality or expression to again manipulate and seek out other properties (p. 3). Figure 2.2 shares a visual to help explain this movement.
Mason (1989) argued that the process of abstracting lies in the “momentary movement from articulating to manipulating” (p. 3). We can also see a shift even prior to students being able to manipulate a generality. As students begin to see and express patterns and ultimately create conjectures, we can hypothesize shifts in their cognition (Mason, 1989). Mason further helps us connect this type of abstraction to ideas of mathematical structure. As students are able to identify and use structure there is a certain level of abstraction required. Thus, the identification and potentially use of structure also signals that a cognitive shift has taken place.

In their book on constructing understanding of fractions, decimals and percents, Fosnot and Dolk (2002) drew attention to overarching “big ideas” which underlie a developmental progression of strategies and the overall structures of mathematics. They viewed these “big ideas” as “characteristic of shifts in learners’ reasoning— shifts in perspective, in logic, in the mathematical relationships they set up” (pp.16-17). Fosnot and Dolk frame the emergence of language such as “the greater the denominator, the smaller the fraction” or “the whole matters, ”
as big ideas that represent an important and substantial shift in thinking for children (p. 17). In keeping with Fosnot and Dolk, I too frame the elucidation of such big ideas as students themselves generate or grasp them as indicative of a cognitive shift. Additionally, I believe that the “smaller ideas” that may emerge as steppingstones to these big ideas may also constitute cognitive shifts.

I draw upon the above definitions to frame how I identify if students have experienced a cognitive shift. Though the nature of the mathematics of this study is less likely to lead to “expressions” suitable for manipulating, Mason’s (1989) concept of a manipulable expression as a marker of abstraction may be connected with students’ growing sense of the magnitude of fractions and decimals. As students construct an internal sense of the magnitude of fractions and decimals, they are able to manipulate, operate upon, compare, and estimate with these numbers without having to slow down and specifically attend to the components making up the number. Also in keeping with Mason, I identify a cognitive shift if students are able to identify and make use of structure as they work with decimals and fractions.

**Theoretical Framework**

Two prominent theories make their way into much of the research on fraction learning and fraction magnitude in particular: the theory of *integrated numerical development* (Siegler et al., 2011) and *Conceptual Change Theories* (DeWolf & Vosniadou, 2015; Stafylidou & Vosniadou, 2004).

**Conceptual Change Framework Theory**

The conceptual change framework was originally developed to explain learning in science but has since been adapted and applied to learning in mathematics as well. The conceptual change framework has three key definitional elements:
1. “The knowledge acquisition process is not always a process of enriching existing conceptual structures. Sometimes the acquisition of new information requires the radical reorganization of what is already known.”

2. “Learning that requires the reorganization of existing knowledge structures is more difficult and time consuming than learning that can be accomplished through enrichment. Moreover, it is likely that in the process of reorganization students will create misconceptions.”

3. “Many misconceptions are synthetic models that reveal students’ attempts to assimilate the new information to their existing knowledge base.” (Stafylidou & Vosniadou, 2004, p. 504).

Theories of conceptual change connect well with ideas such as “whole number bias” (Ni & Zhou, 2005). Children form an initial theory of number based on whole number counting, then encounter fractions and errantly generalize their previous understanding of whole numbers to fractions, encounter serious difficulties, and thus must replace their initial concepts in a “radical reorganization” (Stafylidou & Vosniadou, 2004, p. 504).

One of the key beliefs in a conceptual change approach is perceiving discrepancies between students’ conceptions of fractions and formal mathematical conceptions of fractions as regular stages of transition rather than individual deficits (Bertolone-Smith, 2016). Within conceptual change framework theory, children are seen as possessing “naïve theories” that they generate from their early interactions with social, cultural, and educational constructs (Vosniadou, 2007). Conceptual change is “the process in which a child adjusts their current naïve theory to include new information and build a more thorough framework of knowledge” (Bertolone-Smith, 2016, p. 19).
Integrated Theory of Numerical Development.

Siegler et al.’s (2011) integrated theory of numerical development also recognizes the critical differences between learning whole numbers and fractions, but simultaneously “emphasizes a crucial continuity that unites their acquisition – steadily expanding understanding of the connection between numbers and their magnitudes” (Torbeyns et al., 2015, p. 6).

Within this perspective, development of understanding of rational numbers involves both a gradual expansion of the range of whole numbers whose magnitudes are understood (from smaller to larger) and a conceptual change from an initial understanding of numbers in terms of characteristic features of whole numbers to a later understanding of rational numbers in terms of a single defining feature, their magnitudes. (Torbeyns et al., 2015, p. 6)

Thus, within the integrated theory of numerical development we see recognition of the conceptual change that must occur, but also stability and a deeper mathematical structure as students expand their understanding of numbers and their magnitudes. This differs from conceptual change theories in two main ways. First, this theory recognizes the beneficial role whole number magnitude knowledge may play in learning about fractions (Torbeyns et al., 2015). The second key difference is that the integrated theory views whole number interference as only one of several sources of students’ difficulties in learning fractions (Torbeyns et al., 2015).

Within the integrated theory, the reason fractions are more challenging to learn than whole numbers is the same reason they are critical to numerical development—“A fraction is a ratio or division of two whole numbers, the numerator and denominator, and is thus considerably more complex than a single whole number” (Torbeyns et al., 2015, p. 6). Fractions’ complexity
is what makes them so crucial for students’ mathematical learning. Part of the accompanying “central challenge of numerical development” is learning which “properties of whole numbers apply to all numbers and which do not” (Siegler et al., 2011, p. 343). An example of one of these properties ties back to the density property—whole numbers have exactly one predecessor and one successor, but this is not true for either fractions or decimals. Students may also have errantly come to a generalization when working with whole numbers that multiplying always yields a larger number and dividing always yields a smaller number. This no longer holds true in the fraction world when exploring multiplication and division of fractions.

An important core belief underlies both of these learning theories regarding fraction understanding: building understanding of all types of real numbers requires time and continual development from young children to adults (Ni & Zhou, 2006; Siegler et al., 2011; Stafylidou & Vosniadou, 2004). Both of these theories inform how I approach students’ learning within this teaching experiment. I chose to lean towards Siegler et al.’s (2011) integrated theory of numerical development for its larger, more-encompassing nature, while still drawing upon conceptual change theories to help myself understand the transition students make in in their reasoning.
CHAPTER THREE

METHODOLOGY

This study investigated how fifth grade students develop understanding of fraction and decimal magnitude using a constructivist teaching experiment framework. This chapter presents the research methodology. The chapter begins by revisiting the research purpose and questions. Next, I present an overview of, and rationale for the choice of, the constructivist teaching experiment. This overview is followed by the procedures for the study including participants, setting, data collection methods, and analysis strategies. Lastly, limitations of the study, reflexivity, and ethical issues are discussed.

Research Purpose and Questions

Understanding of fraction and decimal magnitude has been identified as a critical factor in students’ competence with fractions (Booth & Newton, 2012; Siegler et al., 2013). The purpose of this teaching experiment was to investigate fifth grade students’ developing reasoning on this critical concept. In accordance with the essential elements of a teaching experiment, we explored several iterations where I studied student thinking, interacted with students, and engaged in recursive analysis on tasks focused on decimal and fraction magnitude. This study sought to begin to answer the questions below as a means of moving towards better understanding of student reasoning about decimal and fraction magnitude and the instructional experiences that can help develop understanding and generalization.

1. How do fifth grade students reason about the magnitude of fractions and decimals?

2. What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude?
3. What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?

Rationale for Teaching Experiment Design

The teaching experiment fits within the larger research methodology of design research (Molina et al., 2007). I begin with an overview of design research, followed by a discussion of the teaching experiment methodology, as well as why this methodology serves the purposes of this study.

Design Research

As a whole, design research can be described as “a set of methodological approaches in which instructional design and research are interdependent” (Molina et al., 2007, p. 1). This research methodology aims to systematically design and study particular forms of learning, and the means educators use to support them while being sensitive to the unique ecology of learning and education (Cobb et al., 2003; Molina et al., 2007). Design research is frequently used in education settings to deal with complex situations, multiple dependent variables and to develop theories about learning processes while engaging flexibly in design revisions (Lamberg & Middleton, 2009, p. 233). The current study involves developing theories about domain specific learning processes and the instructional experiences that support them and is thus well served by the design research framework.

Cobb et al. (2003) identified five crosscutting features that apply to all subtypes of design research. The first crosscutting feature is the intent to “develop a class of theories about both the process of learning and the means that are designed to support the learning” (p. 9-10). Second, design research is interventionist and participatory in nature, facilitating researchers in investigating possibilities for student learning and educational improvement in natural settings.
The goal is to bring about new forms of learning and improve what is happening on a practical level. Researchers are not just observing but are attempting to engineer learning in some way. Together, the first and second features build to the third crosscutting feature of design experiments: developing theory, while simultaneously being open to revision of said theory. Extrapolating on this characteristic, Cobb and colleagues identified two “faces” of design experiments: “prospective and reflective” (p. 10). In other words, the researcher is simultaneously implementing designs with a hypothesized learning process and tasks to support them (prospective); while also fostering the emergence of alternative pathways for learning by “capitalizing on contingencies that arise as the design unfolds” (reflective) (p. 10). The fifth feature is built on the prospective and reflective aspects of design research. This is the iterative design characteristic of design research. This characteristic of design research gets back to its pragmatic roots— the theory must be accountable to the activity of design. The theory cannot reside in the larger abstract world of theory, but “must do real work” (p. 10). The theory should inform perspective design, not just project a grand theory of learning. Theories that emerge from design research should directly address the types of problems and challenges practitioners encounter in the field. All five of Cobb et al.’s (2003) cross-cutting features apply to the current study as I seek to use instructional experiences to support and cultivate theories about students’ developing reasoning about fraction and decimal magnitude.

**Teaching Experiments**

As previously noted, teaching experiments are a special subtype of design research. The teaching experiment emerged in the United States in the 1970s to address the gap in other models of research between the practice of teaching and the practice of research (Steffe & Thompson,
2000). Educators and researchers found they needed models that included accounting for students’ progress within mathematical interactions (Steffe & Thompson, 2000).

Steffe and Thompson (2000) articulated two major reasons leading to the emergence of teaching experiments. First, models educators might use to make sense of students’ mathematical thinking were developed outside of mathematics education and for other purposes. As educators tried to use these models to study students’ mathematical development, they began to realize models were needed that were grounded in mathematics education. Researchers began to realize they could only get so far through borrowing from other research fields such as philosophy or psychology as they attempted to explain students’ mathematical thinking and learning within the context of teaching. Instead, “models were needed that included an account of the progress students make as a result of interactive mathematical communication” (Steffe & Thompson, p. 270).

Second, there existed a large gap between the practice of research and the practice of teaching (Steffe & Thompson, 2000). According to Steffe and Thompson (2000), experimental methodologies used prior to teaching experiments had their roots in the “agriculture paradigm.” Within this paradigm, researchers select samples from a target population and subject them to “treatments.” The effect of the treatments is then compared in order to find potential differences between treatments. Though this seems to be a reasonable way to proceed, Steffe and Thompson argued that it failed because it “suppressed conceptual analysis in the conduct of research” (p. 270). Steffe and Thompson contended that this model separates the subjects from their learning and ignores how students make meaning and the meanings they make. Steffe and Thompson found the previous strong reliance on psychometrics in classical methods problematic. Tasks and thinking were objects “to be manipulated” in classical experimental research based in
psychometrics (p. 271). In this realm, researchers control students’ environments and in doing so seek to uncover “the reality of their knowledge” (Steffe & Thompson, 2000, p. 271). However, from a constructivist perspective, a “constant” stimulus does not exist; students construct for themselves their understanding of the tasks in which they engage, and it is their constructive process and understanding of the task that are worthy of our study (Steffe & Thompson, 2000). Thus, such design failed to consider the actual mathematics of students. The teaching experiment is intended to allow teachers to focus on the mathematical thinking of students and the interaction between instructional experiences and that thinking.

Four basic elements make up the teaching experiment methodology (Steffe & Thompson, 2000). The first element in a teaching experiment involves the researcher engaging in “exploratory teaching” to become “thoroughly acquainted, at an experiential level, with students’ ways and means of operating” in their particular mathematical domain (Steffe & Thompson, 2000, p. 274). The second element in the methodology is to test research hypotheses. This might include generating hypotheses before and during the teaching episode and may be one or more hypotheses. This is important because the primary job of the teacher-researcher is to hypothesize meanings that inform students’ actions (Hajra, 2013). The third element or component is the teaching itself. This teaching is fundamentally about the interaction between the teacher-researcher and student. The teacher-researcher is to attempt to limit their own prior expectations of what students will do so as to be open to the students’ own conceptions (Steffe & Thompson, 2000). This helps the teacher-researcher move towards the goal of building “living models of students’ mathematics” (Hajra, 2013, p. 11). The final element in a teaching experiment is retrospective analysis and model building. The teacher-researcher carefully reviews video
recordings of teaching and learning interactions to learn more about what went on during the interaction and the students’ thinking to build models (Hajra, 2013; Steffe & Cobb, 2012).

The teaching experiment methodology is rooted in constructivist learning principles (Steffe, 1991). Within the constructivist view, the nature of mathematical learning and knowledge is understood as “coordinated schemes of action and operation” (Steffe, 1991, p. 177). Students’ mathematical understandings are separate from our own and are indicated by their actions and words when they engage in mathematical activities. As researchers, we attempt to construct models of students’ mathematical understandings based on those words and actions. As discussed in chapter two, “Mathematics of students” refers to the models we build, including the “modifications students make in their ways of operating” (Steffe & Thompson, 2000, p. 268). Thus, we must ensure that our research methodology is designed to be a “flexible investigative tool that will enable us to study the mathematics of students” (Steffe, 1991, p. 177).

The teaching experiment offers the opportunity to dig deeply into students’ mathematical thinking and their construction of mathematical concepts. This methodology allowed me to focus on students’ thinking as they interacted with instructional experiences. It further provided the opportunity for me to adjust instruction in response to interactions with students (Steffe & Thompson, 2000). I follow in the footsteps of the researchers of the preeminent Rational Number Project in using a teaching experiment methodology as I seek to learn more about students’ thinking in connection with their instructional experiences with fractions and decimals.

**Conjectures and Hypotheses within Teaching Experiments**

According to Confrey and Lachance (2000), we can understand a conjecture to be an inference that is formed without proof or sufficient evidence. In the context of teaching and learning mathematics, this inference may relate to how “mathematics for educational purposes
should be organized or conceptualized or taught” (Confrey & Lachance, 2000, p. 139). Confrey and Lachance specifically separated out the idea of a conjecture from the idea of a hypothesis in a traditional experimental design approach. They argued that a conjecture within design research is not an assertion one is attempting to prove or disprove. In contrast to hypothesis-driven research where one is seeking to discover if a given intervention or theory worked or not, conjecture-guided research works to revise and develop the conjecture during the research (Confrey & Lachance, 2000). In this way, conjectures not only emerge to address specific practical issues at the beginning of the research, but also to help frame retrospective analysis of data (Cobb, 2000). Confrey and Lachance argued that while a hypothesis remains static throughout the research process, a conjecture may evolve as the research progresses. This conceptualization of the conjecture matches well with my own goals within this study as I worked to develop and revise theories about students’ understanding of fraction and decimal magnitude.

Confrey and Lachance (2000) further explained that the conjecture should have two “significant dimensions” (p. 139). First, all conjectures should possess a mathematical content dimension. In the case of the current study, that content is focused on how students develop understanding of the magnitude of fractions and decimals and the connections between the two. Second, a conjecture must have a pedagogical dimension connected to the content dimension that addresses the question, “How should this content be taught?” (p. 139). The latter dimension of the conjecture is explored through iterations of instructional experiences within this study. In particular, I used discourse, carefully selected numbers, tasks, and the number line to support student learning.
It should be noted that others in the field of design research and teaching experiments such as Steffe and Thompson (2000) do still use the term hypothesis. I find resonance with the idea of the continual revision nature of conjectures as explicated by Confrey and Lachance and thus use conjecture as it is situated within their definition throughout the course of this study. I also use the term conjecture to refer to students’ big ideas as they develop them. Thus, students have conjectures about the nature of fractions and decimals and their relationship to each other, and I have conjectures about the nature of students’ thinking and learning about these concepts.

**Conceptual Framework**

I ground myself in a conceptual framework that focuses on the relationship between students’ understanding of decimal and fraction magnitude and the interaction between their reasoning and instructional experiences. This framework ties back to Simon’s (1995) mathematics teaching cycle and to Steffe and Thompson’s (2000) focus on the interaction between instructional tasks and models of student thinking. Figure 3.2 presents this conceptual framework. The framework focuses on the interactive and cyclical nature of the work in a teaching experiment. Instructional experiences impact student reasoning and evidence and glimpses of students’ thinking inform future instructional experiences. This framework connects back to the larger teaching experiment purpose espoused by Steffe and Thompson (2000): “It is the researchers who are striving to learn what change they can bring forth in their students and how to explain such change” (p. 295).
Participants

Three students from one fifth-grade class in a suburban, Title 1, elementary school in a midwestern state participated in this study. The three students were selected in collaboration with the classroom teacher. Sample selection occurred throughout two rounds. Due to the nature of this study, selection began with convenience sampling. Many teachers are uncomfortable allowing a stranger access to their class. Therefore, I found it prudent to find a class where I already had some connection with the teacher. I began by reaching out to a teacher with a class that fit well with the purposes of this study. From there, I discussed how the research would work and ultimately selected a class where the teacher and I perceived that the teaching experiment methodology would be feasible for, and beneficial to, all parties involved.

My rapport with the teacher enabled me to not only gain me access to their class, but also allowed me to engage in conversations with them as I selected students for the teaching experiment. Additionally, selecting a teacher who was already known to me enabled me to have a filtering system for the type of instruction that was happening within the classroom. I chose to select a class where I knew something of the teacher and their teaching methods in an attempt to
minimize additional issues that could arise from students being unaccustomed to mathematical discourse or constructing their own understandings.

The selected teacher had been teaching for a little over three years. Discussions with the teacher prior to final selection enabled me to have a sense of the instructional approach used for fractions in the classroom. The school district used the Investigations curriculum which is a conceptually-based curriculum. The students had experiences folding and working with fraction strips. Initially, I had planned to conduct observations to learn more about instruction in the classroom, however, due to Covid-19 restrictions I was unable to do so.

Fifth grade was selected as the grade level of study because the content of the Common Core State Standards (CCSSM) at grades four and five aligns most closely with the mathematical content of this study: understanding of fraction and decimal magnitude through the use of comparison and ordering. Fifth grade status meant that students in this study had already experienced third and fourth grade learning regarding fraction and decimal magnitude. Students in the selected class were also regularly engaged in discussion and explanation of their thinking and larger mathematical concepts. This was an important precondition given that a significant portion of this study was investigating what students were thinking and asking students to express their thinking either in written or verbal form.

Within the selected class, I narrowed my focus to specific students. At this stage, I utilized a purposive sampling method with a goal of identifying three to five students who appeared to meet the needs of the study well. Figure 3.3 summarizes the sample selection process.
All students in the classroom took a fraction and decimal baseline assessment (see Appendix A). This assessment addressed basic components of fraction understanding as called for in the grade three and four CCSSM standards. Concepts and standards related to operations with decimals and fractions were not included as criteria to enable further depth on standards relating to fractions as numbers and comparing and ordering of fractions which are more relevant to the focus of this study. I reviewed the assessments of all students looking for which students appeared to be struggling most with understanding fractions as numbers and fraction and decimal concepts.

I used the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP) Fraction Progression Framework to analyze student results on the baseline assessment (Petit et al., 2022). The progression classifies student thinking as moving along a continuum, recognizing that students may move back and forth between levels as they learn new concepts or encounter
new problem situations. A basic overview of the framework can be seen in Figure 3.4 and the full framework can be found in Appendix B.

**Figure 3.4**

*OGAP Fraction Progression (Modified). Adapted from (Petit Consulting, 2014)*

I identified students within the early fractional and transitional strategies stages as identified by the OGAP fraction progression. Students with non-fractional reasoning would likely not have been ready for the magnitude concepts that are the focus of this study and students in the fractional and application stages would already be meeting many goals of the study. It is the developing understanding that occurs during the early fractional and transitional stages that was of interest in this teaching experiment. All of the students in the selected
classroom seemed to be past the non-fractional stage so I looked for the students within the early fractional strategies category who seemed like they were struggling the most with fraction and decimal magnitude concepts.

In addition to the baseline assessment, I consulted the knowledge and expertise of the classroom teacher in identifying students who may be willing subjects for small group instruction. My initial review and selection based on the baseline assessment for the most struggling students was affirmed by the classroom teacher. Three students emerged in this selection and the teacher confirmed they would likely be able and willing to work with me. The fifth-grade class happened to be almost 80% female, and this led to an entirely female participant group. All three students were fluent in the English language and none of the students had an Individualized Education Plan. I did not ask the students to disclose their racial background.

Setting

I had planned to hold small group sessions in a room outside of the students’ regular classroom; however, the Covid-19 pandemic led to significant changes in my facilitation plans. The shutdown of schools for Covid-19 prevented data collection from occurring in the intended semester or intended way. I was able to collect data the following year when students were back in the classroom; however, due to school district policy I was not able to physically be present as a visitor in the building. The classroom teacher and I developed a plan for me to meet with the small group over Zoom. The students were supposed to meet with me using one Chromebook in a meeting area outside of their classroom in order to facilitate them still being able to talk with each other. Unfortunately, on our first meeting day, the teacher realized a group from another class was utilizing that space. We pivoted and set up a plan for each student to be on their own Chromebook with headphones inside the regular classroom. This structure was not ideal and led
to many technology and instructional challenges. Additionally, working on their own Chromebooks in the main classroom made discussion between the students challenging.

All lessons occurred during the students’ regularly scheduled 30-minute intervention time so they would not miss class instruction. Initially, I planned on meeting with the students three times per week; however, once we began meeting and it became apparent how much time technology issues were taking from instruction and how frequently the students wanted to meet, we began meeting four times a week when possible. The teaching experiment lasted for five weeks total, ending on the students’ last day of school. I had planned on ending the teaching experiment earlier, but the students wanted to meet until the last day, and it was evident that we still had additional mathematical growth to pursue.

**Data Collection and Analysis**

Data was collected over five weeks at the end of the students’ Spring Semester. In order to minimize interference from class tasks and discussions regarding fractions and decimals, the experiment took place during a segment of time where the class was engaged in a unit of mathematics study other than fraction and decimals. This timing enabled me to study how the students were reasoning about fraction and decimal magnitude, their skill and comfort in generalization, and their critical ideas about what fractions are without interference from instructional tasks focused on fractions or decimals in the classroom.

**Data Sources**

I utilized multiple sources of data during the duration of the study which are summarized in Figure 3.5. Data sources include a baseline assessment, meetings with the classroom teacher, video recordings and transcriptions of all sessions, student work artifacts, discussions with the witness researcher, and my own research log.
Baseline and Concluding Assessments

This study began by gathering data on students’ beginning understanding and reasoning regarding fractions and decimals as numbers with a targeted focus on their understanding of magnitude. All students in the class from which participants were drawn completed a baseline assessment.

The baseline assessment focused on magnitude but also explored other fraction concepts such as unit fractions, partitioning, and iterating. This helped me have a more holistic picture of students’ baseline understanding of fractions and decimals as numbers. The baseline assessment can be found in Appendix A. I used the OGAP Fraction Progression discussed earlier in this chapter and found in Appendix B as I analyzed students’ baseline assessments.

Baseline assessments were used to both select small group participants and contribute to initial instructional plans. Only the assessments of the students selected for the small group were used for the latter. The baseline assessment also helped inform my knowledge of students’ entering understanding of fractions and decimals so as to be able to ascertain where shifts in understanding may have occurred throughout the course of the teaching experiment.

All three students also completed a concluding assessment during the last week of the teaching experiment. The concluding assessment can be found in Appendix C. The concluding
assessment provided further information on how students were reasoning with fraction and decimal concepts at the conclusion of the teaching experiment.

*Meetings with Classroom Teacher*

I had hoped to observe two to three mathematics sessions in the classroom prior to beginning the teaching experiment. Unfortunately, due to the Covid-19 restrictions I was unable to observe in the students’ classroom. I was able to hold conversations with the classroom teacher to discuss which students would be a good fit for the small group, and the students’ strengths and struggles with mathematics and small group work as perceived by the classroom teacher. Though this was not one of my more significant data sources, I think it is important to be cognizant of the fact that my conversations with the teacher may have impacted my impressions of the students’ and their mathematical thinking.

*Video Recordings and Transcripts*

All sessions were recorded via the Zoom video-conferencing platform. I watched each video recording prior to the following session and during retrospective analysis. I used the reflection protocol found in Appendix D to guide my reflections following each session. Video recordings of the sessions were transcribed during the retrospective analysis after data collection was completed. Recordings and their transcripts were analyzed for patterns and significant events, including, but not limited to, learning shifts, student struggles, and evidence of student generalization. All recordings and transcripts were included within initial examination of data. As the study progressed, I focused more on specific time segments, teaching episodes, or sessions where it appeared a shift in understanding may have taken place or evidence of a particularly important misconception emerged.
**Student Work**

All possible student work artifacts from all sessions and tasks were collected, reviewed, and analyzed continuously after each week and as a whole during retrospective analysis at the conclusion of the data collection phase. Written work included students’ notes, work on, and final answers to tasks. Some work artifacts took the form of screenshots of student papers, number lines, or fraction strips when paper versions of students’ thinking were not produced or turned in. Students also completed some work independently outside of session time which was also collected and included within the data set. I had to rely on the students to save all written work and return it to the class teacher for me to collect weekly so unfortunately, not all written work made it to me.

**Discussions with the Witness-Researcher**

Steffe and Thompson (2000) recommended that a witness-researcher work alongside the teacher-researcher in every teaching experiment. The goal of this study was for the witness researcher to attend the final teaching and learning episode during each week. Unfortunately, due to changes cause by Covid-19 for the initially intended witness-researcher and then unforeseen life events for a later selected witness researcher, in-time observations and meetings were not possible. I did engage the role of a witness researcher retrospectively in reviewing video recording and transcripts. I used the witness-researcher discussion protocol found in Appendix E to structure meetings with the witness researcher following teaching sessions. Both my written notes and recordings of the meetings were also used as data sources.

**Research Log**

I used the protocol found in Appendix D to reflect upon the events and conversations immediately following completion of each teaching episode. These reflections were recorded in
my research log. Data and notes from the transcriptions and my research log played a pivotal role in planning the next session as well as in later retrospective analysis. I also recorded conjectures and ideas as they emerged during data collection and analysis within my research log.

**Teaching Experiment Phases**

According to Gravemeijer and Cobb (2006), three phases make up a design experiment: (1) preparing for the experiment, (2) experimenting during teaching episodes, and (3) conducting retrospective analyses. Figure 3.6 shows the steps of the study color-coded to align with the three phases and their place within the overall timeline of the study.

*Figure 3.6*

*Timeline of Study*

**Phase One**

The first step in preparing for the experiment must be to clarify the mathematical learning goals (Gravemeijer & Cobb, 2006). I began this work in my review of the literature and
continued it as I moved into my interactions with the students. Previous experiences working with students, teachers, and pre-service teachers also contributed to my selection and synthesis of goals. Table 3.2 encapsulates the goals I set forth prior to any work with students.

Table 3.2

Learning Goals for Instructional Work

<table>
<thead>
<tr>
<th>Overarching learning goals for students:</th>
<th>Students will</th>
</tr>
</thead>
<tbody>
<tr>
<td>- understand fractions and decimals as numbers</td>
<td>- have a sense of the magnitude of fractions and decimals</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Targeted learning goals:</th>
<th>Students will</th>
</tr>
</thead>
<tbody>
<tr>
<td>- understand that a fraction is a number with a relative amount conveyed by the relationship between the numerator and denominator (Lamon, 2012)</td>
<td>- understand that fractions are quantities (Petit et al., 2016)</td>
</tr>
<tr>
<td>- understand a fraction as a number on the number line and be able to represent fractions on a number line (NGA &amp; CCSSO, 2010)</td>
<td>- understand that fractions with denominators of 10 and 100 can be written using a decimal point (NGA &amp; CCSSO, 2010)</td>
</tr>
<tr>
<td>- understand that fractions [and decimals] are numbers that extend the number system beyond whole numbers (Petit et al., 2016; Siegler, 2016)</td>
<td>- understand a decimal as a number on the number line and be able to represent decimals on a number line (NGA &amp; CCSSO, 2010)</td>
</tr>
<tr>
<td>- understand “decimals” represent a different type of notation for numbers which can also be represented in fraction form (Lamon, 2012)</td>
<td>- be able to compare two or more fractions by:</td>
</tr>
<tr>
<td></td>
<td>o reasoning about relative magnitude</td>
</tr>
<tr>
<td></td>
<td>o using unit fraction reasoning</td>
</tr>
<tr>
<td></td>
<td>o using benchmark reasoning</td>
</tr>
<tr>
<td></td>
<td>(OGAP Framework, 2012)</td>
</tr>
<tr>
<td>- be able to compare decimals using the meaning of a decimal as a fraction (NGA &amp; CCSSO, 2010)</td>
<td></td>
</tr>
</tbody>
</table>

As an additional preliminary step, I administered the fraction and decimal baseline assessment (see Appendix A) discussed in the previous sections. Lastly, I consulted with the students’ teacher to learn more about the teacher’s impression of the students as mathematicians. I reflected upon all of this information in conjunction with the major ideas currently addressing understanding of fraction and decimal magnitude to develop the initial instructional plans.
**Phase Two**

Phase Two included the teaching, reflecting, planning, and teaching again components of the study. Due to the iterative nature of a teaching experiment, both data collection and analysis were conducted in recursive cycles during this phase. Student work, video-recordings of teaching episodes, research log entries, and notes from and recordings of my conversations with the witness researcher were all collected as a part of this phase. Despite the name “teaching experiment,” the goal of this phase is not to try and prove whether or not the initial plan or theory works. The purpose is rather to test and improve theories about instruction that were developed in the preliminary phase (Gravemeijer & Cobb, 2006).

Phase two consisted of five weeks of instruction organized into six lessons. Each lesson’s work was comprised of three components: two to four teaching sessions, analysis of the data, and planning for the next set of related learning experiences (see Figure 3.8). Each session and lesson informs the planning and instructional experiences of the following episode and session.

**Figure 3.8**

*Overview of Lessons Making up Phase Two*

![Figure 3.8](image)

Each session built to the next session as interactions and student work from the preceding session help inform instructional plans for the following session. Figure 3.9 shares plans from the first week of the experiment as an example of our work. In accordance with the nature of the
teaching experiment, subsequent sessions were responsive to the learning and interactions which occur during the teaching and learning episodes of the first episodes and session.

**Figure 3.9**

*Plans for Lesson One*

**Phase Three**

In accordance with the teaching experiment methodology, data analysis was both ongoing and retrospective (Steffe & Thompson, 2000). Phase three included retrospective analysis on the entire data set collected during the experiment. The goal of this phase was to produce a coherent picture of the students’ progress and the interaction between instructional activities and student thinking over an extended period of time as they engaged in tasks centered on fraction and decimal magnitude. More specifically, I looked to identify shifts in understanding and the characteristics of instructional experiences that led to these shifts in understanding. It is
important that this phase not leave behind the reflections and analyses of phase two but add on new layers and ideas to the ideas generated in the previous phase (Gravemeijer & Cobb, 2006).

Within the retrospective analysis, I began with open coding to explore the initial data pieces (Cobb & Whitenack, 1996; Corbin & Strauss, 2014). I initially explored data in chronological order, treating data as isolated incidents, asking questions about them, and comparing incidents with each other for similarities and differences (Corbin & Strauss, 2014). In my first cycle of coding, I primarily used descriptive with some holistic and simultaneous coding emerging as well (Miles et al., 2014). As I found incidents that were “conceptually similar,” I grouped them together and began to note patterns and developed possible categories and themes while still being open to change (Corbin & Strauss, 2014).

The first round of retrospective analysis helped me develop an initial image of what transpired during the teaching and learning sessions. This first “image” was set “in terms of conjectures” which I then “tested against the whole data set” (Gravemeijer & Cobb, 2006, p. 96). I used the results of this analysis as the foundation for the next round of analysis where I aimed to identify patterns and potentially even explanations. I used the Dedoose computer program to help myself in these efforts. Dedoose is a web-based data management system that enables users to organize, sort, and analyze multiple types of data including, video, audio, and text.

Following the creation of this first “image” I conducted three additional rounds of conjecture testing as I moved towards my goal of developing conjectures of students’ schemes and operations of thought and where shifts in understanding may have occurred (Gravemeijer & Cobb, 2006). I pushed myself to see both the “forest” and the “trees” as I contemplated natural themes that emerged from the data (Merriam & Tisdell, 2016). I created several iterations of a
visual data matrix as I worked to see emergent categories and themes. I moved between focusing on apparent student understandings, misconceptions, and instructional experiences. My later cycles of coding led to significant changes in my initial codes as I managed, filtered, and refined the codes I had previously generated. My final codebook can be found in Appendix F.

**Limitations and Delimitations**

My research design faces limitations in scope and varying demographic variables such as age, gender, and school district. My analysis may be missing out on important factors and their implications for student understanding of fraction and decimal magnitude. This study looked at only one small group of students from one class. The instructional methods used both within the small group sessions in this study and the students’ homeroom adhere to the practices, goals, and vision of the National Council of Teachers of Mathematics (2014) in their text *Principles to Actions* and the National Research Council (2001) in their text *Adding It Up: Helping Children Learn Mathematics*. Both of these organizations and texts promote conceptual understanding-based mathematics learning in the classroom (this understanding and its components are discussed in chapter two). The results of this study may not fit well in a classroom or school less aligned with these goals and where direct instruction is the primary modality of learning.

Additionally, I recognize that students’ learning is impacted by more than just the instructional tasks given. This study did not investigate or account for several important potential factors such as (but not limited to): events occurring in the students’ classroom, students’ prior schooling experiences, larger learning challenges the students might be experiencing, students’ experiences prior to arriving at school each day, and students’ perceptions of interactions with adults and educators. Outside factors often have a significant impact on student learning. If a student is experiencing hardship at home, in school, or elsewhere, their learning is likely to be
impacted. Exploration of the impact of such factors was outside the purview of this study, but that does not mean such factors did not influence results.

Furthermore, every student is an individual and every classroom is unique. We cannot assume that replicating the instructional activities explored within this study would lead to the same interactions or identical learning for all students. This study’s findings, thus, are not generalizable to larger populations of students.

Lastly, this study faces limitations in that it involves inferences about the learners’ observed behaviors and conversations. I drew inferences based on my own background and experiences in the classroom in working with these students which limits the study’s objectivity. The Ethics section addresses methods used to overcome subjectivity and bias, but we can never assume these factors do not impact findings.

**Reflexivity**

I cannot ignore my background and its implications for both how the study unfolded and my own interpretations. My own experiences in this world as a white, female educator, and graduate student impact how I interact with others and how I interpret such interactions, including data. As a white, cisgender woman, I know I have to consider how my presence may impact how students respond, particularly if I am an outsider to a student. My background as a classroom teacher for nine years and as an in-service and now pre-service teacher educator undoubtedly positions me to act in certain ways as a researcher and to interpret data through particular lenses. I bring to the table preconceived notions of what it means to learn and truly “do” mathematics which also impact both my actions and my interpretations.

While my background gives me experiences and certain insights into students’ mathematical learning, it also has the potential to color how I interpret data. I tried to be cautious
to monitor and reflect upon how my biases may impact this study. This included being critical of my initial interpretations, looking for non-examples, and engaging in open dialogue with the witness researcher and other experts in the field to help me monitor and check my biases.

The teaching experiment was designed in order to eliminate the separation between research and the practice of teaching (Steffe & Thompson, 2000). Within a teaching experiment, the teacher’s and students’ actions are “codependent” during all interactions (Steffe & Thompson, 2000, p. 301). This means, however, that the researcher must be cautious about the assumptions and mathematical knowledge they are imputing upon the students. The teacher-researcher must simultaneously use their mathematical knowledge to seek to understand and support students’ thinking while also trying to set aside that understanding in order to learn from the students (Steffe & Thompson, 2000). Thus, I tried to be conscious of my own mathematical knowledge and assumptions as I attempted to build models of students’ mathematical thinking.

Ethical Issues

The following section details ethical and confidentiality considerations within the course of this study. Technical components relating to IRB approval, consent, and confidentiality are shared first, followed by a discussion of credibility and validity.

IRB Approval and Participant Consent

CITI training for IRB-Social and Behavior Researchers 1- Basic Course was completed on April 22, 2019. Human subjects approval was granted by the Institutional Review Board of the University of Wisconsin–Milwaukee in February 2020. Official approval was obtained from the school district under study in January 2020.

Students are considered vulnerable populations, so I was careful to obtain both their assent and their caregivers’ consent. Consent letters were sent home to students’ caregivers in the
caregivers’ home language in May 2021. All three participating students returned consent letters immediately. Study goals and methods were outlined in consent letters and caregivers were informed they may remove their child from the study at any time. Students and caregivers were also told and reminded that participation or nonparticipation in the study would have no bearing on their grade, placement in classes at school, or any other opportunities that may arise. They were also assured that all student information from the study would remain confidential. Verbal assent from participating students was also sought at the beginning and throughout the study.

**Confidentiality**

I engaged in several steps in order to ensure participant confidentiality. First, I deidentified any information and used single letter pseudonyms for all participants. Additionally, I am withholding the name of the teacher, school, and district, and seek to limit identifying descriptors beyond basic demographics for both the school and district. I used either Zoom’s automatic or an online digital transcription service with additional review by only myself and the witness researcher. All transcripts and recordings have been stored on a password protected computer. I was also careful to keep our focus on mathematics and not to collect information that has the potential to cause harm to participants.

**Establishing Trustworthiness**

Trustworthiness and credibility are essential when conducting qualitative research studies (Lincoln & Guba, 1985). Merriam and Tisdell (2016) identified the challenges qualitative research has faced in the past few decades in establishing validity and reliability. In accordance with these challenges, rather than framing an assessment of my study’s validity around whether or not it reflects reality, I used Lincoln and Guba’s (1985) “notion of credibility” which asks, “Are the findings credible, given the data presented?” (p. 242). Lincoln and Guba’s (1989)
fourth generation evaluation model emphasized the “importance of evolving and emerging constructions” which matches well with this constructivist research design (Confrey & Lachance, 2000, p. 156).

Lincoln and Guba (1989) asserted that neither internal nor external validity can truly exist. They argued that internal validity is “nothing more than an assessment of the degree of isomorphism between a study’s findings and the ‘real world,’” and thus “cannot have meaning as a criterion in a paradigm that rejects a realist ontology” (Lincoln & Guba, 1989, p. 236). External validity or generalizability as we may think of it also has little meaning if we consider constructivist notions of “reality” which are different within different minds based upon groups and individuals’ different experiences (Lincoln & Guba, 1989). Lincoln and Guba (1989) used credibility as a parallel to the traditional notion of internal validity. Rather than focus on a presumed reality, the goal is to match the “constructed realities of respondents” and the representations of those realities by the researcher (p. 237).

My first step in establishing credibility was to use persistent observation. I aim to use “sufficient” observation to identify the most critical and relevant characteristics and elements in the situation and to delve into them in detail (Lincoln & Guba, 1989, p. 237). Second, I used peer debriefing. Peer debriefing is the process of engaging in deep discussions of findings, tentative analyses, and conclusions to “test out” the findings with someone who has no vested interest in the results (Lincoln & Guba, 1989, p. 237). The witness researcher is well situated to engage in this debriefing process. Through an outsider posing probing questions it helped me to understand my own biases and values and their impact in the inquiry and test out working hypotheses.
Lincoln and Guba (1989) further replaced the notion of generalizability with that of transferability. As opposed to generalizability, transferability is relative and depends on the extent to which essential conditions match (Lincoln & Guba, 1989). Here, the “burden of proof” is on the inquirer to determine if the situations match enough that they can transfer findings (p. 241). Within my results/conceptual analysis section I attempted to build thick descriptions of what transpired that will allow readers to make their own decisions regarding transferability (Creswell & Poth, 2018). Though the focus is not on my participants or the setting per se, it is on the learning experiences and understandings, my hope is that such thick descriptions will bring the reader to the learning sessions and help them experience the learning that is taking place. Additionally, my goal is that rich, thick description will help the reader determine if their situation matches the interactions here to a high enough degree that they can determine whether the conclusions of this study can be applied to their own situation.

Steffe and Thompson (2000) specifically raised the point that issues relating back to generalizability are significantly different for teaching experiments. The researcher’s analyses are dynamic concepts that may be used and potentially modified in future interactions with students. If the teacher-researcher’s way of thinking about students’ mathematics is useful for interpreting the mathematical activity of students not involved in the original study, then we are provided with “critical confirmation of our way of thinking” (Steffe & Thompson, 2000, p. 300). In this way, we are not generalizing results to others hypothetically; rather, results are useful in organizing and leading our work with students engaged in the mathematics (Steffe & Thompson, 2000). This conceptualization of generalizability seems to fit well with Lincoln and Guba’s (1989) notions of transferability as we consider the structure and goals of a teaching experiment.
Dependability is often considered another important hallmark of trustworthiness and this “concept of dependability speaks to the stability of the data” (Confrey & Lachance, 2000, p. 156). However, in a conjecture-driven teaching experiment we are not necessarily looking for stability of the data. In fact, rather than being a threat to dependability, changes and shifts are indications of a growing and successful inquiry (Guba & Lincoln, 1989). However, these changes and shifts do need to be “both tracked and traceable (publicly inspectable)” so that outsiders may study the process, assess decisions that were made, and understand what critical factors in the situation led the researcher their decisions and interpretations (Guba & Lincoln, 1989). As such, I attempted to keep detailed track of my methodological and analytical decisions within my conceptual analysis chapter. This record of the research process will hopefully enable outsiders to follow and assess the “reasoning behind the methodological path taken through the data” (Confrey & Lachance, 2000, p. 156).

Lastly, within the construct of confirmability we want to ensure that the research findings are “grounded in the data rather than in the whims of the research team” (Confrey & Lachance, 2000, p. 156). An external observer should be able to reconstruct the findings using the data if they are indeed confirmable. To enable this to happen Guba and Lincoln (1989) suggested the researcher provide an audit trail. Through this audit trail I kept evidence of the circumstances in which the methodological decisions were made and the data that were analyzed so that an external auditor might reconstruct the process and findings (Confrey & Lachance, 2000).
CHAPTER FOUR

CONCEPTUAL ANALYSIS OF TEACHING AND LEARNING

The purpose of this chapter is to provide the reader with an image of the teaching and learning which occurred throughout the teaching experiment. The teaching experiment was organized as six larger lessons with each requiring two to five sessions with students. Table 4.1 provides a summary of each lesson’s key tasks.

Table 4.1

Teaching Experiment Lessons

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Tasks</th>
</tr>
</thead>
</table>
| Lesson 1 | Establishing a Foundation  
Task 1: Fraction Strips & Unit Fractions  
Task 2: Representing Fractions |
| Lesson 2 | Task 3: Greater or Less than One-half?  
Task 4: Pushing for Precision |
| Lesson 3 | Task 5: Extending to Tenths and Hundredths  
Task 6: Tenths on the Number Line  
Task 7: Building Understanding of the Number Line  
Task 8: Building the Number Line Another Way  
Task 9: Working with a Number Line from Zero to Ten |
| Lesson 4 | Task 10: Identifying Pre-Marked Locations on the Number Line  
Task 11: Closest to the Whole  
Task 12: Closest to…?  
Task 13: Closest to One |
| Lesson 5 | Task 14: Back to the Number Line  
Task 15: Closest to One Whole  
Task 16: From Tenths to Thousandths  
Task 17: Unfriendly Numbers on the Number Line |
| Lesson 6 | Task 18 Bigger or Smaller  
Task 19 Comparing Fractions, Looking for Flexibility  
Task 20 Fraction and Decimal Pair Comparisons  
Task 21 More Comparisons  
Task 22: Ordering Fractions: 2/7, 1/12, 5/9 |
The following sections are not an exhaustive recounting of what occurred during the teaching experiment, but rather a selection of key tasks, responses, and exchanges to present the reader with a picture of the students’ mathematics as I understood them. Throughout the discussion I am intentional in trying to mirror how fractions were presented and discussed. If fractions were shared verbally, they are represented in words (e.g., 3-tenths). If fractions were represented symbolically such as on a number line, they are written in traditional fraction notation (e.g., 3/10). Some pictures of student work were recreated to increase legibility or to conceal identifying features. Students’ baseline and concluding assessment results are included in Appendix H. Discussion of students responses from the baseline and concluding assessments are integrated throughout this chapter and the following chapter.

**Lesson 1**

Lesson One took place over the first two sessions and the beginning of the third session. I had three goals as we entered into Lesson One: (1) explain to the students how the teaching and learning sessions would work and ask for their assent, (2) build rapport with the students and establish norms for how our time together would proceed, and (3) learn more about the students’ current state of thinking about fractions and decimals.

**Establishing a Foundation**

We began Lesson 1 with introductions, explanations, and a brief getting to know you game, as well as obtaining assent. I then gave the students an open task to begin our work.

Our first task asked the students to tell me everything they knew about the numbers 2/3, 4/3 and 0.37. I clarified that there were no right or wrong answers, but that I would ask them to share their thinking after they had had some time to record. I noted to the students that they might draw pictures, think about what the numbers are like in connection to each other, where
they might see them in a situation, consider how big the numbers were, or where they belonged on a number line. I intended for this task to be as open as possible to provide a window into what the students tended to engage with when presented with fractions.

The students seemed to focus on equivalencies in their thinking. Student Z shared that \( \frac{2}{3} \) is equivalent to \( \frac{4}{6} \) and \( \frac{4}{6} \) is equivalent to \( \frac{8}{12} \). Student E volunteered that \( \frac{4}{3} \) is equal to 1 and \( \frac{1}{3} \). I pressed Student E to explain how she knew \( \frac{4}{3} \) was equivalent to 1 and \( \frac{1}{3} \). She explained, “Because 4 and \( \frac{1}{3} \), is um \( \frac{4}{3} \) the 4 is bigger than the 3 so there would be one whole and then we have one third.” Student J then shared that 0.37 could also be written as \( \frac{37}{100} \).

The time we were able to spend engaged in mathematical thinking and discussion was brief in the first session but provided me a glimpse into the students’ initial thoughts when presented with fractions and decimals. It appeared the students seemed most comfortable and most likely to engage with these numbers by renaming them using equivalency. I had intentionally included two fractions with the same denominator to give students the opportunity to note the special relationships between those two fractions (i.e., \( \frac{2}{3} \) is smaller than \( \frac{4}{3} \), \( \frac{4}{3} \) is twice as big as \( \frac{2}{3} \), etc.), but size did not emerge as a topic during our discussion. This could have been due to a lack of time, or because the first shared thought related to equivalency; however, it is also possible the students felt more comfortable in the world of equivalency rather than addressing the size or magnitude of the numbers.

**Task 1: Fraction Strips & Unit Fractions**

We began our second session by folding fraction strips. Students had previously folded fraction strips during their class unit on fractions. Despite prior experience, I believed engaging in folding work was important for several reasons. First, the students’ fraction unit had been several months before. Second, folding the strips together would provide me a window into how
students engaged in and talked about folding fraction strips. Third, I wanted to use the strips folding as a time to bring forward and establish common fraction language such as partition, unit fraction, size of parts, and number of parts.

All three students were able to handle folding fourths and eighths easily and efficiently. When asked about connections between the strips, Student Z noted that, “Like four if you just fold it one more time you get eighths.” I asked what would happen if we folded our eighths in half again. Student Z responded that we would get sixteenths. When asked how she knew it was sixteenths she responded, “Because it's basically so if you folded the four again, it would just double the amount. So that would be eight and then if you fold it—eighths again, then it would be sixteenths” Student Z’s response seemed to affirm that she had some understanding about the impact of doubling and the relationship between these numbers which allowed us to move to our next topic.

We moved to discussing unit fractions with our fraction strips next. Students indicated they had discussed unit fractions in class, but when pressed they were unable to recall what they were or any examples. I had the students use their fourths strips to find and model how just one of the shares of the whole which had been folded into four equal parts would equal the unit fraction of one-fourth.

Following this, I asked the students to show 3-fourths using their fourths strip. I questioned students as to where we could see unit fractions on our 3-fourths strip representation. Student E responded “in between the lines” while Student Z held up one of the fourths partitions to the computer screen. I then asked all three students to hold up their strips and show me where they saw each fourth. Using their hands and strips, they folded and pointed to show each of the
fourths. All three students correctly responded “three” to the question “How many one fourths are there in 3-fourths?”

To solidify the concept of unit fractions we continued with 4-fourths, 6-fourths, 3-eighths, and 78-hundredths. The students’ affirmative response that they remembered unit fractions, but their inability to recall what unit fractions were, indicated they may have entered the lesson not firm in their knowledge of unit fractions. At the conclusion of our brief unit fraction work, it appeared the students were able to identify the unit fraction and the number of unit fractions used to build each non-unit fraction. I felt comfortable that the students were able to move from unit fractions into our next task.

**Task 2: Representing Fractions**

In order to spend a small amount of one-on-one time with each student, I set the students up with independent work while I pulled each student to a break-out room. I wanted to learn more about how the students were thinking about and visualizing fractions that were not as friendly for drawing. This intent framed our work in the break-out room. Students were each asked to verbally explain how they would show the following fractions: \( \frac{11}{5}, \frac{199}{200} \).

Overall, this one-on-one time provided me with more insight into how the students were thinking about these two “unfriendly” fractions, as well as their comfort in visualizing a physical or pictorial representation. After some questioning to support them in talking and clarifying their thinking, Students J and Student E demonstrated relative ease in explaining how to represent both fractions. Student J needed a bit more prompting connected directly to building a visual model to get started but was able to comfortably explain once given the prompting.

Student Z readily explained how she would show \( \frac{11}{5} \). The second fraction, \( \frac{199}{200} \), however, was not quite as smooth. I questioned Student Z as to what she was envisioning for the
number. Student Z’s and my exchange regarding how to represent 199/200 on fraction strips presented an interesting window into how she was merging her understanding of fraction strips, tenths and hundredths, and a fraction that is one part away from becoming one-whole.

Student Z: Um I would fold tenths 10 times because 10 times 10 is 100. And then I would do it 10, another 10 times and fold one. So, there's just nine.

Teacher: So, you would have just nine of those tenths?

Student Z: Yeah.

Teacher: So, you fold it in tenths first, is that right? And you fold it in tenths again. So what size were your pieces then?

Student Z: My pieces like were tenths. They were in the tenths. They were folded as tenths and so was the other 10-tenths.

Teacher: So, you have two strips? So, you did one in tenths and then another strip in tenths?

Student Z: Yeah. And then I folded one. So, it only looked like there was nine instead of 10.

Teacher: Oh, so then you have one. That's just nine of those tenths and then one that is ten tenths. Is that right? (Student Z nods) So where do we see the two-hundredths in there?

Student Z: If you unfold the last one.

Teacher: Then we'd have two hundred equal pieces?

Student Z: Or if you like, shaded in every single one of those squares up to the last one and don't shade in the last one.

Teacher: Then we would have our 199 for the two-hundredths?
Student Z: Yeah.

This exchange showed that Student Z was still working to solidify her understanding of the relationship between tenths and hundredths, as well as how to represent numbers such as this. I knew that it would be critical to return to work with tenths and hundredths with intentionality.

For our first task back together in the main Zoom room I asked students to share some equivalent numbers for 3-thirds. Student E quickly responded with an answer of 6-eighths. I asked her to explain and how she might prove their equivalency. At that point Student E self-corrected from 6-eighths to 6-sixths. I asked her why she made this change and she explained, “Cause six is equivalent to three. And whatever we do to one part, we have to do to another, so we multiply both of them by two to get 6.” In Student E’s explanation we can see language that is based more in following a procedure rather than visualizing what is happening to the values. She had an awareness that whatever “we do to” one part of the fraction must happen to the other part, but no explanation as to why this was the case. Her initial statement that six was equivalent to three also seems to indicate some sort of gap in true understanding of what it means to be equivalent. We worked together to draw a picture of 3-thirds and then partitioned it to show 6-sixths to support the students in making a connection between the “rule” and what was actually happening with the fraction. Student E then extrapolated to several more equivalent fractions.

Student Z shared “a whole” for her equivalency to 3-thirds. She explained her equivalency saying, “Because 3-thirds if you like made it into a fraction and you shaded all of them in then that would equal one whole.” This explanation suggests Student Z was firm in connecting a visual model to the numerical value of numbers such as 3-thirds and was able to use that thinking to help herself generate multiple equivalencies.
Summary of Learning Lesson 1

Lesson One contributed to a provisional baseline image I was building about the students’ understanding of fractions and decimals and their understanding of magnitude. I noticed the students appeared to be comfortable orienting towards equivalencies and less likely to engage with the magnitude of the numbers independently. I wanted to push the students to engage with the magnitude of fractions and decimals more readily and thus affirmed plans to engage in sorts using the benchmark numbers of zero, one-half, and one-whole in the following lesson.

Lesson 2

Lesson 2 took place over two sessions and was designed to encourage the students to engage with benchmarks and develop in their use and explanation of them. The sessions focused on the benchmarks of zero, one-half, and one-whole. Given the students’ comfort with equivalencies in the first lesson, I conjectured they would be able to draw upon this understanding to assist them in using benchmarks to assess and assign magnitude for relatively friendly fractions. I wondered, however, if, and how, they would generalize this line of thinking to less friendly fractions.

Task 3: Greater or Less than One-half?

Students first engaged with a “Greater or less than ½?” table to support them in their sorting work. Students were asked to place each fraction in either the “less than ½” or “greater than ½” side of the table and explain their choice. Numbers were ordered to support student thinking as they developed a foundation and then tested their reasoning strategies with less friendly numbers. Figure 4.1 shows Student Z’s completed table.
Students handled the first two numbers for the sort, 15/15 and 3/8, with ease. Student E identified 15/15 as equivalent to one-whole and thus greater than one-half. All three students placed 3/8 in the “Less than ½” column. Student Z explained: “Because 4-eighths would be half and um three is one less than four, so it would be less than one-half.” The number 3/8 was intended to allow the students to utilize their knowledge of equivalency as they began to work on comparing to a benchmark. The students had demonstrated strategies for creating equivalent fractions during the baseline assessment and initial discussions so I conjectured they would be comfortable finding the equivalent number of eighths for one-half. Their ease with this number seemed to confirm my initial conjecture.

Next, students were tasked with applying this reasoning to a less friendly number: 4/7. The students took more time to place this fraction on the chart, but ultimately placed it as larger than one-half. Student E explained, “Cuz 4/7 is really close to, I think it's greater than one-half, cuz 4/7 is really close to um, well 7 is really close to 6 and it's 4 so and 4 is really close to 3 so it would be, well actually and both of those numbers are greater than 6 and 7, the 3 and 7 and 6 so...
it would be um greater than.” Student E’s thinking here shows her making a connection to a “friendlier” fraction that is equivalent to one-half (3/6). She was able to consider the relationship of the given numerator and denominator back to this equivalency to one-half and used them to assist in determining if the fraction was greater or less than one-half.

To assess the students’ generalization of this type of thinking I next gave them the number 3/7. I asked students to not only be ready to justify their placement, but also to see if they could make any connections in their work with 4/7 as they were thinking about 3/7.

Student E responded first, explaining that 3/7 is smaller than one-half “because the 4/7 was more than one-half, so it, 3/7 is smaller than 4/7 so it must be less than one-half.” Student E was correct in her statement but had not truly justified 3/7 as smaller than one-half yet. Student Z responded with slightly different thinking than Student E’s, explaining, “I think 3/7 is right next, it's basically 4/7 just one less shaded in. So that means I think it would be less than one half because 4 um 4 times 2 would equal eight. And it's seven, so then you have to do one less.”

Student Z drew on a “friendlier” fraction (4/8) equivalent to one-half similar to Student E. Student Z’s explanation, however, went one step farther as it more directly addressed the relationship between 4/8 and 4/7 and why that makes 4/7 larger than one-half. Both students’ made connections to nearby friendlier fractions to help them reason with these less friendly numbers. In this way, they both appeared to be using benchmarks and equivalency to help themselves grapple with the magnitude of this less friendly fraction.

**Task 3, Part B: Decimals and Benchmarks**

I wanted to see how students would reason with comparing decimal fractions to one-half. I presented 0.501 as their first number. All three students placed this number as larger than one-half. Both Student Z and Student E identified that 500-thousandths was equal to one-half, and
that 501-thousandths was larger than 500-thousandths. Student Z seemed to connect more specifically to her knowledge of whole numbers saying, “I think it's more than one-half because 500 is half of a thousand and it's 501. So that's one over so I think it's more than one half.” In her response, Student Z successfully used and extended her understanding of whole number magnitude to assist her with assessing the decimal’s magnitude.

Next, students correctly identified that 0.489 was less than one half but seemed less comfortable when pressed for justification. Student E tried to draw on place value reasoning but struggled to explain why ultimately. The following vignette shows where Student E began and where we ended with Student Z’s interpretation as they began to explore the distance from the benchmark of 0.500.

Student E: Because the last one was zero and 501 thousandths. So, this one's smaller than the last one, so it must be um less than one half.

Teacher: How do you know this one is smaller than the last one?

Student E: Because um, it because it, the four, and the four is, the four is smaller than the five, so… (long pause) Ah I have no idea.

Teacher: Okay take some time to think about it. Student J, Student Z, what are you thinking about?

Student Z: I think the same thing because. Basically, what Student E said, I agree with her.

Teacher: You agree with her. How do you know that 0.489 is smaller than 0.500?

Student Z: Because um, it's uhh, hmm 10, no 11 smaller, than 500 and 500 is like the benchmark.
Student Z’s thinking could benefit from further flushing out, but in examining her statement, it is evident she is thinking about (1) the equivalent number of thousandths needed for one half and (2) the distance from the number under consideration (0.489) and one-half. Though she indicated agreement with Student E’s initial thoughts, her thinking seems to go in a different direction. Student E seemed to be attempting to use place value knowledge (conceptual or rote) to compare the 4 in 0.489 and the 5 in 0.500 but became stuck when she attempted to explain. Student Z, on the other hand, was beginning to apply residual reasoning and benchmarks as she considered that 0.489 is 11 (thousandths) less than one-half. It is worth noting both students were able to use the equivalency of thousandths to one-half as they worked with this decimal fraction.

In the expectation to explain and justify their thinking, Student E became hung up trying to apply reasoning that she was not completely secure in. It is unknown if that is because the reasoning was not conceptually based or if she had simply forgotten, but the start and eventual tapering off of her explanation provide a window into how she was thinking about comparing these numbers.

We were only able to begin work with our final number of the day: 0.09. Students Z and Student J were having technology issues, but Student E identified that it should be in the “Less than ½” column. When pressed to justify she explained that “Zero and 9 tenths is really small, cuz like it’s in the tenths place.” To clarify, I asked her if it was 9 tenths, or 9 hundredths and she responded, “tenths.” To address this misconception, I asked her what a picture of 9-tenths would look like. Technology and timing issues prevented us from continuing our work and I made a note to return to the number 0.09 to learn a bit more about Student E’s thinking as well as see how the other two students would handle this number.
Task 4: Pushing for Precision

To test and deepen student thinking in Task 4 we added a layer to our placement chart going beyond comparing to one-half to consider if the given numbers were closest to zero, one-half, or one whole. A picture of Student J’s table is included in figure 4.2.

Figure 4.2

Student J Sorting Table, Task 4

Task 4, Part A: Smaller than ½, but is it Closer to Zero or ½?

We began our sort with the fraction 3/7. Though all three students were able to identify that 3/7 was smaller than one-half in the previous session, I was not confident they were fully grappling with its size yet. I hoped to see how they would reason with the number 3/7 when confronted with a more precise sorting table.

Student E placed 3/7 correctly as “Closer to ½,” but Student Z and Student J both initially placed it as “Closer to zero.” Student E explained, “I think it's closer to one-half because 3-sevenths is really close to 3-sixths and 3-sixths is one-half.” I pressed for further justification and Student E seemed to then draw upon her understanding of the impact of partitioning to support her response, explaining, “Because if you divided 3-sixths into, if you like, added one more line or partition it into one more, it would turn into the seventh. So, it would be really close to sixths.”
This reasoning does not mathematically prove her conjecture; however, it does demonstrate consideration of the impact of partitioning, the size of the pieces, and the relationship of these fractions to each other as she grappled with the given number’s size.

Student Z disappeared during Student E’s explanation, but Student J responded that she changed her thinking after hearing Student E’s reasoning and agreed with her. Since Student Z and Student J did not explain why they placed 3/7 where they did, I am not certain what reasoning they were using; however, their initial placement does hint that though they knew 3/7 was smaller than one-half, they possessed a misconception or a gap in their sense of its magnitude.

**Task 4, Part B: What is the Distance from 1/2 and zero? A Critical Moment**

Our next number, 2-tenths, seemed to present little challenge for the students and they all placed it “closest to zero” on the slide. Student E justified her placement saying, “Because anything in a tenths except that’s right by 5-tenths is really small. So, it, the 2-tenths, would be very small, so we could put it, it would be really close to zero.” Though Student E’s thinking is true and certainly worked for this number I was not confident from her explanation that she (or the other two students) was fully thinking about the size of this fraction.

To test this thinking, I offered up a fraction that I had not planned to use: 4-tenths. I asked the students if they thought 4-tenths would be closer to zero or one-half. All three students responded, “zero.” This presented an opportunity to push their thinking as they appeared to potentially be over-generalizing their reasoning from 2-tenths.

I hoped that asking the students to create their own models and using a series of questions would help to address the overgeneralization. I asked the students to make a tape diagram split into tenths and shade in 4 of the tenths. Then I asked how many of the pieces of size one-tenth
we would shade to show half of the tape diagram. Student E directed me to shade 5 of the tenths, so I did so in a different color than 4-tenths. Next, I prompted the students to look at their pictures and consider if 4-tenths appeared to be closer to half full or zero. Student J and Student E then responded, “half of it being full.” Seeing the fraction on the tape diagram alongside the equivalent one-half value appeared to help the students accurately identify its relationship to one-half.

To support and assess students in generalizing this strategy, I next asked them to determine if 3-tenths would be closer to zero or one-half. Without needing to draw, Student E quickly answered, “One-half” because “it’s 2-tenths from one-half.” Modeling 4-tenths on the tape diagram and using the visual representation to explore the distance of 4-tenths from zero and one-half appeared to be beneficial both as they applied it to 3-tenths and later in this session and beyond.

To see how students would engage with fractions with larger denominators I asked them to place 40-hundredths next. Student E wrote the following on her paper and held it up to the camera. Student E appeared to be carrying forward the reasoning we worked on developing with 4-tenths in looking for its distance from one-half.

\[
\frac{40}{100} + \frac{10}{100} = \frac{50}{100} = \frac{1}{2} \text{ or } \frac{50}{100}
\]

Student J expressed similar reasoning verbally, saying “Because if you were to add one more tenth, or 10 more, then you'd get 50-hundredths, which would be half of it.” Both students appeared to be tapping into this idea that it is beneficial to consider the distance of the given
fraction from the benchmark as they determined if the fraction was closer to zero or one-half. This “distance from” reasoning became a common theme as we moved forward in our work.

**Task 4, Part C: Extending to More Unfriendly Numbers**

The students again demonstrated the ability to anchor back to numbers beyond one-half with the number $17/12$. Student E recorded on her paper which she held up: “$17/12 = 1$ and $5/12$.” Student Z verbally shared, “I think it's greater than one because the denominator is 12 and the numerator is 17 and that goes over one so it would be one and …”

The students seemed comfortable with these numbers so I wanted to provide them with a number that might challenge their thinking. I brought back the decimal fraction 0.09 from the end of our previous session. Our time had to end before I was able to have students share their thinking for 0.09; however, I was able to see that both Student E and Student J had placed it in the “Between: zero and $\frac{1}{2}$, Closer to $\frac{1}{2}$” column on the chart (Student Z did not place the number). This placement was intriguing for me, and I knew it would require more than a brief explanation at the very end of our session. Similarly to the end of Session Three, I made a note to return to the number of 0.09 again.

**Summary of Learning Lesson 2**

Lesson Two made clear that the students were beginning to apply benchmark and distance reasoning as they grappled with the size of fractions. Students appeared to be tapping into equivalencies, “distance from” reasoning, and friendlier fractions to help themselves reason with benchmarks and magnitude. Based on the baseline assessment and our beginning work with 0.09, I wanted to push students to work more on the relationship between tenths and hundredths. I also knew it would be critical to continue to press for justification to explore more richly what the students were truly thinking about these numbers.
Lesson 3

Lesson 3 took place over four sessions and was designed to engage the students in reasoning about the relationship between tenths and hundredths, and how they would represent fractions with tenths and hundredths denominators on fraction strips and on number lines. I was curious to learn how students engaged with this critical relationship and if they would be able to apply their understanding of tenths and hundredths to further their magnitude work. This work turned out to be challenging for the students and illuminated several areas of fragility.

Task 5: Extending to Tenths and Hundredths

Task Five focused on exploring tenths and hundredths. Students folded paper fraction strips into tenths and then envisioned making connections to hundredths and beyond. The goal in this work was for students to develop their understanding of the relationship between tenths and hundredths and to establish a foundation as we transitioned to the number line.

Task 5, Part A: Exploring Tenths

We began the session by trying to fold paper fraction strips into tenths. Students appeared frustrated in their folding, so I gently noted that sometimes when I was trying to create a certain number of folds, I thought about how I moved between to other fraction sets such as fourths and eighths. Student Z then excitedly shared that they could fold fifths and then fold each of their fifths in half.

At first, Student Z did not seem to be independently drawing on the strategy of using fifths to make tenths despite making connections between fourths, eighths, and sixteenths in Lesson One. Once given the question of what size partitions she could use to help herself get to tenths she appeared to connect back to that line of thinking and was quickly able to use the relationship between fifths and tenths to help herself fold tenths.
I next asked students to show 3-tenths using their tenths strips. All three students held up their fraction strip partially folded back to reveal 3 one-tenth sized partitions. We then worked on writing 3-tenths using decimal notation. Students E and J were able to do so. Student Z responded, “I forgot;” however, seeing the other two students’ responses of 0.3 appeared to trigger something in her memory and she shared that we could also write 3-tenths as 0.30. Student J added 0.300000. It appeared that the students had some familiarity with using decimal notation to represent fractional values. We did not explore their different ways of writing 3-tenths in this session so it remains uncertain if they fully understood why they could write 3-tenths as 0.3, 0.30, or 0.3000 and have the value remain unchanged.

**Task 5, Part B: Envisioning Hundredths**

I wanted to help students develop an understanding of the relationship between tenths and hundredths by pushing them to make their own partitions rather than providing them with pre-partitioned grids. Additionally, I hoped to support students in making connections back to fractions by continuing with their tenths strips. I asked the students how I could make the fraction strip which was folded into tenths into hundredths. Student Z responded that I would keep folding it “in half.” Student Z’s response to fold the strip in half came as a surprise given her general success thus far in folding fraction strips. I asked her to continue with her response to learn more about what she was thinking.

Teacher: Keep folding it in half? How many times would I need to fold it in half?

Student Z: Ten times.

Teacher: Ten times. So, each of these tenths (holding up strip and showing one partition of size one tenth) we would split into ten equal pieces?

Student Z: Yeah.
In reviewing this interaction, I realized I made a mistake when I assumed Student Z was saying to fold each of the tenths into ten equal pieces. Though this was likely her intention, by assuming it was, I missed the opportunity to ensure she was making this critical connection. Student Z’s understanding, however, did seem more robust than her initial suggestion to “keep folding it in half,” might have indicated. Upon further explanation, it became apparent she was in fact thinking about folding into ten pieces rather than two “halves.” Student Z had used similar language in an early session as well, despite not actually folding the fraction strip in half.

“Folding it in half” may have been Student Z’s colloquial way of discussing folding the partitions in general. Her use of this language, however, does indicate potential fragility or at least a lack of precision in how she was conceptualizing folding fractions in half or otherwise.

Task 5, Part C: Applying tenths and hundredths reasoning to additional fraction values

To further explore and support students’ sense of magnitude, we worked to estimate how we might represent various fractions using their fraction strips. First, I asked the students to pick up an unfolded strip and estimate for me what 50-hundredths would look like on their unfolded strip. All three students pointed to essentially the middle of their unfolded strip and Student Z volunteered, “It would be in the middle because a hundred I mean fifty times two is one hundred and if you fold it in half, you'll have two pieces.”

I asked the students to next estimate 57-hundredths. All three students held up their strips and pointed to slightly over from where they had last shown for 50-hundredths. Student Z volunteered her thinking, saying 57-hundredths belonged, “A little to the --- in the middle.” When I pushed to see if she meant exactly in the middle, she clarified it would be “more to the right” and was able to explain that “Because it’s seven more, and if you put it right exactly in the middle it wouldn't be 57, it would be 50.” I pressed her to elaborate on what she
was referring to with the seven asking, “It's seven more what? It's seven more... Tenths? Hundredths? Wholes?” She then responded, “Oneths.”

Student Z’s explanation for 57-hundredths suggested she had a firm grasp on the relationship between 57-hundredths and 50-hundredths or one-half and was able to use the relationship to help justify her representation of 57-hundredths on the fraction strip. It was interesting that despite this, she was not able to accurately name the hundredths. One possible explanation is that she may have been automatically translating to whole numbers, using them to help her reason, but then struggling to connect back to fraction values. In examining her language closely, it is evident that she spoke in terms of 50, 57 and seven and did not mention hundredths or fractions at all until pushed. When pressed, she ultimately named the hundredths as “oneths,” thus further indicating a sticking point between whole numbers and hundredths. This appeared to be further evidence of Student Z’s fragility in thinking in terms of hundredths.

While Student Z was talking, Students J and E were quiet, but each found a way to share their thinking. Student J appeared to mark half using her finger on her fraction strip and then slowly walked her finger over with little taps as she moved to the right. Student E held up her fraction strip to the camera showing markings for both 50/100 and 57/100.

Students J and E also appeared able to use the relationship between 57-hundredths and the one-half benchmark equivalency of 50-hundredths to help them represent 57-hundredths on their fraction strips. What is unknown from their work in this task is if they were having similar struggles as Student Z in fully conceptualizing these numbers in terms of hundredths. The
students had not been asked to label fraction strips at all, so it was interesting that Student E chose to express her thinking in this way. This way of representing these fractions on the blank strip may indicate she was approaching locating them in a similar way as one might on a number line.

To give the students an opportunity to explore and solidify their thinking for fractions built from hundredths, I next asked them to show 98 one-hundredths on their strips. I intentionally phrased this fraction slightly differently because I wanted to be extra clear that we were working in terms of one-hundredths as opposed to seeing one hundred wholes represented on one fraction strip. This language also connected back to our work with unit fractions in Task One. Student Z and Student J each held up their strip and pointed with their finger close to the right end. Student E held up her strip with 98/100 directly marked on it.

![Fraction strip with 98/100 marked]

Student E: Because it's two fractions away from a hundred hundredths.

Teacher: You're two away from one hundred one-hundredths. Two pieces of size?... Ten?

Or one tenth? Two pieces of size one one-hundredth?

Student Z: The oneths.

Student E neglected to note the size of the two “fractions,” but her specificity in naming her whole as composed of 100 hundredths potentially points to a greater awareness that she is dealing with hundredths rather than whole numbers. I missed the opportunity to learn what she would have named those fractions as Student Z jumped in with her “oneths” response. Student Z seemed to still be fragile in what to name those fractional pieces in this follow-up task and
continued to title them “oneths.” She appeared to be caught in the world between whole numbers and fractions as she was not firm in either what size the fractional parts were or how to name them.

To further support reasoning between tenths and hundredths I next asked the students how they would show 10-hundredths on their strips and to explain their thinking. Student J and Student Z pointed to roughly to roughly at location equivalent to one-tenth of their fraction strip starting at the left side. Student E labeled 10-hundredths at roughly 15-hundredths from the left end of her strip. Student Z explained her placement saying, “Because it's only 10 and there's 90 more little pieces, so it'll be on the way like left.” As Student Z spoke, Student E nodded her head affirmatively which seemed to indicate similar thinking to Student Z and thus likely more of a precision issue rather than a conceptual issue in her representation.

To see how the students would connect their estimate for 10-hundredths back to the actual value of one-tenth, I asked them to show where 10-hundredths would go on their previously folded tenths fraction strip. Student Z held up her fraction strip to show only one 1-tenth sized partition. Student J did not share anything, and Student E showed what appeared to be one tenth of a tenth folded back and labeled with $\frac{10}{100}$.

To ensure my perceptions of their placements were accurate I probed to find out what they were thinking about the relationship between 10-hundredths and one-tenth.

Teacher: Would it be a whole one of these partitions (held up and pointed to one tenth partition on the tenths strip), part of one of these partitions, or more than one of these partitions?

Student E: Exactly one partition (in chat)
Teacher: …Student J, Student Z, what are you thinking? Student J is saying exactly too.
Student Z, what do you think? Exactly? Why do you think it's exactly one partition?
Student E: Cuz, um, because, um, like 10-hundredths is like, kind of like cuz uh (long
pause). It's like one-tenth or like then 10-hundredths and then like a 100-thousandths....
(Very long pause and looks down).
Teacher: So, you're thinking the 10-hundredths is going to equal one-tenth because we
have how many groups of ten-hundredths in here (Teacher holds up and points to one tenth
on fraction strip)? (Student J holds up one finger to indicate one group of ten-hundredths).

When I first viewed Student E’s fraction strip, I thought she was identifying one-tenth of a tenth
on the fraction strip as 10 hundredths. After her selection of 10-hundredths and one-tenth being
equal, and her subsequent explanation, this initial impression appeared to be more tenuous.
Student E was the first student to share that 10-hundredths was equal to exactly one one-tenth
partition. This seems to indicate at least some understanding of the relationship between tenths
and hundredths. My initial impression may have been due simply to not being able to clearly see
what she was aiming for on her fraction strip. It is also possible that viewing her peers’
representations and being questioned about the connections to the one-tenth partition shifted
Student E away from her initial response and my earlier impression was accurate.

Moving between the unfolded paper strip and the tenths strip did not seem to present
additional challenge for students as they envisioned 10-hundredths on their fraction strips. This
ease in transition potentially indicates an emerging understanding of the relationship between 10-
hundredths and one-tenth. I was curious to see if students would be able to work with additional
numbers as readily.
The number 9-hundredths had previously presented itself as an interesting test case for effective generalizing of strategies. I brought this number forward again to the students to represent with their fraction strips. Student Z did not show her fraction strip. Student J held up her fraction strip and pointed to the second one tenth partition from the right end of her strip.

Student J declined to share her thinking verbally so one cannot know exactly what she was thinking, but it is possible she was counting this as the ninth 1-tenth partition from the left-side of the fraction strip. If this interpretation is accurate, then it would indicate Student J was employing a distance use of the fraction strip, similar to a number line-interpreting the ninth partition as representing 9 hundredths. Such an interpretation would indicate confusion between the values of 9 tenths and 9 hundredths as this partition is the ninth tenth from the left-side, but not the ninth hundredth.

Student E held up her folded fraction strip with 9/100 labeled at what appeared to be about the same location as Student J-the ninth 1-tenth partition.

A fraction strip is non-directional so it is possible either student may have been thinking from right to left as they presented their fraction strip. If the students were thinking from right to left then 9/100 as a location would be more reasonably close to zero, though still not accurate in its location. Such an interpretation, however, seems unlikely for Student E as she had labeled...
10/100 on the opposite end of the strip. Taken in conjunction with her earlier work with 9-hundredths in Task 1, I conjectured that it was more likely Student E was conflating the magnitude of 9-hundredths with 9-tenths once more.

To dig deeper into how the students were perceiving the magnitude of 9-hundredths, I drew upon the same questions we had discussed earlier for 10-hundredths—is the value of this number (9/100) greater than a one-tenth partition, less than a one-tenth partition, or exactly equal to a one-tenth partition? In this case, all three students said that 9 hundredths was more than one of our one-tenth partitions; however, when pressed to explain, no one was willing to share. Lack of willingness to explain is potentially revealing in terms of students’ understanding of this fraction’s magnitude. At least one student had always been willing to explain for all previous fractions; however, when presented with 9-hundredths, all three students appeared to shut down after their initial response. This may indicate that they did not know how to explain their selection, or even that their selection had been a guess rather than based on their conception of this fraction’s magnitude. It seems most likely that the lack of justification indicates at least fragility with understanding of this number’s magnitude.

Given the students’ unwillingness to explain their responses, I tried to prompt them to make connections between 9-hundredths to 10-hundredths and thus develop a firmer idea of the fraction’s magnitude. I also hoped that students would draw upon this questioning pattern and connections between 10-hundredths and one-tenth in other similar magnitude situations.

Teacher: Let's think about this—is 9 one-hundredths more or less than 10 one-hundredths?

Student J: Less because ten is greater than nine.

Student E: Less.
Teacher: How do you know it's less?

Student E: Cuz um, isn't that like ten is in tenths place and then nine is in the oneths place?

Student E’s justification contained several interesting components. First, she appeared to be trying to use decimal notation when she asserted that 10-hundredths was in the tenths place. This is incorrect as the digit one is in the tenths place rather than ten; however, her explanation does show a connection between her understanding of these numbers (10/100 and 9/100) and their decimal notation. Such an interpretation would indicate Student E may have been trying to draw upon a rule or partial understanding she remembered from working with decimal comparison rather than a rich conception of the magnitude of these two numbers. It is also interesting that Student E did not seem to make this connection or justification when asked if 9-hundredths was greater than, less than, or equal to one-tenth. Something about renaming one-tenth as 10-hundredths seemed to shift something for her. Student E’s use of the term “oneths” here is of further interest. This was not terminology she had used initially, rather, it appears she may have adopted Student Z’s language for describing hundredths. In the relationship between ones and tens the magnitude change is the same as from tenths to hundredths so the students’ nomenclature may have just been a language issue, or it may be indicative of challenges in moving between these two magnitudes. In hindsight, it would have been valuable to dig further into Student E’s thinking here rather than proceeding to elaborate on Student J’s response.

Student J’s explanation that nine is greater than ten is minimal enough that we cannot impute for certain what she was thinking; however, it does seem to potentially point towards more of a unit-fraction based conception of the comparison of 9-hundredths and 10-hundredths. In comparing 9-hundredths and 10-hundredths we are comparing 9 and 10 pieces of size one-
hundredth respectively. In this sense, it was reasonable for Student J to focus solely on the 9 and 10 and compare those in the same way she might compare whole numbers.

Thinking in terms of 9-hundredths and one-tenth appeared to be challenging for all three students. Once we started to think about one-tenth as 10- hundredths (which had been discussed previously) they seemed to be able to connect the two values more readily.

Task Five presented opportunities for students to explore and connect tenths and hundredths on the fraction strip. The students used unfolded fraction strips to estimate different values of tenths and hundredths. They appeared to draw upon equivalencies, benchmarks and the “distance from” language and reasoning that began to emerge in Lesson One. The students also made connections between the values they estimated on their unfolded fraction strips with their folded tenths fraction strips. This became most relevant as they were pressed to connect the value of tenths to 10-hundredths and 9-hundredths. Instability emerged most clearly in student thinking as we began to work with the number 9-hundredths. Through their selections and our discussion, it became clear they were not secure in their assessment of the magnitude of this number and that areas of fragility remained in their connections between tenths and hundredths.

As I reviewed the transcript later, I realized I may have unintentionally been contributing to fragility in use of the fraction strips to represent fraction values. My language choice with terms such as “put” and “place” may have contributed to the students’ use of the fraction strip in more of a distance, or number line manner. In hindsight, it may have been more effective to ask students to “show” or “represent” the fraction on their fraction strip. Though it is possible this lens on the fraction strip representation could help students make the transition from the fraction strip to the number line, no evidence emerged during this study to support such a move. Further research into language choice here could be potentially valuable.
At the time of this work, it appeared to me that students were making strong connections between tenths and hundredths on the fraction strip and were thus ready to transition to a number line. As I reviewed the recording and transcript later, I realized this move may have been premature and the students may have benefited from more work with fraction strips and additional precision on my part. Clearer language when discussing fractions as represented by the fraction strip, as well as representing fractions greater than one whole using the fraction strips, could have benefitted the students’ understanding.

**Task 6: Tenths on the Number Line**

At the time of this lesson, I felt students were secure in their work with tenths and hundredths on the fraction strip and moved to transition them to the number line next. My goal was for students to use their work on the physical model of the fraction strip to support their conceptualization of fractions on the written number line.

We began with a number line from zero to ten and placed one as our first number. I asked the students to place one-tenth on their number lines next. Student E and Student Z appeared to place one-tenth at one whole. Student J appeared to place one-tenth at roughly two wholes.

**Student J**

![Image of Student J's number line]

**Student Z**

![Image of Student Z's number line]
Student E

Student Z explained, “It would go by the zero still, but like a little tiny bit farther, because if you need 10 tenths to make 10 wholes, I mean, one whole. So, you would have to put it right next to the zero or one.” It was not clear what she meant by “right next to the zero or one” since these are significantly different locations. Seeking clarity, I asked Student Z if she was putting one-tenth between the zero and the one interval on her number line or as bigger than the one or past it. She responded that it was bigger than the one whole. While I was inquiring about the other students’ thinking, Student Z decided to revise her initial response.

Student Z: Actually, I think we should put one in the middle (between 0 and 1).

Teacher: In the middle. Why are you thinking to put it in the middle?

Student Z: Or by the ten because (pause) because one tenth is less than one whole, than one. So, the one-tenth would have to go like in the middle maybe.

Student Z seemed to still be working through the number line and where numbers belonged. All three students appeared to be struggling with locating one-tenth on their zero-to-ten number lines.

The placement of one-tenth at the whole illuminated a misunderstanding of the magnitude of one-tenth; however, the students had accurately identified one tenth of the entire distance they had marked on the number line. I had chosen to begin with a zero-to-ten number line because I thought it would allow the students to ground themselves in a whole-number number line first and then also allow us to explore the relationship between tenths, ones, and
tens. However, the students’ placements of one-tenth indicated that rather than being supportive, this instructional decision may have caused more confusion. In hindsight, a zero-to-two number line may have assisted in preventing the misconceptions that emerged in this task. I decided to backtrack and create a new number line from zero-to-one to remove the additional confusion of ten wholes. My hope was that a zero-to-one number line would more closely line up with the fraction strip we had been working with in the previous task.

Teacher: Where would you put one tenth on your zero to one number line?

Student Z: By the zero?

Teacher: Okay.

Student Z: Because you’re not even halfway there.

[All students placed 1/10 at a reasonable location on their 0-1 number line.]

Teacher: So, if we split our zero to one, kind of like our fraction strip, into ten equal pieces (teacher holds up and points to partitions on tenths fraction strip partitions) we just want one tenth of that distance. You are one tenth away from zero. Now, if we come back to our (first) number line, that is the zero and then the ten is on the end. Each of these we'll call them intervals, each of these intervals or these partitions are one whole (point to intervals on number line) So I want you to imagine like a fraction strip, how would I have to split the space between zero to one to make it into tenths? How many pieces would I split the space from zero to one into?

Student E: Ten

In reviewing the transcript for this interaction, I realized how confusing my language had become. I was attempting to use the fraction strip to support the students as they made the transition to the number line but made several critical mistakes. First, I should not have shifted to
a zero-to-one number line. Though a zero-to-one number line can directly line up with a fraction strip, it forecloses the opportunity for students to think of what needs to happen beyond the whole or the one. A number line from zero to two or another number may have been a better choice here so students could envision lining fraction strips up end to end as they connected the strips to a number line. Second, I made this transition far too quickly for students who were still struggling with these concepts. I tied us back to our fraction strips but did not provide time for students to explore and solidify the connection. We began to lose sight of the whole as we moved between the zero-to-one number line, the paper fraction strip, and the zero-to-ten number line. Third, my attempt to shift the students’ language with partition and interval was premature. My goal was for students to connect partitions on the fraction strip with intervals on the number line—the distance between various locations on the number line; however, more time and discussion was needed to help make this transition.

The students appeared more comfortable placing one-tenth on the zero-to-one number line. I believed they were starting to comprehend that a tenth is one part of 10 equal sized partitions or intervals within one whole. I wanted to assess next if they could think about where one-hundredth would go on a zero-to-one number line partitioned into tenths.

Teacher: What about one 1-hundredth? Where would one 1-hundredth go on there?
(points to zero to one number line)
Student Z: Still by the zero.
Teacher: Would one 1-hundredth be closer to zero or would one-tenth be closer to zero?
Student Z: One-hundredth.
Student E: One-tenth.
(Student J put 1/100 in the chat)
Teacher: Tell me what you're thinking.

Student Z: One-hundredth because it's a big, it's a bigger number, which means the pieces are going to be smaller. So, I think I would be closer to zero.

Teacher: You're thinking since the whole was split into more pieces, those pieces are smaller, so it's going to be closer to zero?

Student E: I think one-tenth would be closer to zero. Because one-tenth is a bigger number than one-hundredth and one-hundredth would be probably closer to a negative number.

Teacher: You think that one-hundredth is closer to a negative number. Student J, I saw you put one-hundredth in the chat, why are you thinking one-hundredth?

Student J: Like Student Z said, the pieces are going to be smaller (with one-hundredth) and with one-tenth, the pieces are going to be bigger.

I was surprised when Student E posited that one-tenth would be closer to zero than one-hundredth because one-hundredth would be “closer to a negative number.” She seemed to understand that the piece of size one-tenth would be larger than the piece of size one-hundredth, but then became muddled on where both fractions lay in relation to zero.

If given more time, I believe the students would have benefitted from further discussion about the connection and relationship between tenths, hundredths, wholes, and tens and how they are represented on number lines and with fraction strips. The students were grappling with the relationship between these different values and did not appear to be stable in their understanding yet. My goal was for students to understand that 10 tenths make a whole, 100 hundredths make a whole, 10 hundredths make one tenth and to be able to extend such reasoning to additional values. Students at first appeared more comfortable with the relationship between tenths and
hundredths and the whole when working with paper fraction strips, but as we worked further it became evident that their reasoning may have been more tenuous than they initially demonstrated. Further exploration would have been valuable here to see if additional work on the fraction strip and then transitioning to a number line of an alternative length such as zero-to-two or zero-to-seven could have helped build conceptual understanding here.

My teaching log from the day indicates that I walked away from this session feeling the need to do more work in three areas: (1) translating paper fraction strips to the number line, (2) the relationship of hundredths to tenths on the number line, and (3) placing fractions on a 0-10 number line.

**Task 7: Building Understanding of the Number Line**

After working with the zero-to-ten and zero-to-one number lines, I conjectured that the students would benefit from additional work with the number line specifically targeted at helping them make connections between the number line and their paper fraction strips.

**Task 7, Part A: Building a Number Line through Partitioning, Looking for Hundredths**

We began the session by laying our fraction strips across paper and marking where our folds fell on the number line. Then we labeled each of the locations to create a number line. Figure 4.3 shows what this looked like in progress for my number line, but all students used their own paper strips to make their own number lines.

**Figure 4.3**

*Construction of a Number Line Using Paper Fraction Strips, Task 7*
After everyone had completed marking halves and tenths on their zero to one number line, I asked how we could make hundredths on the number line.

Student E: I'd split the number into tenths.

Teacher: What part of the number line would you split into tenths?

Student E: The one?

I was uncertain what “one” student E was referring to so I asked her to hold up her number line and show exactly what she would be splitting into tenths. Student E shared her number line and circled the interval she would split into tenths.

Student E appeared to have circled the distance between 9/10 and 10/10. I was surprised by this choice. The distance between 9/10 and 10/10 is still one-tenth and thus an appropriate interval to partition into ten equal pieces if working towards hundredths, but different from where we might typically start between zero and 1/10. This was not incorrect, but I was curious to see what would happen later when we went to place 1/100 on the number line. It is of further note, that though she accurately indicated one partition of size one-tenth as needing to be split into ten pieces, she did not have the correct name for this interval. Her uncertainty and subsequent choice of “the one” as the name for this interval indicates further fragility even within some potential evidence of understanding.
Task 7, Part B: Plotting Tenths and Hundredths on the Zero-to-One Number Line

I was hopeful that building this number line from our fraction strips would give the students support in how they envisioned tenths and hundredths on the number line. I did not want to assume they were already secure in these placements and relationships based on our work in the previous session, so I asked them to begin by placing one-tenth on their number line or adding a star if they had already labeled it. All three students accurately located one-tenth on their number line.

Students were then tasked with placing one-hundredth on their number line. We had already discussed how we would partition the number line to see hundredths in Part A of the task; however, given that Student E had circled the interval between 9/10 and 10/10 as where she would partition hundredths, I was curious to see where she and the other two students would locate it as a number on their number lines. Student Z appeared to label 1/100 correctly. Student J held her pen above her number line but did not add a label till later in our discussion. Student E labeled 1/100 below 10/10 on her number line. This may have been connected to her thinking earlier as she had been working with the interval on that portion of the number line as she thought in terms of hundredths. To address this misconception, I reminded the group we could identify intervals of one-tenth throughout our number line, but that the number 1/10 occupies a specific location one tenth of the distance between zero and one from zero. I then extended this example to hundredths, modeling for the students where the hundredths were.

In retrospect, I should have taken the time here to create another number line for a different distance such as zero to three and asked the students to place 1/10 and 1/100 on that number line. My choice to use the zero-to-one number line once more may have limited students in being able to identify and reference the referent whole. Student E’s answers also presented a
missed opportunity to dig deeper into how she was conceptualizing the relationships between these values on the number line. Unfortunately, I missed this opportunity at the time and forged ahead with the next part of our task.

I wanted to give the students the opportunity to apply what we had just discussed with 1/100 while extending to more than a unit fraction so I asked them to place 10/100. Student E quickly marked 10/100 right above 1/10 on her number line. Student J marked hers in the same location soon after. Student Z pointed to zero on her number line and tentatively asked, “Would we put it here?” This would have been a good opportunity for me to question Student Z about what she was thinking and why she believed 10/100 would go in that location. Unfortunately, I missed the opportunity to learn more about her thinking and instead used a different line of questioning to help direct her to where 10/100 would be asking, “If we count up, if we start at zero and you count up ten jumps of size one 1-hundredth, where would you land?” I could see Student E pointing to 1/10 on her number line once I asked, but Student Z appeared to point to about halfway between 1/10 and 2/10. I pushed Student Z to consider what the different tick marks on her number line represented. She was able to identify her one-tenth partitions and from there we identified our one-hundredth partitions and made 10 hops of size one-hundredth. Student Z almost seemed to be picking her best guess at first. Directing her back to her tick marks and encouraging her to make physical “hops” of the size one-hundredth on her number line appeared to help her cue into the relevant components of the number line and make more connections between the given number and its representation on the number line.

Students J and E appeared to be somewhat secure here in where 10/100 would be located. This may indicate Student E was more comfortable working with a non-unit fraction than she was with one 1-hundredth in the previous prompt, or it may indicate growth in understanding.
The next number to be placed was 5-tenths. Students J and E accurately placed 5-tenths at the fifth 1-tenth marking on their number line which was also marked with one-half from our previous work. Student Z’s fraction strips must have been a little bit off because when she labeled her 5-tenths line it was right next to one-half rather than sharing a tick mark with it. I was surprised by this mistake as Student Z had consistently drawn on equivalencies for one-half in our previous work. This is possible further evidence of the precariousness of her understanding of the number line and the meaning of the location of numbers on the number line. I reminded the students that 5/10 and one-half are equivalent, so they do not each need their own line as they share the same location on the number line.

The students’ understanding of the magnitude of fractions smaller than one-tenth had thus far been tenuous, so I followed up by next asking them to place 5/100 on their number line. Student E had already recorded “5/100” on her number line earlier right above the line marking 5/10 and 1/2. As I asked the students to place 5/100, she began to point to where she already had the number labeled. To address her misconception and nudge all three students to consider what would be a reasonable location for this number, I tried to bring in some estimative thinking. To do this, I used a version of a question that had supported students in our previous work; however, in this case, the students surprised me with their responses, revealing a potential additional misconception. At the very beginning of our discussion Student J independently accurately placed 5-hundredths on her number line. Student Z did not place 5-hundredths until the end of our discussion.

Teacher: Can you put 5-hundredths on your number line? And before you put it on, I want you to think about something for me first. Is 5-hundredths going to be more or less than 10-hundredths?
Student Z: It's going to be more. Because it's smaller and 10 pieces are bigger. So, it would be smaller but otherwise 10 would be bigger.

Teacher: So, you're thinking about the size of the pieces. We're thinking about 5-hundredths versus 10-hundredths, right. Are the pieces different sizes? Are the pieces the same size?

Student E: It's different?

Student Z: Different.

Teacher: Why are they different?

Student Z: Because the five is bigger.

I believe Students Z and E may have been errantly tapping into their knowledge of the impact of partitioning during this work. They had both previously demonstrated understandings that a larger denominator means the whole is partitioned into more pieces and thus the pieces are smaller. When they answered that 5-hundredths would be larger than 10-hundredths, they seemed to be applying this reasoning to compare the five and the ten. This may indicate either a momentary or a larger gap in full understanding of the meaning of fractions.

To support their learning, I tried to bring the students back to a foundation of unit fraction reasoning and the meaning of the numerator and denominator. My goal was to get the students thinking about what each of the fractions represented— for 5-hundredths we were considering five pieces of size one 1-hundredths and for 10-hundredths we were considering ten pieces of size one 1-hundredth. The one-hundredth (the unit fraction) was the consistent unit that indicated the size of the piece under consideration. Because both fractions were built out of the same unit fraction, we could focus our attention on just the numerators and compare five and ten in the
same way that we might compare whole numbers. After this discussion, both students E and Z were able to share that 5-hundredths had more hundredths than 10-hundredths.

The students then counted “hops” of size one-hundredth from zero to place 5/100 on their number lines. I had not anticipated we would go down this path when we first began. My initial question was intended to help the students tap into estimative reasoning and thus realize that 5/100 should be located before 10/100 on the number line; however, through the students’ responses it appeared more needed to be addressed than I initially anticipated.

Task 7, Part C: Placing Unfriendlier Fractions Built from 1/100 on the Number Line

I wanted to give the students the opportunity to apply and extend the thinking we had discussed with 5-hundredths as well as return to a number that had previously presented challenges. The next number I asked them to place was 9-hundredths. Student E quickly marked 9/100 above 9/10 on the number line. Student Z shared verbally rather than writing, saying, “Kind of just a little bit behind one-tenth.” I pressed her to explain her thinking and she elaborated, “Because um all the hundredths [cut out] one tenth and um between one tenth, and it’s one less than 10.” The recording is a bit garbled in this location, however, the parts I can distinguish seem to demonstrate connections between 9-hundredths and 10-hundredths. This potentially shows Student Z engaged in a more unit fraction-based understanding of the fraction 9-hundredths and its relation to 10-hundredths than she had demonstrated when we were placing 5-hundredths. Student E adjusted her 9/100 location after listening to Student Z’s justification.

I was curious to see how the students would handle a number of hundredths greater than 10 hundredths, so I asked them to place 60-hundredths next. All three students accurately marked 60/100 on their number lines with relative speed. When I asked for reasoning behind their placement, Student Z volunteered, “I put it by the 6, because um, by 6/10, because one tenth
equals 10 and 10 times 6 is 60.” I inferred that as she said, “one tenth equals 10,” Student Z was referring to 10 hundredths. Student Z appeared to apply an important understanding that each tenth is equal to 10 hundredths. Student Z’s lack of precision in naming numbers—6 rather than 6/10 initially, and then 10 rather than 10/100 is worth noting. This was not the first time Student Z used just the numerator to explain her thinking. In this case, she self-corrected from 6 to 6/10 as the relationship between 6/10 and 60/100 was critical here and would be lost if she tried to use the whole number 6, but she did not seem to always self-correct in this way. This tendency could indicate an understanding rooted in unit fractions. It seems more likely, however, to point towards a lack of precision in language tied with a fragility in thinking about fraction size as determined by the relationship between the numerator and denominator.

To push the students by using a number that was not a multiple of 10/100 I next tasked them with placing 39-hundredth. Students E and Z placed 39/100 just before 4/10. Student J placed it just before 3/10 which may have been indicative of a misconception or misapplication or may have just been a precision error. Student E began our discussion:

Student E: I put it right by, kind of by 4/10 because it's like one tenth away from 3/10 and it would be like 30...39/10 so then multiplied that by ten and got 39/100.

Teacher: So, you're thinking about that 3-tenths will be 30-hundredths, or the 4-tenths will be 40-hundredths? And this is just one-hundredth less than 4/10 or 40/100 or it's 9-hundredths more than 30/100 or 3/10?

It was difficult to hear Student E during live sessions at times, so I did my best to interpret and re-state her thinking. In listening to the recording later I realized her explanation does not entirely make sense nor is my interpretation accurate. Though Student E’s explanation did not justify her placement of 39/100, I do think her placement and portions of her language indicate at
least partial understanding of the relationship of 39 hundredths to 40 hundredths or 4-tenths. She appears to have gotten a bit lost in her explanation but is at least attempting to tap into the distance of 39/100 to friendlier fractions she already has labeled nearby. She was also able to accurately place 39/100 on her number line so it seems likely at least some conceptual understanding is occurring. Her responses in our follow-up discussion also appeared to indicate understanding. Student Z followed Student E with her own explanation.

Student Z: I kind of put it like by 4/10 and like in between 4/10 and 3/10 because it's 39 so it'd be like right next to 4/10.

Teacher: Um-hm. Why did you put it right next to 4/10 Student Z?

Student Z: Because it's only one away from 40 and um 9 away from 30.

Teacher: Can I push you for a little precision or clarity? One what away? Is it one whole away from four tenths? Is it one tenth away from four tenths? It's one what away?

Student Z: Uh. ----- because one tenth would be a whole, one of these (pinches fingers to show one tenth partition on number line). It's not one tenth.

Teacher: Yeah?

Student Z: Because it's just way bigger so…

Teacher: Yeah? So, it's one what? Thirty-nine hundredths is one what away from 4 tenths?

Student E: One hundredth

Teacher: Student E, you're thinking one hundredth. OK, Student J you jump in too and then let's talk about what we're thinking.

Student J: I think one hundredth too.
Teacher: You think one hundredth too? Student E or Student J can you tell us why you're thinking it's one hundredth?

Student E: It's one hundredth, because we add 1/100 to 39/100 to get to 4/10 or 40/100.

Teacher: So, you're thinking 39/100 plus one more hundredth, well 39 plus one is going to be 40 of those one-hundredths.

Student E: Yeah.

The later part of this discussion seemed to provide evidence that Student Z was making important connections between 39/100 and 40/100 to help her accurately place 39/100 on the number line. She understood that 39 needed one “something” to get to 40/100 but seemed uncertain about what size that one partition needed to be. In my attempt to bring Student E and Student J back into the conversation I may have cut Student Z off from delving further into this line of reasoning and demonstrating if she was able to identify if the piece was size one-hundredth or not— an area which had been challenging for her earlier. I was heartened overall though to see the students making connections between 39/100 and 40/100. Students appeared to be generalizing their “distance from” reasoning to help support their placement and justification of 39-hundredths.

I asked the students to place 75-hundredths on their number lines next. Students E and J accurately placed 75/100 between 7/10 and 8/10 on their number lines. Student Z placed it between 6/10 and 7/10, but later self-corrected during the discussion.

Student Z: I think that goes right in the middle of 7/10 and 6/10, because it's 5 hundredths away.

Teacher: It's 5-hundredths away? Student J, Student E, what are you two thinking?
Student E: I agree with Student Z it would go right in the middle of a jump from 7/10 to 8/10.

Student J: Yeah, I agree with Student Z.

Teacher: I notice something a little different. Student Z said right between 6/10 and 7/10. And then Student E when you said it, you said right between 7/10 and 8/10. So, what are you guys thinking? Is it halfway between 6 and 7 tenths or halfway between 7 and 8 tenths?

Student Z: I meant 8.

Teacher: Oh OK. You meant 7 and 8 tenths. Why exactly halfway?

Student Z: Because it's 5 away, 5 hundredths away from both.

Teacher: Oh, so it's 5 hundredths more than 7/10 or 70/100, and it's 5 hundredths less than 8/10 or 80/100.

Student Z’s initial error did seem to be more of an issue of precision rather than a misconception. It seems significant that for this explanation she was able to accurately speak in terms of hundredths as she was explaining the distance of 75/100 from its surrounding tenths. In previous tasks, she had been stuck on naming those pieces “oneths” or simply not identifying what size they were. The context and task here was somewhat different so this does not necessarily indicate she was accurately identifying hundredths in all situations, but it does seem to signal progress from our previous tasks. That all three students were able to identify the fact that 75/100 was 5 hundredths away from 7/10 and 8/10 (as opposed to only 70/100 and 80/100) demonstrated a potentially growing understanding of the relationship between tenths and hundredths.
The students’ thinking appeared to demonstrate growth during Task 7 as we slowed down and returned to building a basic number line from zero-to-one using the partitions on our paper tenths fraction strip. In hindsight, this number line should have extended beyond one whole to give us the opportunity to identify and mark the referent whole. At the time, however, I was pleased with the students’ progress and moved to continue supporting them with understanding the number line.

It is worth noting that the students seemed more able to accurately place fractions greater than 10/100 on the number line. Both 5/100 and 9/100 presented challenges for the students that 60/100, 39/100 and 75/100 did not. Something about the magnitude of these two numbers being less than one tenth appears to have made them more challenging to place on the number line.

**Task 8: Building the Number Line Another Way**

The goal of Task 8 was to help students build understanding of the number line through an alternative construction method using iteration as opposed to partitioning. After building the number line, we worked to place numbers in the correct proportional locations as we had with previous tasks. Figure 4.4 shows the three student number lines that resulted from the work on Task 8.

**Figure 4.4**

*Completed Student Number Lines, Task 8*
Task 8, Part A: Building a Number Line through Iteration

Students were first tasked with building a number line through iteration of their one-tenth (unit fraction) piece. I asked students to begin by folding their tenths strips so they could only see one-tenth. Then, I modeled laying the folded fraction strip piece of size one-tenth on the paper, marking a tick mark on either side, and drawing a line to connect them. I labeled the tick mark on the left side zero and asked the students what I would label the tick mark on the right side. After a little bit of thinking time, Student Z volunteered, “one-tenth.” I modeled making the next partition and asked again what to label the new tick mark. Student Z quickly answered “2-tenths.” Students then iterated their own one-tenth pieces to build their number lines from zero-to-13/10 or to as far as they could fit on their papers.

Our next step was to label our number lines with the number one. Our process here was opposite all our previous number lines. Prior to this, we had always labeled zero to one first and then partitioned our tenths. In this case, I was asking the students to start by building a number
line through iteration of pieces of size one-tenth and *then* label one whole on their number line. I was pleased to see all three students were able to accurately label their 10/10 tick mark as one. Student E explained her placement saying, “At 10 tenths… because the numerator and the denominator are the same number so that means we have a whole.”

To continue to test and build the students’ understanding of the relationship between tenths and hundredths, I next asked them to place one-hundredth on their number lines. Before writing, Student Z responded, “Right next to zero” while pointing at her number line. To get a sense of how she was engaging in the proportional distance of one-hundredth to zero and one-tenth, I asked her to mark one-hundredth down on her number line. Student Z placed the tick mark in a reasonable location so I pressed her for an explanation as to why one-hundredth would belong right next to zero. Her response showed an attempt to use distance reasoning, but then a muddling in the relationship between hundredths, tenths, and one whole.

Student Z: Because 99-hundredths would be closer to one-tenth and that's... 99-hundredths [pause] um it's 99 hundredths away from one-tenth.

Teacher: It's 99 hundredths away from one-tenth or one whole?

Student Z: One-tenth.

Teacher: One-tenth? So, we would still have 99 hundredths to go in between that zero and one-tenth there?

Student Z: Yeah, right.

Teacher: Okay, so let's think a little bit about hundredths. If I have my whole fraction strip (holds up unfolded fraction strip on screen) how many pieces would I have to split it into to get hundredths?

Student Z: 99
[Student E puts 99 into chat]

Teacher: 99? Tell me why you guys are thinking 99.

Student Z: Because there's only one and nine, 99 plus one equals 100.

Teacher: Oh, so if I wanted to get just the one one-hundredth. If I want to split the whole thing into pieces of size one-hundredth… If I just wanted hundredths, how many pieces would I split my whole into?

Student E: 99

Both Student E and Z demonstrated insecurity in their understanding of the relationship between one-hundredth and one-tenth. They seemed to be attempting to draw on “distance from” reasoning as they considered the distance between the two numbers but were not accurately assessing how many hundredths would be in one tenth. Both students responded incorrectly even when asked how many pieces they would need to split the whole into to make hundredths. This was interesting because their responses contrasted with previous work with the fraction strip where they had consistently been able to explain that the strip would be split into 100 pieces to make hundredths. Their challenge may have been related to understanding how many pieces the whole would need to be split into or potentially is indicative of the complexity of the transition from physical models to the number line. It is also possible I was being unclear, or the students were confused as to the nature of the question.

I brought the students back to our fraction strip that had been split into tenths and holding up one partition of one-tenth asked, “How many splits would we do in one of our one-tenth partitions?” Student Z accurately answered, “ten.” I wanted the students to connect the understanding they had built with the fraction strips, so I laid the one-tenth partition of the
fraction strip on the number line and asked how many hundredths there were between zero and one-tenth.

Student Z: There are 99? Or 100?

Teacher: Think a little bit more about what you told me what the fraction strip (holds fraction strip up on its own and points to folded back one-tenth partition). How many partitions would there be in one of these tenths to make 100.

Student Z: 100.

Teacher: So, let's see…

Student Z: No, 10, 10, 10.

Teacher: Okay, so how many hundredths, are there when we go to the number line (points to zero to 1/10 partition on number line)?

(No response)

Despite being able to say a moment before that 10 one-hundredth partitions would be in the one-tenth partition of the fractions strip; the students did not seem to be able to connect this value to their number line. We had created their number line using this one-tenth partition, but something was still interfering with their being able to recognize that the one-tenth partition on the fraction strip held the same value and the same number of hundredths as the zero to one-tenth partition or interval on their number line. It may be that their understanding of hundredths and tenths on the fraction strip was more tenuous than I realized and thus prevented them from making the connection or it may be that connecting the number line is even more challenging than I had previously realized.

We began to think about thousandths as we worked through what would happen if it indeed required 99 “pieces” to go to get to one-tenth. I was curious to see if the students would
be able to make the necessary connections between the number of partitions and the location of one-thousandth on the number line. Our conversation continued as we explored the location of this next fraction and considered its relationship to one-tenth and one-hundredth. I had noticed the students sometimes placing numbers on number lines without seeming to pause to think so I wrapped the initial request to place one-thousandth on their number lines with an additional question to push them to consider the size of the piece in relation to one-hundredth.

Teacher: Where would one-thousandth be? Can you put one-thousandth on your number line please? Is it going to be greater than or less than one-hundredth?

Student Z: Greater

Teacher: Why are you thinking greater?

Student Z: Because it's um 900 more.

Student Z’s response here is intriguing. At the time, I inferred that she made a computation error here in answering 900 and was attempting to apply once again “distance from” reasoning—focusing on how far one-thousandth would be from the whole. As I reviewed the transcript, however, I began to wonder if she was in fact thinking about the distance between one-hundredth and one-thousandth. Even within this line of thinking, two mistakes appear. First, one-thousandth is nine thousandths away from one-hundredth or 999 thousandths away from one whole. Second, and perhaps even more critically, one-hundredth is larger than one-thousandth.

This work on the number line seemed to illuminate areas of fragility that were not as apparent when we worked solely with fraction strips. The students still were not secure in their understanding of the relationship between tenths and hundredths. I brought in thousandths because of the students’ responses about partitioning tenths into 100 shares, and to explore another proportional power of ten relationship; however, in retrospect, this may have been a
mistake as students were still tenuous in their understanding of the relationship between tenths and hundredths.

**Task 8, Part B: Non-Unit Fractions on the Number Line**

We moved next into placing non-unit fractions on the zero to 13/10 number line. Our first non-unit fraction to place was 5-hundredths. Despite our struggles with the previous unit fractions, the students were all able to accurately place 5-hundredths on their number lines. Student J explained in the chat, “I put it between 0 and 1/10.” I asked for justification for why 5-hundredths would go between zero and one-tenth. Student E responded “Because um a hundred is a large number, and it would be closer to zero than one-tenth because it's only five of those partitions, really small partitions colored in.” Student E demonstrated understanding of the relationship between hundredths and tenths and recognition that pieces of size one-hundredth would be smaller than one-tenth. She then layered on her understanding that five partitions of size one-hundredth would not be very large in relation to the whole because the partitions were small (due to the whole being split into hundredths). This response suggested more understanding of these relationships than had been evident in our work earlier in the session dealing with unit fractions. I am not sure if it was the earlier work that helped support this thinking or if non-unit fractions helped her to grasp the values more concretely.

To push the students to see how they would relate a less friendly fraction to the partitions they had already placed on the number line, I asked them to place 19-hundredths next. Student J and Student E placed the number just after 2/10. I was unable to see Student Z’s placement on her number line at the time but in reviewing her number line later it appears to be just before 1/10. Student E started our discussion off:

Student E: I put it between 2/10 and 3/10.
Teacher: Why did you put it between 2/10 and 3/10?

Student E: Because um, because 19... it's the same thing as the 5 hundredths. It’s like the hundredths are the hundredths are really small and then it's only 19 of them.

Teacher: Ohh so you're thinking about those small hundredths since it's 19 of them. Did you put it- and this can be a question for everyone, not just Student E- did you put it closer, I noticed, you said you put it between 2/10 and 3/10, did you put it closer to 2/10? Or closer to 3/10?

Student E: Closer to 2/10.

Teacher: Why closer to 2/10?

(Technology issues)

Teacher: Student J and Student Z did you put it in a similar spot?

Student Z: I put it um in a similar spot and um Student J, what did you put?

[Student Z and J had begun sharing a Chromebook at this point due to technology issues]

Student J: I put it in a similar spot to Student E.

Teacher: Did you put it before or after 2/10?

Student Z: Before, I mean after. (Student J indicated the same)

Student E’s explanation seems to show evidence of a unit fraction based understanding of the fraction 19-hundredths. She seemed able to picture both the 5-hundredths and the 19-hundredths and understand that each was composed of that number of iterations (5 and 19 respectively) of their unit fraction one-hundredth.

The fact that all three students were using 2-tenths to help themselves place 19-hundredths on their number line but interpreting 2-tenths as coming before 19-hundredths rather than after, was quite fascinating. I had chosen 19-hundredths because I wanted the students to
make connections between these two numbers and give us the opportunity to explore the distance between them, but I did not anticipate students would place 19-hundredths as larger than 2-tenths. Their responses here suggest the students were understanding that 19-hundredths and 2-tenths are close on the number line but may still have been insecure in understanding that the smaller fraction needs to be placed to the left on the number line.

I wanted to help the students explore the connection between 19-hundredths and 2-tenths and their relative placements further, so I pursued the following line of questioning to help them discover their mistake.

Teacher: Okay, so let's think about if we're writing 2 tenths in terms of hundredths, how would we write 2 tenths? It equals how many hundredths? It equals how many hundredths?
(No response)
Teacher: And if I'm stuck on my number line, I can always pick up my fractions strip (picks up fraction strip and holds up 2/10) and think about, “Okay, well, here are 2 tenths, how many hundredths would be in 2 tenths total?”
[Student E puts 20/100 in chat. Students J and Z indicate agreement]
Teacher: Is 19 hundredths going to be more or less than 20 hundredths? Point up or down for me.
[all point down].
Teacher: It's going to be less, right? How much less is 19/00 than 20/100?
Student E: Umm one.

In hindsight I should have asked the students to return to their number lines and consider again their placement of 19-hundredths based on our conversation. I knew students were struggling to
transfer their thinking from fraction strips to the number line and this would have been an additional opportunity for them to apply our thinking from the fraction strip to the number line. This was a missed opportunity for them to make the connection and for me to learn more about if and how they were able to connect the values between the fraction strip and number line.

Our familiar “unfriendly” fraction made an appearance next, though this time in decimal form. I asked the students where they would place 0.09 on their number lines. I refrained from saying this number out loud, opting to write it in the chat instead because I wanted to see how students would make the connection between decimals and their current fraction number line.

Student Z started our conversation off by automatically saying 0.09 as a fraction.

Student Z: It's 9-hundredths. Zero and 9-hundredths, it's basically the same thing.

Teacher: Yeah 9-hundredths so where does 9-hundredths go on your number line? You can think about what you did for one-hundredth and 5-hundredths, 19-hundredths... Is 9-hundredths smaller than those or bigger than those? Where would it be?

[Student E had placed 0.09 around 9/10 on her number line.]

Teacher: Okay, so let's think about that 9-hundredths and let's think about its relationship with one tenth. Is 9-hundredths going to be more or less than 9-tenths.

Student E: More.

[I misheard and thought she said less]

Teacher: It's going to be less? [Student E nods] Why is it going to be less than one-tenth?

(No response)

Teacher: How many hundredths are in one-tenth?

Student Z: Student E said more.
Teacher: Oh, is it More? Okay why is it going to be more? (pause) How many hundredths are in one-tenth?

(No response)

Nine hundredths again seemed to be challenging for the students as they worked to conceptualize its magnitude. It was unfortunate that I initially misheard Student E as she may have been more willing to share an explanation without that mistake and such an explanation may have shed light on her thinking. I again brought the students back to their fraction strips to see if they could access their understanding and apply it to the number line. The students were able to identify that there would be 10 hundredths in one-tenth on their fraction strips and from there that 9-hundredths would be smaller than one-tenth. However, when I then asked if 9-hundredths or one-tenth would be closer to zero, Student E responded that one-tenth would be closer to zero.

This was not the first time Student E struggled with being able to determine which fraction would be closer to zero. Though she was able to think in terms of the connections between tenths and hundredths and compare those accurately with 9-hundredths, this thinking still seemed to fall apart when she was asked to apply it to the number line.

It was interesting to see how students worked with 0.09 after we had (successfully placed it on the number line in the previous session). This number seemed to potentially illuminate fragility again in understanding and applying the magnitude relationship between tenths and hundredths. The fact that the number was written in decimal form did not seem to cause issues, but the relationships between hundredths and the tenths already labeled on the number line seemed to create confusion.
Task 9: Working with a Number Line from Zero to Ten

Given the students’ still apparent fragility in managing the relationship between tenths, hundredths, and wholes on the number line, I developed our next activity to help anchor them with whole numbers on the number line and then tie in fractional values.

Students set up an initial number line from zero to ten on their papers. I asked them to then label one and then nine on their number line. I then asked the students to focus on proportionally placing the numbers on their number lines and consider how far one would be from zero and nine would be from ten. Student E wrote in the chat, “They’re both one part away.” I followed up asking, “What else do we know about that one part?” Student Z then jumped in saying, “They’re both one-tenth away.” Student E jumped in with a similar line of reasoning that they were equal distances (which was an important recognition) and that the distance was one tenth. This may have been a conceptual error potentially indicating the students were struggling to think in terms of whole numbers on a number line. Alternatively, it could have been more of a linguistic issue in adjusting to adding the language “of the number line” to their description of 1/10. My choice to use a zero-to-ten number line here again likely contributed to this issue and may have made the task more ambiguous thus preventing me from fully being able to understand what the students were thinking.

I had returned to whole-number number lines to solidify how the students interacted with smaller and larger numbers on a number line with what I hoped would be less challenging values; however, the shift to thinking in terms of whole numbers appeared to present its own challenges. I brought students back to our discussion from an earlier session about each number signifying two different things on a number line.
I then asked the students to place one-tenth on their number lines. Student Z and Student E placed one-tenth at one whole on their number lines. I wanted to first see if they would be able to address this error with a reminder that our fraction line was not just from zero to one. I reiterated that we were placing one-tenth as a number not just as the fraction of this number line because our number line went “all the way to 10 wholes so it's as if I had 10 fractions strips lined up end to end.” I held up multiple fraction strips and modeled linking them together end to end. This did not seem to have an impact on where students were placing one-tenth, so I asked for a volunteer to talk about where they put their one-tenth. Student E volunteered that she had placed it underneath one. I pressed her for justification asking her to explain why she put it under one. She responded, “I knew the one equaled 10 tenths and then 10 tenths is at least I think it’s equivalent to one tenth.” It is interesting that Student E was able to identify that one whole is equivalent to 10 tenths, but then went on to say she thought it was also equivalent to one-tenth. Student E appeared to have some sort of misconception or gap in understanding, either in believing these two values were equivalent or in her general understanding of one-tenth such that she was unable to build justification and reached for whatever felt most convenient at the end.

I asked Students J and Z to join in with their thinking. Student Z volunteered, “I put it by the, I put it under the one too because that’s one which is one tenth of the way on the number line.” Student J wrote that she thought the same in the chat.

It appeared the students were struggling to think about tenths as a value versus a fraction of the number line, so I asked them to take out their tenths strips. I asked how many tenths they saw on their tenths strip and Student E responded, “10.” I then asked what I would need to do to show one-tenth on my fraction strip and Student E recommended I fold it ten times. I folded the strip into tenths so only one-tenth was showing and then held that up next to an unfolded fraction
strip and asked the students if it looked as if the two were the same size or the same value.

Student E was the first to respond again answering, “no.” I then tried to help the students make the transfer from the strips back to their number line presenting the idea of shrinking their number line down and using it to cover the interval between zero and one whole. Student J was then able to revise her previous placement. She explained she changed one-tenth to closer to zero and wrote in the chat that it was because “10/10 is 1 whole but 1/10 is closer to 0.”

It is possible Student J made a critical connection here in recognizing that even on this number line, 10-tenths is equal to one-whole, and one-tenth would still only be one-tenth of one whole from zero and thus closer to zero. I reiterated that one-tenth would be one tenth of the way from zero on the way to one whole on our number line and modeled splitting the interval between zero and one whole on the number line into pieces of size one-tenth and labeling 1/10 as a location on the number line. In retrospect, rather than encouraging the students to imagine the whole fraction strip shrunk down to fit each whole on the number line, it may have been more beneficial to have them place 10 fraction strips together end to end. I also should have considered a non-one value other than ten for our number line here as once again the selection of ten seemed to present challenges and potential for misconceptions.

We moved next to placing a non-unit fraction on the number line with the placement of 3-tenths. Student E very quickly placed 3-tenths on her number line but did so errantly right above the whole number 3. I wanted to see if some questioning related to benchmarks could help her discover her mistake, so I prompted the students to ask themselves: “Is 3-tenths going to be smaller or greater than one whole?” and reminded them we could consider using their fraction strip to support themselves. Student J then wrote in the chat, “3/10 is closer to zero because it is
7/10 away from 1 whole.” We then modeled iterating our one tenth from the last placement three times and noted that 3/10 is not even half-way from zero to one.

Next, I asked the students to place 1.25 on the number line, assuming they would make a connection to one and one-fourth. This number was chosen because I did not want to neglect numbers greater than one whole when we worked with fraction and decimal values, and I had assumed 0.25 would be a relatively friendly equivalent value outside of tenths for decimals. It soon became apparent to me that this was an incorrect assumption, students did not realize 1.25 was equivalent to one and one-fourth.

I saw that Student E had once again speedily placed this number on her number line and it was now sharing the same location as 5. Student J had placed it about halfway between one and two and Student Z indicated she had made a mistake and needed to remake her number line. Student E shared with us that we could also say the number as one and 25 hundredths. Before we began discussing our placements, I tried again to engage the students in using benchmarks to assist themselves in considering the magnitude of the number. I asked the students if our number was bigger or smaller than one whole. Student E responded, “bigger.” I then asked the students if the number was bigger than two wholes or smaller than two wholes.” This time Student E wrote “smaller” in the chat. This selection was interesting given that she had placed the number at the same location as five on her number line and thus inherently larger than two. Next, I asked the students to consider if the number would be closer to one whole or two wholes. Student Z responded that it would be closer to one whole. Her computer connection became tenuous, but I was able to catch her say “25” and then after some garbling “and if it’s less *** then round down.” I did my best to interpret this and asked her if she was saying that she was thinking about
25/100 being less than half so you would round down to one whole and she nodded affirmatively.

When asked if they knew an equivalent fraction to 25 hundredths that might feel a little friendlier none of the students were able to identify one fourth. I drew the students’ attention to the monetary value of quarters and their relationship to a dollar. Student Z was then able to identify that 25 hundredths was equal to one fourth.

Student Z’s response suggests she was able to think of 1.25 as larger than one whole and thus place it appropriately on the number line. I am not sure if the decimal notation may have helped her and Student J place the number accurately or the fact that it was larger than one, but they seemed comfortable thinking about this number and its relationship to the whole numbers on the number line. Student E still seemed to be struggling with placing fractions and decimals on the whole number line. I was concerned with how quickly she was placing each number. It seemed as if she was not giving herself time to think about their size and their relationship to the other fractions on the number line. I tried to use questioning to help reduce her speed and get her thinking more about each number’s relative magnitude, but I did not feel I was having much success in this endeavor. I noted in my log for the day that I needed to find a way to slow her down.

For our next number line, we changed the level of magnitude to create a zero to 100 number line. I asked the students to place 50, 5, and 89 on their number lines and they were all able to do so accurately. I heard the students draw on distance reasoning as they explained their placements. They also seemed to be more successful in drawing and discussing this number line in terms of whole numbers than they were on the previous number line. It is possible the larger magnitude or the previous experience had an impact on the language and reasoning they were
wrapping around this number line. I wrapped up our session by asking the students to once again place 1/10, this time on their zero to 100 number line. We had not labeled one whole on this number line, but all students placed 1/10 very close to zero. If we had had more time, I would have liked to have asked them to place one whole on the number line and explored the location and distance of the two numbers in relation to each other. Since we were short on time I simply asked if it would be bigger or smaller than one on this number line and Student J responded first that it was “smaller” than one. The Students completed 0-to-10 and 0-to-100 number lines can be viewed in figure 4.5.

**Figure 4.5**

*Student 0-to-10 and 0-to-100 Number Lines, Task 9*

Student J

![Student J's number line](image)

Student E

![Student E's number line](image)
Summary of Learning Lesson 3

This lesson illuminated significant areas of instability in students’ reasoning between tenths and hundredths and the number line as a model. Though the students were able to reason through and indicate appropriate folds on the fraction strips, they appeared to struggle to make the connection of these values and their relationships to each other once on the number line. This suggested that students needed further work developing their understanding of these critical relationships.

Additionally, challenges in transferring thinking from fraction strips to number lines may suggest specific issues with the model of the number line. I had hoped returning to work with a whole-number number line would help the students make the necessary transitions, but their responses suggest that rather than helping, it may have illuminated further areas of fragility and even potentially contributed further to them.

I believed that the students’ work during this lesson indicated they needed further experiences with the number line, but I did not want to create negative associations with the model by an over-emphasis, so I planned a few alternative means to address thinking about magnitude for the next lesson and planned to return to number lines in a following lesson.
Lesson 4

Lesson Four took place over two sessions and was designed to support students in conceptualizing the magnitude of less friendly fractions as well as give them opportunities to apply and extend some of the reasoning strategies we had been developing. Students again utilized benchmarks as they considered the size of various fractions.

Task 10: Identifying Pre-Marked Locations on the Number Line

Students were given pre-marked number lines as an independent task outside of the meeting time. Our previous work had focused on self-generated number lines that we would partition or iterate and then label. I was curious to see how students would interact with a number line where the partitions had already been made and they would need to label the values. In reviewing the students’ work it became apparent they needed further support in this area so our first task in Lesson 4 was to return to the number lines they had labeled independently. Figure 4.6 shows each students’ number line.

Figure 4.6

Students’ Pre-marked Number Lines, Completed Independently

Student J

Student E
Student Z did not label her number line, but did volunteer her thinking for our first shape, identifying the star as “3 fourths of the way to one…because there’s four like little lines and the start is on the third one.” Students J and E both offered agreement with Student Z’s assessment. The students appeared to be confusing the tick marks (lines) for the meaning of the partition. I reviewed the relationship between the tick marks and the “spaces” (partitions) between them with the students. Then I focused the students on counting the spaces using little hops. The students were able to identify three hops rather than four. We labeled our first third and then our second. Student Z explained its name was “2/3 because it's just another third so there's like one third plus one third equals 2 thirds.” It is possible the students had not been looking carefully at the image and assumed the star was at 3/4 because it looked to be in roughly that location and we had been dealing more in fourths than thirds. It is of note, however, that Student Z specifically referenced the third line in her explanation for her 3/4 label. This suggests confusion between the lines and the spaces themselves and the expression of a number’s value based on each.

I assumed the students would be able to apply our work labeling 2/3 to the remaining three numbers; however, Student E’s response for the square surprised me.

Student E: I think it's 2/3. Because if you did the bounces again there would be two of them, so it'd be 2/3.

Teacher: Where would we do the bounces from for the square?

Student E: Two to three.
Teacher: From two to three… so I do my bounce. And this is hard, because you don’t have the paper in front of you to actually do the bounce. Actually, it looks like it was just one bounce, wasn’t it? But I also noticed I’m already all the way at two, do I have to think about that two here?

Student E: Yes?

Teacher: Student J what are you thinking? Should we factor in that two? Student J, I noticed you wrote 2 1/3 in the chat, why are you thinking two and one third?

Student J: Because the two is like a whole number.

Student E’s initial response suggested she was attempting to employ the strategy we had used in labeling the star, but a gap in understanding impacted her application. She referenced doing the “bounces” we had previously done, but even if she started at the number two and hopped to the square’s location, she should only have completed one hop. She also neglected to consider the two wholes prior to the number two. In her previously submitted written copy; however, she did appear to consider these two preceding wholes as she assigned the square’s location a value of 2 1/4. Her answer is further intriguing because we had already labeled a location on this number line as 2/3. It is not possible that two different locations on the same number line can both represent the location of 2/3. This mistake could point again to a deeper gap in understanding of the number line.

The students all agreed the triangle should be labeled with three. Student Z justified this label saying “Three, just three… Because it’s on the whole number which is three.” All students also agreed on the value for the final symbol of the X. Student E explained, “If you did the hops, you’d get to four and then you’d have 2 thirds remaining.” Student E’s thinking seemed to shift as she engaged in this final symbol, and she was able to apply the reasoning we had used for the
previous symbols. It is possible Student E had not worked with labeling pre-made number lines previously and doing so created some additional confusion initially. This confusion is worth noting regardless, however, as it seems to still point to certain gaps in understanding with the number line.

Student J seemed to be able to address her initial mistakes on the submitted number line once we worked through examining the partitions rather than the lines for the number line. This initial error was consistent in her written work if we can assume a slight precision error for the star. Student Z did not complete the task in the written work though she did seem to have spent time with it as evidenced by doodles around the symbols and numbers. Like Student J, she appeared to be able to apply the work we did for the first symbol to accurately label the remaining symbols during the group discussion.

**Task 11: Closest to the Whole**

Task 11 asked students to identify which number (1/2, 1/6, 11/13, 7/9) was closest to one whole and justify their thinking. Students E and J both selected 7/9 as closest to one whole. Student Z selected 11/13. Student E justified her choice by saying, “Because ninths are bigger than… well 11/13 would be really close to 7/9 and 7/9 is bigger than 11/13 almost so then 7/9 would be closer to one because the nine the partitions would be bigger than the 13 and the 7 would be closer to 9.” Student E appears to be applying her understanding of the impact of partitioning to recognize that ninths are bigger than thirteenths. What is less clear is why she believed this would mean the 7 is closer to 9. She appeared to fail to recognize that as we are examining which number is closest to one, we need to consider not only the size of the pieces, but also the number of those pieces. Student E was correct that the size of the pieces is worth our focus here because both fractions are only two pieces away from one whole. Where Student E’s
thinking appears to fall apart is then understanding that if those pieces are bigger, the fraction is farther from one whole.

Student Z was next to explain her thinking. She originally selected 11/13 and during her justification shifted to 7/9.

Student Z: Um I think it's 11/13 because it's only 2 thirteenths away from one whole and 7/9 is 2 ninths away but 11 thirteenths is smaller.

Teacher: So, you said 11/13 is 2 thirteenths away from one whole, 7/9 is 2 ninths away from one whole. And then you said, 2 thirteenths is smaller than 2 ninths?

Student Z: Uh-huh.

I misheard or misinterpreted the last part of Student Z’s explanation, thinking she concluded 2 thirteenths was smaller when she had said 11 thirteenths was smaller. Student Z did not correct me when I re-voiced her thinking, so I do not know if that was her original intent or if my mistake shifted her original statement. As we continued, she did seem to be making a comparison of the residual fractions of 2/13 and 2/9.

Teacher: Why are you thinking 2/13 is smaller than 2/9?

Student Z: Because 2/13 is going to be in smaller pieces because they're split up more into our partitions and 7/9 is in bigger pieces and partitioned bigger.

Teacher: You're saying if I have my strip and I split it into 13 pieces versus that same whole into 9 pieces, those 13 pieces, those pieces of size one-thirteenth are going to be smaller, and so we have two for each. Okay, so, then you said those 2 thirteenths are smaller than 2 ninths, so why does that make 11/13 bigger then? If 2/13 is smaller, then why is 11/13 bigger? And everyone can think about that.

Student Z: (gets distracted) Can I change my answer?
Teacher: You can, but you have to justify it if you do.

Student Z: I'm changing my answer to 7/9

Teacher: Because?

Student Z: Because I realized that 7/9 are bigger pieces, which makes it closer to one.

This exchange is intriguing because Student Z initially seemed to be employing residual reasoning in determining that 11/13 is closer to one. After I re-stated her words and asked a push question for justification, she retracted her previous answer and shifted to 7/9. It is unclear if the cause of this shift is the nature of my question, her wanting to be in alignment with the two other students, or if she did notice at that moment that ninths are bigger pieces and in turn think that inherently made 7/9 larger.

To push all the students’ thinking we drew tape diagrams of each number and focused on the gap we needed to cross to get from our given fraction to one whole. After working through 7/9 and 11/13 on the tape diagrams I asked the students to assess if 19/20 or 11/13 would be closer to a whole to give them an opportunity to apply this new residual reasoning strategy we had just explored. Student E quickly answered 19/20, explaining, “Cuz twentieths, like the partitions are really small so then you'd have one twentieth to go.” In this instance at least, Student E appeared to be able to generalize our residual reasoning work from 7/9 and 11/13 to 19/20.

**Task 12: Closest to...?**

For Task Twelve we returned to sorting fractions and decimals into three columns: “Closest to zero,” “Closest to ½” and “Closest to 1.” Students were tasked with working with a mix of fractions and decimals. The selected numbers presented students with the opportunity to employ multiple reasoning strategies.
Task 12, Part A: Close to the Middle

In order to assess if students had retained our work from previous sessions, we began with the fraction 3/8 once more. Students Z and J repeated the mistake they had made previously and placed 3/8 closest to zero. Student E once again placed it closest to one-half. Student E justified her placement by saying, “I put it closest to one-half because you just need one more eighth to be one half.” Interestingly, Student Z’s thinking seemed to draw on the same premise: “It would be closer to zero, because you need one more eighth to be half,” but she came to the opposite conclusion. Student J indicated she agreed with Student Z. We returned to our tape diagrams to represent the fraction once more and compare it with one half.

We drew 3-eighths and one-half and established it would only be the one-eighth to get to one-half, while it would be 3-eighths to get to zero. Even after this Student Z still asserted 3-eighths would be closest to zero. She explained this arguing, “Because yeah, it's three verses one and three is more than one, but… (pauses and looks up). I think it’s closer to half.” Student Z appeared to adjust her reasoning as she worked to justify her answer. As she reiterated her ideas, she seemed to realize she should reach the opposite conclusion and adjusted her answer accordingly.

We ran out of time that day and picked up our work the next session. Students were next tasked with placing the decimal 0.76 on their chart. Student Z and Student J both placed the
number closest to one while Student E placed it closest to ½. Student E explained her reasoning first.

Student E: I think it's one-half because 76 is close to 50 and 50 is close to, is one half of 100.

Teacher: So, you're thinking about half of 100 would be 50 and 76 isn't very far from that. How far exactly is 76/100 from 50/100?

Student E: 26

Teacher: 26 hundredths. Okay Student Z or Student J how about you, ladies?

Student Z: I mean I chose; I chose closest to one because it’s only 24 away from one whole and about like 16 away no 26 away from one half.

In this exchange, Student E appeared to begin to employ some important reasoning by considering how far 0.76 is from the benchmark of one-half. A gap emerged, however, when she did not fully carry that reasoning through to also consider how far the number is from our other benchmark of one whole. The selected number (0.76) was intentionally chosen as very close to exactly the middle of one half and one whole precisely to push students to have to consider both distances.

The next number 2/ninths followed up on the work and discussion from 0.76. This number is also close to the midpoint between two benchmarks but increases the level of challenge as the equivalent to ½ is not as immediately apparent. All three students placed 2/9 accurately as closest to zero. I was not surprised by this but wanted to see if their justification would go beyond just saying ninths are small and there are only two of them. Student Z’s reasoning did push beyond this frequent default explanation.

Student Z: I put it closest to zero because it's only two away from zero.
Teacher: Two what away from zero?

Student Z: 2 ninths.

Teacher: 2 ninths away from zero okay.

Student Z: And like 7 ninths away from one.

Teacher: Alright, so pretty far from one whole.

Student Z: And there really is no like proper benchmark in between, for like one half of nine.

Teacher: Yeah, how should we deal with the one-half here? What, how many ninths would be equivalent to one half… This is a tricky one. You can all think about this, not just Student Z.

Student Z: I feel like it would be about five.

Teacher: About five…

Student Z: Or like four and a half. Four and a half!

Teacher: Four and a half. Why are you thinking four and a half?

Student Z: Because a half plus a half equals one whole and four plus four equals eight plus one whole because the two halves equals nine.

Teacher: Mm hmm, Student J and E what are you thinking? Four and a half ninths? You think that's half?

(Students E and J nod)

Teacher: So, if four and a half ninths equals one half and we're thinking about two-ninths, how far is two-ninths from one-half?

Student E: Two and a half ninths.
Student Z likely began by relating 2-ninths to just zero and one whole because both of those benchmarks felt more accessible to her. The half equivalency for ninths was not immediately apparent. Through prompting, however, student Z was able to identify what the equivalent value of one-half would be. As a group, we were then able to ascertain the distance of 2-ninths from one-half and thus confirm the accuracy of the classification of 2-ninths as closest to zero.

The goal of both the previous numbers (0.76 and 2/9) was to push students’ reasoning with the distance of the number from two benchmarks and thus refine their conception of its size even further. Retrospectively, I realized it may have been better to first follow 0.76 with a similarly situated number such as 0.24 and then move to 2-ninths. Two-ninths required an additional layer of reasoning and thus shifted focus away from the distance from its two surrounding benchmarks.

**Task 12, Part B: Back to Tenths and Hundredths**

Our next number to place was 0.09. This number had previously surfaced gaps in understanding with the students tending to place it as closer to one whole than zero. This time, however, students’ perception of the number’s magnitude may have shifted because all three students placed it as closest to zero. Student E shared, “Nine hundredths is like hundredths if you would like partition up the fraction strip like really small and then you're only shading in nine of them so there's a small piece of it.” Student Z shared, “I think it's um closest to zero because um it's not even close to one and not really close to one half. We would need more hundredths.” As I was re-stating Student Z’s thinking she jumped back in saying “49-hundredths.” At the time I thought she was saying 49-hundredths was closer to one-half, but when I reviewed the video, it became apparent she was saying we needed 49 more hundredths to get to the half.
Student Z’s choice of 49-hundredths seems likely to indicate that she was correctly utilizing 50/100 as the appropriate equivalency to one-half. What is less clear is where the 49 itself emerged from. Her reasoning here and her reasoning during Lesson 1 when she was working to explain a hypothetical fraction strip representation of 199/200 may be connected. In both situations, a confusion about the size of the share and the relationship of nine to the benchmark emerged. Her responses related to “distance from” in other situations were typically correct so I conjecture that the issue here relates back to fragility regarding the value of tenths and hundredths. Not asking her to explain further here was a missed opportunity to understand more about how she was grappling with the relationship between 0.09 and one-half. I should have spent more time probing students to learn more about their thinking here as this was a shift from their previous classification of 0.09.

Our next number was 0.6. All three students placed the number as closest to one-half. Student J began our discussion.

Student J: I think it’s closest to one-half.

Teacher: Why do you think that?

Student J: Because I think it's close to one-half because it's, 6 tenths is one tenth away from being half.

Teacher: How do you know it’s one-tenth away from being half?

Student J: Because 5-tenths is half of one.

Student J generally shared less in our small group, but her explanation here demonstrated she was drawing upon distance as a helpful way to determine the nearest benchmark to our number. She also appeared to be comfortable finding and using the equivalency in tenths for one half.
The students were successful in classifying both 0.09 and 0.6 in this benchmark sort. They readily renamed these decimal fractions as fractions and accurately placed them in the correct location. This sort was a shift for the students in understanding 0.09 as a number smaller than one-half. It is possible students grew in their understanding of the number’s value here as they did not seem to fall prey to the common mistake of whole numbers interference. The students could have seen 0.6 and 0.09 and assumed 0.09 is larger because nine is larger than six; however, since they were reasoning with each of these individually in relation to the benchmark of one-half, they were able to achieve at least some degree of understanding of their sizes. This may demonstrate a growth in their understanding of the value of tenths and hundredths.

**Task 13: Closest to One**

The next task asked the students to identify which fraction out of 7/3, 7/5, 7/6, and 7/12 was closest to one whole. In the previous iteration of this task all of the numbers were less than one whole. In this iteration, three of the numbers were less than one whole while one was greater. This shifted the task from solely focusing on which fraction is largest, to needing to think about the distance from the whole even if the fraction is greater than one whole.

All three students chose 7/6 as closest to one whole. Student Z called on “distance from” reasoning to explain her choice, “I chose 7/6 because it's only one and 1/6 because it's an improper fraction…So to make it a mixed number, it would be one and 1/6 and the other ones would be like 1 and 2/5 so that one will be closer to one because it's only 1/6 away.” In her explanation, we can see Student Z focused on the distance from the whole rather than deferring to the greatest fraction. She did not, however, fully justify her answer as she failed to address a comparison of 2/5 and 1/6.
It is of note that Student Z found it worthwhile to contrast the distance of \( \frac{7}{5} \) and \( \frac{7}{6} \), but not the other two fractions under consideration. It is possible that these fractions are more “obviously” farther away for her that she does not feel the need to include them in her justification. None of the students fell into the strap of automatically choosing the fraction that is less than one whole.

**Summary of Learning Lesson 4**

Lesson Four presented opportunities for students to revisit, refine, and apply their growing understanding of strategies to assess the magnitude of fractions and decimals. Students labeled pre-marked number lines, continued to label number lines, and began to focus on the distance from one whole with intentionally selected sets of numbers.

Students consistently drew upon benchmarks to justify their magnitude assertions but continued to require pushes to fully consider all relevant information. This appeared both in terms of thinking about the number of shares (or “missing” number of shares) and the size of the share, as well as in considering the distance from more than one benchmark. The students presented potential evidence of growth in their understanding of tenths and hundredths when they were able to accurately discuss the magnitude of 0.09 and recognize it as closest to zero.

It seemed appropriate at this point to return to creating and placing numbers on our own number lines as well as continuing to assess and develop understanding of the relationship between tenths and hundredths. A focus on the relationship between the numerator and denominator as well as the relationship to all surrounding benchmarks appeared valuable.

**Lesson 5**

Lesson 5 brought students back to explicit work on the relationships between tenths, hundredths, and now thousandths. We returned to number lines as we worked to place sets of
unfriendly fractions on the number line and pushed to consider our fraction locations not only in relation to the benchmarks, but also to each other.

**Task 14: Back to the Number Line**

Students were given the following set of numbers to place on the number line as independent work prior to the start of Lesson 5: 2/2, 0.75, 3/8, 0.4, 98/100, 48/50, 0.09, 3/4. Student Z was absent for Session 11, but Students J and E shared their number lines, and we discussed the placement of their numbers. Neither student turned in their paper number lines following the task, but recreated images from screenshots of their computer screens can be found in Figure 4.7.

**Figure 4.7**

*Recreated Images of Student Number Lines, Task 14*

Student J

![Number line diagram](image)

Student E

![Number line diagram](image)

Student E and I began with a one-on-one discussion as we waited for Student J to join us. Student E started by explaining that she had placed 48/50 “right by ¾” because it was “really
close to being one whole.” She was able to explain that she knew it was close to one whole because it only needed 2-fiftieths to “get to one whole.” She explained further that she placed the fraction bigger than ¾. Student E appeared to consider several important components. First, she assessed the distance of 48/50 from one whole noting it only needed 2-fiftieths to get to the whole. She remembered to include the size of the two pieces she was away from the whole. This is important as being two fiftieths away from the whole is very different from being two fourths away from the whole. It is important to consider both the number of pieces and the size of the pieces. When pressed to consider 48/50 in relation to 3/4 she brought in the size of the share. Three-fourths is only one piece away from the whole, however, as Student E noted, the fourths are much larger partitions than fiftieths and thus 48-fiftieths is closer to the whole and larger than 3-fourths. Student E’s response here suggests she was bringing together both understanding of the size of partitions, as well as “distance from” reasoning to help herself place this fraction accurately on the number line.

I included 0.09 on the number line to see if students could apply the magnitude reasoning they had employed when sorting it between benchmarks. Student E placed 0.09 at a reasonable location on her number line—close to zero and smaller than 3/8. She justified her placement saying, “I put 0.09 by zero because hundredths are really, really small so then there's only nine of them, which is really close to zero.” This placement, and explanation, was a significant shift from Task 3 when she named 0.09 as 9-tenths rather than 9-hundredths even when pressed and Task 4 when she placed 0.09 as closest to one-half. This was the second indication that some sort of shift had occurred in how Student E perceived the magnitude of 0.09. In Task 12 she was able to sort 0.09 as closest to zero and in this task, she was able to accurately place 0.09 on the number line.
Student J joined us for the next number: 0.75. Both students had placed 0.75 close to 3/4, but not occupying the same tick mark. I asked the students to consider if 0.75 was equivalent to any of the other numbers on their number lines. Both students responded that they did not think it was. In Task 9 it had also been apparent that the students did not seem to understand the equivalency between 25-hundredths and one-fourth. It appeared that despite working that day to explore that equivalency, students were unable to make the connection to seeing 0.75 as equivalent to 3-fourths independently.

I asked the students to think about the number 100 and consider how much would be in each group if it was split into four equal groups. Student E initially responded “ten.” I attempted to help her discover her error on her own, but even after discussing that four groups of 10 would only equal 40 she continued to answer that 100 split into four groups would lead to 10 in each group. This repeat is potentially indicative that she was not completely understanding either the picture or that we wanted to have 100 total partitions or possibly both. I should have stopped moving forward here to inquire more about what she was thinking and understanding about our picture and our goal. Instead, I pushed ahead to identify how many partitions we would need to re-partition our fourths into to make hundredths. At the time of the lesson, I did not want to focus on this equivalency and thought a quick review would help the students in getting to the necessary understandings. In retrospect, I realized that this was a place to slow down and make sense of what was happening mathematically. It is also likely that struggles to conceptualize the number of hundredths in each fourth were related to struggles in conceptualizing the number of hundredths in each tenth.

We ultimately visualized partitioning each fourth in 25 equal shares to have our whole partitioned into 100 equal shares. We then looked at how many of the four boxes composed of
25/100 we would need to circle to have 75/100. A disruption occurred at this time in the students’ classroom, so it is unclear if they struggled with this question or were distracted by the events happening nearby. I coached them to count by twenty-fives and they identified that it would take three of the boxes. Student J then identified that 3-fourths would be equivalent to 75-hundredths and both students adjusted their number line labels accordingly. Despite the students’ general apparent comfort in using fraction language to describe decimals and readily re-writing decimals as fractions they did not seem secure yet in their understanding of the relationship between hundredths and fourths.

**Task 15: Closest to One Whole**

In Task 15, students were asked to determine which number was closest to one whole out of a set of numbers: 3/4, 0.0989, 5/4, 8/7, 7/8 and justify their decision. Both Student J and Student E selected 8/7. Student J explained that “7-sevenths would be one whole and 8-sevenths is only one seventh more.” Student E’s thinking was parallel to Student J’s with an addition of describing sevenths as small and thus “only a little bit more.” Both students were correct in their assessment of the distance of 8 sevenths to one whole, but it appeared they were failing to fully consider the distance of the other numbers from one. I hoped to encourage the students to think more about the distance of the other fractions.

Teacher: What about 7-eighths? Think a little bit about 7-eighths? How far is that from one whole?

Student E: One-eighth

Teacher: One-eighth. So, is 7-eighths closer to the whole or is 8-sevenths?

Student E: 7-eighths

Teacher: Why do you think 7-eighths?
Student E: It's exactly the same as the 8-sevenths but eight is smaller than the sevenths.

So, like you'd only need one more eighth, which is only a small bit more.

In drawing the students’ attention to 7-eighths, I was able to nudge Student E to engage more closely with the fraction’s magnitude and realize her mistake. Once she was specifically attending to the size of the additional fraction, she appeared to quickly be able to accurately assess the correct size and relationship to one whole. Student J appeared distracted by something nearby and did not join back in for the remainder of the discussion.

**Task 16: From Tenths to Thousandths**

Our work in previous tasks indicated students were still tenuous in their understanding of the relationship between fraction multiples of tenths. Prior to returning to labeling another number line together, I wanted to ground students in their understanding of these relationships using fraction strips.

I asked the students to pick up a blank fraction strip and either fold it in tenths or imagine that it was folded in tenths. I then asked them to consider how we would create thousandths from tenths on their fraction strip. Student E responded that we would split each tenth 100 times.

We had envisioned each tenth as split into 100-thousandths, but I had intentionally skipped over hundredths. I wanted to see how the students would relate hundredths and thousandths to each other and if they would be able to extend their previous thinking.

I asked how many thousandths would be in one hundredth. Student E shared “ten” and then elaborated, “Because if we did the hundredths times ten it would get us thousandths so like if we split 100 into tenths, we could split it you would get thousandths.” Student J indicated that she agreed with Student E. In her explanation, Student E demonstrated flexibility in moving between thousandths and hundredths. She appeared able to envision partitioning hundredths into
ten pieces to create thousandths and then used that understanding to assert that we could then assume 10 thousandths were within each hundredth. I was surprised Student E made this transition so readily given the fragility I had seen in previous tasks when working with tenths and hundredths.

For our next push, I asked the students where 20 thousandths would be located on their fraction strips. Both students drew two lines near the right end of their fraction strip or one whole. Student J explained her thinking first.

Student J: Okay, so I put it there because first, I don't want to split it into thousandths because that'd be a little too much work…And if it was a thousandths square then this would be 20.

Teacher: Why would that be 20?

Student J: Because thousandth pieces are really small.

Teacher: They're really small so they'd be right about there. Student E how about you? Did you have similar thinking, or did you approach it differently?

Student E: Um I just knew that if we were doing thousandths they would be like really small. So then 20, the 20 would be like really close to that first end.

Both students appeared to tap into their understanding of the impact of partitioning and concluded that the pieces would be very small and thus not occupy a large portion of the fraction strip. The students’ representation on the fraction strip seems to indicate that they knew 20 pieces of size one-thousandth would not get them very far. In hindsight, it would have been beneficial to have the students represent this fraction on a fraction strip partitioned into tenths to see how they would relate 20 thousandths to one-tenth.
The students were next tasked with showing where 490-thousandths would be on their fraction strip. Both students anchored back to the benchmark of one half in their placement justification.

Student E: I put it kind of like in the middle cuz that's close to 500 so 500 is like half of 1,000 so I put it right in the middle.

Teacher: How far is 490-thousandths from one-half?

Student E: One thousandth? No, 10 thousandths.

Student J: So, I knew that 500 was one-half away from 1,000 so I just so I went a little bit back. And I knew that would be 490 thousandths.

It appeared both students were comfortable in using an equivalent value of one-half as they estimated the location of 490-thousandths on their fraction strips. Student E initially stated 490-thousandths was only one thousandth from one-half, but quickly corrected herself. Student E’s justification in this lesson demonstrated growth compared to her explanation previously when she became stuck trying to justify her selection of 0.489 as less than one half by comparing the tenths’ place digits. This time, she immediately tapped into the distance from the benchmark of one-half to help herself rather than focusing on the digit in each number’s tenths place to help herself as she had before. Student J also appeared to demonstrate confidence and a conceptually sound reasoning strategy for her placement of 490-thousandths. Student J had chosen not to share when we had discussed 0.489 previously so it is not possible to know what her thinking was at that time. She did appear to be growing in either confidence or comfort in sharing her ideas and explanations as we worked in this lesson.
Task 17: Unfriendly Numbers on the Number Line

Student E and Student J’s work with the fraction strips had been strong thus far in the Lesson. The students had consistently struggled more, however, when dealing with tenths and hundredths and their relational values on the number line. The final task of the lesson was for the students to create a number line from zero to one and place several numbers on it.

Task 17, Part A: 0.3 and 0.25 on the Number Line

The students first placed the decimal 0.3 on the number line. Both students placed the number between zero and 1/2, but slightly closer to 1/2. Both students anchored back to one-half to justify their placement, but the way they explained their thinking differed slightly.

Student J: Well, first I knew that five tenths was one half and fourth tenths was only a little bit behind five tenths and then I did three tenths after four tenths.

Teacher: Student E, how about you?

Student E: I thought of one-half and knew that we would need two tenths, so I put it close to (pointed to 3/10 on number line) close to there.

Teacher: I noticed for both of you, it looks like it's a little bit closer to one-half than zero.

Student J: Can I add onto what I was saying?

Teacher: Of course, always.

Student J: Here I’ll show you what I did so, then I also had 1/10, 2/10, 3/10, and 4/10 (points to where she labeled all the tenths on her number line)

Teacher: Oh, so you're kind of partitioning that whole thing, so you can see how big those pieces besides one-tenth were.

Student J: Um-uh
Teacher: And so, as we think about that, is three tenths going to be closer to one-half or closer to zero?

Student E: One-half

Teacher: Why?

Student E: Because um we only need two tenths to get to one-half, so it equals five tenths which is equivalent to one-half so tenths really small so, then you'd only need two small pieces to get there.

Teacher: And how far are we from zero?

Student E: Three.

Teacher: Three-tenths. So, we're two tenths away from one-half and three tenths away from zero.

Both explanations seemed to indicate the students understood the placement of 3-tenths in relation to the benchmarks of zero and one-half. Their responses here seemed to demonstrate an understanding of the magnitude of 3-tenths and its location on the number line. I was curious to see how they would relate 3-tenths to our next number, 0.25.

Both students placed 0.25 at the same location on their number line as 4/10. I had hoped the students would be able to apply our previous work with fourths and hundredths, but it appeared a misconception was still occurring. I conjectured that because we had been talking about fourths and their equivalency with hundredths, the students may have made an incorrect connection between the “four” in 4/10 and 0.25. As I questioned the students about their placement of 0.25 this conjecture appeared to be at least partially correct.

Student E: I put it close to 4/10 because um, we only need 25 to get to, we only need four twenty-fifths to get to 100.
Teacher: So, I like what you just said there. Student E you mentioned, you only need four twenty-fifths to get to 100 so I'm going to share screen to the picture we did before really quick. I can tell you guys are kind of connecting back to those four pieces, but if we're thinking about this, our 25/100 is right here, how many fourths is 25/100 equivalent to?

Student E: Umm one of them?

Teacher: One of them. So, 25/100 also equals one fourth, which I think might be what you guys were thinking about when you put it by four tenths. But are one-fourth and 4-tenths equal?

Student E: No

Me: So, for thinking about 25-hundredths, I wonder if we could think a little bit more about where that one goes. And the other thing you might think about is, is 25-hundredths going to be smaller or bigger than 3-tenths? How many hundredths would 3-tenths be equal to?

Student E: Umm 30-hundredths.

Teacher: 30-hundredths. So, is 25-hundredths going to be smaller or bigger than 30-hundredths?

Student E: Smaller.

It is interesting that both students demonstrated the same misconception in placing 0.25 at 4/10 on the number line. This seems to indicate that their understanding was still quite fragile even after our second dive into the equivalencies between fourths and hundredths. In choosing to set 0.25 equivalent to 4/10 the students seemed to be attempting to apply an (incorrect) rote method without a conceptual foundation to support it. I was surprised that hundredths and fourths were such a sticking point for the students. In my previous experiences fourths and
hundredths have seemed to be a more accessible equivalency for students, but I was finding that not to be the case for these three students.

**Task 17, Part B: Adding to our Number Lines**

Student Z joined us again in the next session as we resumed work on our number lines. Figure 4.8 shows the completed number lines for Student Z and Student J at the end of the task. Student E did not submit her number line.

**Figure 4.8**

*Student Number Lines, Task 17*

Student Z final number line (3/8 moved locations)

![Student Z number line](image)

Student J number line

![Student J number line](image)

Despite the challenges with 0.25 the previous day, Student E was able to share for Student Z that it was equivalent to one-fourth. As we discussed 0.25 and compared it with 0.3, Student Z indicated she believed they would be “basically like equal.” To help Student Z surface her own mistake I asked her to consider how many hundredths would be in 0.3 and then relate that thinking back to 25-hundredths. Student Z was able to modify her reasoning to recognize that 0.25 would in fact be the smaller number once the equivalent value of tenths was brought forward in hundredths. A small, but important shift, emerged for Student Z during this
conversation. When she was identifying the five pieces difference between 0.25 and 0.3, she identified the five pieces were size one-hundredth. In previous tasks, Student Z had been able to identify the difference in number of pieces, but often became stuck when naming the size of the pieces, repeatedly naming them “oneths” instead when pressed.

The next number to generate discussion was 3-eighths. Student Z explained she placed 3-eighths close to 2/10 because it was “not even halfway to one whole or 8-eighths.” The relationship she noted between 3-eighths and one whole is accurate and can also be extended to address the benchmark of one-half but does not address the relationship to 2-tenths despite her inclusion of 2-tenths in her initial explanation. In this way, Student Z appeared to be accurately engaging with the benchmarks, but still unsure in relating the fraction 3-eighths to other nearby fractions.

Student E placed 3/8 between 3/10 and 1/2. Student E explained that she placed 3/8 “really close to 1/2 because eighths are really small, and you’d only need 1/8 to get one-half.”

I inferred that Student E was envisioning only needing one partition to move from 3/8 to 1/2. After Student E explained her thinking, Student Z announced she had changed her mind and moved 3/8 into the middle of 3/10 and 1/2. She explained that she had noticed “the partitions were wrong” so she moved it closer to 1/2. It is possible Student Z was shifting her location to match her peer’s, but she had previously been comfortable diverging from Student E’s responses, so it is also possible that listening to Student E’s reasoning had impacted her placement.
To explore further how the students were relating 3/8 and 3/10, I asked if 3/8 or 3/10 would be bigger. Student E answered that 3/10 would be larger while Student Z answered that 3/8 would be larger. Their choices were intriguing because their initial number line placements indicated the opposite for each student. By extension, I wonder if Students Z and E were not considering the relationship to 2/10 when they placed 3/8 on the number line despite their proximity.

Student E anchored back to one whole as she explained, saying 3/8 was 5-eighths from the whole and 3/10 was 7-tenths from the whole. Both statements were accurate but did not necessarily justify why she had chosen 3/10 as larger as it is not readily apparent which is larger between 5-eighths and 7-tenths. It is possible Student E was using a strategy that had previously been successful—examining the “distance from” without fully engaging in what it would mean for this problem.

Student J indicated she had similar reasoning but thought about how 3/8 was one-eighth from the half and 3/10 was 2-tenths from the half. Student J concluded that 3/8 was larger because one-eighth was a smaller distance to go than 2-tenths. Again, the statements were correct, but did not yet quite justify the mathematical conclusion that one fraction was larger than the other.

I drew the students’ attention to the fact that both of the fractions shared the same numerator or, *number of pieces*, of three. I asked them to think about a cake that had been split into eight pieces versus ten pieces and consider which pieces would be larger. If the pieces of size one-eighth are larger and we have three pieces in both situations, then three pieces of size one-eighth is going to be larger than having 3 pieces of size one-tenth. This reasoning had
previously emerged in Task 5, but students did not seem to be tapping back into it in this instance.

Students were next tasked with placing 3-hundredths on their number line. I was curious to see their placement of 3-hundredths in comparison to their placement of 3-tenths as the relationship between tenths and hundredths had previously been an issue. Student J placed 3-hundredths “between zero and 1/10” on her number line. She explained that she chose that location because “Because I know hundredth pieces are smaller than tenths.” The students had largely focused on solely the distance from the fraction under consideration to the benchmarks of one half and one whole, so I was glad to hear Student J relate tenths and hundredths in this instance and use those to help herself. This seemed to be a step towards greater use of the relationship between tenths and hundredths; however, her explanation did not actually mathematically justify her placement. Hundredths are smaller than tenths; however, the size of the fraction is determined by the relationship between the numerator and denominator. Without consideration of the number of pieces (the numerator) we cannot speak to a fraction’s size.

I could see on Student J’s number line that 3/100 was about halfway between zero and 1/10 so I wanted to push her to engage in the relationship between 3/100 and 1/10 more directly. I asked all of the students if 3/100 was closer to zero or 1/10. All three students responded that it was closer to zero. Student E explained that “Hundredths are small pieces and there's only 3 of them so it's really small.” This was in keeping with her typical explanations but did not fully justify her response, so I wanted to push her to go deeper. I asked the students how many hundredths we would need to get from 3/100 to 1/10. None of the students answered so to nudge them a bit I asked how many hundredths would be in 1/10. Our recording stopped after this point due to technology issues, so I am limited in retrospective analysis. I did not find this interchange
significant enough to delve into it further in my researcher log which I am interpreting to mean the students were able to successfully identify that we would need 7-hundredths to move from 3/100 to 1/10 and ultimately that 3/100 was closer to zero than 1/10. This conclusion also fits with where they located 3/100 on their number lines.

**Summary of Learning Lesson 5**

Lesson five presented evidence the students were beginning to reason more confidently with the magnitude of fractions and decimals. Something seemed to shift or click in how they were working with the relationship between tenths and hundredths. The decimal fraction 0.09 had presented significant challenges for students in early sessions, but both Students E and J appeared to be able to address its value comfortably in this lesson.

It was a surprise to find the students so unfamiliar with the relationship between fourths and hundredths, but it presented an additional opportunity to work on equivalency reasoning. By the end of the lesson, Student E, at least, appeared comfortable in drawing upon that equivalency.

An additional important shift in this lesson was Student J’s growth in participation in our discussions. Student J began sharing much more often. This may have been due to a growing mathematical confidence or possibly because only two students were present for one of the sessions. Student J’s discussion participation levels were not as high in the following sessions but remained higher than they had been early on in our work together.

As we moved from this lesson into the next, I wanted to present the students with opportunities to use and solidify reasoning we had begun relating to the fractions with the same denominator, focus on proportional partitioning on their number lines, and lastly begin work exploring the density of numbers.
Lesson 6

Lesson 6 took place over the final three sessions. The lesson called for students to apply their concepts of magnitude for fraction comparison sets and extend their concept of numbers and magnitude as we began to explore density concepts. Student Z missed the first ten minutes of the initial tasks of this lesson and Student J was absent for the entirety of Task 18.

Task 18: Bigger or Smaller

Task 18 asked students to consider given fractions, generate fractions larger and smaller than them, and compare them to other given fractions.

Task 18, Part A: Smaller than 4/6

The first component of this task asked Student E to identify several fractions smaller than 4-sixths. This was a shift from previous work where all numbers under consideration were provided. In this instance, the student needed to draw upon her knowledge of 4-sixths and its magnitude, as well as identify numbers with a smaller magnitude.

Student E first shared that one-hundredth would be smaller than 4-sixths “because hundredths are really small so then there's only one of them.” I was curious to see where she would go from here, so I asked her to give me another fraction smaller than 4-sixths. She shared 2-sixths, explaining that “two of the six partitions are filled so then there's four partitions for 4-sixths which is more than 2-sixths.” In her first response, Student E appeared to consider the magnitude of the number and simply draw upon another fraction which she knew was very small. This is reasonable, though it does not address as overtly the size of the given fraction. In her second response, she appeared to engage more directly with the magnitude of 4-sixths and identified that if she had fewer shares of size one-sixth then her fraction would be smaller.
Task 18, Part B: Comparing 4/6 and 3/4

To see how Student E would grapple with comparing 4-sixths to a fraction without the same denominator, I next asked her if 3-fourths would be smaller or greater than 4-sixths. These two fractions do not share a numerator or denominator, and both are greater than the benchmark of one-half. This can create a challenging comparison if students are not relying on pictures or using common denominators.

Student E handled this challenge by drawing tape diagrams and concluded that ¾ was larger.

Despite her correct response, I noted that her tape diagrams contained a critical error— the wholes were not the same size. I reminded Student E that when we are comparing fractions the wholes must be the same size. She then proceeded to redraw the tape diagrams and still identified 3-fourths as larger. This was not the first time Student E had made this error of not drawing the wholes the same size for a comparison problem. In hindsight, it may have been more helpful for me to engage her with a series of questions so she could identify for herself why two different size wholes are problematic for a comparison problem.

Task 18, Part C: Bigger than ¾, Smaller than One

Student E appeared to feel comfortable identifying when fractions were “very small” so I wanted to see how she would proceed with generating a fraction between a given fraction and one whole. We continued with the number 3-fourths, but this time I asked her to identify a fraction larger than 3-fourths and smaller than one whole. This task presents a particular
challenge in that a student cannot just add another share of size one-fourth since that would bring them to the whole. One must shift to a different denominator or think in terms of fractions of fourths. This requires students to draw upon their understanding of partitioning and the ability to partition parts further into smaller and smaller partitions. In this way, this task also begins to get at the density of numbers.

Student E began by answering 60-hundredths as a fraction larger than 3-fourths. As I questioned Student E, she began to justify her answer of 60-hundredths but then shifted in her selection while still maintaining the same justification path.

Student E: Actually, it would be 70-hundredths.
Teacher: 70-hundredths? Why did you decide 70-hundredths?
Student E: Because I compared it to 3-fourths and then I found out 70-hundredths was bigger but it’s smaller than one.
Teacher: Did you draw a picture? How did you do your comparison? (Student E nods yes.) Can I see your picture?
(Student E holds up picture to screen.)

Teacher: So, let's see... I see 3-fourths on the top and then it looks like on the bottom, did you do like 7-tenths because that's equivalent to 70 hundredths? Is that what I'm seeing? (Student E nods yes.)
Student E drew upon a strategy she felt comfortable with for this challenging task—creating tape diagrams. Her tape diagrams brought her close to the correct answer, but a lack of precision in partitioning prevented her from full accuracy. Though her final answer was not correct, Student E’s selection of hundredths does potentially speak to certain understandings. First, Student E understood that she needed to work with a fraction partitioned into more pieces than fourths. Second, her choice in using hundredths may indicate an emerging comfort with hundredths themselves. Third, Student’s E tape diagram representation of 7 tenths to represent 70 hundredths seems to signal growth in the relationship between tenths and hundredths and an understanding that tenths would be a more efficient way to model hundredths.

It is interesting that Student E chose 70-hundredths, a number very close to the equivalent value for 3-fourths of 75-hundredths. That this equivalency did not come to mind, may indicate that Student E remained tenuous in her knowledge of equivalencies between hundredths and fourths. I reminded Student E of our work in the previous session when we identified how many hundredths would be in one-fourth. Similar to the previous day, we drew a diagram of 4-fourths and explored how many hundredths would need to be within each fourth to make 100-hundredths.

Student Z entered just as we were establishing 75-hundredths as an equivalency for 3-fourths. She offered 80-hundredths as a fraction larger than 3-fourths. Student E followed with 90-hundredths. Student Z’s initial response of 80-hundredths may indicate understanding, but it is impossible to know if she would have been able to generate 80-hundredths so readily if she had not entered as we were identifying 75-hundredths as an equivalent fraction for 3-fourths.

I wanted to extend the students’ thinking beyond multiples of one-hundredth in between 3-fourths and one whole. I drew a tape diagram of 3-fourths and split each fourth in half. Student
Z was able to share that I now had eight pieces and 3-fourths was equal to 6-eighths. From there, she was able to identify 7-eighths as between 3-fourths and one whole.

I noted to the students that we could keep splitting. Drawing a picture of one-half, I showed and discussed that I could make fourths, then eighths, then sixteenths and then could go on splitting forever. I shared that in this splitting forever, we would be able to identify infinite numbers between any two numbers.

**Task 19: Comparing Fractions, Looking for Flexibility**

Task 19 presented the students with opportunities to employ various reasoning strategies as they worked to compare pairs of fractions. Each fraction pair was intentionally selected to elicit certain reasoning strategies with the goal that students would move beyond full reliance on pictures.

*Task 19, Part A: Comparing Same Numerators*

Students were given the numbers 4-sixths and 4-sevenths and asked to compare their sizes. I had been surprised to see students did not utilize the same numerators in an earlier comparison task. I was curious to see if they would attend to this shared feature now.

I reminded the students that they could draw pictures but should remember that our drawing can lead to inaccurate pictures so they should also work to bring in some of their reasoning strategies.

Student E chose 4/6 as larger. Student Z started by answering that 4/7 was larger, but in the course of her explanation, self-corrected to select 4/6 as larger.

Student Z: I picked 4/7 because it's one seventh and then 4/6, oh no no no 4/7 no no no 4/6 is bigger than 4/7 because it's a bigger partition.

Teacher: The pieces are bigger?
Student Z: Yeah.

Teacher: And what do we know about the numerator? What do we know about how many pieces we have?

Student Z: There's the same amount for the numerator.

Though I thought it was probable Student Z was taking the numerator into consideration given her focus on the size of the shares, I wanted to bring explicit attention to the same numerators since students had not attended to this aspect in prior work. It came to my attention later in the lesson, that I should perhaps not have assumed Student Z was drawing upon this understanding of the same numerators.

Student E still chose to draw a picture. Her wholes were closer in size than they had been previously, but still not quite the same size. Her partitions were not as precise as is ideal for a comparison problem.

Student E’s note below her picture, which does not seem to be entirely supported by her picture, leads me to wonder if she truly relied on her picture for her answer or created her picture after coming up with her answer. Student E does not explicitly tap into the fact that the two fractions have the same numerators, but her note does address that sixths are bigger than sevenths. It is
possible she is using this information in conjunction with an understanding that both fractions
have the same numerator to assist her in this comparison. It is also possible she is not attending
to the numerator at all and is instead focusing solely on the denominators for her comparison.

**Task 19, Part B: Greater and Less than ½**

The next pair of fractions was 3-tenths and 10-twelfths. We had engaged in fraction sorts
using the benchmarks of zero, one-half, and one whole repeatedly so I hoped students would
readily apply the benchmark of one-half to this pair of numbers. I intentionally selected two
fractions on either side of one-half that did not have the same numerator or denominator as a
means to elicit this strategy. The students, however, did not initially utilize the benchmark of
one-half.

Student E: I think it's 10-twelfths because you only need two more twelfths to get to one
whole and then for 3-tenths, you have to get seven tenths to get to one whole so then 10-
twelfths is closer to one whole than 3-tenths.

Teacher: You're thinking about their distance from one whole. You're thinking 10-
twelfths is only two twelfths away and 3-tenths is seven tenths away and what do you
know about pieces of size tenths and twelfths? Are they really different sizes or are they
kind of close in size?

Student E: Kind of close in size.

Teacher: Kind of close in size, right? Because if we were talking about thirds and
twentieths, we couldn't necessarily just say, “Oh well, it's one third away and it's 2
twentieths away so 2/3 is closer,” but tenths and twelfths aren't very far apart.

It was interesting to see Student E draw upon distance from the whole reasoning here rather than
using a benchmark that would have been, in some ways, clearer. She is correct that 2-twelfths is
smaller than 7-tenths; however, she did not mathematically prove this fact. This may be in part
due to my own interference of a leading question rather than a more open-ended probing
question.

Student Z answered 10-twelfths initially, but then interestingly said this was a mistake
and that 3-tenths was larger. She justified her reasoning by saying she had drawn a picture and
noticed that the pieces in 3-tenths were bigger. I prompted Student Z to examine her picture and
identify which tape diagram was fuller in terms of its shading. From there, she was able to
identify 10-twelfths as the larger fraction.

Student Z’s justification for 3-tenths as larger because tenths are bigger pieces failed to
take the numerator or number of pieces into account. Student Z’s picture was not entirely
accurate, but even within it, it is evident 10-twelfths is larger. Interestingly, the picture does not
actually seem to support her assertion that tenths are larger than twelfths because the partitions
are not all even. Student Z is drawing upon her understanding of the impact of partitioning here,
but not coupling that understanding with an understanding of the impact of the numerator. This
may indicate an underlying continued fragility in her conception of the meaning of fractions.

I wanted to draw the students’ attention to the fact that the benchmark of one-half could
be helpful in comparing 3-tenths and 10-twelfths. At this point, it seemed that I might need to be
more explicit in helping the students make this connection. I prompted the students to ask
themselves benchmark questions in situations like this. Both students were able to identify that
3-tenths is smaller than one-half and 10-twelfths is larger than one-half.
When I explicitly asked the students about the relationship of each of these fractions to one-half, they were easily able to answer. It had not independently occurred to them, however, to draw upon that knowledge. I was surprised given the growth I had seen in their use of benchmarks during our benchmark sorts that they did not use the benchmark of one-half for this comparison. I should not have assumed the students would make a natural connection or transition from their previous work with benchmarks to using benchmarks in this new comparison context. Their lack of benchmark use may indicate that students still had room to grow in generalizing the use of benchmarks to assist with assessing magnitude.

**Task 19, Part C: The Relationship between the Numerator and Denominator**

Student E continued to rely on pictures to support her reasoning so I gave her a pair of fractions that I thought she would be less likely to draw. I also wanted to provide the students with two fractions whose share sizes were not significantly different. I asked the students to compare 12-fiftieths and 8-sixtieths. Despite my goals, Student E still drew tape diagrams for both of these fractions. In hindsight, I wonder what would have happened if I had chosen fractions such as 8/62 and 12/51 which have fewer factors and thus would have been more challenging to draw.

Student Z did not opt to draw for this comparison. Instead, she began to utilize the respective relationships between the numerator and denominator for each fraction.

Student Z: Yeah, so what I did, I didn't split it apart because it was way too many partitions that I would need to make. So, what I did, I actually just knew it off the top of my head because 12 is bigger than 8 and yeah, it's a shorter way to go when you have 50 so it will be longer to make 60 so 12/50 is closer to one or bigger.
Teacher: When you're saying it's a shorter way with the fiftieths and a longer way with the sixtieths- (Student Z jumps in).

Student Z: And the fiftieths are going to be bigger pieces because they're not split up more.

Unlike the previous fraction comparison, Student Z attended to both the numerator and denominator in this fraction pair to assist herself in determining which was largest. Student Z appeared to be drawing upon distance reasoning in conjunction with her knowledge of the impact of partitioning and the number of shares. She noted both that the pieces of size one-fiftieth would be smaller, as well as that fewer pieces would go between 12-fiftieths and 50-fiftieths than 8-sixtieths and 60-sixtieths. Student Z’s understanding and flexibility with these fractions appears to be inconsistent with her reasoning for the previous fraction pair when she ignored the numerator and only attended to the denominator. I am not certain why we might see such inconsistency with the two pairs given in the same lesson. It is possible the discussion for the first pair reminded her to consider the relationship for the second pair.

Student E still chose to draw the fractions, but it is not clear if that was a necessary step for her in making the comparison. She shared her picture and explained, “What I did is like I made the, I did the 12-fiftieths and the 8-sixtieths, and I found out that eight was smaller than the 60, I mean 12.”
In her picture, Student E accidentally represented 3/60 rather than 8/60; however, her written note still refers to the “8” being smaller, indicating she was thinking about how many pieces were under consideration. It is unclear if she is ignoring the denominators in her verbal explanation or has simply decided they do not need to be included for some reason.

**Task 20: Fraction and Decimal Pair Comparisons**

I hoped to explore and support the students’ understanding of the density of numbers. The density of numbers had briefly appeared in the previous lesson as we considered numbers between 3-fourths and one whole. Students had also been exploring critical concepts of partitioning into smaller and smaller parts as they explored the relationship between the whole to tenths, hundredths, and ultimately thousandths. I hoped the work with Task 20 would allow us to explore and discuss these critical concepts further. I wanted students to be able to think about the density of both decimals and fractions, so I elected to bring in decimals for this task.

**Task 20, Part A: Between 0.3 and 0.4**

We began with a task asking if numbers are between 0.3 and 0.4 and if so, how many. All three students responded in agreement that numbers exist between 0.3 and 0.4. Student Z shared that she believed 100 numbers are between 0.3 and 0.4. As I asked her to explain, she shifted her answer, saying, “Because, no, 300. No. Yeah, because you can add as many zeros as you want, and it would still be the same thing.” It appeared Student Z was applying the idea that we can “add zeros” to a decimal without changing the value. For example, 0.3 is the same as 0.3000. A few concerns emerge with this response. First, mathematically speaking we are not “adding” zeros to the number. This is a common way students explain the equivalency of decimals. Second, and more importantly, is the idea that “adding zeros” would create new numbers. Student Z did not seem to be thinking about all of the values that can occur in between these two
numbers, but rather how to rename the two numbers themselves. This still does not quite explain why she chose 100 or shifted to the answer of 300. It is possible the 300 was meant to get at 300 thousandths, but this is not quite clear. Her explanation does not seem to align entirely with her answers which further indicates she was still grappling with this question.

At the time of this lesson, I did not fully consider what Student Z’s statement implied and instead assumed she was referring to being able to split the distance between 0.3 and 0.4 into hundredths based on her initial response and then interpreted her adjustment to mean she recognized that it could be split into additional partitions and that she had chosen 300 as that number for an unknown reason. In hindsight, I wonder if bringing her back to a physical or visual model and asking her to explain further would have shed more light on her thinking for both of us. This would be an interesting and important area to investigate in future study of students’ fraction thinking; however, it was not pursued at this point.

Student J shared that she believed 100 numbers exist between 0.3 and 0.4 but was unable to share her explanation due to technology issues. Student E next shared her answer of “infinity.” Technology challenges also prevented her from sharing her thinking. By the time the technology issues were resolved, Student E changed her answer to 100. She explained, “Because if you add, like if like you were doing the zero and 3 tenths into zero and 4 tenths there'd be at least 100 more to get there.” I believe this may be interpreted in the frame of the distance between 3/10 and 4/10 on the number line with at least 100 “hops” to get from 3/10 to 4/10. I do not know if Student E’s answer of 100 was an attempt to refer to these one hundred partitions between 0.3 and 0.4 as hundredths or as simply 100 partitions between these two numbers. One hundred partitions between 0.3 and 0.4 would lead to thousandths, not hundredths, but it is possible this is what student E was pondering. Alternatively, 100 may have simply represented a “large” number
for her that would result in many partitions. My guess is that her choice of 100 relates back in at least some way to the work we had done with tenths and hundredths. It seems that Student E may still have been grappling with fully understanding partitioning intervals and the relationship between tenths, hundredth, and the whole.

It is further interesting that Student E initially responded “infinity” and then shifted to 100. It is likely she was drawing upon our brief discussion of infinite partitions between numbers from a previous task. Her infinity response indicates she must have seen some connection between our discussion of infinite partitions and this question. This may indicate a growing understanding of the density of numbers. However, if this is true, the question remains of why she changed her answer. It is possible she lost confidence or that she struggled to explain the answer of infinity and decided to switch to a response which she felt more comfortable explaining.

I decided to use Student E’s explanation as a starting point to explore this task further. I drew a number line from zero to one whole, partitioned it into tenths and labeled 0.3 and 0.4 on it. Due to the nature of their number, I did not draw the 100 partitions between the two numbers. We talked about making 100 tiny partitions between 0.3 and 0.4 and I began drawing some example partitions. In retrospect, this would have been a valuable opportunity to talk about what the size of those pieces would be and clarify that they would not be hundredths. I asked the students if I could keep partitioning the number line as much as I wanted. Student E nodded her head affirmatively. Student J was distracted by Student Z trying to get her computer reconnected. I noted here that Student E’s initial response of “infinity” was in fact correct as we could partition for an infinite number of times and thus identify an infinite quantity of numbers.
Task 20, Part B: 0.35 and 0.36

Our next task asked the students if decimals exist between 0.35 and 0.36, and if so, how many. Student Z responded first saying 94, then quickly switching to 95, then to 65 after being asked to explain, and then back again to 95. I was surprised by this response as it did not seem to fit with her thinking from the previous task with 0.3 and 0.4 or the concept of infinity that we had just worked on. Despite her rapidly shifting responses, she did not seem to be guessing, but rather grappling with some understanding or misunderstanding of the concept or question itself. She explained her thinking saying, “Because 94, actually 95 because the five the five is like it's kind of hard to explain but 95 plus five equals 100 which would equal a new number. It would equal 106 hundredths.” It is challenging to understand exactly how Student Z was reasoning within this explanation. I did not thoroughly unpack her thinking at the time; however, in retrospect, I think she may have been attempting to find the remaining (what she perceived to be) hundredths between 0.35 and 0.36. I think she may have been attending to the five hundredths within 0.35 and then adding the five hundredths to another 95 hundredths in order to make 100 hundredths and thus get to a “new” number. In actuality, 0.35 is only one hundredth away from 0.36. Student Z may have been conflating how to make a whole with the partitions and the size of partitions between numbers. Though this thinking is not correct, it is interesting evidence of how Student Z was grappling with the density of numbers at the time and her evolving understanding of the relationship between tenths, hundredths, and the whole. I am still unsure as to where the 106 hundredths at the end of her explanation emerged from. The 6 hundredths within 0.36 may be playing a role in some way here. The recording is somewhat quiet here, however, so it is possible I am not hearing her statement correctly. Student Z did seem to
understand that the five in 0.35 represents five hundredths, but her explanation leads to the question of whether or not she understands that 0.35 represents 35 hundredths.

Student E and Student J both shared “infinity” in the chat in response to the number of numbers between 0.35 and 0.36. Student E explained saying, “I think it's infinity again because like last time we can um split like we did a number line from zero to one and then we labeled like the 3, what the two numbers were, and then we could separate into as many as we want.” Student E was drawing on both our discussion and the visual model we had used as a support and connecting them to this current question. This seems to be evidence of an emerging understanding of the relationship between partitioning and density of numbers. As Student E finished her explanation, Student Z wrote in the chat that she also thought infinity.

It is important to note that all three students had no doubt numbers exist between each of these given sets of decimals. At the time of this work, they were still puzzling out what this meant and would look like and how to explain their thinking, but they seemed to have at least some intuitive notions about partitioning and the density of numbers. Shifting from fractions to decimals provided a different setting for the students to grapple with ideas of partitioning and density. We had not connected decimals as directly to fraction strips or as frequently to the number line. Their thinking here provides a glimpse into how they were reasoning and what was within (and not within) their grasp at the moment as well as what they had the language to describe. The idea of density is a complex concept for fifth graders to grapple with, but beginning to explore these concepts will hopefully position them well for further learning in future grades. In retrospect, it would have been interesting to also ask the students to name one number in between each of these pairs to learn more about their thinking.
**Task 21: More Comparisons**

For Task 21 we shifted back to comparisons between fraction pairs. I reminded the students again that they were always welcome to draw pictures, but that pictures can be hard to make perfect, so it is important to also think about what we know about the numerator and denominator and their relationship.

**Task 21, Part A: 3/5 and 3/8**

Our first task asked the students to compare 3-fifths and 3-eighths. Student E shared first that she had drawn pictures and had seen that 3-fifths was bigger than 3-eighths.

In Student E’s picture it is clear she was again struggling with making the wholes the same size. What is interesting about the lack of precision in this particular picture is it means her picture almost does not support her reasoning. This led me to wonder if Student E may have also applied additional reasoning in her decision and then simply relied on the picture as an explanation method. Regardless, it is concerning that she continued to not make the wholes the same size as this indicates a potential fundamental gap in understanding what it means to compare two fractions.

Student Z shared next that she also chose 3-fifths because “3-fifths are just bigger partitions in general” and 3-fifths is “close to a whole to a whole number because the denominator is five and the numerator is three.” I should have asked Student Z to explain further how this supported her selection, but instead I asked her what she noticed about the numerator of both numbers. This was leading to a strategy of using the same numerator to compare and may have taken away from Student Z’s own reasoning direction. Student Z noted that they were both
the same and we connected back to the idea that if we have three of a smaller thing and three of a larger thing the three of the larger thing would be more.

Student J wanted to add on after Student Z’s explanation. She shared, “I know that 3-fifths is closer to being one whole and 3-eighths is closer to being one half… Because I know that five fifths would be one whole and if 3-fifths is only two fifths away from being one whole and 3-eighths is one eighth from being one-half, but if you wanted to make it one whole it would have to be, you'd have to have five eighths.” Student J appeared to be drawing upon both distance from reasoning and benchmarks in her explanation here, noting that 3-eighths would need additional length to reach the half and a significant amount to go to the whole, whereas 3-fifths did not have very far to go to get to the whole. This explanation could grow with further discussion of how these distances can be mathematically proven to justify her answer, but overall Student J seems to have drawn upon understanding of the magnitude of these numbers and their relationship to one-half and one whole.

I was surprised once more to see that none of the students had fully utilized benchmark reasoning here. All three seemed comfortable comparing to one-half during our sorts, but they did not appear to generalize that strategy when comparing fractions such as these 3-fifths and 3-eighths. I highlighted that Student J was thinking about the half in relation to 3-eighths and reminded the students that we could also use our benchmark of one-half to help us in problems such as this, noting that 3-fifths was more than one-half, and 3-eighths was less than one-half. I hoped that another set of numbers with one less than one-half and one greater than one-half might draw this strategy out.
Task 21, Part B: 7/18 and 11/20

I asked the students to compare 7-eightheenths and 11-twentieths next. In addition to providing an opportunity to employ benchmark reasoning, I also hoped these numbers might lend themselves less to a picture and thus encourage the use of more reasoning strategies.

Student J volunteered to start us off. I noticed in the course of her explanation that she made a precision error that impacted her final result, so we worked to address this, and she adjusted her answer accordingly.

Student J: I think it's 7-eightheenths just because 7/18 is 5 eightheenths away from being one whole and 11-twentieths is 9 twentieths away from being one whole.

I shifted to letting Student E and Student Z share their thinking here, but I should have inquired further as to why determining these distances from the whole led Student J to believe 11-twentieths was bigger. Given that Student J adjusted her response after realizing that 7-eightheenths had farther to go to the whole it does appear she was using distance from reasoning in some manner to help her accurately determine which fraction was largest.

Student Z explained her thinking next, saying she agreed with Student J. She noted first that 7-eightheenths has a smaller denominator and 11-twentieths has a bigger denominator has a smaller denominator. From there she seemed to shift paths saying, “11-twentieths is one twentieth away from being one-half and two eightheenths away from, 7-eightheenths is two eightheenths away from 9-eightheenths which is one-half, so I think 11-twentieths is closer.” I pushed her to clarify if each number was greater or less than one-half and she was able to do so.

Student Z’s initial note comparing the denominator does not appear to be entirely relevant to her eventual explanation but may show that she is still feeling the need to think specifically about the size of the denominator and thus size of the partition when she is
considering the size of fractions. Her explanation seemed to indicate that she was using the benchmark of one-half to help her compare these fractions; however, I wanted to ensure that she had identified not just how far these fractions were from one half, but also if they were greater or smaller than one half.

Student E had shared similar reasoning using the benchmark of one-half in the chat to me, so I invited her to share her thinking next. She explained, “I think it's 11-twentieths because I know that 11-twentieths is more than one-half, so it is really close to one whole and also 7-eighteenths is 11 more eighteenths to get to one. 11-twentieths needs 9 twentieths more to get to a whole, so it is really close to a whole.” Student E seemed to be attempting to draw upon the benchmark one-half here as well as the distance from the whole. Her use of the two may indicate she was still sorting out an effective strategy for comparing numbers such as these. Naming 11-twentieths as “really close to the whole” is not entirely accurate but may have felt reasonable to Student E in comparison to 7-eighteenths which is less than even one-half.

All three students seemed to consistently return to “distance from” reasoning. This reasoning strategy had proven powerful for them in our early lessons, and they seemed to defer to it even in instances when another reasoning strategy may have been more effective. The students seemed to be starting to experiment with utilizing benchmarks, but then defaulting to distance from reasoning as they worked to solidify this understanding.

**Task 21, Part C: 9/10 and 11/12**

To further explore students’ understanding and use of residual reasoning strategies I next asked them to compare 9-tenths and 11-twelfths. All three students shared their responses via private chat first. Student E chose 11-twelfths. Student Z said they were equal. Student J chose 9-tenths.
Student E began our explanation, saying, “I think 11-twelfths because it's only one twelfth more to get to a whole and then 9-tenths needs one tenth and tenths are a smidge bigger than twelfths so, then it would take a little time to get to the whole then.” Student E seemed to apply the residual reasoning strategies we had worked through in previous tasks as she compared these two fractions.

Student Z shared her thinking next, “So I think, at first, I said they were equal because they're both one away and I realized that nine is bigger than 12 wait ... No, no. Ten is bigger than 12 by like two and it's partitioned a little bit bigger than 11-twelfths so that one’s bigger. 9-tenths is bigger.” Initially, Student Z seemed to also have tapped into the key fact that both fractions are one away from the whole, but rather than using this to determine which was larger, she initially thought it meant they were equal. As she continued to reason, she recognized that this did not mean they were equal. Her discussion of nine being larger than 12 and 10 being bigger than 12 is a bit confusing; however, I think she may have been confused as to which numbers she was looking at initially, and then shifted to comparing based on the size of the denominator. Ten is not larger than twelve, but pieces of size one-tenth are larger than pieces of size one-twelfth which may have led her to the conclusion that the fraction expressed in terms of tenths is larger than the fraction expressed in terms of twelfths. Her conception that tenths are larger than twelfths is both true and important within this comparison; however, she failed to consider the numerator and the relationship between the numerators and denominators as she came to her conclusion. Student J indicated that she agreed with Student Z’s thinking.

Due to time constraints, I re-stated the students’ thinking and drew a quick sketch of each fraction, highlighting that each was one piece away from being a whole and then noting the size
of the piece to get to a whole was smaller for 11-twelfths than 9-tenths thus reaffirming Student E’s strategy.

**Task 21, Part D: 0.274 and 0.83**

I wanted to see the types of reasoning strategies students would use when asked to compare a pair of decimals. I tasked the students with comparing 0.274 and 0.83.

All three students responded 0.83 privately in the chat. Student Z shared her reasoning first. She explained, “Basically because if you just add a zero to make it even, then because if you keep adding zero, it won't matter. It will still mean the same thing... Then 830 is bigger than 274.” Student Z’s language of “add a zero” though not mathematically accurate, is common for students as a strategy to compare decimals. As student Z related, in making this change we can think of these values as 274 and 830 (thousandths) and apply our whole number reasoning strategies to make the comparison more accessible. Student Z’s transition to, and use, of whole number language here is important. She seemed to move between the decimal values and the whole number values and remain in the latter but was still able to use this type of thinking to help her within the decimal comparison scenario. Because Student Z did not explicitly connect back to the original values, I wanted to make sure we returned to the fact that we were considering 274 and 830 thousandths respectively. I made this connection myself in my follow-up discussion but should have questioned Student Z more as to the value of the two numbers. What is not clear within Student Z’s explanation is if she understands what she is doing by “adding a zero” and why doing so works to aid in comparisons or if she is just applying a strategy she has learned.

Student E approached the comparison in a different way. She explained that she knew that “if you made zero and 83-hundredths a fraction it would be 83-hundredths and it’s really
close to one whole which is 100 hundredths.” I should have followed up to nudge her to then address the value of 0.274 expressed in fraction form, but instead I took her understanding as implied. Student E consistently said and wrote any number written in decimal form using fraction language. This seemed helpful for her in working with decimals and fractions together and in grappling with the size of decimals. Student J indicated that she agreed with Student E’s reasoning.

**Task 22: Ordering Fractions: 2/7, 1/12, 5/9**

For the students’ last task, I asked them to put three fractions in order from smallest to largest: 2/7, 1/12, and 5/9. Students J and E placed the fractions in the order of 1/12, 2/7, 5/9 in the chat. Student Z held up a number line she had made.

![Number line](image)

Student Z explained that she believed 1/12 was the smallest because it was one twelfth of the way to the whole. Technology issues prevented the capture of the rest of Student Z’s explanation.

Student E explained next that 5/9 was larger than 2/7 because “2/7 is not really close to a half because and then because sevenths are really big, and then you just add one more seventh and it would be almost one-half, and then 5/9 is just one ninth away.” Though Student E could develop her explanation further, it seems reasonable to infer that she was accurately using the benchmark of one-half in her comparison. She noted that 2/7 was not yet close to one half, that sevenths themselves were large pieces, and that even adding one would not yet get you to a half. Whereas 5/9 was only one ninth away from one-half. Mathematically, this is slightly off, but this
precision error does not interfere with her reasoning through the task. She did not explicitly identify if 5/9 was larger or smaller than one-half but given her reasoning with the distance of 2/7 it seems fair for her to assert that 5/9 is larger. Student J did not explain her order.

This final task wrapped up our work together. Though the students and I would have liked to keep going, it was the last day of fifth grade, and they needed to return to their larger class. I had intended to conclude our work prior to the last week of school, but the students requested to keep going as long as possible.

**Summary of Learning Lesson 6**

Lesson 6 provided opportunities to explore the students’ reasoning for comparison and partitioning as they relate to both equivalency and density. I continued to be surprised by how the students dealt with new mathematical situations.

The exploration and understanding of partitioning emerged as critical within this lesson. The students grappled with the impact of partitioning as they explored equivalencies between different fractions and the use of equivalency to identify fractions greater and less than other fractions. Discussions regarding partitioning illuminated continued instability in understanding the relationship between tenths, hundredths, and the whole. The students also continued to grapple with equivalency between hundredths and fourths. This relationship appeared to be particularly challenging for them.

Partitioning and its impact also appeared as we began to explore concepts of fraction and decimal density. The students grappled with partitioning as they considered the existence of numbers between given fraction and decimal pairs. They appeared to understand that by partitioning one could find additional numbers, but were still solidifying how to go about this, what size the partitions would be, and which numbers existed between the given number pairs.
As we worked to compare fraction pairs, the students continued to draw on distance-from reasoning for the majority of their explorations. The students did not appear to fall into the whole-number bias trap where they would select the larger denominator as the larger fraction; however, in their efforts not to do so, they seemed to forget to consider the numerator as well at times. Rather, they focused solely on the size of the shares and neglected to attend to the number of shares. Through their explanations they demonstrated a need to consider both the denominator and numerator and then merge those considerations as they contemplated the fraction as a whole and compared it with another fraction.

It also became apparent throughout this lesson that the students lacked the inclination to use benchmarks when comparing fraction pairs even when this may have been the most efficient strategy. The students appeared able to use this information but did not naturally gravitate towards using it. It is possible the students felt more comfortable using distance from reasoning. It is also possible utilizing benchmarks did not naturally occur to them. This would be an area worth further investigation in future research.
CHAPTER FIVE

RETROSPECTIVE ANALYSIS

This chapter presents a qualitative analysis of the teaching experiment with the aim of answering the three research questions.

1. How do fifth grade students reason about the magnitude of fractions and decimals?

2. What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude?

3. What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?

The teaching experiment engaged three fifth-grade students identified as struggling with decimals and fractions in 15 sessions over the course of five weeks. In seeking to answer these questions I identified three major themes in students’ evolving reasoning: (1) understanding and use of partitioning, (2) understanding and use of relationships, and (3) building and expanding understanding of the number line.

Students’ work across the three themes was not linear or always consistent. Rather, their thinking and strategies oftentimes intersected and overlapped. Figure 5.1 illustrates the intertwined nature of the three themes.
The diagram shows the three major themes and the subthemes. The first major theme, understanding and use of partitioning includes three subthemes: impact of partitioning, density of numbers, and equivalency. The second major theme, understanding and use of relationships includes two subthemes: multiplicative reasoning and relationships between numbers. The final major theme, building and expanding understanding of the number line includes three subthemes: transitions between physical models to the number line, representing powers of ten on the number line, and losing the referent whole on the number line. The arrows between the three major themes represent the interconnected and interdependent nature of the means in which the students’ understanding emerged and developed. This chapter will discuss each theme and provide examples.
Theme 1: Understanding and Use of Partitioning

Throughout the course of the teaching experiment, partitioning was a key component of how students engaged with and deepened their understanding of fractions. It became evident that thinking about and use of partitioning touched multiple areas. Students used partitioning-based thinking in more “obvious” ways such as when they partitioned fraction strips in the second session, and as they applied their understanding of what happens when a whole or parts of a whole are partitioned in more complex situations.

Three subthemes emerged within the students’ work on partitioning: understanding the impact of partitioning, density of fractions, and use of partitioning to support equivalency work. The following sections discuss each of the subthemes and the ways in which they emerged through the students’ work within the teaching experiment.

Subtheme: Impact of Partitioning

This subtheme discusses the three students’ understanding of the impact of partitioning. This type of reasoning was defined by the understanding of how partitioning impacts wholes and partitions to create smaller parts. Such understanding is a key link between the symbolic notation of fractions and knowing what those symbols represent and thus the fraction’s size or magnitude.

The students applied partitioning on the most basic level when they created visual and physical models of fractions using paper strips and pen and paper. They understood that the denominator indicated the number of pieces into which the whole was split. Importantly, they were able to extend partitioning reasoning to then recognize that a larger denominator meant the whole was split into more parts and thus the pieces or unit fractions were smaller. The students were then able to apply their understanding of partitioning to fraction comparisons and ordering. Most times, partitioning-based thinking was beneficial to the students, potentially providing
them with access to a richer mental image of the fraction under consideration. At other times, the 
students seemed to almost overgeneralize or overextend partitioning-based thinking, even in 
instances where it was not as helpful to them.

All three students seemed to enter into the experiment secure in their ability to partition 
wholes in order to create unit fractions. Conversations with the students and the classroom 
teacher indicated they had folded and used fraction strips during their previous class unit on 
fractions. We also folded fraction strips as an anchor to our work during our second session. 
Folding fraction strips allowed me to observe the students as they partitioned strips as well as re-
engage them in this important work. The students appeared comfortable and confident as they 
created evenly partitioned strips of halves, fourths, eighths, and thirds. Folding fraction strips 
was an important precursor to the understanding students would need to apply to more complex 
concepts as the teaching experiment progressed.

Throughout the teaching experiment, students seemed to be secure in their understanding 
of the impact of partitioning upon wholes and partitions. All three students demonstrated 
understanding and use of partitioning as they engaged with a variety of fractions in different 
contexts. Examples of understanding of the impact of partitioning can be seen in the students’ 
discussions of various fractions.

Student Z: “One 1-hundredth because it's a big, it's a bigger number, which 
means the pieces are going to be smaller.” (Session 5)

Student E: “Thousandths if you made that into a fraction bar it would be 
really, really small.” (Session 7)

Student J: “Because I know hundredth pieces are smaller than tenths.”

(Session 12)
The students appeared to consistently understand that the denominator indicates the number of partitions into which the whole should be divided and thus the size of those resulting partitions. This thinking seemed to extend to support a unit-fraction based understanding of fractions—seeing a fraction such as $7/100$ as seven pieces of size one-hundredth. Understanding of the role of the denominator may speak to the impact of the students’ early and consistent experiences partitioning wholes rather than just engaging with fractions symbolically.

The students’ understanding of the impact of partitioning was particularly evident in their work comparing and ordering fractions. The students regularly drew upon their understanding of the size of the partitions within a fraction as a core pillar of their comparison reasoning. In the following example, Student Z considered which fraction would be larger, acknowledging that thirteenths would be smaller than ninths because the whole was partitioned into more pieces. Student E was able to apply similar thinking to compare 4-sixths and 4-sevenths.

Student Z: Because 2-thirteenths is going to be in smaller pieces because they’re split up more into our partitions and 7-ninths is in bigger pieces and partitioned bigger.” (Session 9)

Student E: “Because sevenths are smaller than sixths, then 4-sixths is bigger than sevenths so then 4-sixths would be bigger than 4-sevenths.” (Session 13)

I hypothesize that the students found considerations of the size of the partitions to be a powerful strategy as they worked to assess the size of fractions. Rather than automatically creating equivalent fractions to compare like denominators, the students appeared to draw upon their understanding of the impact of partitioning as an effective method to conceptualize and compare fraction sizes. The students seemed to engage more deeply with the size of each fraction
as they considered the impact of the number of the partitions on the size of the partition.

Applying partitioning reasoning is also more efficient and more feasible than creating common denominators in many cases.

Seeing the relationship between partitioning and its impact on the size of the parts is integral to understanding fractions conceptually. Without emphasis on more rules-based symbolic fraction comparison methods creating common denominators, the students seemed to latch onto the importance of considering the impact of partitioning in determining the size of the fraction.

It is notable that the students did not present evidence of whole number bias in considering fraction size—assuming that the fraction with the larger denominator is the larger number. It is possible their early work physically and visually partitioning wholes to create fractions helped to counteract the emergence of this common misconception.

The students seemed to find a focus on partitioning so valuable for their work with fractions that at times they appeared to be overly reliant on it. Such reliance may have been due to simply forgetting to pursue other methods or it may have been an indication of a lack of full understanding of certain fraction concepts. In such instances, the students’ attention seemed to remain overly fixed on the size of the partition, causing them to miss fully exploring or addressing the relationship between the numerator and denominator. The students’ challenges remaining overly focused on the denominator are discussed further in Theme 2.

Overall, the students’ experiences physically and visually partitioning fraction strips and tape diagrams seemed to provide an important foundation for them as they explored and used an understanding of the impact of partitioning. This understanding provided a powerful launchpad for them as they entered into more complex concepts.
Subtheme: Density of Numbers

This subtheme discusses the students’ understanding and use of partitioning as they worked through concepts of density with fractions and decimals. The students appeared secure in their understanding that numbers existed “between” other numbers. However, they still grappled with the idea of infinite partitions and just how many numbers were between other numbers. Understanding of partitioning appeared to be a particularly critical foundation for the students as they began to investigate how wholes and partitions could be partitioned infinitely.

The three students began to interact with concepts of density as they engaged in physical, visual, and mental explorations of partitioning within the experiment. All three students seemed comfortable with the idea that a whole that was already partitioned could continue to be partitioned. Their comfort was evidenced when they worked to create eighths from fourths, tenths from fifths, and other related fractions. The students were able to verbally explain how to create hundredths from tenths on a paper fraction strip. They were also able to envision what would need to be done to create thousandths on a tenths strip. The students seemed comfortable engaging in partitioning these values to create new sizes of partitions on the fraction strips. I conjecture that their work physically partitioning fraction strips may have helped provide support for their emerging understanding.

The students’ developing understanding of density also surfaced when they were tasked with naming fractions that fell between two given numbers. Their developing density reasoning was particularly prevalent in Session 13 when the students worked to identify a fraction between 3-fourths and one-whole. Student E initially suggested 70-hundredths based on her drawing in Figure 5.2. Her drawing was not precise enough to produce the correct value; however,
conceptually, she appeared to understand she needed to look for a fraction whose shading would fall between the two given values.

**Figure 5.2.**  
*Student E’s Work finding a number between $\frac{3}{4}$ and 1*

Student E’s difficulty identifying an appropriate value appeared to stem from challenges in grasping the equivalence between hundredths and fourths, rather than issues in understanding that numbers exist between 3-fourths and one-whole. The students’ willingness to propose fractions that lie between these two values appears to indicate a solid understanding and comfort with the concept of “numbers between fractions.” While the students needed support in naming a precise value, they engaged in the task of proposing one without hesitation.

As we delved deeper into the topic of density with decimals in Session 14, fragility in the students’ understanding began to surface. When posed the question, “Are there decimals between 0.3 and 0.4? How many do you think?” all three students affirmed the presence of such numbers. Such preliminary understanding is crucial since not all students recognize numbers exist between two fractions or decimals. However, when asked about the quantity of numbers between the two values, all three students encountered difficulties.

Student Z’s reasoning for density questions appeared most unstable. She first suggested 100 and then revised her answer to 300, explaining, “Yeah because you can add as many zeros as you want, and it would still be the same thing” (Session 14). She appeared to become lost in the
notion of “adding” zeros to the 0.3, indicating a potential misunderstanding of what is meant by “numbers in between.” Rather than conceptualizing the numbers “between” the two values, she may have been focusing on renaming the 0.3, even potentially perceiving new names for the same value as new numbers. Her response was particularly interesting in light of her response to the same question previously. Figure 5.3 shows Student Z’s response to the same question weeks earlier in independent written work. It is interesting that Student Z chose the number 1000 here, as her chosen pattern would not lead to 1000 numbers between the two given numbers. It is not clear what prompted the shift between the two times the question was asked, but it is possible that the later misconception is evidence of Student Z attempting to integrate new learning through which she was still working.

**Figure 5.3**

*Student Z Independent Written Work on Decimals Between 0.3 and 0.4*

![Image](image)

I have less insight into Student J’s reasoning because technology issues prevented her from explaining her response. She indicated “100” as an initial answer in the chat which may indicate a connection back to the work we had been doing with partitioning hundredths. Her response differed from her response in earlier independent work shown in Figure 5.4. It is possible that in the independent work she was not entirely clear what the question was asking and gave an example of a number between 0.3 and 0.4. Her understanding of the meaning of the question seemed to have shifted between her early work and her response in Session 14. Student
J’s shift may have been due to the question being asked verbally or changes in her ideas of what “between” meant based on work in the previous session with fractions.

**Figure 5.4**

*Student J Independent Written Work on Decimals Between 0.3 and 0.4*

Student E seemed to be working through merging different areas of understanding. She initially responded “infinity,” but adjusted her answer to 100 after hearing Student J and Student Z’s responses. Student E justified her revised answer saying, “Because…if you were doing the 3-tenths into 4-tenths there'd be at least 100 more to get there.” Student E’s understanding is not entirely clear from her explanation; however, she seemed to possibly be drawing upon the idea of the distance or “hops” that could be created between the two numbers. Reasoning involving partitioning the space between the two decimals would demonstrate an initial grasp of some of the density principle, where further partitions can be created between two values. This is potentially an intermediary step between perceiving the numbers as merely sequential, to recognizing the infinite number of subdivisions between any two numbers. Student E’s written response from earlier in the teaching experiment is shown in Figure 5.5. Similar to Student J, it seems she may have interpreted the question differently than as intended which speaks to a potential need to rephrase how such questions are posed. Her thinking during Session 14 does indicate a significant shift from her earlier thinking.
Following work partitioning space between 0.3 and 0.4 on a number line, Students E and J were able to identify that infinite numbers would exist between another two given decimals. The next task asked students if numbers exist between 0.35 and 0.36 and if so, how many. Student E justified her response saying, “I think it's infinity again because like last time we can, um, split like we did a number line from zero to one and then we labeled like the three, what the two numbers were, and then we could separate it into as many as we want” (Student E, Session 14). Student E’s explanation focused on our ability to partition as much as we would “want” provides an example of her emerging understanding of the density property of numbers and an important shift from her answer following Session 2.

The students appeared to grow in their understanding of density over the course of the teaching experiment. They were able to add to their understanding that numbers exist between numbers to begin to see that they could partition numbers indefinitely to create infinite numbers between other numbers. The students’ work with density is potentially another instance where instructional experiences may have served as a buffer from the interference of whole number bias. Despite challenges in identifying specific numbers between two fractions or decimals, all three students affirmed the existence of such numbers. It is possible their experiences partitioning, and then partitioning their partitions, supported a growing understanding of the concept of density of numbers.
It is important to note, the latter two density questions asked the students to reason with decimals rather than fractions. It is possible the shift to decimal form impacted how the students engaged with the concepts. We worked with the idea of fractions between fractions, but I did not ask the students how many fractions would be between two other fractions. In retrospect, I should have posed questions of this nature and provided opportunities with paper fraction strips and on tape diagrams to explore and support responses. Instead, I shifted the focus from fractions to decimals more rapidly than may have been best. The density of numbers is a complex topic for fifth grade students and additional work investigating the numbers “between” fractions and decimals likely would have benefitted them.

**Subtheme: Equivalency**

This subtheme discusses the students’ use of partitioning as they worked with fraction equivalency concepts. A full understanding of equivalency involves more than use of the procedure of multiplying or dividing the numerator and denominator of a fraction by the same number. Full understanding of equivalency is grounded in an understanding of the relationship between partitions—understanding that re-partitioning a particular area will not change the magnitude of the quantity. The students did not seem to automatically or immediately draw connections between partitioning and equivalency at first, but ultimately were supported by their understanding of partitioning as they worked to understand equivalency conceptually.

At the beginning of the experiment, the students appeared proud to share their knowledge of fractions that were equivalent to other fractions. When asked what they knew about particular fractions, they seemed to fall back on identifying equivalent fractions. However, the students did not seem to independently draw upon their understanding of partitioning as they worked to find equivalencies. Rather, they seemed to focus on using the more traditional procedure of
multiplying both the numerator and denominator by the same number to generate equivalent fractions. This tendency could be heard in Student E’s justification for the equivalency between 3-thirds and 6-sixths: “Cause six is equivalent to three. And whatever we do to one part, we have to do to another, so we multiply both of them by two to get six” (Session 3). Student E’s explanation here appears to be procedurally based. Immediately prior to it, she had made a mistake in naming 6-eighths as equivalent to 3-thirds. She self-corrected as soon as I asked her to justify her equivalency, but the error and her justification speak to a potential gap in her full understanding of equivalency.

I was able to nudge the students to engage with equivalency more conceptually by drawing their attention to their work with fraction strips and partitioning. By folding fraction strips and noting the relationships between the number of folds in the whole, Student Z was able to see and verbalize how additional folds or partitions would create a new size of partitions. She explained, “Because it's basically, so if you folded the four again, it would just double the amount. So that would be eight and then if you fold it eight again then it would be sixteenth” (Session 2). Student Z’s discussion of the connection between her fourths and eighths paper strips speaks to the conceptual underpinnings of equivalence between fractions. However, such explanations did not appear to be the norm for the students at the beginning of the teaching experiment.

Connecting back to partitioning physical and visual models emerged as critical in supporting the students in engaging with equivalency conceptually. The importance of anchoring ourselves in physical and visual models was especially apparent in Session 3. Despite her initial procedural explanation, Student E was able to talk through partitioning 3-thirds to create 6-sixths after being prompted to use a tape diagram. She was then able to extend similar thinking to
create the equivalent fraction of 9-ninths by partitioning a tape diagram again. Student Z noted that 3-thirds was equivalent to one whole and justified her answer saying, “Because 3-thirds if you like made it into a fraction and you shaded all of them in then that would equal one whole.” Following Student Z’s response, the students shared the equivalencies of 4-fourths, 10-tenths, and 11-elevenths for our original fraction of 3-thirds. Student Z’s connection to a whole seemed to spur additional thinking from all three students about this equivalence class of fractions (i.e., all fractions equivalent to one whole). The connection helped the students move away from a more procedurally based focus on multiplying or dividing to find equivalent fractions. The students were able to use their knowledge of what it means for a value to equal a whole or “one” and their knowledge of partitioning to generate many equivalent fractions. Challenging students to consider equivalencies to 3-thirds on a tape diagram appeared important because certain equivalencies could not be found through simple multiplication and division. Their resulting work potentially speaks to a stronger grasp of the meaning of equivalency supported by partitioning.

The students seemed able to handle equivalencies that could readily be represented by partitioning tape diagrams and through fraction strips. As equivalencies became less easily represented physically or as we shifted to the number line, components of their understanding appeared to break down. Instability was evident specifically for the relationships between fourths and hundredths, and tenths, hundredths, and other powers of ten. It is possible a deeper focus on the role and assistance of partitioning when working on such equivalencies may have benefitted the students in navigating these more challenging relationships. Theme 2 discusses the students’ challenges with these specific equivalent relationships further.
Equivalence between fractions was not initially a focus of this study but emerged as significant within the results. Equivalency was part of the puzzle as the students thought about and developed their understanding of fractions and their magnitude. The students’ early affinity for generating equivalent values perhaps spoke to their comfort in applying a procedure they had learned and found success with—multiplying the numerator and denominator by the same number. As we progressed through the experiment, the students seemed less likely to generate an equivalent fraction unless it was useful to the problem, such as when they wanted to compare to one-half and needed to know the equivalent number of partitions.

**Summary of Theme 1: Understanding and Use of Partitioning**

Throughout the teaching experiment, all three students used their understanding of partitioning in important ways. Understanding ranged from simply partitioning wholes to extending that thinking to partition partitions. Their work with comparisons, ordering, equivalency, and density was supported through their understanding of and ability to use partitioning as an action upon wholes and partitions. Their work exploring and using partitioning also seemed to support their developing understanding that all numbers have magnitudes which can be ordered. It seems possible that their fundamental understanding of partitioning served as a buffer to the impact of whole number bias. It should also be acknowledged that the students sometimes struggled to master more challenging, higher-level work with partitioning. They seemed to recognize the power of partitioning but were still working to fully solidify their use and understanding of it in more complex situations.

**Theme 2: Understanding and Use of Relationships**

Certain relationships emerged as critical to the students’ developing understanding of fraction and decimal magnitude: the multiplicative relationship between equivalent fractions, the
relationship between the numerator and denominator, the relationship of given fractions to benchmark fractions, and the relationship between decimals and fractions. At times, the students were able to use a growing understanding of these critical relationships to assist themselves. At other times, the critical relationships seemed to elude them or confound them further. Three subthemes emerged within the students’ work with relationships: (1) multiplicative reasoning, (2) relationships between fractions and benchmarks, and (3) relationships between fractions and decimals. The subthemes will be presented and discussed further in the following sections.

**Subtheme: Multiplicative Reasoning**

This subtheme discusses the students’ understanding and use of proportional relationships, specifically, how they were able to think multiplicatively to understand fractions. Students need to use multiplicative reasoning to fully understand the relationship between the numerator and denominator, and as one way to create or identify equivalent fractions. The section begins by discussing students’ work with the multiplicative relationship between the numerator and denominator and then moves to discussion of students’ developing understanding of equivalent fractions.

**The Relationship between the Numerator and Denominator**

The numerator and denominator have a special relationship in fractions. The size of a fraction is defined by the *multiplicative relationship* between the numerator and the denominator. The fraction 2/3 is defined as 2 parts of size 1/3. This can also be interpreted as 2/3 is a multiple of the unit fraction 1/3. The magnitude of the quantity is expressed multiplicatively. The relationship between the numerator (the 2 parts or copies) and the denominator (the part sized 1/3) conveys the size of the fraction. Without the relationship between the numerator and denominator each number lacks meaning in this context.
Throughout the teaching experiment, the students demonstrated some awareness and understanding of the relationship between the numerator and denominator but were inconsistent in their attentiveness to the relationship. At times, they addressed this relationship as they considered a fraction’s magnitude. At other times, they seemed to forget the relationship existed.

Student E demonstrated evidence of attending to both the numerator and denominator when justifying her placement of 3-hundredths as close to zero on a number line.

Student E: Hundredths are really small pieces and there’s only three of them so it’s really small (Session 12).

Student Z provided a strong example of what attention to the numerator-denominator relationship might sound like when comparing 12-fiftieths and 8-sixtieths as she contemplated the size of both numerators and denominators and the distance from the numerator to the denominator to conclude that 12-fiftieths had a shorter distance to complete to form one whole.

Student Z: I didn't split it apart because it was way too many partitions that I would I need to make. So, what I did, I actually just knew it off the top of my head because 12 is bigger than 8 and yeah, it's a shorter way to go when you have 50 so it will be longer to make 60 so 12-fiftieths is closer to one or bigger. (Session 13)

In the above example, Student Z appeared to be using “distance reasoning” which is discussed later in this section. Within “distance reasoning,” she was also paying close attention to the relationship between the numerator and denominator as she assessed which fraction would be larger. The students seemed to attend to the relationship between the numerator and denominator most often when engaging in distance reasoning.

At other times, the students seemed to focus solely on the denominator and forget about the relationship between the numerator and denominator. When Student Z compared 3-tenths
and 10-twelfths in Session 13, her focus remained fixed on only the denominator. She justified her conclusion of 3-tenths as the larger fraction saying, “I drew fraction bars, and I drew one as tenths and then I shaded in three and I noticed that the pieces were bigger in 3-tenths, so I think that 3-tenths [is larger] because the pieces are bigger.” Despite drawing a picture showing 10-twelfths as the larger fraction, she became preoccupied comparing the size of the denominator and forgot to consider the relationship between the numerator and denominator.

In the previous and following examples, the students’ attention seemed to remain overly fixed on the size of the partition, causing them to miss fully exploring or addressing the relationship between the numerator and denominator which ultimately determines the size of the fraction. Student E seemed to become muddled in the relationship between numerator and denominator when comparing 7-ninths and 11-thirteenths focusing on the size of the partitions and getting lost in the relationship of the numerators.

Student E: “Because ninths are bigger than well… 11-thirteenths would be really close to 7-ninths and 7-ninths is bigger than 11-thirteenths almost so then 7-ninths would be closer to one because the nine, the partitions would be bigger than the 13 and the 7 would be closer to 9” (Session 9).

Student E was correct in her assessment that ninths would be bigger than thirteenths but seemed to become lost when trying to navigate the relationship between the numerator and denominator of each number and their comparison to each other.

At times, the students’ reasoning was not off, but needed further elaboration. This frequently happened in regard to the connection between the numerator and denominator. One example of the need for further justification can be seen in Session 14 when Student J compared 7-eighteenths and 11-twentieths. She noted that 7-eighteenths was 11-eighteenths away from
being one whole and 11-twentieths was 9-twentieths away from being one whole and concluded 11-twentieths was larger. Student J was correct that 11-twentieths was larger, but she essentially created a comparison of two new fractions, 11-eighteenths and 9-twentieths, rather than truly exploring the relationship between the numerator and denominator.

A failure to properly consider the relationship between the numerator and denominator was also apparent in Session 4 when all three students identified 4-tenths as closer to zero than one-half. Student E had justified 2-tenths as closest to zero a moment earlier explaining, “Because anything in a tenths except that’s right by 5-tenths is really small. So, it’s, the two tenths would be very small, so we would put it really close to zero.” Students J and Z nodded their assent. I asked the students about 4-tenths next to see if they were considering the relationship between the numerator and denominator fully or just categorizing anything less than one-half as closest to zero. All three students placed 4-tenths as closest to zero, likely indicating they were assuming that any fraction less than one-half was closest to zero. The students were addressing both the numerator and denominator but were not truly considering the relationship between them. Four would be a small numerator if we were working with hundredths, but when working with tenths, then proportionally, it is not nearly as far from making a half or even a whole. The students’ classification of 4-tenths seemed to illuminate the misconceptions caused by the students’ failure to consider the relationship between the numerator and denominator.

The apparent neglect of the relationship between the numerator and denominator at times calls into question the extent to which the three students were truly considering the magnitude of the fractions in the study. Without consideration of the relationship between the numerator and denominator, it is impossible to truly assess the size of a fraction. The students may have benefitted from additional time spent discussing, exploring, and representing the relationship
between the numerator and denominator. In the future, lessons focused on the relationship between the numerator and denominator towards the beginning of the instructional sequence may help develop students’ sense of the size of fractions.

**Equivalent Fractions**

This subtheme discusses the students’ understanding and use of multiplicative reasoning as it relates to equivalent fractions. The students began the study with a tendency to share equivalent fractions when asked about any fraction. Throughout the teaching experiment the students appeared to grow in their ability to work conceptually with equivalent fractions as discussed in Theme 1.

Throughout the teaching experiment, the students demonstrated at least a surface level understanding of multiplicative reasoning as it relates to equivalent fractions. They were able to perform arithmetic to generate equivalent fractions and were able to make connections back to visual models when prompted. They appeared comfortable identifying and using fractions equivalent to one-half when working with benchmarks. As discussed in Theme 1, with the use of physical and visual models such as fraction strips or tape diagrams, the three students seemed able to discuss the relationship between particular equivalent fractions, noting the changing number of partitions. However, the strengths and fragilities of their conceptual understanding of equivalency began to emerge when we worked with a few particular equivalent relationships. These three relationships are (1) equivalencies to one half, (2) equivalencies between powers of ten, and (3) equivalencies between fourths and hundredths. Each relationship will be discussed in the sections that follow.

**Equivalent to One-Half.** The students’ use of multiplicative reasoning came through in their work with finding and using equivalencies for one-half. Throughout the teaching
experiment, the students regularly found and used equivalent fractions for one-half to aid their reasoning during tasks.

The students seemed to find identifying and using equivalencies for one-half useful when engaged in sorting activities, for example determining whether a fraction was more or less than one-half. They appeared comfortable identifying the equivalency to one-half in the size of the shares of many different fractions and even decimals. For example, Student J used her knowledge of equivalency to one half when classifying 0.6 within a benchmark sort in Session 10. She was able to think flexibly between decimals and fractions to help herself identify the appropriate equivalency for one-tenth and justify her classification of 0.6 as closest to one-half.

Student J: I think it's close to one half because it's, 6-tenths is one tenth away from being half.

Teacher: How do you know that it’s one tenth away from being half?

Student J: Because 5-tenths is half of one.

Student Z was able to navigate a less-familiar equivalency to one-half in Session 10 when working with 7-ninths. She initially focused on the distance of 7-ninths from zero and one-whole and then continued to share her thinking as she considered its relationship to one-half.

Student Z: And there really is no like proper benchmark in between, for like one-half of nine.

Teacher: Yeah, how should we deal with the one-half here? How many ninths would be equivalent to one-half? This is a tricky one.

Student Z: I feel like it would be about five.

Teacher: About five...

Student Z: Or like four and a half… Four and a half!
Teacher: Four and a half. Why are you thinking four and a half?

Student Z: Because a half plus a half equals one-whole and four plus four equals eight plus one whole because the two halves equals nine.

At first, Student Z bypassed the one-half benchmark believing there was no “proper benchmark” due to the odd denominator. After prompting, she began to consider how many ninths would be equivalent to one-half and was able to determine the appropriate equivalency. She used her understanding of what a half means and applied the understanding to identify the appropriate ratio between the denominator and its numerator.

The students also drew upon their understanding of equivalencies for one-half when working on the number line. For example, both Students E and J used their understanding of 500-thousandths as equivalent to one-half to help themselves place 0.490 on the number line.

Student E: So, I put it right in the middle (gesturing to number line).

Teacher: How far is 490/1000 from one-half?

Student E: One thousandth? No, 10-thousandths.

Student J: So, I knew that 500 was one-half away from 1,000 so I just went a little bit back and I knew that would be 490/1000.

Both students recorded 0.490 in its fraction form of 490/1000 on their number line. It is possible renaming the decimal in terms of a fraction helped them to deal with its equivalency in thousandths to one-half.

The students seemed comfortable identifying and using equivalencies to one-half throughout the teaching experiment. Their apparent understanding of the one-half equivalence class seemed to benefit the students in their work comparing fractions and placing them on the number line.
Tenths, Hundredths, and Other Powers of Ten. The relationships between tenths, hundredths, wholes, and other powers of ten emerged as both important, and frequently challenging for the students throughout the teaching experiment. The students’ relationship with this critical class of equivalencies appeared complicated. The students’ responses frequently included evidence of understanding—correct responses and explanations that made it appear the students understood and could use the relationship between tenths and hundredths. Then their responses to another task or question would reveal gaps in their understanding.

The students were typically able to identify and explain the relationship between tenths and hundredths when working with fraction strips. They could identify the number of partitions needed in a whole or a tenth to create hundredths. They were also typically able to identify how many hundredths would be in a given number of tenths even when working without the aid of the fraction strip. However, the fragility of their understanding often began to surface as we moved to number lines or more complex situations.

Evidence of understanding and use of the relationship between tenths and hundredths emerged in different ways throughout the study. At times, understanding was shown through focusing on the relationship between tenths and hundredths directly such as when Student E explained how to make hundredths on a number line from zero to one that was partitioned into tenths. Student E was able to show understanding of another power of ten relationship when we extended our work to thousandths in Session 11. When asked how we could create thousandths on a fraction strip that was partitioned in ten equal shares, she responded that we would split each tenth 100 times. She was also able to explain that 10-thousandths was equivalent to one 1-hundredth. She explained her equivalency noting, “Because if we did the hundredths times ten it would get us thousandths. So, like if we split 100 into tenths, we could split it, you would get
thousandths.” Student E began by envisioning a fraction strip partitioned into hundredths and then splitting each hundredth into ten pieces in order to create 1000 pieces on the strip. Student E’s thinking seemed to indicate a certain flexibility in moving between the size of the shares and thus some level of understanding of their relationship.

The students also demonstrated use of the relationship between tenths and hundredths in situations where they were not asked directly to focus on the relationship. In Session 6, all three students were able to accurately place 60-hundredths at the same location on their number line as 6-tenths. Student Z supported their placement by explaining, “I put it by the six, because um, by 6-tenths because one-tenth equals 10 and 10 times six is 60.” The students also used their understanding of equivalencies between tenths and hundredths when placing 75-hundredths on the same number line in Session 6. Despite some hiccups in accuracy, all three students agreed on the connection of 75-hundredths to its neighboring tenths.

Student Z: I think that goes right in the middle of 7-tenths and eight because it's five hundredths away.

Teacher: It's five hundredths away? Student J, Student E, what are you two thinking?

Student E: I agree with Student Z. It would go right in the middle of a jump from seven tenths to eight tenths.

Student J: Yeah, I agree with Student Z.

As students worked to place numbers expressed in terms of hundredths on a number line partitioned into tenths, they seemed comfortable using equivalencies between tenths and hundredths for numbers larger than one-half. Figure 5.6 shows Student J’s number line from Session 6 and her placement of 39/100, 60/100, and 75/100.
Student J further demonstrated understanding of the relationship between tenths and hundredths when explaining that 40-hundredths would be closer to one-half than zero in Session 4.

Student J: I think it’s closer to one-half... because if you were to add one more tenth, or 10 more, then you'd get 50-hundredths, which would be half of it. (Session 4)

Student J’s explanation appears to demonstrate flexibility in thinking between hundredths and tenths.

The students’ understanding of the relationship between tenths, hundredths, and other powers of ten appeared to vary based on the context and the day. Certain tasks provided a window into fragility in their reasoning. When asked how they would show 199/200 in Sessions 2 and 3 Students J and E were able to verbalize how they would create a visual image to show the fraction.

Student J: “So you'd make a rectangle split into two hundred equal parts and then you'd color in one hundred and ninety-nine.” (Session 2)

Student E: “You can have two hundred, but a hundred ninety-nine are filled, and um then um you would have one left over, one left over that's not part of it.” (Session 3)

Student Z struggled more in answering the question.
Student Z: I would fold tenths 10 times because 10 times 10 is 100. Mm hmm. And then I would do it 10, another 10 times and fold one. So, there's just nine.

Teacher: So, you would have just nine of those tenths.

Student Z: Yeah.

Teacher: So, you fold it in tenths first, is that right? And you fold it in tenths again. So what size were your pieces then?

Student Z: My pieces like were tenths. They were in the tenths they were folded as tenths, and so was the other 10 tenths.

Teacher: So, you have two strips? (Nods yes) So, you did one in tenths and then another strip in tenths?

Student Z: Yeah, and then I folded one. So, it only looked like there was nine instead of ten.

Teacher: So, then you have one. That's just nine of those tenths and then one that is 10 tenths. Is that right? Yeah? So where do we see the two hundredths in there?

Student Z: If you unfold the last one.

Teacher: Then we'd have two hundred equal pieces?

Student Z: Or if you like, shaded in every single one of those squares up to the last one and don't shade in the last one.

Student Z seemed to try to draw on the relationship between tenths and hundredths but became muddled in how to create pieces of size two-hundredths. Student Z shifted from one strip to two as she worked to create two-hundredths indicating she likely lost the whole. Even within that shift, she did not create hundredths. Rather, she seemed stuck in partitioning her wholes into
tenths. Student Z’s reasoning here likely indicates gaps in a full understanding of hundredths and what comprises the whole.

Fragility in the students’ understanding of the relationships between tenths, hundredths, and other powers of ten seemed particularly likely to emerge when we worked on the number line. This often presented in the form of mislabeling where a multiple of ten belonged on the number line. All three students struggled with where to mark one-tenth on a number line from zero to ten in Session 5. In Session 7 we made another number line through iteration of our one-tenth pieces from zero to 13-tenths. All three students were able to place one-tenth accurately on the zero-to-13-tenths number line; however, locating fractions with a denominator of 100 exposed additional fragility for all three students.

Challenges with the naming the size of fractional parts was a recurring issue for Student Z. At times, this was simply due to a precision issue, and she was readily able to identify the size of the piece when asked. At other times, particularly when dealing with tenths and hundredths, she struggled to identify the size of the piece she was discussing even when pressed. Student Z’s struggle with the size of pieces was evident as she tried to explain her representation of 57-hundredths on a fraction strip in Session 5.

Student Z: Because it's seven more, and if you put it right exactly in the middle it wouldn't be 57, it would be 50.

Teacher: It's seven more what? It's seven more... Tenths, hundredths, wholes…?

Student Z: Those little pieces.

She had similar struggles later in the session as she jumped into Student E’s explanation of how to represent 98-hundredths on the fraction strip.

Student E: Because it's two fractions away from 100 hundredths.
Teacher: You're two away from 100 one-hundredths. Two pieces of size... Ten? Or one tenth?...

Student Z: The oneths.

Student Z was able to represent both fractions appropriately on her fraction strip, yet still struggled to accurately identify the size of the shares she was considering. She appeared to simultaneously present evidence of understanding while still becoming lost in the size of the shares when working with hundredths in particular. Her challenges in naming the size of a piece seemed to continue through sessions 6 and 7. In Session 8 she also seemed to become lost between the tenth and the whole on a number line from zero to ten but her struggles here were likely due to the use of a zero-to-ten number line. No precise moment emerged when Student Z suddenly gained clarity in naming the size of the shares under consideration, but her struggles in doing so seemed to dissipate following Session 8. Though I still needed to prompt her to address the size of the share at times in later sessions, she was able to accurately name the size of the share in Sessions 9 through 14. It is possible our additional work in understanding the relationship between tenths and hundredths helped Student Z gain clarity or that she simply paid more attention to the size of the share.

The challenges exposed in our work with tenths, hundredths, and other powers of ten potentially indicate some sort of gap or breakdown in understanding of these values. It is possible the number line itself was creating the challenge or that the number line was simply revealing gaps in understanding that were not apparent when working with the fraction strips. Another possible interpretation is that the students were not making the same connections between partitioning and the number line as they were with partitioning and the paper fraction strips. We engaged in experiences partitioning number lines throughout the teaching experiment,
but the students may have needed further time with physical relationships using objects such as paper strips prior to moving to the more abstract visual form of the number line.

In hindsight, as the instructor and designer of the instructional sequence, my choices in lengths for our number lines of zero-to-one and zero-to-ten, likely contributed to some of the student challenges with understanding and representing the relationship between tenths, hundredths, and the whole. Moving between lengths of one whole and ten potentially contributed to students’ confusion and loss of the whole on the number line and the value of tenths and hundredths. This is discussed further in Theme 3.

**Fourths and Hundredths.** One particular equivalent relationship emerged as problematic for the students several times throughout the teaching experiment—the relationship between fourths and hundredths. I did not anticipate this particular difficulty and paid it minimal attention when it emerged during the sessions. However, upon reviewing the transcripts and videos, I realized the students’ challenge with fourths and hundredths was more significant than I initially recognized.

Challenges with the equivalence between fourths and hundredths first arose as we placed 1.25 on a zero-to-ten number line in Session 8. Student E placed 1.25 as equivalent to 5 and Student J placed it midway between 1 and 2. I hoped to make a connection to fourths to help the students as they worked on revising their placements, but the students struggled with the 1.25 and 1¼ equivalency.

Struggles with the equivalency between fourths and hundredths resurfaced in Session 11 when I asked the students to place 3-fourths and 0.75 on the same number line. None of the students were able to identify this equivalency even when asked directly about it. To address the continuing challenge with fourths and hundredths, I had the students work with me to partition a
tape diagram both into fourths and hundredths, determining the number of hundredths equivalent to one-fourth and subsequently 3-fourths.

Fundamental challenges with the equivalency between fourths and hundredths seemed to contribute to challenges with a seemingly unrelated number: 4-tenths. Both Students J and E placed 4-tenths at the same location as 0.25 on their number line during Session 11. Student E explained her placement saying, “I put it under 4-tenths because um I put it close to 4-tenths because um, we only need 25 to get to, we only need 4 twenty-fifths to get to 100.” Her explanation seemed to indicate an attempt to recall and apply the work we had done connecting fourths and hundredths. Student E appeared to remember that we were dealing with four 25s in some way but was not able to integrate her recollection in a meaningful way with what she understood about hundredths and tenths and fourths.

The students failed to consider 4-tenths as 40-hundredths and 0.25 as 25-hundredths. Had they done so, they likely would have been better able to navigate the relationship between the numbers. Instead, they seemed to draw on a tertiary memory of the number four being related to 0.25 and related it to 4-tenths rather than one-fourth. Struggles with 4-tenths and 0.25 seem to indicate an attempt to draw upon previous work, but a lack of understanding leading to partial recall and misapplication. The students likely needed further experiences partitioning a whole first into fourths and then into hundredths themselves to fully explore and solidify the connection.

Entering into the teaching experiment, I had assumed the relationship between fourths and hundredths would be a familiar and comfortable relationship for the students. I found, however, that it was not only unfamiliar, but that the students struggled to make connections to it and recall it in later use. The students’ challenges here likely speak to fundamental gaps in
understanding of equivalencies and how to generate equivalencies beyond more simple applications of the procedural rules. It also suggests teachers should use caution in assuming students’ comfort with the equivalency between 25-hundredths to one-fourth.

**Subtheme: Relationships between Numbers**

This subtheme discusses the students’ understanding and use of specific relationships between numbers throughout the course of the teaching experiment. The section will first discuss relationships of fractions to benchmarks fractions and then will discuss the relationships between decimals and fractions.

**Relationships to Benchmarks**

Over the course of the study, the students increasingly began to use benchmarks of zero, one-half, and one-whole as they worked through various fraction tasks. One example of the students’ shift towards use of benchmarks was noted in a shift from the baseline to the concluding assessment. The students were asked to place the following numbers on a number line from zero to two for both assessments: 25%, 0.09, 0.4, 1.25, 0, 0.09, 0.4, 1.25, 0.09, 0.4. None of the students added benchmarks to their number line for the baseline assessment. All three students added the benchmark of one-half to their number line on the concluding assessment. The inclusion of one-half as a benchmark potentially indicates an increase in the perceived value and utility of benchmarks for the students as they worked to place fractions and decimals on the number line. Pictures of the students’ baseline and concluding assessment number lines can be seen in Figure 5.11 in Theme 3.

We began Lesson 2 with sorting activities using benchmarks. Using benchmarks appeared to become a favorite reasoning strategy for the students as the teaching experiment progressed. The students seemed to draw upon relationships back to the benchmarks of one-half and one
whole more automatically beginning around about Session 4. Their use of benchmarks as a primary strategy became particularly prominent in Sessions 10 and beyond.

The students used benchmarks to help themselves compare fractions and place fractions on the number line. The examples that follow show how Student J used benchmarks when explaining a number line placement and Students E and Z used benchmarks when comparing two fractions.

Student J: I knew that 500 was one half away from 1,000 so I just, so I went a little bit back and I knew that would be 490/1000. (Session 11)

Student E: Two-sevenths is not really close to a half because sevenths are really big, and then you just add one more seventh and it would be almost one-half, and then 5-ninths is just one-ninth away. (Session 15)

Student Z: Eleven-twentieths has a bigger denominator, but the numerator 11-twentieths is one-twentieth away from being one-half and 2-eighteenths away from. Seven-eighteenths is 2-eighteenths away from 9-eighteenths which is one half, so I think 11-twentieths is closer. (Session 14)

In all three examples, the students drew upon the relationship of the fractions under consideration back to the benchmark of one-half. Their conception of the fractions’ magnitudes seemed to be potentially predicated on the fraction’s distance from the one-half benchmark.

Interestingly, the students seemed to tend to focus on each fraction’s distance from the benchmark rather than if the fraction was greater or less than the benchmark when comparing fractions. In the preceding comparison examples, I had selected the numbers to elicit the use of the one-half benchmark, but the students did not use the benchmark in the way I anticipated. Rather than simply using their classification as greater or less than one-half, the students
addressed the specific distance from one-half and used that to aid in their comparison. Upon further probing, it became apparent they were aware of which fraction was smaller and which was larger than one-half, but they still did not seem to gravitate towards use of this strategy.

The concept of gauging distance from a benchmark seemed to resonate with the students, particularly as they grappled with assessing fraction magnitudes. This became a preferred justification path for students when comparing fractions and when placing them on number lines. As noted earlier, I named this type of thinking “distance reasoning” and it appeared to become a significant framework through which the students engaged with fractions and decimals throughout the study. Distance reasoning emerged when the students focused on benchmarks such as during our work on benchmark sorts, but also when benchmarks were not the immediate focus.

A further illustration of distance reasoning was evident within Student Z’s thinking when categorizing 2-ninths as closest to zero in Session 10 because it was “two away from zero” and “7-ninths away from one.” Student Z looked at the distance of the fraction from zero and one whole and used the comparison of those two distances to help justify her classification of the fraction. She could have simply said 2-ninths was small, but instead of doing so she drew upon distance reasoning to help her justify why it was closer to zero. Student E used distance reasoning when classifying 40-hundredths during a benchmark sort sharing her thinking in her written work for the day shown in Figure 5.7. Though she does not explicitly address the comparative distance from zero she seemed confident when sharing that showing that 40/100 only required 10/100 to equal a half as justifying her conclusion.
Figure 5.7

*Student E explanation for 40/100*

\[
\frac{40}{100} \quad \frac{10}{100} = \frac{1}{2} \text{ of } \frac{50}{100}
\]

Student Z again applied distance reasoning in Session 10 when classifying 0.76. She assessed that 0.76 would be “24 away from one whole” and 26-hundredths away from one half which helped her determine that it would be closest to one whole.

The students also frequently drew upon distance reasoning when justifying estimates of fractions on a fraction strip or their location on their number lines. Student E used distance reasoning when sharing where she believed 98-hundredths would be represented on a fraction strip representing one whole in Session 10. All three students held their fraction strips up to demonstrate an appropriate value and Student E explained her choice saying, “Because it's two fractions away from 100-hundredths.” In a follow-up representation of 10-hundredths, Student Z explained, “Because it's only 10 and there's 90 more little pieces, so it'll be on the way like left” (Session 10). In both explanations, the students seemed to lose the *size* of the piece, representing some of the challenges experienced in navigating the relationship between tenths and hundredths; however, they still seemed able to draw upon and use distance reasoning.

At times, the students seemed to potentially overuse distance reasoning, applying it even when it was not the most effective or efficient strategy. Alternatively, they would use it appropriately, but not fully work through it. Student E and Student J both engaged in distance reasoning, but in doing so did not fully justify their response in Session 12 when comparing 3-eighths and 3-tenths.
Student E: I think it's 3-eighths because like I say for 3-eighths is really close to, is closer to one half than 3-tenths because we only need 7-tenths which is more to get to one half [meant one whole] which is a lot and then we need only 5-eighths which is not that much more than 7-tenths.

Student J: I thought about it similarly because 3-eighths is only one eighth away from being one half and 3-tenths is 2-tenths from being one-half.

Both students were attempting to use the distance from the benchmark to justify their response but did not fully mathematically support their choice. Rather, they seemed to create a new comparison between the distances from the half or the whole. It is worth noting that both fractions have the same numerator of three. The students could have focused on the presence of the same numerator and noted the difference between the size of the partitions in justifying their comparison. The students seemed to default to distance reasoning even in cases such as the previous examples when it was not as efficient. Use of distance reasoning even when it was not most efficient potentially speaks to the students needing additional development of flexibility in strategy use when comparing fractions.

**Relationships between Fractions and Decimals**

The students were asked to work with both fractions and decimals throughout the teaching experiment. The students seemed comfortable overall moving between fraction and decimal notation but seemed to gravitate towards expressing all numbers in fraction form. Despite their apparent comfort with both decimals and fractions, certain challenges emerged which potentially indicated the use of decimals as a complicating factor.

The students tended to rename decimals as fractions both in their speech and their written work throughout the teaching experiment. This tendency is potentially reflective of their...
classroom instruction where they were taught to read decimals conceptually: for example, saying 3-tenths rather than zero-point-three when reading the notation 0.3. I was surprised, though not concerned, to see that the students carried this language into their written work, frequently re-writing given decimals as fractions on their papers. It is possible expressing decimals as fractions helped the students to conceptualize their size.

The students’ tendency to rename decimals as fractions typically seemed to aid them in working with decimals. This benefit was apparent in several of our tasks early in the teaching experiment. In Session 3, Student E adeptly identified that 0.501 is larger than one-half and Student Z was able to justify 0.489 as smaller than one-half.

Student E: More than one-half…. Because zero and 501-thousandths is really close to one half, but it's one more than one half.

Student Z: Because um, it's hmm, ten, no eleven smaller, than 500 and 500 is like the benchmark.

Expressing the decimals as fractions and using their knowledge of an appropriate fraction benchmark to one-half seemed to support the students in their work classifying decimals as greater or less than one-half. In this way, their work renaming decimals as fractions supported their ability to engage with the given decimals.

Despite the students’ apparent comfort with the relationship between decimals and fractions, certain tasks in the teaching experiment seemed to illuminate fragility in their understanding. For example, the students’ struggles with the equivalency between fourths and hundredths may have been due in part to the expression of the numbers 0.75 and 1.25 in decimal form. Student E and J’s conflation of one-fourth and 0.4 seems to indicate they were not consistently secure in how to express 0.4 as a fraction.
Our shift from fractions to decimals in our work identifying numbers “between” numbers as we explored density may have complicated the students’ thinking. It is possible the use of decimals created or contributed to fragility in the students’ understanding of how many and what numbers would exist between two other numbers. The students may have benefitted from additional time investigating density concepts with fractions and then making more direct connections between fractions and decimals as we shifted to exploring numbers between specific decimals.

One of the goals of this teaching experiment was to help students understand the magnitude of both decimals and fractions through their relationship to each other. At times, the students appeared to move comfortably between decimals and fractions and use their understanding of fractions to assist themselves with decimals. At other times, the use of decimals seemed to complicate the students’ thinking. The inclusion of decimals in the study seemed to cause additional challenges for students, but also illuminate gaps or misconceptions that might otherwise have remained hidden. I conjecture that overall, the students’ sense of magnitude and ability to work with decimals was enhanced by their connections to fractions. As a result, I believe magnitude work that includes both fractions and decimals is valuable to future instructional sequences.

**Summary of Theme 2: Understanding and Use of Relationships**

Understanding and use of relationships between numbers emerged as critical for the students in this study. The students worked to relate numbers to each other and use their understanding of their relationships to help themselves grapple with the magnitude of numbers. Consideration of the distance from benchmarks became a key framework for the students as they engaged with less friendly numbers throughout the teaching experiment. Having to engage with
the relationships between numbers challenged, confounded, and seemed to benefit the students all at the same time.

**Theme 3: Building and Expanding Understanding of the Number Line**

The number line emerged as critically important throughout the teaching experiment. The number line also proved to be more challenging for the students than I initially anticipated. It is not clear if the number line itself created more challenges for the students or simply illuminated hidden fragilities that might not otherwise have been apparent. Much of what occurred on the number line intersected with other themes; however, the number line itself is worthy of discussion because of its particular importance within this study.

Three subthemes emerged as significant within the students’ work on the number line: transitions between physical models to the number line, representing power of ten on the number line, and losing the whole on the number line. The following sections discuss each of the subthemes and the ways in which they surfaced through the students’ work within the teaching experiment.

**Subtheme: Transitions between Physical Models and the Number Line**

This subtheme discusses students’ work and struggles transitioning from a physical model—paper fraction strips, to a written number line. As discussed in earlier sections, the students appeared comfortable working with physical models such as paper strips and were typically able to represent various numbers including tenths and hundredths on paper fraction strips. As the students moved to the number line, however, previously perceived understandings and skills appeared to become more tenuous.

As we began our work with number lines in Session 5, it became apparent all three students needed support in thinking about numbers less than one whole on the number line.
Given the tactile nature of the paper fraction strips and the students’ previous comfort with them, I believed directly connecting the strips to the written number line would allow the students to use their previous understanding to support the new model. We engaged in this transition in two ways. First, the students laid their pre-partitioned fraction strips down on a paper and marked number line locations on their paper using the folds of the fraction strip in Session 6. Second, the students created a separate number line through iterating the one-tenth partition of their fraction strip end-to-end on their paper to create a number line from zero to 13-tenths in Session 7. The students were successful in constructing number lines using both methods; however, further tasks revealed continuing gaps in their understanding of magnitudes as represented on the number line.

The connection between paper fraction strips and a written number line appeared challenging for the students. This challenge was evident in Student Z’s struggles with one-hundredth on the number line in Session 7. Student Z stated that one-hundredth belonged near the zero, but then followed up by explaining that it belonged there “because 99-hundredths would be closer to one-tenth and that's 99-tenths away from one-tenth.” At first, Student Z appeared to have a grasp of the location of one-hundredth on the number line, but her follow-up explanation revealed the fragile nature of her understanding of the relationship between the whole, tenths, and hundredths. I brought Student Z back to the paper fraction strips and she was able to elucidate that ten one-hundredth pieces would be in one-tenth. However, she was not able to maintain the relationship between tenths and hundredths when we then returned back to the number line.

Teacher: If I'm thinking about my fractions strip and I'm going back to my number line that I created, how many hundredths are there from zero to one-tenth?
Student Z: There are 99? Over 100?

Teacher: Think a little bit more about what you told me with the fraction strip. How many partitions would there be in one of these tenths to make hundredths?

Student Z: 100

Despite successfully indicating that 10 hundredths were in one-tenth a moment prior on the paper strips, Student Z still appeared lost when describing the relationship on the number line. Student Z’s struggles with tenths and hundredths likely indicate fragility in her understanding of hundredths either increased, or illuminated, by her work on the number line.

Another example of the students’ struggles to connect understandings on the fraction strip to the number line emerged in Session 8. After struggles locating one-tenth on the number line and then successful work using the fraction strips to identify the relationship between tenths and one whole, Student E was able to place one-tenth on the number line. However, she became lost shortly after when placing 3-tenths on the same number line. Rather than iterating her one-tenth piece three times, she continued to place 3-tenths at 3 wholes on the number line.

We returned to more direct work placing tenths and hundredths on the number line in Session 11. Something seemed to have clicked with the students when engaged in number line work in this session. The number 0.09 had previously been problematic for all three students, but Student E and Student J were able to locate the number and explain its location on the number line in Session 11 (Student Z was not present for Session 11). The students’ later number lines were not perfect, but all three students appeared to have a clearer sense of the relationship between tenths, hundredths, and the whole on the number line for the remainder of the teaching experiment. It is not clear if our previous work making connections between the paper fraction
strips and the number line eventually contributed to their growth or if it was due to another factor, but something seemed to have shifted.

**Subtheme: Representing Powers of Ten on the Number Line**

This subtheme further discusses the particular challenge the three students appeared to experience in navigating the relationships between powers of ten on the number line. The number line emerged as a key place where this challenge appeared. It is not clear if the number line created more challenges for students with tenths and hundredths or simply illuminated hidden fragilities that might not otherwise have been apparent.

The tenuous nature of the students’ understanding of the relationship between tenths, hundredths, and the whole became apparent almost immediately when working on the number line. As noted in the previous section, the students seemed to be able to handle these relationships when working with paper fraction strips, but then struggled when we moved to number lines. The students’ struggles with powers of ten on the number line first appeared when none of the students were able to accurately place one-tenth or one-hundredth on the zero-to-ten number line in Session 5. Even after working to support the students in making the transition from paper fraction strips to the number line, the number line continued to reveal gaps in the students’ understanding in Sessions 6, 7, and 8. Figure 5.8 shows Student Z’s placement of 10/100 as approximately right at zero (Session 6).
Student E experienced similar struggles with the placement of tenths on the number line. Figure 5.9 shows one example of Student E’s struggles with placing and labeling tenths on the number line. In addition to other errors, one-tenth is located in the same place as one whole and 10-tenths, and 13-tenths, 3-tenths and 3 all share the same location.

Student J was not immune from challenges with placing powers of ten on the number line either. Figure 5.10 illustrates revisions in Student J’s thinking during Session 7 as she placed hundredths and tenths on the number line. It is difficult to see all of her original placements due to cross-outs, but her struggles with 9/100 are still visible.
Portions of the students’ struggles with placement of tenths and hundredths on the number lines may also have been influenced by my design features—particularly the use of a number line from zero-to-ten in Sessions 5 and 8. The use of the zero-to-ten number line complicated our work as the students grappled with thinking about one whole as both one whole and as one-tenth of the zero-to-ten number line. Though the zero-to-ten number line may have helped to surface certain misconceptions, this complication did not overall enhance the students’ learning at the time of the teaching experiment. The students likely needed to solidify their understanding of tenths, hundredths, and the whole and their relationship on the number line prior to adding in this layer of complexity.

As noted previously, by the end of the teaching experiment, the students appeared better able to handle placement of tenths and hundredths on the number line. Figure 5.11 presents each student’s independent number line from their baseline and concluding assessments. The students’ number lines are not perfect, but growth can be seen in their placement of many of the numbers. Of particular note is the shift in all three students’ placements of 0.09 and 0.4. Their concluding assessment locations of both numbers may indicate a greater conceptual understanding of their magnitude and location on the number line.
**Figure 5.11**

*Students’ Baseline and Concluding Assessment Number Lines*

<table>
<thead>
<tr>
<th>Student Z</th>
<th>Baseline Assessment:</th>
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</thead>
<tbody>
<tr>
<td><img src="image1" alt="Baseline Assessment" /></td>
<td><img src="image2" alt="Concluding Assessment" /></td>
</tr>
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<table>
<thead>
<tr>
<th>Student J</th>
<th>Baseline Assessment:</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3" alt="Baseline Assessment" /></td>
<td><img src="image4" alt="Concluding Assessment" /></td>
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</tbody>
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<table>
<thead>
<tr>
<th>Student E</th>
<th>Baseline Assessment:</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image5" alt="Baseline Assessment" /></td>
<td><img src="image6" alt="Concluding Assessment" /></td>
</tr>
</tbody>
</table>
Subtheme: Losing the Referent Whole on the Number Line

This subtheme discusses issues with loss of the referent whole when working on the number line. Some of the students’ challenges working with the number line may be attributable to losing the referent whole or the “one” when working on the number line. Identification of the referent whole is critical when working with fractions, particularly on the number line. The referent whole is defined by the interval between zero and one whole on the number line. The “referent whole” allows the students to grasp the size of all other numbers in reference to it. One of the students’ challenges with working on a number line was recognition that a number line can represent multiple wholes, but that the referent whole itself is still just one.

Many of the students’ struggles in placing tenths and hundredths on the number line appeared to relate back to loss of the referent whole. The students’ work in Session 5 working with a zero-to-ten number line is particularly revealing of their struggles with the referent whole. The students seemed to equate the number one-tenth with one tenth of the whole zero-to-ten number line, thus locating one-tenth as equivalent to one whole. The number line from zero-to-ten seemed to become their referent whole rather than one whole itself. The students’ struggles in identifying the referent whole on the zero-to-ten number line likely indicates that the use of zero-to-ten for two of our early number lines was particularly problematic.

Struggles with the referent whole emerged again when we returned to another zero-to-ten number line later in Session 8. All three students continued to conflate one-tenth of the entire written number line with the number one-tenth, placing one-tenth right at or near one whole. Student Z explained, “I put it under the one too because that's one, which is one tenth of the way on the number line.” Student Z was correct that the distance was one-tenth of the total distance represented on the number line but missed the actual value of one-tenth. Student E appeared to
lose understandings she had explained at other times, explaining her placement of one-tenth at
one whole saying, “Because I knew the one equaled 10-tenths and then 10-tenths is at least I
think it's equivalent to one-tenth.” The zero-to-ten number line thus appeared particularly
problematic for the students. It potentially both illuminated and contributed to the students’ loss
of the whole on the number line.

The students did not always lose the referent whole on the number line. All three students
were able to accurately locate one whole at 10-tenths when we created our number line from zero
to 13-tenths using iteration of the one-tenth partition in Session 7. However, despite her
successful placement of one-whole on the zero-to-13-tenths number line, Student Z’s
understanding of the relationship between the whole and other values still appeared tenuous
when she explained that one-hundredth would be 99-hundredths away from one-tenth on the
same number line.

The students’ loss of the referent whole on the number line is likely indicative of larger
issues or gaps in their understanding of the number line and the relationships between tenths,
hundredths, and the referent whole. The students likely would have benefitted from additional
time and work to solidify their understanding of what the referent whole is and its relationship to
other numbers. The number line may be more challenging than other models for students who
are insecure in their sense of the referent whole because of its contiguous nature. In hindsight,
future instructional designs should strive to be more attentive to and proactive in supporting the
students’ sense of the referent whole when transitioning to the number line.

Understanding of where the referent whole is on the number line is critical to
understanding the magnitude and placement of numbers on the number line. Without definition
of the referent whole via the interval between zero and one-whole, we cannot have a sense of the
size of any other numbers on the number line. The students could not truly be successful in their work with hundredths and tenths on the number line unless they understood and could identify the referent whole to which tenth and hundredth partitions related. The students’ apparent growth in success on the number line by the end of the teaching experiment likely indicates less loss of the referent whole; however, it would be imprudent to assume they were all fully secure in this area. Understanding and maintaining the referent whole will be critical for the students as they move forward with rational number work in future grades.

Summary of Theme 3: Building and Expanding Understanding of the Number Line

One of the goals of the current study was for students to build and expand their mental number line. I hoped to both support students’ sense of fraction and decimal magnitudes through the number line, as well as be able to see evidence of their understandings on the number line. Throughout the course of the teaching experiment, the number line was a key component of how students engaged with and deepened their understanding of fractions and decimals. The number line also appeared to complicate the students’ learning at times. The students’ struggles with powers of ten and losing the referent whole on the number line revealed that their understanding was more fragile than I initially realized.

Ultimately, the students appeared to grow in their ability to place numbers in reasonably accurate locations on the number line over the course of the study, but progress was not without growing pains. The students also appeared to find additional value in the number line as a representation by the end of the teaching experiment. The students only included a number line on their baseline assessment when the question directly asked them to place numbers on the number line. However, on the concluding assessment, Students J and Z used a number line to justify their thinking for questions involving ordering and distance from one whole.
Moving between paper fraction strips and the number line did not initially appear as helpful as I had hoped but may ultimately have benefitted the students. Partitioning the number line further and iterating hops of one-hundredth and one-tenth seemed beneficial at times and then at other times did not seem to transfer between tasks. No clear moments of “aha” or leaps in understanding emerged, but some sort of cognitive shift did seem to have taken place by the end of the teaching experiment as evidenced by the students’ work with tenths and hundredths on their later number lines.

**Summary of the Retrospective Analysis**

Three fifth-grade students were engaged in studying fraction and decimal magnitude over the course of 15 sessions spread throughout five weeks. Three major themes surfaced as I worked to answer the research questions: (1) understanding and use of partitioning, (2) understanding and use of relationships, and (3) building and expanding understanding of the number line.

The three themes intersected and overlapped in their emergence through the students’ thinking but all three appeared to be critical pieces of the students’ developing understanding of decimal and fraction magnitude. Each theme was its own piece of the students’ puzzle as we worked to develop their conceptual understanding of the size of numbers and their locations on the number line. The students’ work within these three themes illuminated places where their thinking grew and places where growth was still needed. The following chapter discusses these three themes and their link with the literature on fraction and decimal magnitude.
CHAPTER SIX

DISCUSSION AND CONCLUSIONS

The purpose of this teaching experiment was to investigate how upper elementary students think mathematically when presented with fraction and decimal magnitude tasks and the means teachers might use to support their learning. Three questions guided this inquiry:

4. How do fifth grade students reason about the magnitude of fractions and decimals?
5. What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude?
6. What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?

This chapter begins by revisiting two theoretical frameworks that informed interpretation of the data: the integrated theory of numerical development (Siegler et al., 2011) and conceptual change framework theories (DeWolf & Vosniadou, 2015; Stafylidou & Vosniadou, 2004). The study’s findings are then discussed with connections to the extant literature. Next, conclusions are presented in relation to the research questions. The chapter then discusses implications for instruction. Finally, the chapter addresses limitations of this study and implications for future research.

This study included three fifth-grade students all from the same class in a mid-size, Title 1 public school. The three students were selected based on a baseline assessment and conversations with their classroom teacher. The students were identified as experiencing the most challenges with fractions and decimals in their fifth-grade class. Teaching and learning took place over 15 sessions across five weeks. All sessions were conducted virtually via Zoom with each student.
typically working on their own Chromebook. Analysis was conducted both during the teaching experiment and retrospectively after all teaching sessions had been completed. Data gathered from the students’ written work and recordings and transcripts of the teaching sessions were analyzed. This analysis generated three major themes. These themes create a picture of the three students’ developing understanding of fraction and decimal magnitude over the course of the teaching experiment.

**Theoretical Frameworks**

Two theoretical frameworks informed interpretation of the data throughout this teaching experiment: *conceptual change theory framework* and the *integrated theory of numerical development*. I present a brief overview of these two theories in the following paragraphs. Chapter 2 provides a more thorough discussion of both frameworks.

In the conceptual change theory framework, students adjust their current “naive” theories as they gain more information (Vosniadou, 2007). Stafylidou and Vosniadou (2004) put forth that knowledge acquisition is not always a process of “enriching existing conceptual structures” (p. 504). Rather, sometimes new information “requires the radical reorganization” of previously known ideas (p. 504). Additionally, Stafylidou and Vosniadou (2004) noted that learning which requires “reorganization” of previously existing knowledge structures is more challenging and cumbersome than learning via enrichment. Students will often create misconceptions within this process of reorganization. Many of these misconceptions may be “synthetic models” that reveal the student’s attempts to “assimilate” new information to their current knowledge framework (p. 504).

The integrated theory of numerical development asserts that children’s developing understanding of numerical magnitudes is the “unifying theme of numerical development”
(Sigler, 2016, p. 341). Within the integrated theory, students’ mental number lines and understanding of magnitude is critical to their mathematical development (Siegler, 2016). Also important to the integrated theory is the idea that all rational numbers have a magnitude that is represented on a mental number line, a “dynamic structure” that begins with small whole numbers and expands rightward and leftward as well as interstitially to rational numbers (Siegler, 2016). An important facet of the integrated theory is that rational numbers are just as important as whole numbers within students’ numerical development.

**Major Findings**

The retrospective analysis surfaced three major themes from the teaching experiment: (1) understanding and use of partitioning, (2) understanding and use of relationships, and (3) building and expanding understanding of the number line. This section explores each theme’s intersection with the relevant literature.

**Discussion of Theme 1: Understanding and Use of Partitioning**

This study found that students’ understanding, and use of partitioning was a significant and critical means through which they engaged with and built concepts of fraction magnitude. The students’ work actively partitioning physical and visual models for themselves appeared foundational to their resulting understanding of the meaning of a fraction’s denominator. This critical value of early experiences partitioning further supports Post and Behr’s (1992) findings that opportunities to partition are a key component of students’ full understanding of fractions.

I entered into this study fully invested in providing students with opportunities to partition wholes for themselves. I was still unprepared for just how significant partitioning would become for the students, or how frequently and consistently the students would root their justifications in their understanding of the impact of partitioning. Many students are not given
opportunities to partition wholes for themselves. Many curriculum programs often offer pre-partitioned fraction strips and other models which remove the cognitive demand of students engaging in their own partitioning. This study provides further evidence, that as Post and Behr (1992), Siebert and Gaskin (2006), and Siegler et al. (2010) have called for, educators should offer opportunities for students to partition wholes for themselves.

Throughout the study, experiences partitioning even appeared to serve as a buffer from certain misconceptions. One common type of a whole number bias misconception—assuming that a larger denominator indicates a larger fraction, did not appear to be an issue for the students during this teaching experiment. Post et al. (1995) found misconceptions related to the size of fractions based on whole number interference to be common with students and to even endure after substantial instruction. DeWolf and Vosniadou (2015), Gomez and Dartnell (2011), Gonzalez-Forte et al. (2020), Obersteiner et al. (2017) and Rinne et al. (2017) have also identified gap thinking and reverse bias to be common misconceptions for students working with fraction magnitude. All three students in this study were identified by their classroom teacher as struggling with fractions and decimals, but none of the students seemed to fall into the common traps of whole number bias or gap thinking. It seems probable that the students’ work with physical partitioning contributed to their avoidance of these types of misconceptions and supported a firmer understanding of the meaning of the denominator.

Partitioning also surfaced as critical to supporting students’ reasoning as they grappled with density concepts. Physically, visually, and even mentally picturing partitions enabled the students to conceptualize fractions and decimals between other fractions, decimals, and whole numbers. Understanding the density of numbers has been consistently identified as challenging for students (McMullen et al., 2015; Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011;
Vamvakoussi & Vosniadou, 2010). Students in this study, however, appeared ready to confront ideas of density. Though the students were still developing in their ability to name specific values between given fractions and decimals, all three appeared confident that they could continuously partition the space between two fractions or decimals. I conjecture that the students’ work partitioning provided them with an essential entryway into density concepts. The connection between experiences physically partitioning and building density concepts did not appear in my review of the literature. Further study of how physical experiences partitioning tie into developing understanding of the density property may provide insight to supporting students in this critical yet challenging concept.

All three students appeared to overcome limitations of the whole number system to be able to perceive numbers between whole numbers and fractions. Their work imagining fractions “between fractions” seemed to demonstrate evidence of an expanding mental number line as called for within the integrated theory (Siegler, 2016). The students’ work with density is important as it is a key component of an expanding understanding of rational numbers. Remaining struggles in naming the number of decimals or fractions between two values may be understood via the conceptual change framework theory as the students worked to assimilate new information into their existing knowledge frameworks (Stafylidou & Vosniadou, 2004).

Discussion of Theme 2: Understanding and Use of Relationships

Use of key relationships emerged as a critical pillar of students’ developing reasoning about fraction and decimal magnitude in the teaching experiment. Relationships within fractions and between fractions and decimals were both challenging and important for the students’ developing understanding of magnitude. Relationships within fractions included the multiplicative relationship between the numerator and denominator. Relationships between
fractions and decimals included the relationship between tenths and hundredths, relationships of fractions and decimals back to benchmarks, and relationships between fractions and decimals as two different notation systems.

The multiplicative relationship of the numerator and denominator defines a fraction’s magnitude. Fuchs et al. (2017) and Neagoy (2017) identified multiplicative understanding of fractions and their magnitude as significant barriers to student understanding. The students in this study seemed to recognize the need to consider both the numerator and denominator as they grappled with questions of fraction size. This was an important first step; however, they did not seem to fully integrate the numerator and denominator relationship into a full conception of a fraction’s magnitude. Rather, the students seemed to rely on the distance of the fraction to either its whole or the benchmark of one-half in helping them assess size. Even though students attended to both the numerator and denominator, it is unclear if they truly understood that the relationship between the numerator and denominator determined one single number with a specific magnitude. Additional work supporting understanding the relationship between the numerator and denominator as representing a single number could support the students, not only in their magnitude work, but also within fraction arithmetic and other areas of mathematics (Siegler & Pyke, 2013).

Relating fractions and decimals to the benchmarks of one-half and one-whole emerged as a compelling strategy for the students. Thinking about the distance to one-half and one-whole seemed to give the students access to a magnitude picture they may not have had otherwise. The students’ work with benchmarks did not unfold in the way I had anticipated. Post et al. (1986) presented benchmarking as a transitive strategy where students use fraction relations to an external value (the benchmark fraction) to assist themselves. The students in this study seemed to
use a combination of transitive or benchmark reasoning and residual reasoning as they focused on the distance of fractions from the benchmark and used the distance to help them assess which fraction was greater.

Clarke and Roche (2009) found that benchmarking and residual reasoning tended to be used by students who displayed a stronger conceptual understanding of fractions. That the students in this study, who were identified as struggling the most in their class, gravitated to their version of residual thinking with benchmarks seems important. It is possible that instruction on benchmarks helped support these three students in developing their benchmarking and residual strategies and build their conceptual understanding of fraction magnitude.

The relationships between tenths, hundredths, and other powers of ten seemed to be a hurdle the students needed to overcome during the teaching experiment. The students’ gaps in understanding of the relationship between tenths and hundredths was most apparent when they were tasked with placing fractions and decimals on the number line. The students’ struggles here were not entirely surprising and fit with the findings of Baturo (1998) and Martinie and Bay-Williams (2003) on students’ lack of proficiency with tenths and hundredths on the number line. Students in the current study seemed to struggle more with placement of hundredths than tenths, also in keeping with Baturo’s (1998) findings. Further, the students seemed to struggle specifically with values less than one-tenth, similar to Martinie and Bay-Williams’s (2003) findings.

Connected to their work with tenths and hundredths, the students were challenged to make critical connections between fractions and decimals throughout the teaching experiment. The students entered the study supported by their previous classwork reading decimals using fraction language (e.g., 0.3 read out loud as 3-tenths). Their baseline assessment number lines,
however, revealed that they were not yet able to accurately grapple with the magnitude of both fractions and decimals on the same number line. By the end of the teaching experiment, they readily engaged with placement of both notations on one number line, showing improvement in the placement of both.

I conjectured that tasks which demanded the students to work with both decimals and fractions would be more challenging for the students and may reveal misconceptions. This conjecture was in line with Braithwaite et al. (2022) who found that tasks which stay fully within one notation may fail to “capture important aspects” of students’ rational number magnitude knowledge (p. 26). The students’ work on number line tasks throughout this teaching experiment seemed to lend credence to this conjecture—placing decimals and fractions on the same number line revealed fragilities for all three students. I hypothesize that working with fractions and decimals simultaneously supported the students in developing a fuller and richer understanding of their magnitudes. The students’ work connecting the magnitude of fractions and decimals fits with Schiller and Seigler’s (2023) recent calls for more intentional cross-notational work in order to develop this type of number sense as a new dimension of the integrated theory.

**Discussion of Theme 3: Building and Expanding Understanding of the Number Line**

The number line emerged as both a complex and important component of students’ development of fraction and decimal magnitude reasoning throughout this teaching experiment. A central tenet of the integrated theory requires the expansion of students’ mental number lines interstitially to embrace fractions and decimals (Siegler, 2016). At the beginning of this study, it seemed the students were uncertain of where to place fractions and decimals on their visual representations of their mental number lines—they were insecure in this domain of the integrated theory. Though the path was not smooth or easy, by the end of the teaching experiment, the
students appeared better able to include decimals and fractions in appropriate locations on their written number lines.

The transition to number lines from paper fraction strips was challenging for all three students. Concepts and magnitudes they seemed to understand when working with paper fraction strips did not seem to transfer to their written number lines. Challenges transferring understanding between paper fraction strips and the number line were especially apparent when working with tenths, hundredths, and other powers of ten. Even after using paper fraction strips to mark tenths on a number line and then finding a given fraction or decimal such as 3-tenths on their strips, the students still struggled to locate such values appropriately on their number lines. The students’ misconceptions guiding inaccurate placement of tenths and hundredths on the number line may be “synthetic models” revealing their attempts to “assimilate” new information to their existing mental number lines as explained by the conceptual change theory (Stafylidou & Vosniadou, 2004).

In the last third of the teaching experiment, a shift seemed to occur for the students. All three students appeared to have a stronger understanding of how and where to place both fractions and decimals on the number line. I hypothesize that the combination of experiences partitioning, relating fractions and decimals to benchmarks, and creating number lines many times eventually supported the students in a stronger understanding of the number line and the location of tenths and hundredths upon it. The students appeared able to adjust their initial theories to include new information and a more thorough framework of tenths and hundredths on the number line.

One reason the number line appeared to be challenging for the students was due to loss of the referent whole. At times, the students seemed to conceive of the entire number line,
regardless of its length, as their whole unit. This led to errors such as placing 4-tenths at the number 4 on a zero-to-ten number line. The students’ struggles with the referent whole on the number line in this study are in keeping with challenges many students face on the number line (Ni, 2001; Novillis-Larson, 1980; Wong & Evans, 2008). The students in this study likely would have benefited from more explicit attention to and discussion of the referent whole for both their number lines and their paper fraction strips.

Despite the challenges it brought, the number line appeared to be a valuable component of the teaching experiment. First, the number line appeared to reveal gaps and misconceptions in the students’ thinking that may otherwise have gone unnoticed such as challenges with the values of tenths and hundredths. This finding is in line with Pearn and Stephen’s (2004) and Martinie and Bay-William’s (2003) works demonstrating that the number line as a representation exposed gaps in students’ understanding. The number line also seemed to play a role supporting the students in this study in addressing misconceptions. The number line seemed to not only help expose misconceptions, but also to support the students’ developing fraction and decimal reasoning. These results are in alignment with Hamdan and Gunderson (2016) who found a number line intervention improved students’ sense of fraction magnitudes as assessed through multiple measures.

Students’ work placing decimals and fractions on the number line potentially helped them integrate their knowledge of both notations. The students’ initial struggles placing fractions and decimals accurately on the same number line fit with Schiller and Siegler’s (2023) recent findings studying cross-notational understanding. Schiller and Siegler found many students, regardless of grade level, lack an integrated understanding of different rational number notations. The three students’ growth in this study is a positive sign that they were moving towards
integration of their understanding of fraction and decimal notations and thus towards achievement of this dimension of the integrated theory.

Conclusions

Fractions and decimals are a widely acknowledged important, yet persistently, challenging subject for many students. I offer these conclusions as an ongoing piece of the puzzle as we work to understand and support students’ in their work with critical fraction and decimal magnitude concepts.

Question 1: How do fifth grade students reason about the magnitude of fractions and decimals?

Conclusion 1: The three students in this study drew upon different components of their understanding and different strategies with which they felt comfortable. They tried to use a combination of their understanding of partitioning, some attention to the multiplicative relationship between the numerator and denominator, and consideration of the distance of given fractions and decimals from benchmarks such as one-half and one-whole as they grappled with magnitude concepts.

The students’ understanding of the impact of partitioning, which led them to the conclusion that a larger denominator results in smaller pieces, seemed to serve as a foundation. The students drew upon this understanding again and again throughout the teaching experiment.

Distance reasoning also emerged as a preferred and foundational strategy for the three students. Once the students developed a framework using distance from benchmarks, they seemed to find power in it and would consistently examine and discuss the distance of fractions
under consideration to the benchmarks of one-half and one-whole. The students had to grapple with both the number of parts and the size of parts while employing distance reasoning.

The students were also pushed to attend to the relationship between the numerator and denominator through their use of distance reasoning. They had to consider the value of the numerator and denominator simultaneously, as well their relationship to each other, as they identified an equivalent benchmark and determined the distance of a given fraction from the equivalent benchmark. Distance reasoning appeared to be both supported by and in turn support simultaneous attention to the numerator and denominator.

**Conclusion 2:** The students were still grappling with which strategies to use and when to use them. Their still-developing flexibility seems indicative that additional work and time was needed to support them in a full, flexible, and deep understanding of the magnitude of fractions and decimals.

Partitioning and distance reasoning provided a conceptual foundation for the students as they solved problems related to fraction and decimal magnitude. However, within their use of these strategies, the students were not yet stable in their ability to select and apply the most efficient or effective strategy for a given task. The students were still in the process of generating and mastering a full conceptual framework for dealing with fractions and decimal magnitude.

**Question 2:** What are the shifts in mathematical thinking that occur with students’ evolving understanding as they progress towards generalization of fraction and decimal magnitude?

**Conclusion 1:** The students shifted from a more automatic reliance on procedurally based strategies to more broadly applying self-developed conceptual strategies. As the students’ thinking evolved over the course of the five weeks, their justifications became more grounded in
a conceptual foundation rather than automatic recall and application (or misapplication) of a memorized procedure. The students shifted towards drawing more readily upon the relationship of fractions to benchmark fractions, their understanding of the size of partitions, and the relationship between the numerator and denominator. As they made these shifts, the students also grew in their willingness to engage with more challenging, or “unfriendly” fractions and decimals and to share their reasoning.

**Conclusion 2:** Throughout the experiment, the students grew in their attention to the magnitude of the fractions and decimals under consideration. At the beginning of the teaching experiment, they engaged with the numbers more globally, tending to create equivalent values when asked about fractions. As we progressed, the students shifted to focusing more on the actual magnitude of the fraction—attending to how much smaller or larger it was than one-half, zero, or one-whole. This shift also extended to their growth on the number line where they were more able to relate fractions and decimals to each other and benchmarks and place them proportionally on the number line. The students’ closer attention to the relationship between the numerator and denominator and between fractions and decimals to other fractions and decimals supported important growth in their placement of fractions and decimals on the number line.

**Conclusion 3:** The three students’ attention to and ability to grapple with the relationships between tenths, hundredths, and other powers of ten shifted over the course of the teaching experiment. Their understanding of these relationships appeared unstable during the first nine sessions of the teaching experiment. A shift occurred during the last six sessions and the students were better able to manage the relationship between tenths and hundredths and appropriately place them on the number line. The students shifted from misconceptions and errors when dealing with numbers such as 0.09 or 2-hundredths and their relationship to one-
tenth and one-whole, to readily being able to discuss their magnitudes and place them appropriately on the number line.

**Conclusion 4:** Over the course of the teaching experiment, the students’ overall knowledge of fraction and decimal magnitude became more integrated. The students grew in accuracy when locating both fractions and decimals on the number line, comfort using fraction language to express decimals, and adaptability working with both notations interchangeably.

**Conclusion 5:** At the conclusion of the teaching experiment, the students were still growing in several areas. I had hoped to see more flexibility in strategy use develop, particularly as the students added more strategies to their toolboxes. The students did shift towards using benchmarks which was one of my goals. However, they would still become stuck in distance reasoning at times, perseverating on it as opposed to using it flexibly when the situation called for it.

**Question 3: What are the characteristics of instructional experiences that lead to shifts in students’ mathematical understanding of fraction and decimal magnitude?**

**Conclusion 1:** Physical and visual models and exploring the connections between them emerged as key to the students’ developing understanding of fraction and decimal magnitude. In particular, physically partitioning fraction strips was pivotal to students’ growing conceptual understanding. A critical feature of our partitioning work was providing unmarked paper strips and number lines which the students had to partition for themselves. The students latched onto the understandings they developed through determining how to create appropriately sized
partitions and the appearance of those resulting partitions. This physical experience gave them a foundation to return to again and again as they grappled with unfriendly fractions.

**Conclusion 2:** The students were particularly supported by our work relating fractions and decimals to benchmark fractions such as one-half and one-whole. The students’ development and use of distance reasoning seemed to be directly related to their benchmark work. Asking the students to not only determine if fractions or decimals were greater than or less than one-half and one-whole, but also the size of the fractions in relation to those benchmarks helped nudge them to attend more closely to the magnitude of the fractions and decimals under consideration.

**Conclusion 3:** The number line emerged as a complicated and crucial component of our work in the teaching experiment. I initially viewed the number line as solely supporting students’ reasoning. Over the course of the teaching experiment, I realized the number line was significantly more challenging for this group of students than I anticipated. The challenge provided a complication to our work, but also an opportunity for me to uncover fragilities in the students’ thinking I may have otherwise missed. Pushing the students to place *both* decimals and fractions on the same number line was also critical for helping them integrate their developing understanding of both of these notations and their magnitude, as well as expanding and deepening their mental number line.

**Conclusion 4:** Lastly, number choice mattered for tasks. Certain numbers such as 0.09 illuminated fragilities in the students’ thinking as well as helped push them to think more complexly about the size of the number. Unfriendly fractions pushed the students to grapple with how to apply strategies and concepts in novel situations. When students had to work with
numbers such as 17-twelfths and 11-thirteenths, they had to dig deeper into the meaning of the numerator and denominator and their relationship to each other.

**Limitations**

This study has some limitations. The study sample size was small. Different students, a larger sample, or even students experiencing different instruction in their regular classroom might produce different results. Implementation of the teaching and learning sessions via Zoom likely had a significant impact on the students’ experiences and thus their learning. Technology difficulties often delayed or even prevented students from sharing their thinking or hearing the thinking of the other two students. Engaging in the lessons while on headphones in a larger classroom seemed to impact the students’ opportunities to engage in discourse with each other. The students may have also been influenced by their relationship to me as a person perceived to have power. They may have tried to craft answers or follow lines of reasoning they might not have otherwise to please me.

The goal of any teaching experiment is not generalization, but rather to provide a window into a particular set of teaching and learning experiences (Steffe & Thompson, 2000). The presentation of the students’ thinking within a teaching experiment is conducted through the lens of the teacher-researcher. The teacher-researcher’s ways of thinking about the students’ mathematics may be helpful to others in interpreting the mathematics of students but should not be assumed to transfer to all similar students. The goal of this teaching experiment was to provide more depth and nuance to how these three fifth-grade students were reasoning about
fraction and decimal magnitude. Results should not be viewed as an explanatory model for all students struggling with fractions and decimals.

It is important to note that I selected the three participating students because they were identified as struggling the most with fractions and decimals in their class. Their fraction and decimal foundation and the ways they approached and worked with new strategies may vary from other students that enter fifth grade with a stronger foundation in fractions and decimals. I believe their engagement in the teaching experiment provides insight into potential reasoning of students struggling with these concepts but am cautious of overgeneralization to all fifth-grade students.

Since I designed and implemented the instruction and collected and analyzed the data, my own perspective and prior experiences influence the results. I disclosed my own background and theoretical perspective in Chapter 3 to convey to the reader my potential biases and assumptions. In an attempt to mitigate my own biases, I engaged in persistent observation and peer debriefing (via a witness researcher). I also attempted to provide an audit trail through thick description within the conceptual analysis provided in Chapter 4.

**Recommendations**

The three students who participated in this study appeared to grow in their understanding of fraction and decimal magnitude. Their growth, however, was not always linear or firm. Reasoning about fraction and decimal magnitude is challenging. Building and expanding a mental number line is challenging.

Experiences physically partitioning wholes appeared critical to laying a conceptual foundation for students’ work developing their understanding of decimal and fraction magnitude.
Future students may also benefit from multiple opportunities to engage in their own partitioning work early on and throughout fraction and decimal instruction.

Consideration of fractions and decimals in relation to benchmarks such as one-half and one-whole seemed to support students’ developing a conceptual understanding of their magnitude. Multiple opportunities to sort fractions and decimals as greater and less than one-half or one-whole, as well as opportunities which press students to determine which benchmark is closest, may help students attend to these important relationships and use them to develop a richer sense of fraction and decimal magnitudes.

Curriculum developers and instructors should also be mindful of their number selection for tasks. Limiting students to “friendly” denominators may conceal hidden gaps in understanding. Employing the use of unfriendly fractions and decimals may help reveal fragilities in understanding as well as deepen and extend the students’ sense of magnitude for fractions and decimals.

The transition between fraction strips and the written number line was not easy or smooth for the students in this study. These students, and likely many others, would benefit from additional time, work, and discussion supporting this critical transition. It is also important to use number lines that include multiple wholes (e.g., one-to-two) that help push students to identify and attend to the referent whole.

Overall, the three students in this study seemed to benefit from connecting decimals and fractions. These two topics are often taught separately in schools. Instructors may find value in tying these two topics together more closely and regularly.
Future Research

In this study, I explored the development of students’ reasoning about fraction and decimal magnitudes and the learning experiences that supported them. Over the course of the study, several areas emerged that I believe may present an opportunity for further research to benefit the teaching and learning of fractions and decimals.

The Common Core State Standards have called for a greater use of number lines for fraction instruction. A shift towards greater use of number lines has the potential to benefit many students. Experiences in this study indicate further research in supporting students’ work on the number line may be beneficial for teaching and learning. The three students in this study did not make a seamless transition from physical fraction strips to the number line. Additional exploration of how to support the transition from physical models to the number line may provide insight and utility to teachers of fractions and decimals.

Student understanding of the relationships between tenths, hundredths, and other powers of ten is another topic which may benefit from additional exploration. Baturo (1998) and Martinie and Bay-Williams (2003) have contributed important work on the relationship between tenths and hundredths, but additional investigation of how students build their understanding of these critical relationships and how to support their developing understanding is needed.

Distance reasoning emerged as a critical and favorite reasoning strategy for all three students in the present study. My search of the literature was not exhaustive and other researchers may employ different terminology, but I was unable to find distance reasoning as used by the students in this study in the current body of literature. Distance reasoning is closely tied with benchmark and residual reasoning but appears to extend beyond placement in either category. The development and heavy use of distance reasoning may be particular to this small
group of students, or it may be a transitional step in developing conceptual understanding for more students than we realize. Additional research into students’ use of benchmarks when working with fraction and decimal magnitude questions may also provide important information related to or beyond distance reasoning.

Overall, much of the literature on fraction magnitude is quantitative in nature—focused more on correct answers in comparison problems and response times. Though such studies provide important information, they do not give us full insight into how students are reasoning. Thus, we miss a piece of the puzzle as we consider how to best support student understanding of fraction and decimal magnitude. More research is needed addressing this critical gap.

Lastly, recent work by Schiller and Seigler (2023) has started to call attention to the value of instruction that addresses fractions and decimals together. Research into the connected teaching of these two concepts, however, is still limited. Additional studies investigating units of study which combine and separate these two topics and their differences may help provide important guidance to the teaching of fractions and decimals.

**Concluding Remarks**

Fractions and decimals are challenging for students, but also pivotal to a full and rich understanding of the magnitude of numbers. I entered into this study with certain hypotheses about what would best support students’ development of magnitude concepts for fractions and decimals. My experiences in the teaching experiment supported some of these hypotheses but also led me to new questions and conjecture as I delved deeply into the understanding and reasoning of the students.

Three major themes related to the students’ reasoning emerged from the five-week teaching experiment. Experiences physically partitioning emerged as even more central to
students’ developing reasoning than I initially anticipated. Relationships between the numerator and denominator and relationships between fractions to other fractions and decimals were critical, but also highly challenging for the students. The number line was more challenging and less supportive than I initially anticipated but vital in revealing misconceptions. All three themes present direction for further study to learn more about how students grapple with the magnitude of fractions and decimals. By better understanding how students are thinking about fractions and decimals, we will be able to support their learning more effectively.
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1. Shade $\frac{1}{3}$ of the rectangle below.

2. Draw a picture to that shows the fraction $\frac{1}{4}$.

   Draw a picture to that shows the fraction $\frac{6}{4}$.

3. Name a fraction that is larger than $\frac{6}{8}$ but smaller than 1:

   Name a fraction that is smaller than $\frac{4}{6}$.

4. Describe two ways to compare the fractions below. Circle the larger fraction.

   $\frac{2}{3}$ $\frac{1}{4}$
5. Order the fractions from smallest to largest: \( \frac{6}{8} \quad \frac{2}{2} \quad \frac{1}{3} \)

6. Which fraction below has a value closest to \( \frac{1}{2} \)?

A. \( \frac{5}{8} \)  B. \( \frac{1}{6} \)  C. \( \frac{2}{2} \)  D. \( \frac{2}{7} \)

Explain your thinking.

7. Place these fractions and decimals on the number line: \( \frac{5}{8} \quad \frac{2}{3} \quad \frac{5}{4} \quad 1.25 \quad 0.09 \quad 0.4 \)

8. Put the numbers in order from smallest to largest: 6.79  6.4  6.786
9. Mark says $\frac{1}{4}$ of his candy bar is smaller than $\frac{1}{5}$ of the same candy bar. Is Mark right? 
Draw a picture and/or use words to explain why you think Mark is right or wrong.

10. Of the following, which is closest in value to 0.52?

- A. $\frac{1}{50}$
- B. $\frac{1}{5}$
- C. $\frac{1}{4}$
- D. $\frac{1}{3}$
- E. $\frac{1}{2}$

How do you know?

11. Which is larger 0.4 or $\frac{6}{100}$?

Explain how you figured out your answer.
Appendix B

OGAP Fraction Progression Framework

Fraction Progression

Note: The examples provided do NOT represent the full set of possible solutions that represent each level.

Middle school topics and concepts in which rational number understanding and procedures are applied:

- Percent
- Proportions
- Ratios
- Similarity
- Transformations
- Probability
- Functions
- Expressions and equations
- Scaling
- Measures of Central Tendency
- Others

OGAP Fraction Progression Framework

Accurately locates fractions on a number line of any length, compares and orders fractions using a range of strategies, finds equivalent fractions, and operates efficiently when solving mathematical and contextual problems.

- Uses reasoning about relative magnitude
- Uses benchmark reasoning
- Uses and fraction reasoning
- Uses equivalence
- Uses efficient algorithms
- Uses properties of operations to demonstrate understanding of concept
- Equations given visual model

Magnitude Reasoning

- A cup is a unit of measure that holds one cup of liquid. The school cafeteria recipe asks for 1/4 cup of sugar, and 1/2 cup of jello. Is the jello large enough to fill the cup?

- Fractional Reasoning

- Evidence of fractional thinking, but reasoning or strategy is not efficient

- Uses Unit Fraction Strategy

- A fruit punch recipe calls for 2 2/3 cups of limeade. How many times would Jim need to fill a 1/3 cup to measure the correct amount of limeade to put in the fruit punch?

- Uses a fractional or transitional strategy (like partitioning visual models) or an operation appropriate for the situation, but the solution includes an error (e.g., partitioning, size of whole, concept error in part of problem)

- Whole number reasoning, not fractional reasoning

- Incorrect operation given context

- Jim is making decorations for a school dance. He has 4 1/4 yards of wire. Each decoration needs 3/4 of a yard of wire. How many full decorations can Jim make?

- Applies rules without evidence of understanding, inappropriate whole number reasoning, or uses an incorrect operation given the problem context

- Non-Fractional

- Underlying Issues/Errors

- Skill, without understanding
- Inappropriate whole number reasoning
- Inappropriate number sense
- Errors in concept
- Errors in procedure
- Errors in calculation
- Errors in fraction

This is a derivative product of the Van de Walle Mathematics Partnership Oregon Assessment Project (OGAP) which was funded by NSF (DUE 0121767) and the US DOE (G060A00002). © 2012 Marje Pett Consulting, CPC, L. Huibert, R. Land, Weile in April 2014.
Appendix C

Concluding Assessment

1. Sort the numbers below into the best location in the chart:

| 0.09  | \( \frac{2}{5} \) | \( \frac{13}{11} \) | 0.489 | \( \frac{3}{7} \) | 0.74 | \( \frac{20}{100} \) |

| Closest to Zero | Closest to \( \frac{1}{2} \) | Closest to 1 |

2. Which fraction below has a value closest to 1?

A. \( \frac{5}{8} \)  B. \( \frac{4}{6} \)  C. \( \frac{11}{12} \)  D. \( \frac{6}{5} \)

Explain your thinking.

3. Place these fractions and decimals on the number line: \( \frac{5}{8} \)  \( \frac{2}{3} \)  \( \frac{5}{4} \)  1.25  0.09  0.4
4. Put the numbers in order from smallest to largest: 0.09 0.4 0.39

5. Of the following, which is closest in value to 0.48?

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
<th>How do you know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>$\frac{1}{48}$</td>
<td></td>
</tr>
<tr>
<td>B.</td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>C.</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>D.</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
<tr>
<td>E.</td>
<td>$\frac{7}{8}$</td>
<td></td>
</tr>
</tbody>
</table>

6. Pretend the picture below is a fraction strip. Shade $\frac{5}{100}$ on the fraction strip as best you can. Explain how you decided to shade below.
7. How many hundredths are in 0.7? How do you know?

8. Place these fractions and decimals on the number line (notice the number line is from 0 to 100): 52, 1, 89, 0.09, 24

9. Name a fraction that is larger than 4/6 but smaller than 1.
Appendix D

Post-teaching episode reflection protocol

1. What were my initial overall impressions from today’s teaching episode?

2. What surprised me in students’ work or conversations?

3. What did students struggle with today? [Misconceptions? Significant mistakes?]

4. Where did I see students growing today? [What shifts in understanding I can identify?]

5. What tasks, experiences, tools, or conversations seemed particularly supportive (or not supportive) for students’ learning today?
   a. Within tasks, were any specific numbers or questions particularly effective?

6. What next steps make sense for the group? For individual students?
Appendix E

Witness-Researcher Meeting Protocol

1. What were your initial impressions of today’s teaching episode?
2. What surprised you today?
3. What seemed like it might be a current struggle for students?
4. How do you see students’ thinking evolving from the last episode you witnessed?
5. What evidence of student understanding or learning did you see today?
6. What tasks, experiences, tools, or conversations seemed particularly supportive (or not supportive) for students’ learning today?
   a. Within tasks, were any specific numbers or questions particularly effective?
7. What next steps do you see as potentially beneficial for the students right now?
# Appendix F

## Code Book

### Theme 1: Understanding and Use of Partitioning

<table>
<thead>
<tr>
<th>Subtheme</th>
<th>Code</th>
<th>Descriptor</th>
<th>Examples</th>
</tr>
</thead>
</table>
| Impact of Partitioning    | General: Understanding of impact of partitioning | Understanding of the impact of partitioning a whole or piece (can demonstrate understanding or gap in understanding) | Session 2: Student Z: Like four if you just fold it one more time you get 8ths
Session 12: Student E: I put mine like first I put mine close to one to one but then I noticed that fourths are really big and I only needed one of them, so then I put it closer to the middle in between 1/2 and 1 |
|                           | Over-generalization of partitioning | Over-focus on impact of partitioning even when not relevant to the context or explanation | Session 9: Student E: because ninths are bigger than well 11/13 would be really close to 7/9 and 7/9 is bigger than 11/13 almost so then 7/9 would be closer to one because The nine the partitions would be bigger than the 13 and the 7 would be closer to 9.
Session 13: Student Z: And I drew one as tenths and then I shaded in 3 and I noticed that the pieces were bigger in 3/10 so I think that 3/10 because the pieces are bigger |
|                           | Equivalency through partitioning | Understanding/exploring/using equivalency between two or more values       | Session 4 (in regards to 40/100) Student J: Because if you were to add one more tenth, or 10 more than you'd get 50 hundredths, which would be half of it.
Session 11: Student E: Where did I put it? Oh I put 2/2 at one whole because halves is 2 twoths which is equal to um one whole |
|                           | Density of Numbers            | More partitions or splits are possible, more numbers exist between two numbers - can be evidence of where it is happening, but also where it is not | Session 2: Student Z: because it's basically so if you folded the 4 again, it would just double the amount. So that would be 8 and then if you fold it eighth again then it would be 16ths.
Session 14: Student E: I think it's infinity again because like last time we can um split like we did a number line from zero to one and then we labeled like the 3 what the two numbers were and then we could separate into as many as we want. |
### Theme 2: Understanding and Use of Relationships

<table>
<thead>
<tr>
<th>Subtheme</th>
<th>Code</th>
<th>Descriptor</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Multiplicative Reasoning</strong></td>
<td>Relationship between the numerator and Denominator</td>
<td>Attending to relationship between numerator and Denom - can be evidence of where it is happening, but also where it is not</td>
<td>Session 13: Student Z: And I drew one as tenths and then I shaded in 3 and I noticed that the pieces were bigger in 3/10 so I think that 3/10 because the pieces are bigger (not attending to relationship between) Session 14: Student Z: So I think at first, I said they were equal because they're both one um away and I realized that 9 is bigger than 12 wait... no no 10 is bigger than 12 by like two and it's partitioned a little bit bigger than 11/12 so that ones bigger. 9/10 is bigger (not attending to relationship between)</td>
</tr>
<tr>
<td><strong>Equivalency between fractions</strong></td>
<td>Understanding or use of multiplicative reasoning when generating, identifying, or justifying equivalencies between fractions (can be evidence of where it is happening but also where it is not)</td>
<td>Session 10: Student Z: And there really is no like proper benchmark in between, for like one-half of nine. Teacher: Yeah, how should we deal with the one-half here? How many ninths would be equivalent to one-half? This is a tricky one. Student Z: I feel like it would be about five. Teacher: About five... Student Z: Or like four and a half... Four and a half? Teacher: Four and a half. Why are you thinking four and a half? Student Z: Because a half plus a half equals one-whole and four plus four equals eight plus one whole because the two halves equals nine.</td>
<td>Session 11: Student E (on 0.25 being placed with 4/10 on the number line) “I put it under 4-tenths because um I put it close to 4-tenths because um, we only need 25 to get to, we only need 4 twenty-fifths to get to 100.”</td>
</tr>
<tr>
<td><strong>Relationships between numbers</strong></td>
<td>Benchmarks: Effectively using distance from a benchmark (typically one whole) to assess or compare the size of a fraction (or decimal)</td>
<td>Using distance from a benchmark</td>
<td>Session 6: Student E: I put it right by, kind of by 4 tenths because it's like one tenth away from three tenths and it would be like thirty, thirty nine tenths so then multiplied that by ten and got 39/100 Session 10: Student Z I mean I chose I chose closest to one because 76 is only 24 away from one whole and About like (goes on to self-correct for distance from 1/2)</td>
</tr>
<tr>
<td></td>
<td>Benchmarks: Over-</td>
<td>Relying on distance</td>
<td>Session 12:</td>
</tr>
<tr>
<td>Subtheme</td>
<td>Code</td>
<td>Descriptor</td>
<td>Examples:</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
<td>------------</td>
<td>-----------</td>
</tr>
<tr>
<td>Transitions between Physical Models and the Number Line</td>
<td>Transitions between paper strips and a written number line</td>
<td>Able to work on physical model (fraction strips) falls apart when transition to a drawn number line</td>
<td>Session 7: Teacher: So how many of how many splits would we do in one of our one 10th partitions? Student Z: 10 Teacher: 10, Okay So if I'm thinking about my fractions strip and I'm going back to my number line that I created how many 100ths are there from zero to 1/10? Student Z: There are 99? over 100</td>
</tr>
<tr>
<td>Representing Powers of Ten on the Number Line</td>
<td>Successes and challenges of representing powers of ten on the number line</td>
<td>Evidence of number line emerging as own challenge/misconception creator</td>
<td>Session 6, Student J</td>
</tr>
<tr>
<td>Losing the Referent Whole on the Number Line</td>
<td>Losing the Referent Whole on the Number Line</td>
<td>Ambiguity emerges in relation to where the referent whole is</td>
<td>Session 7: (placing 1/100 on the 13/10 number line) why are you thinking right next to the zero? Student Z: Because 99/100 would be closer to one 10th and that's... 99/100 [pause] um it's 99/100 away from one tenth Teacher: It's 99/100 away from 1/10 or 1 Student Z: One tenth</td>
</tr>
<tr>
<td>Relationship between fractions and decimals</td>
<td>Seeing, using, or confusion on the relationship between equivalent fraction and decimal values</td>
<td></td>
<td>Session 7: Student Z: It's 9 hundredths zero and 9/100, it's basically the same thing Session 14: Student E: What I was saying was, I knew that if you made zero and 83 hundredths a fraction it would be 83/100 and it's really close to one whole which is 100/100</td>
</tr>
<tr>
<td>Generalization of distance reasoning</td>
<td></td>
<td></td>
<td>Student J: I thought about it similarly because three 8ths is only one 8th away from being one half. Student Z: And 3/10 would be 2/10 from being 1/2</td>
</tr>
<tr>
<td>Student E: I think it's 10/12 Because You only need two more 12ths to get to one whole and then for 3/10, you have to get seven tenths to get to one whole so then 10, 10/12 is closer to one whole than three tenths.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Baseline and Concluding Assessment Student Result Tables

### Baseline Assessment Results

<table>
<thead>
<tr>
<th>Task</th>
<th>Student Z</th>
<th>Student J</th>
<th>Student E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Shade 1/3 of the rectangle</td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
</tr>
<tr>
<td>2. Draw picture to show ¼, picture to show 6/4</td>
<td><img src="image4" alt="Image" /></td>
<td><img src="image5" alt="Image" /></td>
<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
<td>3. Describe two ways to compare 2/3 and 1/4</td>
<td>Circled 2/3 “because 2/3 is 1 away from 1 hole”</td>
<td><img src="image7" alt="Image" /></td>
<td>Circled 2/3 but no explanation</td>
</tr>
<tr>
<td>4. Order the fractions from smallest to largest</td>
<td>1/3 6/8 2/2</td>
<td>1/3 6/8 2/2</td>
<td>6/8, 1/3, 2/2</td>
</tr>
<tr>
<td>5. Which fraction has a value closest to ½?</td>
<td>“1/6 because it’s the farthest away”</td>
<td><img src="image8" alt="Image" /></td>
<td>“5/8. 4/8 is close to 5/8 and 4/8 is ½”</td>
</tr>
<tr>
<td>6. Place the fractions and decimals</td>
<td><img src="image9" alt="Image" /></td>
<td><img src="image10" alt="Image" /></td>
<td><img src="image11" alt="Image" /></td>
</tr>
<tr>
<td>Question</td>
<td>Answer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------------------------------------</td>
<td>------------------------------------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Put the numbers in order from smallest to largest</td>
<td>![Number Line Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Of the following which is closest in value to 0.52</td>
<td>1/50 “because it is only 2 off”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/2 (scratched out “0.5 = 1/2”)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“0.52 is small and so is 1/5 so is closest”</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Which is larger 0.4 or 6/100</td>
<td>Name a fraction that is larger than 3/4 but smaller than 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“Note: I am not sure on this one”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“4/4 because it is ¼ higher than ¾”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“But smaller than 1 cuz missing 1/3”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6/100 is too small</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Name a fraction that is smaller than 4/6</td>
<td>“3/6 bc it is one away from one and its one smaller than 4/6”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“3/6 is 1/6 smaller than 4/6”</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Are there decimals between 0.3 and 0.4? How many do you think?</td>
<td>“Yes there’s up to 1,000 because it goes 0.31 0.32 0.34 0.35 0.36 and so on all the way 1,000”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“0.31 because there is one hundredth add to 0.3”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>“yes: 0.1 0.3 + 0.1 = 0.4”</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Which number is closest to 1? How do you know?</td>
<td>“1.2” “Because its only 2 tenths away from one and the rest are either way over or way under”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7/6 “because it is 1/6 over 7/6”</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task</td>
<td>Student Z</td>
<td>Student J</td>
<td>Student E</td>
</tr>
<tr>
<td>----------------------</td>
<td>------------------------------------------------</td>
<td>------------------------------------------------</td>
<td>------------------------------------------------</td>
</tr>
<tr>
<td>1. Sort in table</td>
<td><img src="281x175" alt="Table" /></td>
<td><img src="404x175" alt="Table" /></td>
<td><img src="555x175" alt="Table" /></td>
</tr>
<tr>
<td>0.09</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2/5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13/11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.74</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20/100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Goes with #5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Place on Number Line</td>
<td><img src="141x248" alt="Number Line" /></td>
<td><img src="404x209" alt="Number Line" /></td>
<td><img src="525x209" alt="Number Line" /></td>
</tr>
<tr>
<td>5/8</td>
<td>2/3</td>
<td>5/4</td>
<td></td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td></td>
</tr>
<tr>
<td>0.09</td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Goes with #6

<table>
<thead>
<tr>
<th>4. Order smallest to largest</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
</tr>
</tbody>
</table>

Goes with #7 on pre-but before all were with whole numbers, might have impacted

<table>
<thead>
<tr>
<th>0.09, 0.39, 0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>9/100, 39/100, 0.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5. Of the following, which is closest in value to 0.48?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
</tr>
<tr>
<td>48/100 would need 2/00 to be a half</td>
</tr>
<tr>
<td>48/100 is close to 1/2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>¼, Because ¼ = 0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>½. 48/100</td>
</tr>
<tr>
<td>½. 48/100 is close to ½</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6. Pretend the picture below is a fraction strip. Shade 5/100 on the fraction strip as best you can. Explain how you decided to shade below.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wrote 10/100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7. How many hundredths are in 0.7? How do you know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wrote 10/100</td>
</tr>
</tbody>
</table>
8. Place these fractions and decimals on the number line (notice the number line is from 0 to 100): 52, 1, 89, 0.09, 24

9. Name a fraction that is larger than $\frac{4}{6}$ but smaller than 1. $\frac{5}{6}$