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The Fattened Davis Complex and the Weighted L^2-(Co)Homology of Coxeter Groups

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The Fattened Davis Complex and the Weighted $L^2$–(co)homology of Coxeter Groups

by

Wiktor J. Mogilski

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

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ABSTRACT

THE FATTENED DAVIS COMPLEX AND THE
WEIGHTED $L^2$–(CO)HOMOLOGY OF COXETER GROUPS

by

Wiktor J. Mogilski

The University of Wisconsin–Milwaukee, 2015
Under the Supervision of Boris L. Okun

Associated to a Coxeter system $(W, S)$ there is a contractible simplicial complex $\Sigma$ called the Davis complex on which $W$ acts properly and cocompactly by reflections. Given a positive real multiparameter $q$ indexed by $S$, one can define the weighted $L^2$–(co)homology groups of $\Sigma$ and associate to them a nonnegative real number called the weighted $L^2$–Betti number. Unfortunately, not much is known about the behavior of these groups when $q$ lies outside a certain restricted range, and weighted $L^2$–Betti numbers have proven difficult to compute. We propose a program to compute the weighted $L^2$–(co)homology of $\Sigma$ by introducing a thickened version of this complex which we call the fattened Davis complex. A salient feature of this complex is that our construction produces a homology manifold with boundary possessing $\Sigma$ as a $W$–equivariant retract. This allows us to use many standard algebraic topology tools such as Poincaré duality for computing the $L^2$–(co)homology of $\Sigma$, and we successfully perform computations for many examples of Coxeter groups.

Within the spectrum of weighted $L^2$–(co)homology there is a conjecture of interest called the Weighted Singer Conjecture. The conjecture claims that if $\Sigma$ is an $n$–manifold (equivalently, the nerve of the corresponding Coxeter group is an $(n–1)$–sphere), then the weighted $L^2$–(co)homology groups of $\Sigma$ vanish above dimension $\frac{n}{2}$.
whenever $q \leq 1$. We present a proof of the conjecture in dimension three that encompasses all but nine Coxeter groups. Then, under some restrictions on the nerve of the Coxeter group, we obtain partial results whenever $n = 4$ (in particular, the conjecture holds for $n = 4$ if the nerve of the corresponding Coxeter group is a flag complex). We also prove a version of this conjecture in dimensions three and four whenever $\Sigma$ is a manifold with (nonempty) boundary, and then extend our results in dimension four to prove a general version of the conjecture for the case where the nerve of the Coxeter group is assumed to be a flag triangulation of a 3–manifold.
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Chapter 1

Introduction

A recurring theme in geometric group theory is the investigation of properties of a group by studying a space on which the group acts (or vice-versa). This space is usually reasonably nice (for example, a metric space), hence geometric group theory closely interacts with algebraic topology, discrete geometry, and geometric topology. The groups that this thesis focuses on are Coxeter groups, and a construction by Davis equips us with a natural simplicial complex called the Davis complex (denoted by $\Sigma$) on which the Coxeter group acts on.

Within the spectrum of Coxeter groups is the theory of weighted $L^2$–(co)homology of Coxeter groups, which is the central subject of this thesis. The main idea is to use the Davis complex to attach to the Coxeter group an equivariant cohomology theory where the objects are Hilbert spaces. Let $S$ denote the generating set of the Coxeter group $W$ and begin with an $S$–tuple $q = (q_s)_{s \in S}$ of positive real numbers, where $q_s = q_t$ if $s$ and $t$ are conjugate in $W$. One uses the $S$–tuple $q$ to assign weights (real numbers) to each of the cells of $\Sigma$ in a way that is compatible with the $W$–action. Elements of $L^2_q C_k(\Sigma)$ are then infinite $W$–equivariant real valued $k$–chains that are square-summable with respect to the weights. In other words, they are real valued functions on the $k$–cells that are square-summable when taking into account the weights on the $k$–cells. These spaces are Hilbert spaces, and there is a weighted boundary operator which is adjoint to the ordinary coboundary operator with respect to the inner product (these operators are bounded). One then defines the
reduced $L^2$–(co)homology spaces $L^2_qH_*(\Sigma)$ (the homology and cohomology spaces are isomorphic as Hilbert spaces).

These cohomology groups are generally infinite dimensional when nonzero, but a striking feature of this cohomology theory is that one can assign to these groups a nonnegative real number called the weighted $L^2$–Betti number (hence they are distinguishable). The predominant goal of this subject is to completely determine the weighted $L^2$–Betti numbers $L^2_qb_*(\Sigma)$ for any Coxeter group. Weighted $L^2$–Betti numbers not only tie the theory of weighted $L^2$–(co)homology to algebraic properties of the Coxeter group, but they also intertwine it with many other topics such as Hecke algebras, growth series, the Euler characteristic conjecture, and operator theory. Outside of the ties to these topics, one of the most important applications of weighted $L^2$–Betti numbers is that they can be used to compute the ordinary $L^2$–Betti numbers of buildings of finite thickness.

Weighted $L^2$–Betti numbers are also notoriously difficult to compute, very little being known when $q \notin \mathcal{R} \cup \mathcal{R}^{-1}$, where $\mathcal{R}$ denotes the region of convergence of the growth series of the Coxeter group. In fact, just computing the ordinary $L^2$–Betti numbers of Coxeter groups still proves troublesome to this day. To illustrate the difficult nature of these invariants, a formula of Atiyah shows that ordinary $L^2$–Betti numbers can be used to compute the orbihedral Euler characteristic. So, if one considers the fundamental group of a closed aspherical $n$–manifold, with $n$ even, then knowing the vanishing of the $L^2$–(co)homology groups of the universal cover outside of the middle dimension implies the Hopf conjecture on the sign of the Euler characteristic of that manifold. Thus $L^2$–(co)homology theory can be thought of as a formidable attack on the Euler characteristic conjecture and has proven to be successful in many situations (for example, in the case of locally symmetric manifolds).

This thesis is structured as follows. In Chapter 2, we introduce some preliminaries and definitions needed for the content of the thesis. For example, we discuss Coxeter groups, growth series, and explain how to construct the Davis complex. To construct the Davis complex, one needs a notion due to Davis called the basic con-
struction. The idea is to start with a certain type of space $X$ called a mirrored space and use a Coxeter group $W$ to build a new space on which the Coxeter group acts on. An important result of this chapter is that we give new conditions on $X$ so that the basic construction produces a homology manifold with boundary with a proper and cocompact Coxeter group action. This result will be especially important in Chapter 6.

Chapter 3 is dedicated to weighted $L^2$–(co)homology theory. We first introduce the weighted $L^2$–(co)homology groups and define weighted $L^2$–Betti numbers. We then introduce an alternate definition of weighted $L^2$–Betti numbers and discuss some previous results for the weighted $L^2$–(co)homology theory of the Davis complex. The new results of this chapter are as follows. We first observe that it is possible to use any acyclic complex on which the Coxeter group acts properly and cocompactly by reflections to compute the weighted $L^2$–Betti numbers of Coxeter groups. In particular, we can use a vcd$W$–dimensional complex of Bestvina to compute weighted $L^2$–Betti numbers. An immediate consequence of this is that $L^2_q b_k(\Sigma) = 0$ for $k > \text{vcd} W$, and in many cases this allows us to obtain vanishing of high-dimensional $L^2_q$–Betti numbers, as Bestvina’s complex is usually of much lower dimension than the Davis complex. We then show that top-dimensional $L^2_q$–Betti numbers behave monotonically in $q$. More precisely, we show that if the top-dimensional $L^2_q$–Betti vanishes for $q = 1$, then it must have been zero for all $q \leq 1$. This later allows us to “push” many of our computations (as well as previously known computations) from $q = 1$ to $q \leq 1$.

Chapter 4 focuses on specific $W$–stable subcomplexes of the Davis complex called ruins, which were used in proofs by Davis, Dymara, Januszkiewicz, and Okun in [8]. By considering a particular exact sequence for the $L^2_q$–(co)homology involving these complexes (also used in [8]), we are able to show new concentration theorems for the $L^2_q$–(co)homology of ruins. Using a spectral sequence appearing in [10], we are then able to show that for a certain range of $q$, $L^2_q H_*(\Sigma, \Sigma^{(k)})$ is concentrated in dimension $k + 1$ (here $\Sigma^{(k)}$ denotes the $k$–skeleton of $\Sigma$). We then proceed derive some consequences, one being that we are able to generalize a theorem of Dymara...
[12, Theorem 10.3] which states that if \( q \in \mathcal{R} \), then \( L^2_q H_*(\Sigma) \) is concentrated in dimension zero.

In Chapter 5 we consider the Weighted Singer Conjecture, which was formulated in [8]. It states that if \( \Sigma \) is an \( n \)-manifold and \( q \leq 1 \), then \( L^2_q H_k(\Sigma) \) vanishes for \( k > \frac{n}{2} \). We first discuss progress on this conjecture, and then we use the results of Chapter 3 to prove that \( L^2_q H_k(\Sigma) = 0 \) whenever \( k \geq n - 1 \) and \( \Sigma \) is an \( n \)-manifold with (nonempty) boundary. For the case when \( n = 3, 4 \), this proves a version of the Weighted Singer Conjecture whenever \( \Sigma \) is an \( n \)-manifold with (nonempty) boundary. We then adapt an argument appearing in [8] and combine it with our results to prove the Weighted Singer Conjecture in dimension three. Then, we prove the conjecture in dimension four under some additional restrictions on the nerve of the corresponding Coxeter group. A consequence of this is that the conjecture holds in dimension four if the nerve is assumed to be flag complex. Lastly, we generalize the result in dimension four and show that \( L^2_q H_k(\Sigma) = 0 \) for \( k > 2 \) whenever the nerve is a flag triangulation of any 3–manifold.

In Chapter 6, we construct a complex that we call the fattened Davis complex. The idea is to “fatten” the Davis complex to a (homology) manifold with boundary so that we have standard algebraic topology tools (such as Poincaré duality) at our disposal. We carefully perform this fattening in such a way so that we can understand the weighted \( L^2 \)–(co)homology of the boundary. In fact, understanding the weighted \( L^2 \)–(co)homology of the boundary will simply amount to understanding the weighted \( L^2 \)–(co)homology of certain infinite special subgroups of \( W \). A large portion of this chapter is dedicated to studying the structure and algebraic topology of the fattened Davis complex. In Chapter 7, we then use the fattened Davis complex (combined with results from previous chapters) to perform new computations of \( L^2_q \)–Betti numbers for many examples of Coxeter groups. Of note is that mostly all of the computations are performed for \( q \geq 1 \), and hence they compute the ordinary \( L^2 \)–(co)homology of buildings associated to these Coxeter groups with integer thickness vector \( q \).
Chapter 2

Coxeter Groups and Preliminaries

2.1 Coxeter systems and Coxeter groups

A Coxeter matrix $M = (m_{st})$ on a set $S$ is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1 & \text{if } s = t \\ \geq 2 & \text{otherwise}. \end{cases}$$

One can associate to $M$ a presentation for a group $W$ as follows. Let $S$ be the set of generators and let $\mathcal{I} = \{(s, t) \in S \times S \mid m_{st} \neq \infty\}$. The set of relations for $W$ is

$$R = \{((st)^{m_{st}})_{(s, t) \in \mathcal{I}}. \}

The group defined by the presentation $(S, R)$ is a Coxeter group and the pair $(W, S)$ is a Coxeter system. If all off-diagonal entries of $M$ are either 2 or $\infty$, then $W$ is right-angled.

Given a subset $T \subset S$, define $W_T$ to be the subgroup of $W$ generated by the elements of $T$. Then $(W_T, T)$ is a Coxeter system. Subgroups of this form are special subgroups. $W_T$ is a spherical subgroup if $W_T$ is finite and, in this case, $T$ is a spherical subset. If $W_T$ is infinite, then $T$ is non-spherical. We will let $\mathcal{S}$ denote the poset of spherical subsets (the partial order being inclusion).

Given $w \in W$, call an expression $w = s_1 s_2 \cdots s_n$ reduced if there exists no integer $k < n$ with $w = s'_1 s'_2 \cdots s'_k$. We define the length of $w$, denoted by $l(w)$, to be the integer $n$ so that $w = s_1 s_2 \cdots s_n$ is a reduced expression for $w$. Given a subset $T \subset S$
and an element $w \in W$, the special coset $wW_T$ contains a unique element of shortest length. This element is said to be $(\emptyset, T)$–reduced.

### 2.2 Growth series

Suppose that $(W, S)$ is a Coxeter system. Let $t := (t_s)_{s \in S}$ denote an $S$–tuple of indeterminates, where $t_s = t_{s'}$ if $s$ and $s'$ are conjugate in $W$. If $s_1s_2\cdots s_n$ is a reduced expression for $w$, define $t_w$ to be the monomial

$$t_w := t_{s_1}t_{s_2}\cdots t_{s_n}.$$ 

Note that $t_w$ is independent of choice of reduced expression due to Tits’ solution to the word problem for Coxeter groups (see the discussion at the beginning of [6, Chapter 17]). The growth series of $W$ is the power series in $t$ defined by

$$W(t) = \sum_{w \in W} t_w.$$ 

The region of convergence $\mathcal{R}$ for $W(t)$ is defined to be

$$\mathcal{R} := \{t \in (0, +\infty)^S \mid W(t) \text{ converges}\}.$$ 

For each $T \subset S$, we denote the growth series of the special subgroup $W_T$ by $W_T(t)$, the respective region of convergence by $\mathcal{R}_T$, and define $t^{-1} := (t_s^{-1})_{s \in S}$. We record the following formula for later computations.

**Theorem 2.2.1** ([6, Theorem 17.1.9]).

$$\frac{1}{W(t)} = \sum_{T \in \mathcal{S}} (-1)^{|T|} W_T(t^{-1}).$$ 

Note that if $W$ is finite, then $W(t)$ is a polynomial with integral coefficients. Thus an immediate consequence of the above formula is that $W(t)$ is a rational function in $t$. 
2.3 Homology manifolds

A space $X$ is a homology $n$–manifold if it has the same local homology groups as $\mathbb{R}^n$, i.e. that for each $x \in X$

$$H_k(X, X - x) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

The pair $(X, \partial X)$ with $\partial X$ closed in $X$ is a homology $n$–manifold with boundary if it has the same local homology groups as does a manifold with boundary, i.e., the following conditions hold:

- $X - \partial X$ is a homology $n$–manifold,
- $\partial X$ is a homology $(n - 1)$–manifold,
- for each $x \in \partial X$, the local homology groups $H_*(X, X - x)$ all vanish.

$X$ is a generalized homology $n$–sphere, abbreviated $GHS^n$, if it is a homology $n$–manifold with the same homology as $S^n$. Similarly, the pair $(X, \partial X)$ is a generalized homology $n$–disk, abbreviated $GHD^n$, if it is a homology $n$–manifold with boundary with the same homology as the pair $(D^n, S^{n-1})$. Note that the cone on a generalized homology sphere is a generalized homology disk.

2.4 Mirrored spaces

A mirror structure over a set $S$ on a space $X$ is a family of subspaces $(X_s)_{s \in S}$ indexed by $S$. Then $X$ is a mirrored space over $S$. Put $X_\emptyset = X$, and for each nonempty subset $T \subseteq S$, define the following subspaces of $X$:

$$X_T := \bigcap_{s \in T} X_s, \quad X_T := \bigcup_{s \in T} X_s.$$  

If $(W, S)$ is a Coxeter system and $X$ is a mirrored space over $S$, then the mirror structure $(X_s)_{s \in S}$ is $W$–finite if $X_T = \emptyset$ for all non-spherical $T \subseteq S$. 

2.4.1 Mirrored homology manifolds with corners

Suppose that $X$ is a mirrored space over a finite set $S$. $X$ is an $S$–mirrored homology $n$–manifold with corners if every nonempty $X_T$ is a homology $(n−|T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supset T} X_U$. By taking $T = \emptyset$, this definition implies that the pair $(X, \partial X)$ is a homology $n$–manifold with boundary.

Given a Coxeter system $(W, S)$, we set $S' = S \cup \{e\}$, where $e$ is the identity element of $W$. We now say that $T \subseteq S'$ is spherical if and only if $T - \{e\}$ is spherical. A mirrored space $X$ over the set $S'$ with $W$–finite mirror structure $(X_s)_{s \in S'}$ is a partially $S$–mirrored homology $n$–manifold with corners if every nonempty $X_T$ is a homology $(n - |T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supset T} X_U$. To summarize, we simply have defined the non–$S$–mirrored part of $X$ to be an auxiliary mirror corresponding to the identity element of $W$.

2.5 Basic construction

Suppose that $(W, S)$ is a Coxeter system and that $X$ is a mirrored space over $S$. As before, for each nonempty subset $T \subset S$, let $W_T$ be the subgroup of $W$ generated by $T \subset S$. Put $S(x) := \{ s \in S \mid x \in X_s \}$. Define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', y)$ if and only if $x = y$ and $w^{-1}w' \in W_{S(x)}$. Give $W \times X$ the product topology and let $U(W, X)$ denote the quotient space:

$$U(W, X) = (W \times X)/\sim.$$ 

$U(W, X)$ is the basic construction and $X$ is the fundamental chamber. There is a natural $W$–action on $W \times X$, and this action respects the equivalence relation, hence the $W$–action on $W \times X$ descends to a $W$–action on $U(W, X)$.

We will be interested in conditions on $X$ which guarantee that the basic construction produces a homology $n$–manifold with boundary. But first, we consider the following proposition, as the proof is similar to the main result of this subsection.

Proposition 2.5.1 (Compare [6, Proposition 10.7.5]). Suppose that $(W, S)$ is a Coxeter system and that $X$ is an $S$–mirrored homology $n$–manifold with corners
with $W$–finite mirror structure. Then $\mathcal{U}(W, X)$ is a homology $n$–manifold.

Proof. Without loss of generality suppose that $x \in X$. By excision, we need to show that the local homology groups $H_*(U, U - x)$ are correct for some neighborhood $U$ of $x$ in $\mathcal{U}(W, X)$. If $x \in X - \partial X$ then we are done since $X - \partial X$ is a homology $n$–manifold and $x$ does not lie in any mirror. As before, set $S(x) = \{ s \in S \mid x \in X_s \}$ and suppose that $|S(x)| \geq 1$.

Let $V$ be a neighborhood of $x$ in $X$. For each $s \in S(x)$, set $V_s = V \cap X_s$, and give $V$ the mirror structure $\{ V_s \}_{s \in S(x)}$. Note that, for each $T \subseteq S(x)$, $V_T = V \cap X_T$, where as before, $X_T = \bigcap_{s \in T} X_s$. Now, $x \in X_{S(x)}$, so for each $T \subset S(x)$, $x \in \partial X_T$ ($X_T$ is by assumption a homology $(n - |T|)$–manifold with boundary and $X_{S(x)} \subseteq \partial X_T$). Furthermore, $x$ does not lie in $\partial X_{S(x)}$. Therefore by excision, it follows that for each $T \subset S(x)$, the local homology groups $H_*(V_T, V_T - x)$ vanish, and $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$ and $Z$ in that dimension.

Now, define

$$Z := V \cup \text{Cone}(V - x)$$
$$Z_s := V_s \cup \text{Cone}(V_s - x)$$

So, $Z$ has the mirror structure $\{ Z_s \}_{s \in S(x)}$. Since $V$ is a neighborhood of $x$ in $X$, and $x \in \partial X$, it follows that the local homology groups $H_*(V, V - x)$ vanish. In particular, $H_*(V) \cong H_*(V - x)$, and the Mayer–Vietoris sequence, along with the five lemma, implies that $Z$ is acyclic. Similarly, for each $T \subset S(x)$, since the local homology groups $H_*(V_T, V_T - x)$ vanish, it follows that $Z_T$ is acyclic. Since $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$ and $Z$ in that dimension, that Mayer–Vietoris sequence again implies that the same is true for $H_*(Z_{S(x)})$. In particular, $Z_{S(x)}$ has the same homology as $S^{n-|S(x)|}$.

We now finish the proof by applying the following lemma:

**Lemma 2.5.2** ([6, Corollary 8.2.5]). $\mathcal{U}(W_{S(x)}, Z)$ has the same homology as $S^n$ if and only if there is a unique spherical subset $R \subseteq S(x)$ satisfying the following three conditions:
(a) $W_{S(x)}$ decomposes as $W_{S(x)} = W_R \times W_{S(x)-R}$.

(b) For all spherical $T' \subseteq S(x)$ with $T' \neq R$, $(Z, Z^{T'})$ is acyclic.

(c) $(Z, Z^R)$ has the same homology as $(D^n, S^{n-1})$.

We apply the lemma to $R = S(x)$. Condition (a) is then satisfied vacuously, so we wish to show (b) and (c). For $T \subseteq R$, consider the cover of $Z^T$ by the mirrors $\{Z_s\}_{s \in T}$. Note that for each $U \subset R$, the intersection of mirrors $Z_U$ is acyclic. The nerve of this cover is a simplex on $U$, and in particular is contractible. The Acyclic Covering Lemma [3, Theorem 4.4, Ch VII] then implies that $Z^U$ is acyclic. Note that $Z_R$ has the same homology as $S^{n-|R|}$, so a similar spectral sequence argument also implies that $Z^R$ has the same homology as $S^{n-1}$.

Now, set $U = U(W, X)$. Since $U(W, Z) = U \cup \text{Cone}(U-x)$ and $U(W, Z)$ has the same homology as $S^n$, it follows that $H_\ast(U, U-x)$ is concentrated in dimension $n$ and $Z$ in that dimension. Therefore $U$ is our desired neighborhood.

Proposition 2.5.3. Suppose that $(W, S)$ is a Coxeter system and suppose that $X$ is a partially $S$–mirrored homology $n$–manifold with corners. Set $Y = X_e$ and give $Y$ the induced mirror structure $(Y_s)_{s \in S}$, where $Y_s := Y \cap X_s$. Then $U(W, X)$ is a homology $n$–manifold with boundary $\partial U(W, X) = U(W, Y)$.

Proof. Set $U = U(W, X)$ and $\partial U = U(W, Y)$. Proposition 2.5.1 guarantees that $\partial U$ is a homology $(n-1)$–manifold. This is because $Y = X_e$, and $X_e$ (with its induced $S$–mirror structure) is an $S$–mirrored homology $(n-1)$–manifold with corners. Similarly, Proposition 2.5.1 implies that $U - \partial U$ is a homology $n$–manifold, since $U - \partial U = U(W, Z)$, where $Z = X - Y$ (with its induced $S$–mirror structure) is an $S$–mirrored homology $n$–manifold with corners. It remains to show that for each $x \in \partial U$, the local homology groups $H_\ast(U, U-x)$ vanish.

Suppose that $x \in \partial U$. Without loss of generality, we can assume that $x \in Y \subset \partial X$. If $x$ does not lie in any mirror $(X_s)_{s \in S}$, then we are done by excision. So, suppose $|S(x)| \geq 1$ (recall $S(x) = \{s \in S \mid x \in X_s\}$) and let $V$ be a neighborhood of $x$ in $X$. We now give $V$ the $S$–mirror structure as in the proof of Proposition 2.5.1, noting that
the only difference between that proof and the current situation is the fact that the local homology groups $H_\ast(V_{S(x)}, V_{S(x)} - x)$ vanish. This is because, since $x \in Y$ and $|S(x)| \geq 1$, it follows that $x \in \partial X_{S(x)}$. Now, following the proof of Proposition 2.5.1 line by line, the only difference now is that $Z_{S(x)}$ is acyclic (as opposed to having the homology of $S^{n-1}$ as before). This then implies that $U(W_{S(x)}, Z)$ is acyclic [6, Corollary 8.2.8], which in turn implies that the local homology groups $H_\ast(U, U - x)$ vanish.

### 2.6 Posets, abstract simplicial complexes, and geometric realizations

A **poset** is a partially ordered set. Given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, set

$$\mathcal{P}_{\geq p} := \{ x \in \mathcal{P} \mid x \geq p \}.$$ 

Define $\mathcal{P}_\leq, \mathcal{P}_<,$ and $\mathcal{P}_>$ similarly. The **opposite** or **dual** poset to $\mathcal{P}$ is the poset $\mathcal{P}^{op}$ with the same underlying set but with the order relation reversed.

An **abstract simplicial complex** consists of a set $S$ (called the **vertex set**) and a collection $\mathcal{S}$ of finite subsets of $S$ such that

(i) for each $s \in S$, $\{s\} \in \mathcal{S}$ and

(ii) if $T \in \mathcal{S}$ and if $T' \subset T$, then $T' \in \mathcal{S}$.

An abstract simplicial complex $\mathcal{S}$ is a poset, the partial order being inclusion. An element of $\mathcal{S}$ is called a **simplex**. If $T$ is a simplex of $\mathcal{S}$ and $T' \leq T$, then we call $T'$ a **face** of $T$. The **dimension** of a simplex $T$ is defined by

$$\dim T := \text{Card}(T) - 1.$$ 

A subset $\mathcal{S}'$ of an abstract simplicial complex $\mathcal{S}$ is a **subcomplex** if it is also an abstract simplicial complex. The subcomplex $\mathcal{S}'$ is a **full subcomplex** if whenever $T \in \mathcal{S}$ such that the vertices of $T$ are contained in $\mathcal{S}'$, then $T \in \mathcal{S}'$. 
2.6.1 Flag complexes

An *incidence relation* is a symmetric and reflexive relation. Suppose that \( \mathcal{P} \) is a poset. We can symmetrize the partial order to obtain an incidence relation on \( \mathcal{P} \) as follows: two elements \( p, q \in \mathcal{P} \) are incident if and only if \( p \leq q \) or \( q \leq p \). Any set of incident elements in a poset is totally ordered. A *flag* in \( \mathcal{P} \) is a finite chain, i.e., a totally ordered subset. When \( \mathcal{P} \) is a poset, \( \text{Flag}(\mathcal{P}) \) denotes the abstract simplicial complex of all flags in \( \mathcal{P} \). It is called the *flag complex* of \( \mathcal{P} \).

2.6.2 Geometric realizations

Suppose that \( \mathcal{S} \) is an abstract simplicial complex with vertex set \( S \). Let \( \mathbb{R}^S \) denote the vector space of all finitely supported functions \( S \to \mathbb{R} \). For each \( s \in S \) let \( e_s \) denote the characteristic function of \( \{s\} \). The *standard simplex* on \( S \), denoted by \( \Delta^S \), is the convex hull of the standard basis \( \{e_s\}_{s \in S} \) of \( \mathbb{R}^S \).

For each nonempty finite subset \( T \subset S \), let \( \sigma_T \) denote the face of \( \Delta^S \) spanned by \( T \). Define a subcomplex \( \text{Geom}(\mathcal{S}) \) of \( \Delta^S \) by

\[ \sigma_T \in \text{Geom}(\mathcal{S}) \text{ if and only if } T \in S_{>0}. \]

The simplicial complex \( \text{Geom}(\mathcal{S}) \) is called the *standard geometric realization of \( \mathcal{S} \).*

The *geometric realization of a poset* \( \mathcal{P} \) is now defined to be the geometric realization of the simplicial complex \( \text{Flag}(\mathcal{P}) \). We use the notation

\[ |\mathcal{P}| := \text{Geom}(\text{Flag}(\mathcal{P})). \]

2.7 The \((\Lambda, S)\)-chamber

A *cell* is the convex hull of finitely many points in \( \mathbb{R}^n \). A *cell complex* is a collection of cells \( \Lambda \) where

(i) if \( C \in \Lambda \) and \( F \) is a face of \( C \), then \( F \in \Lambda \),
(ii) for any two cells \( C_1, C_2 \in \Lambda \), either \( C_1 \cap C_2 = \emptyset \) or \( C_1 \cap C_2 \) is a common face of \( C_1 \) and \( C_2 \),

(iii) \( \Lambda \) is locally finite, i.e. each cell in \( \Lambda \) is contained in only finitely many other cells of \( \Lambda \).

Suppose that \( \Lambda \) is a cell complex with vertex set \( S \) and let \( \mathcal{F}(\Lambda) \) denote the poset of cells of \( \Lambda \), including the empty set. Let \( P := |\mathcal{F}(\Lambda)| \) denote the geometric realization of the poset \( \mathcal{F}(\Lambda) \). For each \( T \in \mathcal{F}(\Lambda) \), define \( P_T := |\mathcal{F}(\Lambda)_{\geq T}| \) and \( \partial P_T := |\mathcal{F}(\Lambda)_{> T}| \), so each \( P_T \) is the cone on \( b\text{Link}(T, \Lambda) \), the barycentric subdivision of \( \text{Link}(T, \Lambda) \). In particular, taking \( T = \emptyset \), we have that \( P \) is the cone on \( b\Lambda \), with cone point corresponding to \( \emptyset \). For each \( s \in S \), put \( P_s := P_{\{s\}} \). This endows \( P \) with the mirror structure \( (P_s)_{s \in S} \). \( P \) is the \((\Lambda, S)\)–chamber.

\[ \text{Figure 2.1: } (\Lambda, S)\text{–chamber when } \Lambda \text{ is the boundary complex of an octahedron} \]

Note that if \( \Lambda \) is a \( GHS^{n-1} \), then the link of every cell \( \sigma \) in \( \Lambda \) is a \( GHS^{n-\dim \sigma - 2} \). It follows that \( P \) is a \( GHD^n \) and that for each \( T \in \mathcal{F}(\Lambda) \), the pair \((P_T, \partial P_T)\) is a \( GHD^{n-\dim \sigma_T - 1} \), where \( \sigma_T \) is the geometric cell in \( \Lambda \) spanned by \( T \).

### 2.7.1 Neighborhoods of faces

Let \( \sigma_T \) denote the geometric cell spanned by the vertex set \( T \) in \( \Lambda \), and let \( b\sigma_T \) denote its barycentric subdivision. By definition, \( b\sigma_T \) is the \((\partial \sigma_T, T)\)–chamber, and in particular, \( \sigma_T \) has a natural mirror structure over \( T \).

\( P \) is itself a flag simplicial complex, and for each \( T \in \mathcal{F}(\Lambda) \), \( P_T \) is a subcomplex of \( P \). Hence \( P_T - \bigcup_{U \supseteq T} P_U \) has a neighborhood of the form \( \sigma_T \ast P_T \), the join of \( \sigma_T \)
and $P_T$. Following the join lines for a little while, it follows that $P_T - \bigcup_{U \supseteq T} P_U$ has neighborhoods of the form $\text{Cone}(\sigma_T) \times P_T$. We record this fact, as we will use it in an upcoming construction.

## 2.8 The Davis complex

Suppose that $(W, S)$ is a Coxeter system and, as before, denote by $S$ the poset of all spherical subsets of $S$, partially ordered by inclusion. $S$ is an abstract simplicial complex with vertex set $S$. Let $L$ be the geometric realization of the abstract simplicial complex $S$ and $K$ be the $(L, S)$–chamber. In this special situation, $K$ is called the Davis chamber and $L$ is called the nerve of $(W, S)$.

For each $s \in S$ define

$$K_s := |S_{\geq \{s\}}|.$$ 

So, $K_s$ is the union of simplices in $K$ with minimum vertex $\{s\}$. The family $(K_s)_{s \in S}$ is a mirror structure on $K$.

The Davis complex $\Sigma_L$ associated to the nerve $L$ is now defined to be $\Sigma_L := \mathcal{U}(W, K)$.

Figure 2.2: $\Sigma_L$ whenever $W = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$
Note that $\Sigma_L$ is naturally a simplicial complex, the simplicial structure of $K$ inducing a simplicial structure on $\Sigma_L$, and moreover, it is proved in [7] that $\Sigma_L$ is contractible. Furthermore, if $L$ is a triangulation of an $(n - 1)$–sphere, then $\Sigma_L$ is an $n$–manifold.

2.8.1 The labeled nerve

There is a natural way to label the edges of $L$ so that the Coxeter system $(W, S)$ can be recovered (up to isomorphism) from $L$. Let $E(L)$ denote the set of edges of $L$. We define the labeling map $m : E(L) \to \{2, 3, \ldots\}$ by sending $\{s, t\} \to m_{st}$, where $m_{st} \in \mathbb{N}$ and $(st)^{m_{st}} = 1$. $L$ with this labeling map is the labeled nerve.

2.8.2 Right-angled cones and suspensions

Let $c$ denote a point and let $L$ be the labeled nerve. Consider the join $L' = c \ast L$, where all of the new edges are labeled by 2. $L'$ is called the right-angled cone on $L$. Note that the corresponding Coxeter system to $L'$ is $(W \times \mathbb{Z}_2, S \cup \{c\})$, and $\Sigma_{L'} = \Sigma_L \times [-1, 1]$.

If $P$ is a collection of two points, not joined by an edge, then we call the right-angled join $P \ast L$ the right-angled suspension of $L$. If the points of $P$ are $c_1$ and $c_2$, then the corresponding Coxeter system to the right-angled suspension of $L$ is $(W \times D_\infty, S \cup \{c_1, c_2\})$, where $D_\infty$ is the infinite dihedral group.

2.8.3 The Coxeter cellulation

The Davis complex also admits a decomposition into Coxeter cells. For each $T \in \mathcal{S}$, let $v_T$ denote the corresponding barycenter in $K$. Let $c_T$ denote the union of simplices $c \in \Sigma_L$ such that $c \cap K_T = v_T$. The boundary of $c_T$ is then cellulated by $wc_U$, where $w \in W_T$ and $U \subset T$. With its simplicial structure, the boundary $\partial c_T$ is the Coxeter complex corresponding to the Coxeter system $(W_T, T)$, which is a sphere since $W_T$ is finite. It follows that $c_T$ and its translates are disks, which are called Coxeter cells of type $T$. We denote $\Sigma_L$ with this decomposition into Coxeter cells by $\Sigma_{cc}$. Note that $\Sigma_{cc}$ is a regular CW complex with with poset of cells that can
be identified with $WS := \{wW_U \mid w \in W, T \in S\}$. The simplicial structure on $\Sigma_L$ is the geometric realization of the poset $WS$, hence $\Sigma_L$ is the barycentric subdivision of $\Sigma_{cc}$. The properties of the Coxeter cellulation can be summarized as follows:

**Proposition 2.8.1** ([6, p.130, Proposition 7.3.4]). $\Sigma_{cc}$ has the following properties:

(i) its vertex set is $W$, its 1–skeleton is the Cayley graph, $Cay(W,S)$, and its 2–skeleton is a Cayley 2–complex,

(ii) the link of each vertex is isomorphic to $L$,

(iii) a subset of $W$ is the vertex set of a cell if and only if it is a coset of a spherical subgroup,

(iv) the poset of cells is $WS$.

### 2.8.4 Twisted products

Suppose that $H$ acts on $Y$ and that $H$ is a subgroup of $G$. The twisted product $G \times_H Y$ is the quotient space of $G \times Y$ by the action $h(g,x) = (gh^{-1}, hx)$. The natural $G$–action on $G \times Y$ descends to a $G$–action on $G \times_H Y$. Hence one can view $G \times_H Y$ as a $G$–bundle over $G/H$, and if $G/H$ is discrete, then it follows that $G \times_H Y$ is just a disjoint union of copies of $Y$, one for each element of $G/H$.

Now, suppose that $(W,S)$ is a Coxeter system with Davis complex $\Sigma_L$ and that $T \subset S$. Let $\Sigma_T$ denote the Davis complex corresponding to the subgroup $W_T$. It follows that the subcomplex of $\Sigma_L$ corresponding to $W_T$ is $W\Sigma_T := W \times_{W_T} \Sigma_T$. In particular, $\Sigma_L$ contains a copy of $\Sigma_T$ for every coset of $W_T$.

### 2.9 Virtual cohomological dimension

The cohomological dimension of a group $\Gamma$ is

$$cd\Gamma := \sup\{n \mid H^n(\Gamma; M) \neq 0 \text{ for some } \mathbb{Z}\Gamma - \text{module } M\}.$$ 

If $\Gamma$ is virtually torsion free, then its virtual cohomological dimension, denoted by $vcd\Gamma$, is the cohomological dimension of any torsion-free subgroup of finite index.
Since Coxeter groups are virtually torsion free, it makes sense to talk about their virtual cohomological dimension, denoted by $\text{vcd} W$. In fact, given a Coxeter system $(W, S)$, one can determine $\text{vcd} W$ simply by looking at the nerve. Given a spherical $T \in S$, let $\sigma_T$ denote the corresponding closed simplex in the nerve $L$.

**Proposition 2.9.1** ([6, Corollary 8.5.5]).

$$\text{vcd} W = \max \{ n | \overline{H}^{n-1}(L - \sigma_T) \neq 0, \text{ for some } T \in S \}$$

Note that the dimension of $\Sigma_L$ is usually much larger than $\text{vcd} W$. For example, when $W$ is finite, $\dim \Sigma_L = |S|$, while $\text{vcd} W = 0$. 
Chapter 3

Weighted $L^2$–(co)homology

In this chapter we present a brief introduction to weighted $L^2$–(co)homology. Further details can be found in [6, 8, 12]. We then compile some new and old results pertaining to the weighted $L^2$–(co)homology of the Davis complex $\Sigma_L$.

Let $(W, S)$ be a Coxeter system. For the remainder of this thesis, let $q = (q_s)_{s \in S}$ denote an $S$–tuple of positive real numbers satisfying $q_s = q_{s'}$ whenever $s$ and $s'$ are conjugate in $W$. Set $q^{-1} = (q_s^{-1})_{s \in S}$. If $w = s_1 \cdots s_n$ is a reduced expression for $w \in W$, we define $q_w := q_{s_1} \cdots q_{s_n}$.

3.1 Hecke–von Neumann algebras

Let $\mathbb{R}(W)$ denote the group algebra of $W$ and let $\{e_w\}_{w \in W}$ denote the standard basis on $\mathbb{R}(W)$ (here $e_w$ denotes the characteristic function of $\{w\}$). Given a multiparameter $q$ of positive real numbers as above, we deform the standard inner product on $\mathbb{R}(W)$ to an inner product

$$\langle e_w, e_{w'} \rangle_q = \begin{cases} q_w & \text{if } w = w' \\ 0 & \text{otherwise.} \end{cases}$$

Using the multiparameter $q$, one can give $\mathbb{R}(W)$ the structure of a Hecke algebra. We will denote $\mathbb{R}(W)$ with this inner product and Hecke algebra structure by $\mathbb{R}_q(W)$, and $L^2_q(W)$ will denote the Hilbert space completion of $\mathbb{R}_q(W)$ with respect to $\langle , \rangle_q$. There is a natural anti-involution on $\mathbb{R}_q(W)$, which implies that there is
an associated Hecke–von Neumann algebra $\mathcal{N}_q(W)$ acting on the right on $L^2_q(W)$. It is the algebra of all bounded linear endomorphisms of $L^2_q(W)$ which commute with the left $\mathbb{R}_q(W)$–action.

Define the von Neumann trace of $\phi \in \mathcal{N}_q(W)$ by $\text{tr}_{\mathcal{N}_q}(\phi) = \langle e_1 \phi, e_1 \rangle_q$, and similarly for an $(n \times n)$–matrix with coefficients in $\phi \in \mathcal{N}_q(W)$ by taking the sum of the von Neumann traces of elements on the diagonal. This allows us to attribute an non-negative real number called the von Neumann dimension for any closed subspace of an $n$–fold direct sum of copies of $L^2_q(W)$ which is stable under the $\mathbb{R}_q(W)$–action, called a Hilbert $\mathcal{N}_q$–module. If $V \subseteq (L^2_q(W))^n$ is a Hilbert $\mathcal{N}_q$–module, and $p_V : (L^2_q(W))^n \to (L^2_q(W))^n$ is the orthogonal projection onto $V$ (note that $p_V \in \mathcal{N}_q(W)$), then define

$$\dim_{\mathcal{N}_q} V := \text{tr}_{\mathcal{N}_q}(p_V).$$

### 3.1.1 Induced Hilbert $\mathcal{N}_q$–modules

Suppose that $T \subset S$ and that $V_T$ is a Hilbert $\mathcal{N}_q(W_T)$–module. The induced Hilbert $\mathcal{N}_q$–module $V$ is defined to be the completion of the tensor product

$$L^2_q(W) \otimes_{\mathbb{R}_q(W_T)} V_T.$$

A standard fact is that its dimension is given by

$$\dim_{\mathcal{N}_q} V = \dim_{\mathcal{N}_q(W_T)} V_T.$$

### 3.2 Weighted $L^2$–(co)homology

Suppose $(W, S)$ is a Coxeter system and that $X$ is a mirrored finite CW complex over $S$. Set $\mathcal{U} = \mathcal{U}(W, X)$. We first orient the cells of $X$ and extend this orientation to $\mathcal{U}$ in such a way so that if $\sigma$ is a positively oriented cell of $X$, then $w\sigma$ is positively oriented for each $w \in W$.

We define a measure on the $W$–orbit of an $i$–cell $\sigma \in X$ by

$$\mu_q(w\sigma) = q^u,$$
where \( u \) is \((\varnothing, S(\sigma))\)-reduced and \( S(\sigma) := \{ s \in S | s \leq X_s \} \). This extends to a measure on the \( i \)-cells \( U^{(i)} \), which we also denote by \( \mu_q \).

Figure 3.1: Weights on the 1–cells of \( \Sigma_L \) whenever \( W = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \) and \( q = q \), a positive real number

Define the \( q \)-weighted \( i \)-dimensional \( L^2 \)-(co)chains on \( U \) to be the Hilbert space:

\[
L^2_q C_i(U) = L^2_q C^{(i)}(U) = L^2(U^{(i)}, \mu_q).
\]

These are infinite \( W \)-equivariant square summable (with respect to \( \mu_q \)) real-valued \( i \)-chains. The inner product is given by

\[
(f, g)_q = \sum_{\sigma} f(\sigma)g(\sigma) \mu_q(\sigma),
\]

and we denote the induced norm by \( \| \|_q \).

The boundary map \( \partial_i : L^2_q C_i(U) \to L^2_q C_{i-1}(U) \) and coboundary map \( \delta^i : L^2_q C_{i-1}(U) \to L^2_q C_{i+1}(U) \) are defined by the usual formulas, however there is one caveat: they are not adjoints with respect to this inner product whenever \( q \neq 1 \). Thus one remedies this issue by perturbing the boundary map \( \partial_i \) to \( \partial^q_i \):

\[
\partial^q_i(f)(\sigma^{i-1}) = \sum_{\sigma^{i-1}, \alpha^i} [\sigma : \alpha] \mu_q(\alpha) \mu_q^{-1}(\sigma) f(\alpha).
\]
A simple computation shows that $\partial^q_i$ is the adjoint of $\delta^i$ with respect to the weighted inner product, hence $(L^2_q C_*(U), \partial^q_i)$ is a chain complex. We now define the reduced $q$–weighted $L^2$–(co)homology by

$$L^2_q H_i(U) = \text{Ker} \partial^q_i / \text{Im} \partial^q_{i+1},$$

$$L^2_q H^i(U) = \text{Ker} \delta^i / \text{Im} \delta^{i-1}.$$

The Hodge Decomposition implies that $L^2_q H^i(U) \cong L^2_q H^i(U) \cong \ker \partial^q_i \cap \ker \delta^i$ and versions of Eilenberg–Steenrod axioms hold for this homology theory. There is also a weighted version of Poincaré duality: If $U$ is a locally compact homology $n$–manifold with boundary $\partial U$, then

$$L^2_q H_i(U) \cong L^2_q H_{n-i}(U, \partial U).$$

One can also assign the von Neumann dimension to each of the Hilbert spaces $L^2_q H_i(U)$ (as they are Hilbert $\mathcal{N}_q$–modules). We denote this by $L^2_q b_i(U)$ and call it the $i$–th $L^2_q$–Betti number of $U$. We then define the weighted Euler characteristic of $U$:

$$\chi_q(U) = \sum (-1)^i L^2_q b_i(U).$$

Figure 3.2: An element of $L^2_q H_1(\Sigma_L)$ whenever $W = \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$
3.2.1 \( L_q^2 \)-Betti numbers and twisted products

Suppose that \( X \) is a mirrored finite CW complex and let \( T \subset S \). Recall the twisted product \( W \times_{W_T} \mathcal{U}(W_T, X) \). It follows that the \( \mathcal{N}_q \)-module

\[
L_q^2 H_*(W \times_{W_T} \mathcal{U}(W_T, X))
\]

is induced from the \( \mathcal{N}_q(W_T) \)-module \( L_q^2 H_*(\mathcal{U}(W_T, X)) \). Thus

\[
L_q^2 b_*(W \times_{W_T} \mathcal{U}(W_T, X)) = L_q^2 b_*(\mathcal{U}(W_T, X)).
\]

3.2.2 An alternate definition of \( L_q^2 \)-Betti numbers

As discussed in [10, §6], there is an alternate approach in defining \( L_q^2 \)-Betti numbers using the ideas of Lück [15]. The main point is that there is an equivalence of categories between the category of Hilbert \( \mathcal{N}_q \)-modules and projective modules of \( \mathcal{N}_q \). Hence one can define \( \dim_{\mathcal{N}_q} M \) for a finitely generated projective \( \mathcal{N}_q \)-module \( M \) which agrees with the dimension of the corresponding Hilbert \( \mathcal{N}_q \)-module. So, \( \dim_{\mathcal{N}_q} M \) for an arbitrary \( \mathcal{N}_q \)-module is then defined to be the dimension of its projective part.

As before, suppose \((W, S)\) is a Coxeter system and that \( X \) is a mirrored finite CW complex over \( S \). Set \( \mathcal{U} = \mathcal{U}(W, X) \). As in [15], define \( H_*^W(\mathcal{U}, \mathcal{N}_q(W)) \) to be the homology of the \( \mathcal{N}_q(W) \)-chain complex

\[
C_*^W(\mathcal{U}, \mathcal{N}_q(W)) := \mathcal{N}_q(W) \otimes_{\mathbb{R}_q(W)} C_*(\mathcal{U}),
\]

where \( C_*(\mathcal{U}) \) is the cellular chain complex of \( \mathcal{U} \) with the induced \( \mathbb{R}_q(W) \)-structure. Similarly, define the cohomology groups \( H_*^W(\mathcal{U}, \mathcal{N}_q(W)) \) to be the cohomology of the complex

\[
C_*^W(\mathcal{U}, \mathcal{N}_q(W)) := \text{Hom}_W(C_*(\mathcal{U}), \mathcal{N}_q(W)).
\]

It then follows that

\[
L_q^2 b_1(\mathcal{U}) = \dim_{\mathcal{N}_q} H_1^W(\mathcal{U}, \mathcal{N}_q(W)) = \dim_{\mathcal{N}_q} H_1^W(\mathcal{U}, \mathcal{N}_q(W)).
\]

The advantage of these definitions is that we do no need to take closures of images as in the definition of reduced \( q \)-weighted \( L^2 \)-(co)homology (this is particularly useful when dealing with spectral sequences).
3.3 New and old results for $\Sigma_L$

In this section we begin by stating some previous results on the weighted $L^2$–(co)homology of $\Sigma_L$. We start with the following result of Dymara, which explicitly computes $L^2_q b_0(\Sigma_L)$.

**Proposition 3.3.1** ([12, Theorem 7.1, Theorem 10.3]). $L^2_q b_0(\Sigma_L) \neq 0$ if and only if $q \in \mathcal{R}$. Moreover, when $q \in \mathcal{R}$, $L^2_q b_k(\Sigma_L) = 0$ for $k > 0$.

Dymara also computes the weighted Euler characteristic of $\Sigma_L$, revealing the connection between weighted $L^2$–(co)homology of $\Sigma_L$ and the growth series of the corresponding Coxeter group $W$.

**Proposition 3.3.2** ([12, Corollary 3.4]).

$$\chi_q(\Sigma_L) = \frac{1}{W(q)}.$$ 

Recall that $\Sigma_{cc}$ denotes $\Sigma_L$ with the Coxeter cellulation (see Section 2.8.3). The following proposition states that if we compute the weighted $L^2$–(co)homology with respect to either cellulation, then we get the same answer.

**Proposition 3.3.3** ([12, Theorem 5.5]).

$$L^2_q H_*(\Sigma_L) \cong L^2_q H_*(\Sigma_{cc}).$$ 

In conjunction with Proposition 3.3.2, the following theorem explicitly computes the weighted $L^2$–(co)homology of Coxeter groups which act properly and cocompactly by reflections on Euclidean space.

**Theorem 3.3.4** ([8, Corollary 14.5]). Suppose that $W$ is a Euclidean reflection group with nerve $L$.

- If $q \leq 1$, then $L^2_q H_*(\Sigma_L)$ is concentrated in dimension 0.
- If $q \geq 1$, then $L^2_q H_*(\Sigma_L)$ is concentrated in dimension $n$. 

The following lemma says that we can compute the weighted $L^2$–Betti numbers of any acyclic complex of the form $U(W, X)$, with $X$ finite, on which $W$ acts properly, and get the same answer. Thus we will sometimes write $L^2_q b_k(W)$ instead of $L^2_q b_k(\Sigma_L)$ to denote the $k$–th $L^2_q$–Betti number of $W$.

**Lemma 3.3.5.** Let $(W, S)$ be a Coxeter system and suppose that $X$ and $X'$ are finite mirrored CW complexes with $U(W, X)$ and $U(W, X')$ both acyclic and both admitting proper $W$–action. Then for every $k \geq 0$,

$$L^2_q b_k(U(W, X)) = L^2_q b_k(U(W, X')).$$

**Proof.** Set $U = U(W, X)$ and $U' = U(W, X')$. Since $U$ and $U'$ are both acyclic, it follows that the respective cellular chain complexes $C_*(U)$ and $C_*(U')$ are are chain homotopic. This chain homotopy induces a chain homotopy of the chain complexes $C^W_*(U, N_q(W))$ and $C^W_*(U', N_q(W))$. 

In fact, Bestvina constructed such a complex for any finitely generated Coxeter group.

**Theorem 3.3.6 ([2]).** Let $W$ be a finitely generated Coxeter group. Then $W$ acts properly and cocompactly on an acyclic $\text{vcd} W$–dimensional complex of the form $U(W, X)$.

**Corollary 3.3.7.** Let $(W, S)$ be a Coxeter system. Then

$$L^2_q b_k(W) = 0 \text{ for } k > \text{vcd } W.$$

**Proof.** We can use the acyclic $\text{vcd} W$–dimensional complex of Theorem 3.3.6 to compute the weighted $L^2$–Betti numbers of $W$. Lemma 3.3.5 now completes the proof.

We now prove a lemma which is crucial for later computations.

**Lemma 3.3.8.** Let $n = \text{vcd } W$ and suppose and that $L^1_1 b_n(W) = 0$. Then

$$L^2_{q} b_k(W) = 0 \text{ for } k \geq n \text{ and } q \leq 1.$$
Proof. By Corollary 3.3.7, we obtain vanishing for \( k > n \). Now, suppose for a contradiction that \( L^2_{q}b_n(W) \neq 0 \) for \( q < 1 \). Let \( B_W \) denote the complex of Theorem 3.3.6. Lemma 3.3.5 says that we can compute weighted \( L^2 \)-Betti numbers of \( W \) with respect to the complex \( B_W \). In particular, \( L^2_{q}b_n(W) = L^2_{q}b_n(B_W) \) and we can choose a nontrivial element \( \psi \in L^2_{q}H_n(B_W) \). Thus \( \psi \) is a cycle under the weighted boundary map \( \partial^q \). Consider the isomorphism of Hilbert spaces

\[
m_q : L^2_{q}C_n(B_W) \to L^2_{q-1}C_n(B_W)
\]
defined by \( m_q(f(\sigma)) = \mu_q(\sigma)f(\sigma) \). In particular, \( m_q\psi \in L^2_{q-1}C_n(B_W) \) and since \( q^{-1} > 1 \),

\[
\|m_q\psi\|_{1} \leq \|m_q\psi\|_{q^{-1}} < \infty.
\]
Hence \( m_q\psi \in L^2_1C_n(B_W) \).

![Figure 3.3: Schematic for the proof of Lemma 3.3.8](image)

Now, a simple computation shows that \( \partial = m_q\partial^q m_q^{-1} \) and since \( \psi \) is a cycle under \( \partial^q \), \( m_q\psi \) is a cycle under \( \partial \), the standard \( L^2 \)-boundary operator. Moreover, since \( B_W \) is \( n \)-dimensional, \( m_q\psi \) is trivially a cocycle. Thus we have produced a nontrivial element of \( L^2_1H_n(B_W) \), a contradiction.

Remark 3.3.9. Note that the statement of Lemma 3.3.8 holds in the more general setting for \( L^2_{q}b_n(U(W,X)) \) (here \( X \) is finite and \( n = \dim X \)). In fact, we obtain the same statement for relative \( L^2_{q}-(co)\)homology as long as we are working in the top dimension.
Chapter 4

Ruins and Weighted \( L^2 - (co)\)homology

4.1 Some Hilbert \( \mathcal{N}_q(W) \)–submodules of \( L^2_q(W) \)

We begin by considering the following self-adjoint idempotents in \( \mathcal{N}_q(W) \):

**Lemma 4.1.1** ([6, Lemma 19.2.6]). Given a subset \( T \subset S \) and and \( q \in R_T^{-1} \), there is an idempotent \( h_T \in \mathcal{N}_q(W) \) defined by

\[
h_T := \frac{1}{W_T(q^{-1})} \sum_{w \in W_T} \varepsilon_w q_w^{-1} e_w,
\]

where \( \varepsilon_w = (-1)^{l(w)} \).

Thus the maps defined by \( x \to h_T x \) are orthogonal projections (whenever \( h_T \) is defined) from \( L^2_q(W) \) onto Hilbert \( \mathcal{N}_q(W) \)–submodules, denoted by \( H_T \). Note that, by [6, Lemma 19.2.13],

\[
H_T = \bigcap_{s \in T} H_s.
\]

Using these submodules, we define a chain complex as follows. For a spherical subset of cardinality \( k \), \( T \in S^{(k)} \), put

\[
C_i(H_T) := \bigoplus_{U \in (S_T^{(i+k)})} H_U.
\]
Fix some ordering of \( \{ s \in S - T \mid T \cup \{ s \} \in S \} \). Whenever \( U \subset V \), we have an inclusion \( i^U_V : H_V \to H_U \), and thus the boundary map \( \partial : C_{i+1}(H_T) \to C_i(H_T) \) corresponds to a matrix \( (\partial_{UV}) \), where \( \partial_{UV} = 0 \) unless \( U \subset V \), and is equal to \( (-1)^j i^U_V \) if \( U \) is obtained by deleting the \( j \)th element of \( V \). This turns \( C_*(H_T) \) into a chain complex of Hilbert \( \mathcal{N}_q(W) \)-modules. Similarly, whenever \( U \subset V \) we have the projection \( p^U_V : H_U \to H_V \). Thus we have a coboundary map where the matrix entries consist of projections, and we get a cochain complex \( C^*(H_T) \) of Hilbert \( \mathcal{N}_q(W) \)-modules.

4.2 Ruins

As before, \( \Sigma_{cc} \) is \( \Sigma_L \) with the Coxeter cellulation. Let \((W,S)\) be a Coxeter system and for \( U \subset S \), set \( S(U) := \{ T \in S \mid T \subset U \} \). Define \( \Sigma(U) \) to be the subcomplex of \( \Sigma_{cc} \) consisting of all (closed) Coxeter cells of type \( T \) with \( T \in S(U) \). Note that \( \Sigma(U) = W \times_{W^U} \Sigma_U \), where \( \Sigma_U \) is the Davis complex corresponding to the group \( W_U \).

Given \( T \in S(U) \), we define the following subcomplexes of \( \Sigma(U) \):

\[ \Omega_{UT} : \text{the union of closed cells of type } T', \text{ with } T' \in S(U)_{\not\subset T}, \]
\[ \partial \Omega_{UT} : \text{the cells of } \Omega_{UT} \text{ of type } T'', \text{ with } T'' \not\in S(U)_{\not\subset T}. \]

The pair \((\Omega_{UT}, \partial \Omega_{UT})\) is the \((U,T)\)-ruin. Note that if \( T = \emptyset \), then \( \Omega_{UT} = \Sigma(U) \) and \( \partial \Omega_{UT} = \emptyset \). Ruins can also be expressed in terms of the basic construction. Define \( K(U,T) := \Omega_{UT} \cap K \) and \( \partial K(U,T) := \partial \Omega_{UT} \cap K \), where \( K \) is the Davis chamber. Then \( K(U,T) \) and \( \partial K(U,T) \) have an induced mirror structure, and it follows that

\[ \Omega_{UT} = U(W,K(U,T)), \]

and

\[ \partial \Omega_{UT} = U(W,\partial K(U,T)). \]

The \((S,T)\)-ruin has a chain complex that looks like this:
Proposition 4.2.1 ([6, Lemma 20.6.21]). For $T \in S^{(k)}$, the chain complexes $C_*(H_T)$ and $L_2^q C_{*+k}(\Omega_{ST}, \partial) \oplus \Omega_{ST}$ of $N_q(W)$-modules are isomorphic. In particular,

$$L_2^q C_m(\Omega_{ST}, \partial) = 0 \text{ for } m < k.$$ 

For brevity, we write $(\Omega_{UT}, \partial)$. For $s \in T$, set $U' = U - s$ and $T' = T - s$. As in [8, Proof of Theorem 8.3], we have the following weak exact sequence:

$$\cdots \rightarrow L_2^q H_*(\Omega_{U'T'}, \partial) \rightarrow L_2^q H_*(\Omega_{UT'}, \partial) \rightarrow L_2^q H_*(\Omega_{UT}, \partial) \rightarrow \cdots \tag{4.1}$$

For the special case when $U = S$ and $T = \{s\}$ the above sequence becomes:

$$\cdots \rightarrow L_2^q H_*(\Sigma(S - s)) \rightarrow L_2^q H_*(\Sigma(S)) \rightarrow L_2^q H_*(\Omega_{S\{s\}}, \partial) \rightarrow \cdots$$

4.3 $L_2^q$-(co)homology of ruins

Given a Coxeter system $(W, S)$, for $T \in S$ and $T \subseteq V \subseteq S$, define

$$St(T, V) := \bigcup_{U \subseteq V \atop U \cap T \in S} U,$$

and

$$Lk(T, V) := St(T, V) \setminus T.$$

If $V = S$, then we write $St(T)$ and $Lk(T)$ instead of $St(T, S)$ and $Lk(T, S)$. If $T = \emptyset$, we make the convention that $S(T, U) = U$.

Lemma 4.3.1. Suppose that $(W, S)$ is right-angled. Then for $T \in S$,

$$\Omega_{ST} = \Sigma(Lk(T)) \times \Sigma(T),$$

$$\partial \Omega_{ST} = \Sigma(Lk(T)) \times \partial \Sigma(T).$$
Proof. For $U \in \mathcal{S}$, recall the Coxeter cell $c_U$ from Section 2.8.3. As $W$ is right-angled, all Coxeter cells are cubes of appropriate dimension, and thus it follows that $c_U$ is just a direct product of all the Coxeter cells $c_s$ with $s \in U$.

Now, by definition,

$$K(S, T) = \Omega_{ST} \cap K = \bigcup_{U \in \mathcal{S}_{ST}} c_U \cap K,$$

$$K(Lk(T)) = \Sigma(Lk(T)) \cap K = \bigcup_{U \in \mathcal{S}(Lk(T))} c_U \cap K,$$

and

$$K(T) = \Sigma(T) \cap K = c_T \cap K.$$

Now, let $U \in \mathcal{S}_{ST}$. Then

$$c_U = c_T \times c_{U-T}.$$  

Since $U \in \mathcal{S}_{ST}$, it follows that $U \subset St(T)$, as $U \cup T = U \in \mathcal{S}$. Thus $U - T \subset \mathcal{S}(Lk(T))$. Therefore

$$K(S, T) \subseteq K(Lk(T)) \times K(T) = \bigcup_{U \in \mathcal{S}(Lk(T))} (c_T \times c_U) \cap K.$$

For the reverse inclusion, let $U \in \mathcal{S}(Lk(T))$. Then $U \cup T \in \mathcal{S}_{ST}$. This is because the only way that $U \cup T$ could fail to be spherical is if there would exist $u \in U$ and $t \in T$ with $m_{tu} = \infty$ ($W$ is right-angled), and if this happened then it would contradict the fact that $U \subset Lk(T)$. Therefore we have shown that

$$K(S, T) = K(Lk(T)) \times K(T),$$

and thus

$$\Omega_{ST} = U(W, K(S, T)) = U(W, K(Lk(T))) \times U(W, K(T)) = \Sigma(Lk(T)) \times \Sigma(T).$$

The proof for $\partial \Omega_{ST}$ follows a similar unwinding of definitions. Begin by noting that:

$$\partial \Omega_{ST} \cap K = \bigcup_{V \in \mathcal{S}_{ST}} c_V \cap K.$$
and
\[ \partial \Sigma(T) \cap K = \bigcup_{U \subseteq T} c_U \cap K. \]

To conclude the proof, observe that for \( W \) right-angled, a subset \( V \) satisfying \( V \notin S_{\geq 2T} \) is a disjoint union \( V = A \cup B \), where \( A \in S(Lk(T)) \) and \( B \notin T \). Conversely, any such disjoint union \( A \cup B \notin S_{\geq 2T} \) (by definition it cannot contain \( T \)), and satisfies \( A \cup B \subset A \cup T \in S_{\geq 2T} \).

\[ \square \]

**Theorem 4.3.2** (Compare [6, Theorem 20.6.22]). Suppose that \( T \in S^{(k)} \) and that \( q \in R_{St(T)} \). Then \( L^2_q H_*(\Omega_{ST}, \partial \Omega_{ST}) \) is concentrated in dimension \( k \). If \( (W, S) \) is assumed to be right-angled, then the same statement holds if we replace \( R_{St(T)} \) by \( R_{Lk(T)} \).

**Proof.** For the proof we temporarily switch notation and denote the \((U,T)\)-ruin by \( \Omega(U,T) \). We first make an observation about ruins. We note that for every \( V \subseteq S \), \( \Omega(V,T) = \Omega(St(T,V),T) \), the point being that \( \Omega(V,T) \) consists of Coxeter cells corresponding to spherical subsets of \( V \) containing \( T \), and \( St(T,V) \) is the union of all such subsets. In particular, \( \Omega(U,T) = \Omega(St(T,U),T) \), and hence \( \Omega(U,T) \) is a subcomplex of \( \Sigma(St(T,U)) \).

The proof is now by induction on \( k \). We will show that for \( U \in S \) and \( T \in S(U)^{(k)} \), \( L^2_q H_*(\Omega(U,T), \partial) \) is concentrated in dimension \( k \). For the base case \( k = 0 \), note that \( St(\emptyset, U) = U \), \( \Omega(U,\emptyset) = \Sigma(U) \), and \( \partial \Omega(U,\emptyset) = \emptyset \). Hence, for \( k = 0 \), the theorem asserts that for \( q \in R_U \), \( L^2_q H_*(\Sigma(U)) \) is concentrated in dimension 0, which is Proposition 3.3.1.

Now, suppose the theorem is true for \( k - 1 \) and let \( T \in S(U)^{(k)} \). Let \( s \in T \), \( V = T - s \) and consider the long exact sequence:

\[ L^2_q H_*(\Omega(St(T,U) - s,V), \partial) \to L^2_q H_*(\Omega(St(T,U),V), \partial) \to L^2_q H_*(\Omega(St(T,U),T), \partial) \]

Note that
\[ \Omega(St(T,U),V) = \Omega(St(V,St(T,U)),V) \]
\[ \Omega(St(T, U) - s, V) = \Omega(St(V, St(T, U) - s), V). \]

Since \( St(V, St(T, U)) \subseteq St(T, U) \) and \( St(V, St(T, U) - s) \subseteq St(T, U) \), it follows that \( \mathcal{R}_{St(T, U)} \subseteq \mathcal{R}_{St(V, St(T, U))} \) and \( \mathcal{R}_{St(T, U)} \subseteq \mathcal{R}_{St(V, St(T, U) - s)} \). Since \( q \in \mathcal{R}_{St(T, U)} \), it follows by induction that the left-hand term and the middle term of the exact sequence are both concentrated in dimension \( k - 1 \). Since \( L_q^2 H_*(\Omega(St(T, U), T), \partial) \) vanishes for \( * < k \) (Proposition 4.2.1), it follows that \( L_q^2 H_*(\Omega(St(T, U), T), \partial) = L_q^2 H_*(\Omega(U, T), \partial) \) is concentrated in dimension \( k \).

Now, suppose that \((W, S)\) is right-angled. Consider the long exact sequence of the pair \((\Omega(S, T), \partial \Omega(S, T))\):

\[ \cdots \rightarrow L_q^2 H_*(\partial \Omega(S, T)) \rightarrow L_q^2 H_*(\Omega(S, T)) \rightarrow L_q^2 H_*(\Omega(S, T), \partial) \rightarrow \cdots \]

Since \( W \) is right-angled, it follows from Lemma 4.3.1 that \( \Omega(S, T) = \Sigma(Lk(T)) \times \Sigma(T) \) and \( \partial \Omega(S, T) = \Sigma(Lk(T)) \times \partial \Sigma(T) \). By the the Künneth formula,

\[
L_q^2 b_i(\Omega(S, T)) = \sum_{i+j=m} L_q^2 b_i(\Sigma(Lk(T))) \cdot L_q^2 b_j(\Sigma(T)),
\]

\[
L_q^2 b_m(\partial \Omega(S, T)) = \sum_{i+j=m} L_q^2 b_i(\Sigma(Lk(T))) \cdot L_q^2 b_j(\partial \Sigma(T)).
\]

Since \( W_T \) is finite and \( q \in \mathcal{R}_{Lk(T)} \), Proposition 3.3.1 implies that \( L_q^2 b_*(\Sigma(Lk(T))) \) and \( L_q^2 b_*(\Sigma(T)) \) are both concentrated in degree 0, and hence \( L_q^2 H_*(\Omega(S, T)) \) is concentrated in dimension 0. Similarly, \( L_q^2 H_*(\partial \Omega(S, T)) \) vanishes above dimension \( k - 1 \) (this is because, since \( W_T \) is finite, \( \Sigma(T) \) is topologically a disjoint collection of \( k \)-disks with boundary \( \partial \Sigma(T) \)). As Proposition 4.2.1 implies that \( L_q^2 H_*(\Omega(St(T, U), T), \partial) \) vanishes for \( * < k \), the long exact sequence for the pair implies that \( L_q^2 H_*(\Omega(S, T), \partial) \) is concentrated in dimension \( k \). \( \blacksquare \)

**Remark 4.3.3.** Suppose that \( T \in S \) and that \( q \in \mathcal{R}_{St(T)} \). Then, for \( U \in S_{ST} \), Theorem 4.3.2 implies that \( L_q^2 H_*(\Omega(S, U), \partial) \) is concentrated in dimension \( |U| \). This is because, if \( T \subset U \), then \( St(U) \subset St(T) \) (similarly, \( Lk(U) \subset Lk(T) \)). Therefore
\( q \in \mathcal{R}_{S(T)} \subseteq \mathcal{R}_{S(U)} \). The analogous statements hold when \( W \) is assumed to be right-angled and \( q \in \mathcal{R}_{Lk(T)} \).

Given a subset \( T \subset S \), define

\[
W^T := \{ w \in W \mid l(ws) < l(w) \text{ for } s \in T \}
\]

and

\[
W^T(t) = \sum_{w \in W^T} t_w.
\]

The following is now a consequence of Theorem 4.3.2 and [6, Corollary 20.6.6].

**Corollary 4.3.4.** Suppose that \( T \in S^{(k)} \) and that \( q \in \mathcal{R}_{S(T)} \). Then \( L^2_q H_* (\Omega(S,T), \partial) \) is concentrated in dimension \( k \) and

\[
L^2_q b_k (\Omega(S,T), \partial) = \frac{W^T(q)}{W(q)}.
\]

If \((W, S)\) is assumed to be right-angled, then the same statement holds if we replace \( \mathcal{R}_{S(T)} \) by \( \mathcal{R}_{Lk(T)} \).

### 4.4 A spectral sequence

In this section, we define a spectral sequence following the line laid down in [10, § 2].

A poset of coefficients is a contravariant functor \( \mathcal{A} \) from a poset \( \mathcal{P} \) to the category of abelian groups. In other words, it is a collection \( \{\mathcal{A}_a\}_{a \in \mathcal{P}} \) of abelian groups together with homomorphisms \( \phi_{ba} : \mathcal{A}_a \rightarrow \mathcal{A}_b \), defined whenever \( a > b \), such that \( \phi_{ca} = \phi_{cb} \phi_{ba} \), whenever \( a > b > c \). The functor \( \mathcal{A} \) gives us a system of coefficients on the cell complex \( \text{Flag}(\mathcal{P}) \): it associates to the simplex \( \sigma \) the abelian group \( \mathcal{A}_{\min(\sigma)} \). Hence, we get a cochain complex

\[
C^j(\text{Flag}(\mathcal{P}); \mathcal{A}) := \bigoplus_{\sigma \in \text{Flag}(\mathcal{P})^{(j)}} \mathcal{A}_{\min(\sigma)},
\]

where \( \text{Flag}(\mathcal{P})^{(j)} \) denotes the set of \( j \)-simplices in \( \text{Flag}(\mathcal{P}) \).
Let $Y$ be a CW complex. A *poset of spaces* in $Y$ over $\mathcal{P}$ is a cover $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ of $Y$ by subcomplexes indexed by $\mathcal{P}$ so that if $N(\mathcal{V})$ denotes the nerve of the cover, then

(i) $a < b \implies Y_a \subset Y_b$,

(ii) the vertex set $\text{Vert}(\sigma)$ of each simplex in $N(\mathcal{V})$ has the greatest lower bound $\wedge \sigma$ in $\mathcal{P}$, and

(iii) $\mathcal{V}$ is closed under taking finite nonempty intersections, i.e., for any simplex $\sigma$ of $N(\mathcal{V})$,

\[
\bigcap_{a \in \sigma} Y_a = Y_{\wedge \sigma}.
\]

Note that any cover leads to a poset of spaces by taking all nonempty intersections as elements of the new cover and removing duplicates. The resulting poset is the set of all nonempty intersections, ordered by inclusion.

The following lemmas appearing in [10] define a spectral sequence associated to a poset of spaces, and give conditions for the sequence to degenerate.

**Lemma 4.4.1** ([10, Lemma 2.1]). Suppose $\mathcal{V} = \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for $Y$ over $\mathcal{P}$. There is a Mayer–Vietoris type spectral sequence converging to $H^*(Y)$ with $E_1$–term:

\[
E_1^{i,j} = C^i(\text{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),
\]

and $E_2$–term:

\[
E_2^{i,j} = H^i(\text{Flag}(\mathcal{P}); \mathcal{H}^j(\mathcal{V})),
\]

where the coefficient system $\mathcal{H}^j(\mathcal{V})$ is given by $\mathcal{H}^j(\mathcal{V})(\sigma) = H^j(Y_{\min(\sigma)})$.

**Lemma 4.4.2** ([10, Lemma 2.2]). Suppose that $\mathcal{V} := \{Y_a\}_{a \in \mathcal{P}}$ is a poset of spaces for $Y$ over $\mathcal{P}$. If for every $a \in \mathcal{P}$, the induced homomorphism $H^*(Y_a) \to H^*(Y_{<a})$ is the zero map, then the spectral sequence degenerates at $E_2$ and

\[
H^*(Y) = \bigoplus_{a \in \mathcal{P}} H^i(\text{Flag}(\mathcal{P}_{\geq a}), \text{Flag}(\mathcal{P}_{> a}), H^j(Y_a)).
\]
4.5 \( L_q^2 \)-\((co)\)homology of \((\Sigma, \Sigma^{(k-1)})\)

To simplify notation, write \(\Sigma\) for \(\Sigma_{cc}\) and let \(\Sigma^{(k-1)}\) denote the \((k-1)\)-skeleton of \(\Sigma_{cc}\). For the proofs in this section, we will also write \(H_q^*(\mathcal{U})\) for \(H_q^*(\mathcal{U}, N_q(W))\) (see Section 3.2.2 for the definition of \(H_q^*(\mathcal{U}, N_q(W))\)).

**Lemma 4.5.1.** Let \(\Delta\) denote the standard \(n\)-simplex and let \(\Delta^{(k)}\) be its \(k\)-skeleton, \(k < n\). Then the reduced homology \(\tilde{H}_*(\Delta^{(k)})\) (with coefficients in \(\mathbb{R}\)) is concentrated in dimension \(k\). Furthermore, \(\tilde{b}_k(\Delta^{(k)}) = \binom{n}{k+1}\).

**Proof.** \(\Delta\) is contractible, so it follows that the augmented chain complex \(C_*(\Delta)\) is exact. In particular, \(C_*(\Delta^{(k)})\) has the same homology as \(C_*(\Delta)\) whenever \(l < k\), so \(\tilde{H}_l(\Delta^{(k)}) = 0\) whenever \(l < k\). We now must show that \(\tilde{H}_k(\Delta^{(k)})\) has the claimed dimension. First, note that \(C_k(\Delta^{(k)}) = C_k(\Delta)\) has dimension \(\binom{n+1}{k+1}\), as there is a \(k\)-simplex for each set of \(k+1\) vertices. We now proceed by induction on \(k\). If \(k = 0\), then \(\Delta^{(0)}\) is just \(n+1\) vertices, so \(\tilde{H}_0(\Delta^{(0)})\) has dimension \(\binom{n}{1} = n\). Suppose that the claim is true for \(k - 1\), and consider the chain complex:

\[
0 \longrightarrow C_k(\Delta) \longrightarrow C_{k-1}(\Delta) \longrightarrow \cdots
\]

By induction, \(\ker \partial_{k-1}\) is has dimension \(\binom{n}{k}\). Since \(\tilde{H}_{k-1}(\Delta) = 0\), it follows that \(\text{im} \partial_k\) also has dimension \(\binom{n}{k}\). So, \(\ker \partial_k\) has dimension \(\binom{n+1}{k+1} - \binom{n}{k} = \binom{n}{k+1}\). \(\blacksquare\)

**Theorem 4.5.2.** Let \(k \geq 1\). Suppose that for every \(T \in S^{(k)}\), \(q \in \mathcal{R}_{St(T)}\), and let \(\Sigma^{(k-1)}\) denote the \((k-1)\)-skeleton of \(\Sigma\). Then \(L_q^2 H_*(\Sigma, \Sigma^{(k-1)})\) is concentrated in dimension \(k\). Furthermore,

\[
L_q^2 b_k(\Sigma, \Sigma^{(k-1)}) = \sum_{U \in S^{(k)}} \binom{|U| - 1}{k-1} L_q^2 b_{|U|}(\Omega_U, \partial).
\]

If \((W, S)\) is assumed to be right-angled, then the same statement holds if we replace \(\mathcal{R}_{St(T)}\) by \(\mathcal{R}_{Lk(T)}\).

**Proof.** We will show that \(H_q^*(\Sigma, \Sigma^{(k-1)})\) is concentrated in dimension \(k\).
Consider the relative cochain complex $L^2_qC^\ast(\Sigma, \Sigma^{(k-1)})$. We have that

$$L^2_qC^i(\Sigma, \Sigma^{(k-1)}) = \begin{cases} 0, & i \leq k-1; \\ \bigoplus_{\xi \in \mathcal{C}(i)} H_U, & i > k-1. \end{cases}$$

Set $C_{-i}(\Sigma, \Sigma^{(k-1)}) = L^2_qC^i(\Sigma, \Sigma^{(k-1)})$. Then $C_{-i}(\Sigma, \Sigma^{(k-1)})$ is a chain complex. Now, for every $T \in \mathcal{S}$, set $\tilde{\Omega}_T = \Omega_{ST}/\partial\Omega_{ST}$ and set $C_{-i}(\tilde{\Omega}_T) = L^2_qC^i(\tilde{\Omega}_T) = L^2_qC^i(\Omega_{ST}, \partial\Omega_{ST})$. In this way, we have made the cochain complex of every $(S, T)$-ruin a subcomplex of the re-indexed relative cochain complex $C^\ast(\Sigma, \Sigma^{(k-1)})$ (a similar trick works using ordinary cochain complexes).

Let $\mathcal{P}$ be the poset $S^{(2k)}$ with the order reversed. By the above re-indexing of cochain complexes, $\{\tilde{\Omega}_T\}_{T \in \mathcal{P}}$ is a poset of spaces over $Y = \Sigma/\Sigma^{(k-1)}$, and hence we have the spectral sequence of Lemma 4.4.1.

We first establish the condition of Lemma 4.4.2. So, we claim that for every $U \in \mathcal{P}$, the induced map $H^\ast_q(\tilde{\Omega}_U) \to H^\ast_q(\tilde{\Omega}_{<U})$ is the zero map. To prove the claim we will show that $H^{|U|}_q(\tilde{\Omega}_{<U}) = 0$, as this implies $H^\ast_q(\tilde{\Omega}_U) \to H^\ast_q(\tilde{\Omega}_{<U})$ is the zero map since $H^\ast_q(\tilde{\Omega}_U)$ is concentrated in dimension $-|U|$ (see Remark 4.3.3). Recall that $\tilde{\Omega}_{<U} = \bigcup_{T \in \mathcal{P} \setminus U} \tilde{\Omega}_T$. The proof is by induction on the number of elements in the union. For the base case, note that for every spherical $V$ properly containing $U$, the induced map $H^\ast_q(\tilde{\Omega}_U) \to H^\ast_q(\tilde{\Omega}_V)$ is the zero map. This is because of Theorem 4.3.2, which states that $H^\ast_q(\tilde{\Omega}_U)$ and $H^\ast_q(\tilde{\Omega}_V)$ are concentrated in dimension $-|U|$ and $-|V|$, respectively, and $-|V| < -|U|$. Now, let $\mathcal{C}$ be a subcollection of elements of $\mathcal{P} \setminus U$ and let $B = \bigcup_{T \in \mathcal{C}} \tilde{\Omega}_T$. We wish to show that $H_q^{\pm|U|}(B) = 0$. Write $B = A \cup \tilde{\Omega}_V$, where $V \in \mathcal{C}$ and $A = \bigcup_{T \in \mathcal{C} \setminus V} \tilde{\Omega}_T$. Then we have the Mayer–Vietoris sequence:

$$H_q^{-|U|-1}(A \cap \tilde{\Omega}_V) \to H_q^{-|U|}(B) \to H_q^{-|U|}(A) \oplus H_q^{-|U|}(\tilde{\Omega}_V)$$

By induction, $H_q^\ast(A)$ vanishes for $* \geq -|U| - 1$, and by Theorem 4.3.2, $H_q^\ast(\tilde{\Omega}_V)$ is concentrated in $-|V| < -|U|$. We now claim $H_q^\ast(A \cap \tilde{\Omega}_V)$ vanishes for $* \geq -|U| - 1$, as this implies $H_q^{-|U|}(B) = 0$. We observe that
\[ A \cap \tilde{\Omega}_V = \bigcup_{T \in \mathcal{C}} \tilde{\Omega}_T \cap \tilde{\Omega}_V \]
\[ = \bigcup_{T \cup V \in \mathcal{S}} \tilde{\Omega}_{T \cup V} \]

The last inequality follows from the fact that \( H_T \cap H_V = H_{T \cup V} \) whenever \( T \cup V \) is spherical (see Section 4.1). Thus \( A \cap \tilde{\Omega}_V \) is the union of elements corresponding to a subcollection of \( \mathcal{C} \). Therefore the claim follows by induction.

We have established the condition in Lemma 4.4.2, and hence

\[ H_q^{-n}(Y) = \bigoplus_{U \in \mathcal{P}} H^{[U]-n}(\text{Flag}(\mathcal{P}_U), \text{Flag}(\mathcal{P}_U); H_q^{-[U]}(\tilde{\Omega}_U)) \quad (4.2) \]

The strategy of the proof now is as follows. By Theorem 4.3.2, for every \( U \in \mathcal{P} \), \( H_q^*(\tilde{\Omega}_U) \) is concentrated in dimension \(-|U|\). So, we are done if we show that for every \( U \in \mathcal{P} \), \( H^*(\text{Flag}(\mathcal{P}_U), \text{Flag}(\mathcal{P}_U)) \) is concentrated in dimension \(|U| - k\). This implies \( E_2^{i,j} = 0 \) unless \( i + j = -k \), and by (4.2), \( H_q^i(Y) \) is concentrated in dimension \(-k\). Re-indexing our complexes, it follows that the cohomology of the complex \( L_2^q C^*(\Sigma, \Sigma^{(k-1)}) \) is concentrated in dimension \( k \).

We now claim that for \( U \in \mathcal{P} \) with \( m = |U| \), \( H^*(\text{Flag}(\mathcal{P}_U), \text{Flag}(\mathcal{P}_U)) \) is concentrated in dimension \( m - k \) and free of rank \( \binom{m-1}{k-1} \). Since the geometric realization of \( \text{Flag}(\mathcal{P}_U) \) is contractible, by the long exact sequence for the pair it suffices to show that the reduced cohomology \( \tilde{H}^*(\text{Flag}(\mathcal{P}_U)) \) is concentrated in dimension \( m - k - 1 \). Note that if \( m = k \), then we are done since \( \text{Flag}(\mathcal{P}_U) = \emptyset \). Also, note that for the special case where \( m - k = 1 \), the map \( H^0(\text{Flag}(\mathcal{P}_U)) \to H^0(\text{Flag}(\mathcal{P}_U)) \) in the long exact sequence for the pair \( (\text{Flag}(\mathcal{P}_U), \text{Flag}(\mathcal{P}_U)) \) is injective (\( \text{Flag}(\mathcal{P}_U) \) is the cone on \( \text{Flag}(\mathcal{P}_U) \)), so showing \( \tilde{H}^*(\text{Flag}(\mathcal{P}_U)) \) is concentrated in dimension \( m - k - 1 = 0 \) does in fact suffice.

Consider the poset \( S^{op} \), which is the poset \( S \) of spherical subsets with order reversed. Note that \( \text{Flag}(S^{op}_U) \cong \text{Flag}(S_U) \). The geometric realization of \( \text{Flag}(S_{<U}) \) is \( b\Delta \), where \( b\Delta \) is the barycentric subdivision of the \((m-1)\)-dimensional simplex \( \Delta \). This is because \( S_{<U} \) is the poset of proper subsets of \( U \). Note that \( \text{Flag}(\mathcal{P}_U) \) is a
subcomplex of Flag($S_{>U}^\text{op}$), and more precisely, the geometric realization of Flag($P_{>U}$) is the subcomplex of barycentric subdivision of $\partial \Delta$ (recall that $k \geq 1$) obtained by removing barycenters corresponding to spherical subsets of cardinality less than $k$.

These barycenters correspond to faces of $\partial \Delta$ of dimension less than or equal to $k - 2$. Hence

$$\widetilde{H}^* (\text{Flag}(P_{>U})) \cong \widetilde{H}^* (\partial \Delta - \Delta^{(k-2)}),$$

where $\Delta^{(k-2)}$ denotes the $(k-2)$-skeleton of $\Delta$. Note that if $k = 1$, then $\Delta^{(k-2)} = \emptyset$, and we are done as the reduced homology $\widetilde{H}^* (\partial \Delta)$ is concentrated in dimension $m - 2$. So, suppose $k > 1$. By Alexander Duality,

$$\widetilde{H}_* (\Delta^{(k-2)}) \cong \widetilde{H}^{m-*+3} (\partial \Delta - \Delta^{(k-2)}).$$

By Lemma 4.5.1, the reduced homology $\widetilde{H}_*(\Delta^{(k-2)})$ is concentrated in dimension $k - 2$. Furthermore, $\widetilde{H}_*(\Delta^{(k-2)})$ has dimension $\binom{m-1}{k-1}$. It follows that $\widetilde{H}^* (\partial \Delta - \Delta^{(k-2)})$ is concentrated in dimension $m - k - 1$ and of dimension $\binom{m-1}{k-1}$, and therefore the same holds for $\widetilde{H}^* (\text{Flag}(P_{>U}))$.

Thus, we have shown that $H_q^k (Y)$ is concentrated in dimension $-k$ and by (4.2)

$$H_q^k (Y) = \bigoplus_{U \in P} H^{[U]-k} (\text{Flag}(P_{\geq U}), \text{Flag}(P_{>U}); H_q^{[U]} (\tilde{\Omega}_U))).$$

In particular, since $H^{[U]-k} (\text{Flag}(P_{\geq U}), \text{Flag}(P_{>U}))$ is free and of dimension $\binom{[U]-1}{k-1}$,
The assertion now follows after recalling that we are making computations with respect to re-indexed complexes. Specifically, \( L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) = L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) \) and \( L^2_{q^b} (\Omega_U) = L^2_{q^b} (\Omega_U, \partial) \).

Using Corollary 4.3.4, we obtain the following formula for \( L^2_{q^b} \)-Betti numbers.

**Corollary 4.5.3.** Let \( k \geq 1 \). Suppose that for every \( T \in S^{(k)} \), \( q \in R_{St}(T) \). Then

\[
L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) = \left\{ \begin{array}{ll}
\sum_{U \in S^{(k)}} (|U|-1) L^2_{q^b} (\Omega_U) & \text{if } n = k; \\
0 & \text{otherwise}.
\end{array} \right.
\]

If \((W, S)\) is assumed to be right-angled, then the same statement holds if we replace \( R_{St}(T) \) by \( R_{Lk}(T) \).

**Remark 4.5.4.** Note that a formula for \( L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) \) could also be derived from an Euler characteristic argument, and it is the same as the formula above by [6, Lemma 17.1.8].

### 4.6 Some Consequences

**Corollary 4.6.1.** Suppose that for every \( T \in S^{(k)} \), \( q \in R_{St}(T) \). Then

\[
L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) = \sum_{U \in S^{(k)}} (|U|-1) L^2_{q^b} (\Omega_U) = 0 \quad \text{for } n > k.
\]

If \((W, S)\) is assumed to be right-angled, then the same statement holds if we replace \( R_{St}(T) \) by \( R_{Lk}(T) \).

**Proof.** For the case where \( k = 0 \), this is just Proposition 3.3.1, so suppose \( k \geq 1 \). By Theorem 4.5.2, \( L^2_{q^b} (\Sigma, \Sigma^{(k-1)}) \) is concentrated in dimension \( k \). Therefore the long exact sequence for the pair \((\Sigma, \Sigma^{(k-1)})\) implies the assertion. \( \blacksquare \)
Example 4.6.2. Suppose that \((W, S)\) is right-angled and that the corresponding nerve \(L\) is a flag triangulation of \(S^2\). Furthermore, suppose that \(q = q\), a positive real number. Let \(\rho\) and \(\rho_T\) denote the radii of convergence of the growth series \(W(t)\) and \(W_T(t)\), respectively. In [6, Example 17.4.3], it was computed that

\[
\rho = \frac{(f_0 - 4) - \sqrt{(f_0 - 4)^2 - 4}}{2},
\]

where \(f_0\) is the number of vertices of \(L\) (note that since \(L\) is flag, \(f_0 \geq 6\)).

The link of every vertex of \(l\) is a \(k\)-gon, with \(k \geq 4\) and \(k < f_0\). If \(W_{Lk(v)}\) denotes the special subgroup corresponding the the link of a vertex \(v\), and \(Lk(v)\) has \(k\) vertices, then by an easy computation (see [6, Example 17.1.15]) we have that

\[
\rho_{Lk(v)} = \frac{(k - 2) - \sqrt{k^2 - 4k}}{2}.
\]

Note that \(\rho_{Lk(v)}\) is a decreasing function of \(k\) when \(k \geq 4\). If \(v_0\) is the vertex of \(L\) whose link has the most vertices, then Corollary 4.6.1 implies that \(L_2^q H_*(\Sigma_L)\) is concentrated in dimension 1 whenever \(\rho < q < \rho_{Lk(v_0)}\). This is was already known, as \(L_2^q b_2(\Sigma_L) = 0\) for \(q \leq 1\) [8, Theorem 16.13].

We present some further consequences of Theorem 4.5.2. But first, some definitions. A locally finite cell complex \(\Lambda\) is an \(n\)-dimensional pseudomanifold if each maximal cell of \(\Lambda\) is \(n\)-dimensional and each \((n-1)\)-cell is a face of precisely two \(n\)-cells. A pseudomanifold \(\Lambda\) is orientable if one can choose orientations for the top-dimensional cells so that their sum is a (possibly infinite) cycle. We now say that a Coxeter system \((W, S)\) is type \(PM^n\) if its nerve \(L\) is an orientable \((n-1)\)-dimensional pseudomanifold with the property that the complement of the codimension-two skeleton of \(L\) is connected.

Corollary 4.6.3. Suppose that \((W, S)\) is right-angled and of type \(PM^n\). Then for \(q \leq 1\),

\[
L_2^q b_k(\Sigma^{(n-2)}) = \left\{ \begin{array}{ll}
\sum_{U \subseteq S^{(n-1)}} \binom{|U| - 1}{n-2} \frac{W_U(q)}{W(q)} + L_2^q b_{n-2}(\Sigma) - L_2^q b_{n-1}(\Sigma), & k = n-2; \\
L_2^q b_k(\Sigma), & \text{otherwise.}
\end{array} \right.
\]
Proof. Since \((W, S)\) is of type \(PM^n\), the nerve \(L\) is a \((n-1)\)-pseudomanifold. Let \(T \in \mathcal{S}^{n-1}\). Then the corresponding geometric simplex \(\sigma_T\) in \(L\) has dimension \(n-2\), and since \(L\) is an \((n-1)\)-pseudomanifold, it follows that \(\sigma_T\) is contained in precisely two \((n-1)\)-simplices \(\sigma_U\) and \(\sigma_V\), where \(U = T \cup \{s\}\) and \(V = T \cup \{t\}\) for some \(s, t \in S\).

Since \(L\) has dimension \(n-1\) and \((W, S)\) is right-angled, it follows that \(m_{st} = \infty\) (otherwise, \(U \cup V\) would span an \(n\)-simplex in \(L\)). It follows that \(L_{k(T)} = \{s, t\}\), and that \(W_{L_k(T)} = D_\infty\). Thus, if \(q \in \mathbb{R}_{L_k(T)}\), then \(q < 1\). Since \(T \in \mathcal{S}^{n-1}\) was arbitrary, it follows that if for every \(T \in \mathcal{S}^{n-1}\), \(q \in \mathbb{R}_{L_k(T)}\), then \(q < 1\).

By Theorem 4.5.2, \(L^2_q H_n(\Sigma, (\Sigma^{(n-2)})\) is concentrated in dimension \(n-1\). Consider the long exact sequence for the pair \((\Sigma, (\Sigma^{(n-2)})\):

\[0 \rightarrow L^2_q H_{n-1}(\Sigma) \rightarrow L^2_q H_{n-1}(\Sigma, (\Sigma^{(n-2)}) \rightarrow L^2_q H_{n-2}(\Sigma^{(n-2)}) \rightarrow L^2_q H_{n-2}(\Sigma) \rightarrow 0\]

A dimension count and continuity of weighted \(L^2_q\)-Betti numbers now implies the assertion.

When \((W, S)\) is of type \(PM^3\), we have the following computation thanks to Corollary 4.6.3. Recall, that \(\Sigma^{(1)}\) is the Cayley graph of \(W\) (Proposition 2.8.1), so the following corollary gives a formula for weighted \(L^2\)-Betti numbers of the Cayley graph when \(W\) is of type \(PM^3\).

**Corollary 4.6.4.** Suppose that \((W, S)\) is right-angled and of type \(PM^3\). Furthermore, suppose that \(q \leq 1\) and \(q \notin \mathcal{R}\). Then

\[L^2_q b_1(\Sigma^{(1)}) = -\frac{1}{W(q)} + \sum_{U \in \mathcal{S}^{(2)}} (|U| - 1) \frac{W(U)(q)}{W(q)}\]

Proof. Since \(\Sigma\) is a pseudomanifold, it follows that \(L^2_q b_3(\Sigma) = 0\) and since \(q \notin \mathcal{R}\), \(L^2_q b_0(\Sigma) = 0\). The assertion follows from Corollary 4.6.3, as \(\chi_q(\Sigma) = L^2_q b_2(\Sigma) - L^2_q b_1(\Sigma) = \frac{1}{W(q)}\).
Example 4.6.5. Suppose that \((W, S)\) is right-angled and of type \(PM^3\), and let \(q = q_0\), a positive real number. Recall the \(f\)-polynomial \(f_L(t)\) of \(L\). It is defined by

\[
f_L(t) := \sum_{i=0}^{3} f_{i+1} t^i,
\]

where \(f_m\) is the number of \(m\)-simplices of \(L\) and \(f_{-1} = 1\). By [6, Proposition 17.4.2], we have the following formula:

\[
\frac{1}{W(t)} = (1 + t)^3 f_L\left(\frac{-t}{1 + t}\right).
\]

This simplifies to

\[
\frac{1}{W(t)} = 1 - (f_0 - 3)t + (f_1 - 2f_0 + 3)t^2 - (f_0 - f_1 + f_2 - 1)t^3
\]

\[
= 1 - (f_0 - 3)t + (f_0 + 3 - 3\chi(L))t^2 - (\chi(L) - 1)t^3.
\]

Here \(\chi(L)\) is the Euler characteristic of \(L\). Note that the second equality follows by using the facts that \(\chi(L) = f_0 - f_1 + f_2\) and \(3f_2 = 2f_1\) (this second formula holds because each edge is contained in exactly two 2-simplices, and each 2-simplex contains exactly three edges).

The radius of convergence \(\rho\) of \(W(t)\) is the smallest modulus of a root of the above polynomial. Since \(W\) is of type \(PM^3\), the link of every vertex of \(L\) is 1-pseudomanifold (in particular, just homeomorphic to \(S^1\)). Thus, just as in Example 4.6.2, the radius of convergence \(\rho_{Lk(v)}\) of the special subgroup \(W_{Lk(v)}\) has the formula:

\[
\rho_{Lk(v)} = \frac{(k - 2) - \sqrt{k^2 - 4k}}{2},
\]

where \(Lk(v)\) is the link of the vertex \(v\) and \(k\) is the number of vertices in \(Lk(v)\).

If \(v_0\) is the vertex of \(L\) whose link has the maximal number of vertices, then Corollary 4.6.1 implies that \(L^2qH_*(\Sigma_L) = 0\) is concentrated in dimension 1 whenever \(\rho < q < \rho_{Lk(v_0)}\).

The main point is that \(\rho\) is explicitly computable. For example, if \(L\) is a flag triangulation of a torus (or, more generally, a flag triangulation of a surface of genus \(g \geq 1\)), it is still an open conjecture that \(L^2q_0b_*(\Sigma_L) = 0\) is concentrated in degree 2.
for $q = 1$ (see [9, Conjecture 11.5.1]), but on the other hand Corollary 4.6.1 allows us to conclude that $L_q^2 b_2(\Sigma_L) = 0$ for $q < \rho_{Lk(v_0)}$. 
Chapter 5

The Weighted Singer Conjecture

Appearing in [8], the following is the appropriate reformulation of the the Singer Conjecture for Coxeter groups [9] for weighted $L^2$–(co)homology:

**Conjecture 5.0.1** (Weighted Singer Conjecture). Suppose that the nerve $L$ is a triangulation of $S^{n-1}$. Then

$$L^2_q H_k(\Sigma_L) = 0 \quad \text{for} \quad k > \frac{n}{2} \quad \text{and} \quad q \leq 1.$$  

By weighted Poincaré duality, this is equivalent to the conjecture that if $q \geq 1$ and $k < \frac{n}{2}$, then $L^2_q H_k(\Sigma)$ vanishes. The conjecture is known for elementary reasons for $n \leq 2$, and in [8], it is proved for the case where $W$ is right-angled and $n \leq 4$. Furthermore, it was shown in in [8] that Conjecture 5.0.1 for $n$ odd implies Conjecture 5.0.1 for $n$ even, under the assumption that $W$ is right-angled.

The original Singer Conjecture for Coxeter groups was formulated for $q = 1$ in [9] and concluded that the $L^2$–(co)homology is concentrated in dimension $\frac{n}{2}$. The original conjecture is known for elementary reasons for $n \leq 2$ and holds by a result of Lott and Lück [14], in conjunction with the validity of the Geometrization Conjecture for 3–manifolds [17], for $n = 3$. It was proved by Davis–Okun [9] for the case where $W$ is right-angled and $n \leq 4$. It was later proved for the case where $W$ is an even Coxeter group and $n \leq 4$ by Schroeder [18], under the assumption that the nerve $L$ is a flag complex. Due to recent work of Okun–Schreve [16, Theorem 4.9], the conjecture is now known in full generality whenever $q = 1$ and $n \leq 4$. In
fact, using induction and [16, Theorem 4.5, Lemma 4.6, Corollary 4.7] proves the following theorem.

**Theorem 5.0.2.** Suppose that the nerve \( L \) is an \((n - 1)\)-sphere or an \((n - 1)\)-disk. Then

\[
L^2_1 H_k(\Sigma_L) = 0 \text{ for } k \geq n - 1.
\]

In this chapter, we present a proof of Conjecture 5.0.1 in dimension three that encompasses all but nine Coxeter groups. Then, under some restrictions on the nerve of the Coxeter group, we obtain partial results whenever \( n = 4 \) (in particular, the conjecture holds for \( n = 4 \) if the nerve of the corresponding Coxeter group is a flag complex). We then extend our results in dimension four to prove a general version of the conjecture for the case where the nerve of the Coxeter group is assumed to be a flag triangulation of a 3–manifold.

### 5.1 The case where \( L \) is a disk

Note that if \( L \) is a triangulation of the \((n - 1)\)-disk, then \( \Sigma_L \) is an \( n \)-manifold with boundary. We now obtain the following theorem, which whenever \( n = 3, 4 \) can be thought of as a version of Conjecture 5.0.1 for the case where \( \Sigma_L \) is an \( n \)-manifold with boundary.

**Theorem 5.1.1.** Suppose that the nerve \( L \) is an \((n - 1)\)-disk. Then

\[
L^2_q H_k(\Sigma_L) = 0 \text{ for } k \geq n - 1 \text{ and } q \leq 1.
\]

**Proof.** By Theorem 5.0.2, we have that \( L^2_1 H_k(\Sigma_L) = 0 \) for \( k \geq n - 1 \). Furthermore, Proposition 2.9.1 implies that \( \text{vcd} W \leq n - 1 \), and hence we are done by Lemma 3.3.8.

### 5.2 A cell structure on \( K \)

Suppose that \( L \) is the labeled nerve of a Coxeter system, homeomorphic to the \( n \)-sphere. For every \( T \in \mathcal{S} \), define \( K_T \) to be the geometric realization of the poset
$S_{zT} = \{ U \in S \mid T \subseteq U \}$. In other words, $K_T$ is the union of all closed simplices in $bL$ with minimum vertex $T$, so $K_T$ is the cone on the barycentric subdivision of the link of $T$ in $L$. If $L$ is a triangulation of the $n$–sphere, then it follows that links of simplices $T$ of $L$ are spheres of dimension $n - |T|$. Thus it follows that each $K_T$ is a $(n - |T| + 1)$–disk, hence $\{K_T\}_{T \in S}$ yields a cellulation of $K$. We denote $K$ with this cellulation by $K_d$. Note that this cellulation extends to $\Sigma_L$, and the simplicial structure on $\Sigma_L$ coincides with the barycentric subdivision of this cell structure.

The codimension-one faces of $K_d$ correspond to vertices of $L$, and we assign dihedral angles to $K_d$ as follows. If $\{s,t\}$ is an edge of $L$, then we assign the dihedral angle $\pi/m_{st}$ between the faces $K_s$ and $K_t$.

Figure 5.1: $K_d$ when $W$ is right-angled and the labeled nerve $L$ is the boundary complex of an octahedron

### 5.3 Andreev’s theorem

In [1], Andreev listed necessary and sufficient conditions for abstract three-dimensional polytopes with assigned dihedral angles $(0, \frac{\pi}{2}]$ to be realized as convex polytopes in $\mathbb{H}^3$. For these polytopes to tile $\mathbb{H}^3$, these angles must be integer submultiples of $\pi$. We now state the theorem.

**Theorem 5.3.1** ([1, Theorem 2]). Let $P$ be an abstract three-dimensional simple polyhedron, not a simplex. The following conditions are necessary and sufficient for the existence in $\mathbb{H}^3$ of a convex polytope of finite volume of the combinatorial type
with the dihedral angles $\alpha_{ij} \leq \frac{\pi}{2}$ (where $\alpha_{ij}$ is the dihedral angle between the faces $F_i, F_j$):

(i) If $F_1, F_2$ and $F_3$ are all the faces meeting at a vertex of $P$, then $\alpha_{12} + \alpha_{23} + \alpha_{31} > \pi$.

(ii) If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of the intersection satisfy $\alpha_{12} + \alpha_{23} + \alpha_{31} < \pi$.

(iii) Four faces cannot intersect cyclically with all four angles$=\frac{\pi}{2}$ unless two of the opposite faces intersect.

(iv) If $P$ is a triangular prism, then the angles along the base and the top cannot all be $\frac{\pi}{2}$.

Our goal is to use the above theorem to formulate conditions on the labeled nerve $L$ so that $\Sigma_L = \mathbb{H}^3$. Note that, for the rest of this thesis, we use the notation $\Sigma_L = X$ whenever $\Sigma_L$ admits a $W_L$–invariant metric making it isometric to $X$.

We say that a vertex in $L$ is a Euclidean 3–vertex if its link has three pairwise connected vertices, and if $v_0, v_1, v_2$ are the vertices, then the labelings on the corresponding edges satisfy:

$$\frac{\pi}{m_{v_0v_1}} + \frac{\pi}{m_{v_0v_2}} + \frac{\pi}{m_{v_1v_2}} = \pi.$$

Similarly, we say a vertex $v$ in $L$ is a Euclidean 4–vertex if $Lk(v)$ is a 4–gon with all edges labeled by 2.

Let $C$ be an empty circuit in $L$ and suppose that $C$ is not the link of some vertex of $L$. If $C$ consists of three vertices $v_0, v_1, v_2$, then we say that $C$ is a Euclidean 3–circuit if the labelings on the edges of $C$ satisfy:

$$\frac{\pi}{m_{v_0v_1}} + \frac{\pi}{m_{v_0v_2}} + \frac{\pi}{m_{v_1v_2}} = \pi.$$

Similarly, if $C$ consists of four vertices and is not the boundary of two adjacent simplices, then we say that $C$ is a Euclidean 4–circuit if the labels on the edges of $C$ are all equal to 2.
Figure 5.2: The two figures on the left show Euclidean vertices, while the far right is not a Euclidean circuit.

The nerve $L$ of a Coxeter system $(W, S)$ has a natural piecewise spherical structure, and under this structure, if $s, t \in S$ are connected by an edge in $L$, then the edge has length $\pi - \pi/m_{st}$, where $(st)^{m_{st}} = 1$. Hence $L$ inherits the structure of a metric flag complex [6, Lemma 12.3.1], meaning that any collection of pairwise connected edges of $L$ spans a simplex if and only if there exists a spherical simplex with the corresponding edge lengths. It follows that if $v$ is a Euclidean 3- or 4-vertex, then $Lk(v)$ is a full subcomplex of $L$. Similarly, Euclidean circuits are full subcomplexes. Thus the corresponding subgroups are in fact special subgroups of $W$.

Suppose that $L$ is the labeled nerve of a Coxeter system, homeomorphic to $S^2$, and let $K_d$ have the prescribed dihedral angles $\pi/m_{st}$ as in 5.2. It follows that if $K_d$ satisfies the conditions of Theorem 5.3.1, then $\Sigma_L = \mathbb{H}^3$. The following theorem now becomes a special case of Theorem 5.3.1.

**Theorem 5.3.2.** Suppose that $L$ is the labeled nerve of a Coxeter system, homeomorphic to $S^2$, but not the boundary of a 3-simplex. Furthermore, suppose that

- $L$ has no Euclidean 3- or 4-circuits.
- $L$ has no Euclidean vertices.
- $L$ is not the right-angled suspension of a 3-gon.

Then $\Sigma_L = \mathbb{H}^3$.

**Proof.** We must show that $K_d$ satisfies the conditions of Theorem 5.3.1. First note that condition (i) is vacuous in our case. Condition (ii) on $K_d$ is equivalent to saying...
that \( L \) has no Euclidean 3–vertices and no Euclidean 3–circuits. Similarly, condition (iii) on \( K_d \) is equivalent to saying that \( L \) has no Euclidean 4–vertices or no Euclidean 4–circuits. Finally, condition (iv) on \( K_d \) is equivalent to saying that \( L \) is not the right-angled suspension of a 3–gon.

For convenience we restate the above theorem in terms of special subgroups.

**Theorem 5.3.3.** Suppose that the nerve \( L \) is a triangulation of \( S^2 \), but not the boundary of a 3–simplex, and let \((W,S)\) be the corresponding Coxeter system. Furthermore, suppose that

- For every \( T \subset S \), \( W_T \) is not a Euclidean reflection group.
- \( W \neq W_T \times D_\infty \), where \( T \subset S \) spans empty triangle in \( L \) and \( D_\infty \) is the infinite dihedral group.

Then \( \Sigma_L = \mathbb{H}^3 \).

### 5.4 Equidistant hypersurfaces

Suppose that the Coxeter group \( W \) has nerve \( L \) that is a triangulation of \( S^2 \) and that \( \Sigma_L = \mathbb{H}^3 \). Let \( D \) denote the Davis chamber (in \( \mathbb{H}^3 \)) and let \( W_M \) be a special subgroup of \( W \). We now consider the (possibly infinite) convex polytope \( W_M D \) in \( \mathbb{H}^3 \).

For \( t > 0 \), let \( S_t \) denote the \( t \)–distant surface from a component \( S \) of \( \partial W_M D \). Then \( S_t \) is a smooth surface (see [4, Proposition II.2.2.1]). In fact, \( S_t \) is a union of pieces of which there are three types: hyperbolic, Euclidean, and spherical, each of which are the equidistant pieces from faces, edges, and vertices of \( S \), respectively. The Euclidean pieces look like rectangles that are each adjacent to two hyperbolic pieces and two spherical pieces, and the spherical pieces are adjacent to Euclidean pieces.

As \( W_M D \) is convex, the nearest point projection \( p: \mathbb{H}^3 \cup \partial \mathbb{H}^3 \to W_M D \) is defined. If we fix \( t > r > 0 \), then \( p \) induces a map \( p_{tr}: S_t \to S_r \).
Lemma 5.4.1. The map $p_{tr} : S_t \rightarrow S_r$ induced by nearest point projection is quasiconformal.

Proof. It suffices to check what $p_{tr}$ does on each of the three types of pieces. First, note that a face of $S$ is simply the intersection of $\partial W M$ with a hyperbolic plane in $\mathbb{H}^3$. Thus $p_{tr}$ simply scales the corresponding hyperbolic pieces on $S_t$ and $S_r$ by a constant factor. Hence $p_{tr}$ is conformal there. Similarly, the map $p_{tr}$ is conformal on the spherical pieces.

Second, we consider the Euclidean piece in $S_t$ equidistant from an edge of $S$. A Euclidean piece looks like a rectangle adjacent to two hyperbolic pieces at two parallel edges (parallel in the intrinsic Euclidean geometry), and the the map induced by nearest point projection $S_t \rightarrow S$ scales by a factor of $1/\cosh(t)$ in the direction of those edges. The other two edges of the Euclidean piece are each adjacent to a spherical piece. An edge like this is the arc of a circle with radius $t$ centered at a vertex in $S$. Thus the edge has length $\theta \sinh(t)$, where $\theta$ is the dihedral angle at the corresponding edge of $S$. Hence the map $p_{tr}$ scales by a factor of $\cosh(r)/\cosh(t)$ in the direction of the edges adjacent to the hyperbolic pieces, and scales the edges adjacent to the spherical pieces by a factor of $\sinh(r)/\sinh(t)$. Therefore $p_{tr}$ is quasiconformal on the Euclidean pieces. \qed

5.5 The conjecture in dimension three

In this section, we prove the following theorem.

Theorem 5.5.1. Suppose that the nerve $L$ of a Coxeter group is a triangulation of $S^2$ not dual to a hyperbolic 3-simplex. Then

$$L_q^2 H_k(\Sigma_L) = 0 \text{ for } k > 1 \text{ and } q \leq 1.$$ 

Suppose that $M$ is a complete smooth Riemannian manifold. Given a a nonnegative measurable function $f : M \rightarrow [0, \infty)$, we define a new norm on the $C^\infty$ $k$–forms called the $L_f^2$ norm by

$$\|\omega\|_{L_f^2}^2 = \int_M \|\omega\|_p^2 f(p) dV,$$
where $\|\omega\|^2_p$ is the pointwise norm and $dV$ is the volume form of $M$. Let $L^2_fC^*(M)$ denote the weighted $L^2$ de Rham complex defined using the $L^2_f$ norm.

**Lemma 5.5.2.** Let $M$ and $N$ be smooth surfaces and suppose that $\phi : M \to N$ is a $K$–quasiconformal diffeomorphism. Let $g : N \to [0, \infty)$ be the function defined by $g(p) = f(\phi^{-1}(p))$. Then for every $\omega \in L^2_gC^1(N)$, we have that

$$\frac{1}{K} \|\omega\|^2_p \leq \|\phi^*(\omega)\|^2_f \leq K \|\omega\|^2_g.$$

**Proof.** The pointwise norm of a 1–form is $\|\omega\|_p = \sup_{x \in T_pM} \omega(x)$, where $T_pM$ is the tangent space of $M$ at $p$. Since $\phi$ is $K$–quasiconformal, its differential $d\phi$ maps the circle $\{ x \in T_pM \mid \|x\| = 1 \}$ to an ellipse with semi-axis $b(p) \leq a(p)$ satisfying $\frac{a(p)}{b(p)} \leq K$. Thus for any $\omega \in L^2_gC^1(N)$,

$$b(p)\|\omega\|_{\phi(p)} \leq \|\phi^*(\omega)\|_p \leq a(p)\|\omega\|_{\phi(p)}.$$

Now, let $dV_M$ and $dV_N$ be the respective volume forms of $M$ and $N$. We have that

$$(fdV_M)_p = \frac{(g(\phi)\phi^*(dV_N))_p}{a(p)b(p)},$$

so for $L^2_f$ norms we have

$$\|\phi^*(\omega)\|^2_f = \int_M \|\phi^*(\omega)\|^2_p f(p)dV_M$$

$$\leq \int_M \frac{a(p)}{b(p)}\|\omega\|^2_{\phi(p)} g(\phi(p)) \phi^*(dV_N)$$

$$\leq K \int_M \|\omega\|^2_{\phi(p)} g(\phi(p)) \phi^*(dV_N)$$

$$= K \int_N \|\omega\|^2_g x dV_N = K\|\omega\|^2_g$$

The remaining inequality follows similarly. 

Suppose that the nerve $L$ of $W$ is a triangulation of $S^2$ and that $\Sigma_L = \mathbb{H}^3$. Define $f$ to be the function $f(p) = q_w$, where $w \in W_L$ is a word of shortest length such that
$p \in wD$ (here $D$ is the Davis chamber). Let $L^2_q\mathcal{H}^*(\mathbb{H}^3)$ denote the weighted $L^2$ de Rham cohomology defined using this $f$.

Let $W_M$ be an infinite special subgroup of $W$ and let $S$ be one of the components of $\partial W_M D$. Put coordinates $(x, t)$ on $\mathbb{H}^3$ so that $t \in \mathbb{R}$ is the oriented distance from $p \in \mathbb{H}^3$ to the closest point $x \in S$. Fix $r > 0$, and for $t \geq r$ let $S_t$ denote the hypersurface consisting of points of (oriented) distance $t$ from $S$. Let $p_{tr} : S_t \to S_r$ be the map induced by nearest point projection, and let $\phi_{tr}$ denote the inverse of $p_{tr}$. By Lemma 5.4.1, $p_{tr}$ is $K(t)$–quasiconformal, with $K(t) = \frac{\tanh(t)}{\tanh(r)}$, and hence so is its inverse $\phi_{tr} : S_r \to S_t$. Let $i_r : S_r \to \mathbb{H}^3$ and $i_t : S_t \to \mathbb{H}^3$ be the inclusions. Then $i_r$ and $i_t \circ \phi_{tr}$ are properly homotopic.

We now adapt the argument after [8, Theorem 16.10] to prove the following lemma.

**Lemma 5.5.3.** If $q \geq 1$, then the map $i_r^* : L^2_q\mathcal{H}^1(\mathbb{H}^3) \to L^2_q\mathcal{H}^1(S_r)$ induced by the inclusion $i_r$ is the zero map.

**Proof.** Set $g(x, y) = f(x, 0)$, so $f(x, y) \geq g(x, y)$, and let $\omega$ be a closed $L^2_g$–form on $\mathbb{H}^3$. We now show that the restriction $i_r^*(\omega)$ to $S_r$ represents the zero class in its reduced $L^2_g$-cohomology. For the remainder of the proof, we will use the notation $\|\alpha\|_g$ and $\|\alpha\|_x$ to denote the respective $L^2_g$ norm and pointwise norm of the harmonic representative of the cohomology class $[\alpha]$.

Suppose for a contradiction that $[i_r^*(\omega)] \neq 0$. Then $\|i_r^*(\omega)\|_g \geq \|[i_r^*(\omega)]\|_g > 0$. By Lemma 5.5.2, it follows that $\|\phi_{tr}^*(i_t^*(\omega))\|_g^2 \leq K(t)\|i_t^*(\omega)\|_g^2$, and since $i_r$ and $i_t \circ \phi_{tr}$ are properly homotopic, $[i_t^*(\omega)] = [\phi_{tr}^*(i_t^*(\omega))]$. Therefore

$$K(t)\|i_t^*(\omega)\|_g^2 \geq \|[i_t^*(\omega)]\|_g^2 > 0.$$  

Now, $i_t^*(\omega)$ is just a restriction of $\omega$, so we have the pointwise inequality $\|\omega\|_x \geq \|i_t^*(\omega)\|_x$. Using Fubini’s Theorem, we compute

$$\|\omega\|_g^2 = \int_{\mathbb{H}^3} \|\omega\|^2 g(x, y) dV \geq \int_r^\infty \int_{S_t} \|\omega\|^2 g(x, y) dAdt \geq \int_r^\infty \int_{S_t} \|i_t^*(\omega)\|^2 g(x, y) dAdt = \int_r^\infty \|i_t^*(\omega)\|_g^2 dt \geq \int_r^\infty \frac{\tanh(t)}{\tanh(r)} \|[i_t^*(\omega)]\|_g^2 dt = \infty.$$
Since \( \|\omega\|_f \geq \|\omega\|_g \), this contradicts the assumption that the \( L^2_f \) norm of \( \omega \) is finite. ■

Suppose that \( L \) is the nerve of a Coxeter group \( W_L \) and that \( A \) is a full subcomplex of \( L \). For the proofs that follow, note that \( \dim_q L^2_q H_k(W_L, \Sigma_A) = L^2_q b_k(\Sigma_A) \) (see [6, pg. 352 (vi)]).

**Lemma 5.5.4.** Suppose that the nerve \( L \) is a triangulation of \( S^2 \) and that there exists a full subcomplex 1–sphere \( M \) of \( L \) that separates \( L \) into two full 2–disks \( L_1 \) and \( L_2 \) with boundary \( M \). Furthermore, suppose that one of the following holds:

1. \( \Sigma_M = \mathbb{R}^2 \).
2. \( \Sigma_L = \mathbb{H}^3 \).

Then

\[
L^2_q H_k(\Sigma_L) \text{ for } k \geq 2 \text{ and } q \leq 1.
\]

**Proof.** Since \( \Sigma_L \) is a 3–manifold, it follows that \( L^2_q b_3(\Sigma_L) = 0 \) [6, Proposition 20.4.1]. Hence we must show that \( L^2_q b_2(\Sigma_L) = 0 \). Consider the following Mayer–Vietoris sequence applied to \( L = L_1 \cup_M L_2 \):

\[
\cdots \to L^2_q H_2(W_L, \Sigma_{L_1}) \oplus L^2_q H_2(W_L, \Sigma_{L_2}) \to L^2_q H_2(\Sigma_L) \to L^2_q H_1(W_L, \Sigma_M) \to \cdots
\]

By Theorem 5.1.1, we have that \( L^2_q H_2(W_L, \Sigma_{L_1}) = L^2_q H_2(W_L, \Sigma_{L_2}) = 0 \). If (i) holds, then Theorem 3.3.4 implies that \( L^2_q H_1(\Sigma_M) = 0 \), and we are done. If (ii) holds, we argue that the connecting homomorphism \( \partial_\ast : L^2_q H_2(\Sigma_L) \to L^2_q H_1(W_L, \Sigma_M) \) is the zero map. By [8, Lemma 16.2], we reduce the proof to showing that the map induced by inclusion \( i_\ast : L^2_{q-1} H_1(W_L, \Sigma_M) \to L^2_{q-1} H_1(\Sigma_L) \) is the zero map, and since \( W_L, \Sigma_M \) is a disjoint union of copies of \( \Sigma_M \), it is enough to show that the restriction of \( i_\ast \) to one summand \( L^2_{q-1} H_1(\Sigma_M) \) is zero.

Consider the infinite convex polytope \( W_M D \), where \( D \) is the Davis chamber for \( W \). We have that \( W_M \) acts properly and cocompactly on \( W_M D \) by isometries. In particular, if \( S \) is one of the components of \( \partial W_M D \), then \( W_M \) acts properly and
cocompactly on $S$, and therefore $L^2_{q^{-1}}H^* (\Sigma_M) \cong L^2_{q^{-1}}H^* (S)$. Hence we are done if we show that map $i^* : L^2_{q^{-1}}H^1 (\mathbb{H}^3) \rightarrow L^2_{q^{-1}}H^1 (S)$ induced by the inclusion $i : S \rightarrow \mathbb{H}^3$ is the zero map.

Fix $r > 0$, and let $S_r$ be the $r$–distant surface from $S$. $S_r$ and $S$ are properly homotopy equivalent, and this equivalence induces a weak isomorphism between $L^2_{q^{-1}}H^* (S)$ and $L^2_{q^{-1}}H^* (S_r)$. Thus we have reduced the proof to showing that the map $i_r^* : L^2_{q^{-1}}H^1 (\mathbb{H}^3) \rightarrow L^2_{q^{-1}}H^1 (S_r)$ induced by the inclusion $i_r : S_r \rightarrow \mathbb{H}^3$ is the zero map, and therefore we are done by Lemma 5.5.3.

Remark 5.5.5. In [8, Section 16] $W$ is strictly assumed to be right-angled, but the proof of [8, Lemma 16.2] does not use this, as it only uses properties of weighted $L^2$–(co)homology.

**Proof of Theorem 5.5.1.** We first suppose that $\Sigma_L = \mathbb{H}^3$. We need to find a full subcomplex $M$ of $L$ satisfying the hypothesis of Lemma 5.5.4. First we suppose that $L$ is a flag complex. Let $v$ be a vertex of $L$ and set $M = Lk (v)$. Since $L$ is flag, $M$ is a full subcomplex of $L$, and since $L$ is a triangulation of the 2–sphere, it follows that $M$ is a 1–sphere, and we are done. Now suppose that $L$ is not flag. Since $L$ is not the boundary of a 3–simplex, there exists an empty 2–simplex in $L$. Let $M$ denote this empty 2–simplex. Then $M$ separates $L$ into two full 2–disks, both with boundary $M$, and we are done. We now suppose that $\Sigma_L \neq \mathbb{H}^3$ and use Theorem 5.3.3 to perform a case-by-case analysis.

**Case I:** $W$ contains a Euclidean special subgroup $W_T$. Let $M$ be the full subcomplex of $L$ corresponding to $W_T$. Then $M$ separates $L$ into two full 2–disks both with boundary $M$, and hence Lemma 5.5.4 (i) implies the assertion.

**Case II:** $W = W_T \times D_\infty$, where $T \subset S$ spans empty triangle in $L$. Either $\Sigma_L = \mathbb{R}^3$ or $\Sigma_L = \mathbb{H}^2 \times \mathbb{R}$. In both cases we are done by the weighted Künneth formula.

**Case III:** $L$ is the boundary of a 3–simplex. By assumption, $L$ is not dual to a hyperbolic simplex, so $\Sigma_L = \mathbb{R}^3$. Therefore we are done by [8, Corollary 14.5].
5.6 The conjecture in dimension four

In dimension four, we prove the following case of the Weighted Singer Conjecture:

**Theorem 5.6.1.** Suppose that the nerve $L$ of a Coxeter group is a triangulation of $S^3$. Furthermore, suppose that there exists a vertex of $L$ such that its link is a full subcomplex of $L$ and not dual to a hyperbolic 3-simplex. Then

$$L_q^2 H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } q \leq 1.$$  

**Proof.** In this case, $\Sigma_L$ is a 4-manifold, and hence $L_q^2 b_4(\Sigma_L) = 0$ [6, Proposition 20.4.1]. It remains to show that $L_q^2 b_3(\Sigma_L) = 0$. Suppose that the nerve $L$ is a triangulation of $S^3$ and let $s \in L$ be a vertex. We make the following observations:

- The nerve $L_{S-s}$ of the Coxeter system $(W_{S-s}, S-s)$ is a 3-disk.
- The nerve $St(s)$ of the Coxeter group $W_{St(s)}$ is a 3-disk.
- The nerve $Lk(s)$ of the Coxeter group $W_{Lk(s)}$ is a 2-sphere.

This is because the subcomplexes $St(s)$, $Lk(s)$, and $L_{S-s}$ of $L$ correspond to the closed star of the vertex $s$, link of the vertex $s$, and complement of the open star of $s$, respectively, which are all by assumption full subcomplexes of $L$.

Consider the following Mayer–Vietoris sequence:

$$\cdots \to L_q^2 H_3(W_L \Sigma_{L_{S-s}}) \oplus L_q^2 H_3(W_L \Sigma_{St(s)}) \to L_q^2 H_3(\Sigma_L) \to L_q^2 H_2(W_L \Sigma_{Lk(s)}) \to \cdots$$

By Theorem 5.1.1, $L_q^2 b_3(\Sigma_{St(s)}) = 0$ and $L_q^2 b_3(\Sigma_{L_{S-s}}) = 0$, and by Theorem 5.5.1, $L_q^2 b_2(\Sigma_{Lk(s)}) = 0$. Therefore by the above sequence, $L_q^2 b_3(\Sigma_L) = 0$.  

We obtain the following corollary.

**Corollary 5.6.2.** Suppose that the nerve $L$ of a Coxeter group is a flag triangulation of $S^3$. Then

$$L_q^2 H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } q \leq 1.$$
Proof. Since $L$ is flag, it follows that the link of every vertex is a full subcomplex of $L$. Furthermore, the link of every vertex is not the boundary of a 3–simplex (and in particular, not dual to a 3–simplex). Theorem 5.6.1 now completes the proof. ■

5.7 The case where $L$ is a 3–manifold

In this section, we prove the following generalization of Corollary 5.6.2.

Theorem 5.7.1. Suppose that $L$ is a flag triangulation of a 3–manifold. Then

$$L_q^2 H_k(\Sigma_L) = 0 \text{ for } k > 2 \text{ and } q \leq 1.$$ 

Note that, in this case, $\Sigma_L$ is a 4–pseudomanifold (i.e. every 3–cell of $\Sigma_L$ is contained in precisely two 4–cells). We will prove the theorem using ruins.

Lemma 5.7.2. Suppose that $L$ is a flag triangulation of a 3–manifold. Then for every $t \in L$, $L_q^2 H_*(\Omega(S,t), \partial \Omega(S,t)) = 0$ for $* > 2$ and $q \leq 1$.

Proof. First, for $t \in L$, recall that the $(S,t)$–ruin has the property that

$$\Omega(S,t) = \Omega(St(t), t),$$

where $St(t) = \{ s \in S \mid m_{st} < \infty \}$. Recall that $Lk(t) = St(t) - t$, and so we have the following weak exact sequence (see sequence (4.1)):

$$\cdots \rightarrow L_q^2 H_*(\Sigma(Lk(t))) \rightarrow L_q^2 H_*(\Sigma(St(t))) \rightarrow L_q^2 H_*(\Omega(S,t), \partial \Omega(S,t)) \rightarrow \cdots$$

Note that

$$L_q^2 b_*(\Sigma(St(t))) = L_q^2 b_*(\Sigma_{St(t)}) \text{ and } L_q^2 b_*(\Sigma(Lk(t))) = L_q^2 b_*(\Sigma_{Lk(t)}),$$

where $\Sigma_{St(t)}$ and $\Sigma_{Lk(t)}$ are the Davis complexes corresponding to the subgroups $W_{St(t)}$ and $W_{Lk(t)}$, respectively. Since $L$ is flag, the respective nerves of the groups $W_{St(t)}$ and $W_{Lk(t)}$ are a 3–disk and a 2–sphere. Furthermore, the nerve of $W_{Lk(t)}$ is
not the boundary of a 3-simplex (again, $L$ is flag). By Theorem 5.1.1, $L^2_{q}b_k(\Sigma_{St(t)}) = 0$ for $k > 2$, and by Theorem 5.5.1, $L^2_{q}b_k(\Sigma_{Lk(t)}) = 0$ for $k > 1$. Therefore weak exactness of the sequence implies that $L^2_{q}H_*(\Omega(S,t),\partial\Omega(S,t)) = 0$ for $* > 2$. 

We now adapt the argument of [18] to complete the proof of Theorem 5.7.1. The main point is that we are able to prove the following lemma for $q \leq 1$.

**Lemma 5.7.3** (Compare [18, Lemma 4.1]). For every $T \in \mathcal{S}^{(2)}$ and $U \subset S$ with $T \subset U$, we have $L^2_{q}H_4(\Omega(U,T),\partial\Omega(U,T)) = 0$ for $q \leq 1$.

**Proof.** Once we establish the lemma for $q = 1$, we apply the argument in Lemma 3.3.8 to obtain the result for $q \leq 1$ (see Remark 3.3.9).

Assume that $\Omega(U,T)$ contains 4-dimensional cells, otherwise we are done. Then every codimension-one face of a 4-cell in $\Omega(U,T)$ is either free (not the face of another 4-cell) or contained in precisely one other 4-cell ($\Sigma_{cc}$ is a 4-pseudomanifold).

If every codimension-one face of a 4-cell is free, then this cell has faces not contained in $\partial\Omega(U,T)$. Thus a relative cycle cannot be supported on this cell.

So, we assume that cells of type $T' \in \mathcal{S}(U)_{s,T}$ have a codimension-one face of type $R$ that is not free. This face must be contained in another 4-cell of type $T'' \in \mathcal{S}(U)_{s,T}$. Thus $T' = R \cup \{t\}$ and $T'' = R \cup \{s\}$ for some $s,t \in S$. Since $L$ is flag and 3-dimensional, $m_{st} = \infty$. Hence we obtain a sequence of adjacent 4-cells $W_{T'}, W_{T''}, sW_{T'}, stW_{T''}, stsW_{T'}, \ldots$. Furthermore, a relative 4-cycle must be constant on adjacent cells of type $T'$ and $T''$, and since we have an infinite sequence of such adjacent cells, this constant must be zero. 

The rest of the argument now follows [18] line by line. We repeat it for the sake of completeness.

**Lemma 5.7.4** ([18, Proposition 4.2]). For every $t \in T$ and $U \subset S$ with $t \in U$, we have $L^2_{q}H_*(\Omega(U,t),\partial\Omega(U,t)) = 0$ for $* > 2$ and $q \leq 1$.

**Proof.** The proof is by induction on $\text{Card}(S-U)$, Lemma 5.7.2 serving as the base case. Let $s \in S$ and set $V = U \cup \{s\}$. If $m_{st} = \infty$, then $\Omega(U,t) = \Omega(V,t)$ and we
are done by induction. Otherwise, consider the weak exact sequence (see sequence (4.1):)

\[ \cdots \rightarrow L_2^q H_*(\Omega(U, t), \partial) \rightarrow L_2^q H_*(\Omega(V, t), \partial) \rightarrow L_2^q H_*(\Omega(V, \{s, t\}), \partial) \rightarrow \cdots \]

By Lemma 5.7.3, \( L_2^q H_4(\Omega(V, \{s, t\}), \partial) = 0 \) and by induction, \( L_2^q H_*(\Omega(V, t), \partial) = 0 \) for \( * > 2 \). Therefore \( L_2^q H_*(\Omega(U, t), \partial) = 0 \) for \( * > 2 \).

**Proof of Theorem 5.7.1.** For every \( U \subset S \) and \( t \in U \), we have the following weak exact sequence:

\[ \cdots \rightarrow L_2^q H_*(\Sigma(U - t)) \rightarrow L_2^q H_*(\Sigma(U)) \rightarrow L_2^q H_*(\Omega(U, t), \partial) \rightarrow \cdots \]

By Lemma 5.7.4, \( L_2^q H_*(\Omega(U, t), \partial) = 0 \) for \( * > 2 \), and hence by weak exactness,

\( L_2^q H_*(\Sigma(U - t)) \cong L_2^q H_*(\Sigma(U)) \) for \( * > 2 \).

It follows that \( L_2^q H_*(\Sigma(S)) \cong L_2^q H_*(\Sigma(\varnothing)) \) for \( * > 2 \), and hence the theorem. ■
Chapter 6

The Fattened Davis Complex

We will now construct a complex which is a “fattened” version of the Davis complex. This new thickened complex will be a homology manifold with boundary possessing the Davis complex as a $W$–equivariant retract. For the remainder of this thesis we suppose that $W$ is an infinite Coxeter group.

6.1 Construction

Given a Coxeter system $(W,S)$, we find a compact $P$ with mirror structure $(P_s)_{s \in S}$ as follows. Let $P^*$ be a cell complex with vertex set $S$ that is a $GHS^{n-1}$, with $n - 1 > \dim L$, such that the nerve $L$ is a subcomplex of $P^*$. Take $P$ to be the $(P^*,S)$–chamber.

Denote by $\mathcal{P}$ the collection of proper nonempty subsets $T$ of $S$ with $P_T \neq \emptyset$. We denote by $\mathcal{N}_\mathcal{P}$ the subcollection of $\mathcal{P}$ corresponding to non-spherical subsets. For $T \in \mathcal{P}$, we denote a neighborhood of the face $P_T$ by $N(P_T)$ and the corresponding closed neighborhood by $\bar{N}(P_T)$.

We begin by building a regular neighborhood of $\partial P$ in $P$. Start by choosing neighborhoods of codimension–$n$ faces so that for any two codimension–$n$ faces $P_U$ and $P_V$ we have $\bar{N}(P_U) \cap \bar{N}(P_V) = \emptyset$. Then we choose neighborhoods of codimension–$(n - 1)$ faces so that for any two codimension–$(n - 1)$ faces $P_U$ and $P_V$ we have:
\[ \bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V). \]

If \( U \cup V \notin \mathcal{P} \), then we take \( N(P_U \cap P_V) = \emptyset \). We proceed inductively, employing condition (6.1) at each step until we obtain the collection \( \{ N(P_T) \}_{T \in \mathcal{P}} \). This collection gives us a regular neighborhood of \( \partial P \).

Finally, we realize the neighborhoods \( \{ N(P_T) \}_{T \in \mathcal{P}} \) in the above construction as \( \{ N_T \times P_T \}_{T \in \mathcal{P}} \), where \( N_T \) is a neighborhood of the cone point in \( \text{Cone}(\sigma_T) \) and \( \sigma_T \) is the geometric cell in \( P^* \) spanned by \( T \) (note that we can always do this, see the discussion in Section 2.7.1).

We now define
\[ K^f := P - \bigcup_{T \in \mathcal{N}_P} N(P_T). \]
We call \( K^f \) the fattened Davis chamber.

Note that the mirror structure \( (P_s)_{s \in S} \) on \( P \) induces a mirror structure \( (K^f_s)_{s \in S} \) on \( K^f \). Define \( \Phi_L := \mathcal{U}(W, K^f) \). We call \( \Phi_L \) the fattened Davis complex.

Given a \( T \in \mathcal{N}_P \), we denote by \( K^f(T) \) the fattened Davis chamber corresponding to \( \sigma_T \) and Coxeter system \( (W_T, T) \) (recall that the geometric cell \( \sigma_T \) has a natural \( W_T \) mirror structure).

**Remark 6.1.1.** For any Coxeter system \( (W, S) \), one can always find a \( P^* \) for the above construction: simply let \( P^* \) be the boundary of the standard \( (|S| - 1) \)-dimensional simplex \( \Delta^{|S|-1} \). Then \( P \) is the barycentric subdivision of \( \Delta^{|S|-1} \), and the Davis chamber \( K \) can then be viewed as a subcomplex of the barycentric subdivision of \( P \) spanned by the barycenters of spherical faces. One can see this using the language of posets. Note that \( K \) is the geometric realization of the poset \( S \) and \( P \) is the geometric realization of the poset of proper subsets of \( S \). The natural inclusion of posets now induces the desired inclusion of \( K \) into \( P \). The mirror structure \( (K_s)_{s \in S} \) on \( K \) is now induced by the mirror structure \( (P_s)_{s \in S} \) on \( P \). In this case \( \mathcal{U}(W, P) \) is the traditional Coxeter complex, and we are essentially viewing \( \Sigma_L \) as a subcomplex of the barycentric subdivision of the Coxeter complex.
6.2 Properties of $\Phi_L$

$W$ is assumed to be infinite, so via the choice of $P$ for construction, the Davis chamber is the subcomplex of $P$ spanned by vertices of $P$ corresponding to spherical faces. Hence we have the following inclusions: $K \subset K^f \subset P$ (See Figure 6.1).

Figure 6.1: $K \subset K^f \subset P$ when $W = D_\infty \times D_\infty$ and $P = \Delta^3$

Note that there is a face preserving deformation retraction of $K^f$ onto $K$, thus we have the following:

**Proposition 6.2.1.** $\Phi_L$ $W$–equivariantly deformation retracts onto $\Sigma_L$.

**Proposition 6.2.2.** $\Phi_L$ is a locally compact contractible homology $n$–manifold with boundary $\partial \Phi_L$.

**Proof.** Since $\Sigma_L$ is contractible, it follows from Proposition 6.2.1 that $\Phi_L$ is contractible. Moreover, $K^f$ is compact since it is closed in $P$ ($P$ is compact), so $\Phi_L$ is locally compact.

As before, give $K^f$ the mirror structure $(K^f_e)_{e \in S}$ induced from $P$, and declare $K^f_e = \partial K^f - \bigcup_{T \in S_{\neq}} (K^f_T - \partial K^f_T)$, where $e$ is the identity element of $W$. According to Proposition 2.5.3, it remains to show that $K^f$ is a partially $S$–mirrored homology manifold with corners. Let $S' = S \cup \{e\}$ and note that by construction $K^f_T = \emptyset$ if and only if $T$ is not spherical. So, we are done if we show that for every spherical $T \subset S'$, $K^f_T$ has dimension $n - |T|$.

If $e \notin T$, then we are done since $(P_T, \partial P_T)$ is a $GHD_{n-|T|}$. This is because $P$ is by definition the $(P^*, S)$–chamber and the nerve $L$ was assumed to be a subcomplex
of $P^*$. Hence, since $T$ is spherical, $\sigma_T$, the geometric cell in $P^*$ corresponding to $T$, is a simplex of dimension $|T| - 1$. Therefore the dimension of $P_T$ is equal to $n - \dim \sigma_T - 1 = n - |T|$.

If $e \in T$, then $U = T - \{e\}$ is spherical, and by the above discussion $K^f_U$ has dimension $n - |U| = n - |T| + 1$. Then $K^f_T = K^f_U \cap K^f_e = \partial K^f_U$ has dimension $n - |T|$. 

**Remark 6.2.3.** If $P = \Delta^{[S]^{-1}}$, then the Coxeter complex $U(W, P)$ is a PL-manifold away from faces with infinite stabilizers. This is because the links of faces corresponding to spherical subsets $T$ are homeomorphic to the Coxeter complex of the corresponding group $W_T$. Since $W_T$ is finite, this Coxeter complex is homeomorphic to a sphere of appropriate dimension. Since we obtain $\Phi_L$ by removing neighborhoods of non-spherical faces (faces with infinite stabilizers), it follows that $\Phi_L$ is a PL-manifold with boundary.

### 6.3 The structure of $\partial \Phi_L$

The main goal of this section is to understand the structure of $\partial \Phi_L$. The first proposition will tell us that $\partial K^f$ can be broken up into pieces, each of which has a nice product structure. This decomposition of $\partial K^f$ then leads us to a cover of $\partial \Phi_L$ which will be used to study the algebraic topology of $\partial \Phi_L$.

For $T \in \mathcal{N}_P$, define

$$C_T = \partial N(P_T) - \bigcup_{U \in \mathcal{N}_P} N(P_U),$$

$$\Lambda_T = P_T - \bigcup_{U \in \mathcal{N}_P \cap U} N(P_U).$$

**Proposition 6.3.1.** (i) Suppose that $U, V \in \mathcal{N}_P$. Then $C_U \cap C_V \neq \emptyset$ if and only if $U \subseteq V$ or $V \subseteq U$.

(ii) If $T \in \mathcal{N}_P$ then

$$C_T \approx K^f(T) \times \Lambda_T.$$ 

(iii) Suppose that $T_1, T_2 \in \mathcal{N}_P$ with $T_1 \subseteq T_2$. Then

$$C_{T_1} \cap C_{T_2} \approx K^f(T_1) \times \Lambda_{T_2}.$$
Proof. For (i), one implication is obvious. If \( U \subset V \), then \( P_V \) is a face of \( P_U \). Thus \( C_U \cap C_V \neq \emptyset \). For the reverse implication, suppose that \( U \notin V \) and \( V \notin U \). By construction and condition (6.1), either \( \bar{N}(P_U) \cap \bar{N}(P_V) = \emptyset \) or \( \bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V) \). The former case immediately implies that \( C_U \cap C_V = \emptyset \), and the latter case implies that the intersection \( \partial N(P_U) \cap \partial N(P_V) \) is removed at some point in the construction of the fattened Davis chamber, hence \( C_U \cap C_V = \emptyset \).

For (ii), recall that we have realized the collection \( \{N(P_T)\}_{T \in N_P} \) as neighborhoods \( \{N_T \times P_T\}_{T \in N_P} \), where \( N_T \) is a neighborhood of the cone point in \( \text{Cone}(\sigma_T) \).

Now, for each \( U \subset T \), let \( \alpha_U \) denote the face in \( \sigma_T \) corresponding to \( P_U \). More precisely, \( \sigma_T \) has a \( W_T \) mirror structure, and \( \alpha_U \) is the intersection of mirrors corresponding to \( U \subset T \). We can express the neighborhoods in the construction of \( K_f(T) \) as neighborhoods \( \{\alpha_U \times N'_U\}_{U \in N_P} \), where \( N'_U \) is a neighborhood of the cone point in \( \text{Cone}(\text{Lk}(\alpha_U, \sigma_T)) \). Here \( \text{Lk}(\alpha_U, \sigma_T) \) denotes the link of the face \( \alpha_U \) in \( \sigma_T \). In particular,

\[
K_f(T) = \sigma_T - \bigcup_{U \in N_P} \alpha_U \times N'_U.
\]

Now, we have that \( \text{Lk}(\alpha_U, \sigma_T) \approx \sigma_U \), so \( N'_U \approx N_U \). Hence

\[
K_f(T) \approx \sigma_T - \bigcup_{U \in N_P} P_U \times N_U.
\]

Moreover, we can write \( \Lambda_T \) and \( C_T \) as

\[
\Lambda_T = P_T - \bigcup_{T \in N_P} P_U \times N_U,
\]

\[
C_T = (\sigma_T \times P_T) - \bigcup_{U \in N_P} P_U \times N_U.
\]

We now show that \( C_T \approx K_f(T) \times \Lambda_T \). Note that \( K_f(T) \times \Lambda_T = (K_f(T) \times P_T) \cap (\sigma_T \times \Lambda_T) \), so we begin unwinding definitions. We first observe that
\[ K^f(T) \times P_T \cong \left( \sigma_T - \bigcup_{U \in \mathcal{N}_P} P_U \times N_U \right) \times P_T \cong (\sigma_T \times P_T) - \bigcup_{U \in \mathcal{N}_P} P_U \times N_U. \]

This is because \( P_T \) is a face of each of the \( P_U \)'s. Similarly, we have

\[ \sigma_T \times \Lambda_T = \sigma_T \times \left( P_T - \bigcup_{U \in \mathcal{N}_P} P_U \times N_U \right) \cong (\sigma_T \times P_T) - \bigcup_{U \in \mathcal{N}_P} P_U \times N_U. \]

This follows from the fact that \( P_U \)'s are faces of \( P_T \). Thus we have shown that \( K^f(T) \times \Lambda_T = (K^f(T) \times P_T) \cap (\sigma_T \times \Lambda_T) \cong C_T \), therefore proving (ii).

We now prove (iii). By (ii),

\[ C_{T_1} \cap C_{T_2} \cong (K^f(T_1) \cap K^f(T_2)) \times (\Lambda_{T_1} \cap \Lambda_{T_2}). \]

It now simply remains to unwind the definitions. Since \( T_1 \subset T_2 \), it follows that \( P_{T_2} \) is a face of \( P_{T_1} \). In particular, \( \sigma_{T_1} \cap \sigma_{T_2} = \sigma_{T_1} \) and hence

\[
K^f(T_1) \cap K^f(T_2) \cong \sigma_{T_1} \cap \sigma_{T_2} - \bigcup_{U \in \mathcal{N}_P} N(P_U) \cup N(P_V)
\]

\[
\cong \sigma_{T_1} - \bigcup_{U \in \mathcal{N}_P \atop V \subset T_2} N(P_U)
\]

\[
\cong K^f(T_1)
\]

A similar computation shows that \( \Lambda_{T_1} \cap \Lambda_{T_2} \cong \Lambda_{T_2} \), thus completing the proof of the proposition.

\[ \partial \Phi_L = \bigcup_j \bigcup_{T \in \mathcal{N}_P^{(j)}} \mathcal{U}(W, C_T), \]

where \( \mathcal{N}_P^{(j)} = \{ T \in \mathcal{N}_P \mid \text{Card}(T) = j \} \).

\textbf{Proof.} The fact that one can decompose \( \partial \Phi_L \) in this way is clear by construction, and the second union is in fact a disjoint union by Proposition 6.3.1 (i).
6.4 Algebraic topology of $\Phi_L$ and $\partial \Phi_L$

We now turn our attention to studying the algebraic topology of $\Phi_L$ and $\partial \Phi_L$. We first begin with a corollary of Proposition 6.2.1.

Corollary 6.4.1.

$$L^2_q H_*(\Phi_L) \cong L^2_q H_*(\Sigma_L).$$

Not only does $\Phi_L$ have the same weighted $L^2$–(co)homology as $\Sigma_L$, but by Proposition 6.2.2, $\Phi_L$ is a locally compact homology manifold with boundary. Thus we have weighted Poincaré duality for $\Phi_L$ at our disposal. With this in mind, we prove the following lemma.

Lemma 6.4.2. Suppose that $(W,S)$ is a Coxeter system with $\text{vcd} W = m$ and that $\Phi_L$ is a homology $n$–manifold with boundary with $L^2_q b_1(\partial \Phi_L) = 0$.

(i) If $n - m = 1$ and $L^2_{q-1} b_m(\Phi_L) = 0$ then $L^2_q b_1(\Sigma_L) = 0$.

(ii) If $n - m \geq 2$ then $L^2_q b_1(\Sigma_L) = 0$.

Proof. Consider the long exact sequence for the pair $(\Phi_L, \partial \Phi_L)$:

$$\cdots \to L^2_q H_1(\partial \Phi_L) \to L^2_q H_1(\Phi_L) \to L^2_q H_1(\Phi_L, \partial \Phi_L) \to \cdots$$

By weighted Poincaré duality

$$L^2_q H_1(\Phi_L, \partial \Phi_L) \cong L^2_q H_{n-1}(\Phi_L).$$

Now, by assumption $L^2_q H_1(\partial \Phi_L) = 0$, so by weak exactness we must show that $L^2_{q-1} H_{n-1}(\Phi_L) = 0$. We will then be done by Corollary 6.4.1, which says that $L^2_q H_1(\Sigma_L) = L^2_q H_1(\Phi_L) = 0$.

For (i), we have that $L^2_{q-1} b_m(\Phi_L) = 0$. Since $n - m = 1$, we have that $m = n - 1$, so it follows that $L^2_{q-1} H_{n-1}(\Phi_L) = 0$. For (ii), we have that $n - m \geq 2$, so $n - 1 \geq m + 1$. Since $\text{vcd} W = m$, Corollary 3.3.7 implies that

$$L^2_{q-1} H_{n-1}(\Sigma_L) = L^2_{q-1} H_{n-1}(\Phi_L) = 0.$$
We devote the remainder of the section to studying the algebraic topology of $\partial \Phi_L$. The following is a corollary of Proposition 6.3.1.

**Corollary 6.4.3.** (i) If $T \in \mathcal{N}_P$, then for every $k \geq 0$

$$L^2_q b_k(U(W, C_T)) = L^2_q b_k(\Phi_{LT}) = L^2_q b_k(\Sigma_{LT}),$$

where $L_T$ is the subcomplex of $L$ corresponding to the subgroup $W_T$.

(ii) Suppose that $T_1, T_2 \in \mathcal{N}_P$ with $T_1 \subset T_2$. Then for every $k \geq 0$

$$L^2_q b_k(U(W, C_{T_1}) \cap U(W, C_{T_2})) = L^2_q b_k(\Phi_{LT_1}) = L^2_q b_k(\Sigma_{LT_1}),$$

where $L_{T_1}$ is the subcomplex of $L$ corresponding to the subgroup $W_{T_1}$.

**Remark 6.4.4.** The $L^2_q$–Betti numbers on the center and the right of the equations in (i) and (ii) are computed with respect to the special subgroups $W_T$ (respectively $W_{T_1}$) of $W$, while the ones on the far left side of the equations are computed with respect to $W$.

**Proof.** We prove only (i) as the proof of (ii) is similar. Proposition 6.3.1 implies that $C_T \approx K^f(T) \times \Lambda_T$ as mirrored spaces, where $\Lambda_T$ is contractible and has no mirror structure. Therefore $U(W, C_T)$ is $W$–equivariantly homotopy equivalent to $U(W, K^f(T))$. Now, $L^2_q H_* \left( U(W, K^f(T))\right)$ is just the completion of

$$L^2_q(W) \otimes \mathbb{R}q(W_T) L^2_q H_* \left( U(W_T, K^f(T))\right),$$

so for every $k \geq 0$,

$$L^2_q b_k \left( U(W, K^f(T)) \right) = L^2_q b_k \left( U(W_T, K^f(T)) \right) = L^2_q b_k(\Phi_{LT}).$$

Consider the cover $\mathcal{V} = \{U(W, C_T)\}_{T \in \mathcal{N}_P}$ of $\partial \Phi_L$ in Proposition 6.3.2. The cover $\mathcal{V}$ will have intersections of variable depth, so we obtain a spectral sequence following [3, Ch. VII, §3,4]:
Proposition 6.4.5. There is a Mayer–Vietoris type spectral sequence converging to $H^W_s(\partial \Phi_L, \mathcal{N}_q(W))$ with $E_1$-term:

$$E_1^{i,j} = \bigoplus_{\sigma \in \text{Flag}(\mathcal{N}_P)} H^W_j(U(W, C_{\text{min} \sigma}), \mathcal{N}_q(W)).$$

Proof. Let $N(V)$ denote the nerve of the cover $V$. It is the abstract simplicial complex whose vertex set is $\mathcal{N}_P$ and whose simplices are the non-empty subsets $\sigma \subset \mathcal{N}_P$ such that the intersection $V_\sigma = \bigcap_{T \in \sigma} U(W, C_T)$ is non-empty. Following [3, Ch. VII, §3,4], there is a Mayer–Vietoris type spectral sequence converging to $H^W_s(\partial \Phi_L, \mathcal{N}_q(W))$ with $E_1$-term:

$$E_1^{i,j} = \bigoplus_{\sigma \in N(V)} H^W_j(V_\sigma, \mathcal{N}_q(W)).$$

We have that $V_\sigma \neq \emptyset$ if and only if $\bigcap_{T \in \sigma} C_T \neq \emptyset$, and applying Proposition 6.3.1 inductively, this happens if and only if the vertices of $\sigma$ form a chain $T_{i_1} \subset T_{i_2} \subset \cdots \subset T_{i_k}$. This observation shows that $N(V) = \text{Flag}(\mathcal{N}_P)$. Now, applying Proposition 6.3.1 inductively, it follows that $V_\sigma \approx U(W, C_{T_{i_1}})$. Hence $H^W_s(V_\sigma, \mathcal{N}_q(W)) = H^W_s(U(W, C_{T_{i_1}}), \mathcal{N}_q(W))$, so the terms in the spectral sequence are the ones claimed.

For later computations, note that Corollary 6.4.3 implies:

$$L^2_b(U(W, C_{\text{min} \sigma})) = \dim_{\mathcal{N}_q} H^W_s(U(W, C_{\text{min} \sigma})) = L^2_b(\Phi_{\text{min} \sigma}) = L^2_b(\Sigma_{\text{min} \sigma}).$$
Chapter 7
Computations

In this section we will use the fattened Davis complex to make concrete computations. We first begin by considering the case where the nerve $L$ of the Coxeter system $(W, S)$ is a graph. Note that for this special case $\Sigma_L$ is two-dimensional. We then briefly discuss how we can use our computations to produce examples of Coxeter groups for which the Weighted Singer Conjecture holds. We then direct our attention to quasi-Lämmer groups, and finish with computations for 2-spherical Coxeter groups whose corresponding nerves are no longer restricted to be graphs.

Let $K_n$ denote the complete graph on $n$ vertices. Recall that a Coxeter system is 2-spherical if the one-skeleton of its nerve is $K_n$ for some $n$. For the purpose of figures and examples, we will distinguish the special case where the labeled nerve $L = K_n(3)$, where $K_n(3)$ denotes the complete graph on $n$ vertices with every edge labeled by 3.

Unless stated otherwise, the standing assumption in this chapter is that $q \geq 1$.

7.1 The case where $L$ is a graph

Suppose that the labeled nerve $L$ is the one-skeleton of an $n$-dimensional cell complex $\Lambda$, where $n \geq 2$. We say that a 2-cell of $\Lambda$ is Euclidean if the corresponding special subgroup generated by the vertices of that cell is a Euclidean reflection group. Note that the only possible labels on a Euclidean cell are $m_{st} \in \{2, 3, 4, 6\}$.

Before proving the main theorem of this section, we begin with a lemma. The
special case of the lemma when $q = 1$ is closely related to a result of Schroeder [19, Theorem 4.6]. We provide an argument which is analogous to that of Schroeder in his proof.

**Lemma 7.1.1.** Suppose that the labeled nerve $L$ is the one-skeleton of a cellulation of $S^2$. Then

\[ L_q^2b_2(\Sigma_L) = 0 \text{ for } q \leq 1. \]

**Proof.** In light of Lemma 3.3.8, we must show that $L_1^2b_2(\Sigma_L) = 0$. We begin by building $L$ to a triangulation of $S^2$ by coning on empty 2–cells and labeling the new edges by 2’s, at each step keeping track of the $L_1^2$–(co)homology with a Mayer–Vietoris sequence. More precisely, start with $T_1 \subset S$ corresponding to an empty 2–cell $L_{T_1}$ in $L$ and denote by $CL_{T_1}$ the right-angled cone on $L_{T_1}$. The corresponding special subgroup $W_{T_1}$ is infinite, and it acts properly and cocompactly by reflections on either $\mathbb{R}^2$ or $\mathbb{H}^2$. In both cases $L_1^2H_2(\Sigma_{L_{T_1}}) = 0$ and hence the K"unneth formula implies that $L_1^2H_2(\Sigma_{CL_{T_1}}) = 0$. We have the following Mayer–Vietoris sequence:

\[ \cdots \rightarrow L_1^2H_2(\Sigma_{L_{T_1}}) \rightarrow L_1^2H_2(\Sigma_{CL_{T_1}}) \oplus L_1^2H_2(\Sigma_L) \xrightarrow{f_1} L_1^2H_2(\Sigma_{L\cup CL_{T_1}}) \rightarrow \cdots \]

In particular, the map $f_1$ is injective. We then choose another $T_2 \subset S$ corresponding to an empty 2–cell $L_{T_2}$ in $L$ and denote by $CL_{T_2}$ the right-angled cone on $L_{T_2}$. By a similar argument, the map $f_2$ in the following Mayer–Vietoris sequence is injective:

\[ \cdots \rightarrow L_1^2H_2(\Sigma_{CL_{T_2}}) \oplus L_1^2H_2(\Sigma_{L\cup CL_{T_2}}) \xrightarrow{f_2} L_1^2H_2(\Sigma_{L\cup CL_{T_1}\cup CL_{T_2}}) \rightarrow \cdots \]

Proceed inductively until all empty 2–cells have been coned off and denote the newly promoted nerve by $L'$. The $f_i$’s yield a sequence of injective maps:

\[ L_1^2H_2(\Sigma_L) \hookrightarrow L_1^2H_2(\Sigma_{L\cup CL_{T_1}}) \hookrightarrow \cdots \hookrightarrow L_1^2H_2(\Sigma_{L'}) \]
Since \( L' \) is a triangulation of \( S^2 \), it follows that \( \Sigma_{L'} \) is a 3–manifold. Now, a result of Lott and Luck [14], in conjunction with the validity of the Geometrization Conjecture for 3–manifolds [17], implies that \( L^2_1 H_\ast(\Sigma_{L'}) \) vanishes in all dimensions. In particular, \( L^2_1 b_2(\Sigma_L) = 0. \)

**Remark 7.1.2.** Schroeder proves a more general theorem for \( q = 1 \) [19, Theorem 4.6]. A metric flag complex \( L \) is planar if it can be embedded as a proper subcomplex of a triangulation of the 2–sphere (see the discussion before Theorem 5.3.3 for the definition of metric flag complex). Schroeder proves that if the nerve \( L \) of a Coxeter system is planar, then \( L^2_1 b_k(\Sigma_L) = 0 \) for \( k \geq 2 \). If \( L \) is planar and \( W \) is the corresponding Coxeter group, then Proposition 2.9.1 implies that \( v_c(W) \leq 2 \). Therefore we can use Lemma 3.3.8 to deduce that \( L^2_q b_k(\Sigma_L) = 0 \) for \( k \geq 2 \) and \( q \leq 1 \).

**Theorem 7.1.3.** Suppose that the labeled nerve \( L \) is the one-skeleton of a cell complex that is a \( GHS^\ast \) \( n \geq 2 \), where all 2–cells are Euclidean, and let \( (W, S) \) denote the corresponding Coxeter system. Then \( L^2_q b_k(\Sigma_L) \) is concentrated in degree 2.

Furthermore,

\[
L^2_q b_2(\Sigma_L) = 1 - \sum_{s \in S} \frac{q_s}{1 + q_s} + \sum_{s, t \in S} \frac{q_s q_t}{1 + q_s + q_t + q_s q_t} + \sum_{s, t \in S} \frac{q_s^3}{1 + 2q_s + 2q_t^2 + q_s^3} +
\]

\[
+ \sum_{s, t \in S} \frac{q_s^2 q_t^2}{1 + q_s + q_t + 2q_s q_t + q_s^2 q_t + q_s q_t^2 + q_s^2 q_t^2} + \frac{q_s^3 q_t^2}{1 + q_s + q_t + 2q_s q_t + q_s q_t^2 + q_s^2 q_t^2 + q_s^3 q_t^2} + \frac{q_s^3 q_t^3}{1 + q_s + q_t + 2q_s q_t + q_s q_t^2 + q_s^2 q_t^2 + q_s^3 q_t^2 + q_s^3 q_t^3}.
\]

**Proof.** Proposition 3.3.1 implies that \( L^2_q b_0(\Sigma_L) = 0 \). Proposition 3.3.2, along with Theorem 2.2.1, explicitly compute the formula for \( L^2_q b_2(\Sigma_L) \). We now turn our attention to showing \( L^2_q b_1(\Sigma_L) = 0 \). For the construction of the fattened Davis complex, we will use the given cell complex as \( P^\ast \).

We prove the theorem by induction on \( n \). For the base case \( n = 2 \), first note that for every \( T \in N_P \), \( \sigma_T \) is Euclidean. Hence Proposition 6.3.1 implies that each \( C_T \) appearing in \( \partial K \) corresponds to a set \( T \in N_P \) where \( W_T \) is Euclidean reflection
group. Thus Corollary 6.4.3 and Theorem 3.3.4 imply that $L^2_q b_1(\mathcal{U}(W, C_T)) = 0$. This and Proposition 3.3.1 imply that the $E^{0,1}_1$ and $E^{1,0}_1$ terms in the $E_1$ sheet of the spectral sequence in Proposition 6.4.5 are zero, which in turn implies that $L^2_q b_1(\partial \Phi_L) = 0$. Now, note that $\Phi_L$ is three-dimensional and $\text{vcd} W = 2$. Moreover, by Lemma 7.1.1, $L^2_{q-1} H_2(\Sigma_L) = 0$. Therefore, via Lemma 6.4.2 (i), we reach the conclusion that $L^2_q b_1(\Sigma_L) = 0$.

Now, suppose the theorem is true for $m < n$. Since $\Sigma_L$ is two-dimensional, Lemma 6.4.2 (ii) tells us that we are done if we show that $L^2_q b_1(\partial \Phi_L) = 0$. Let $T \in \mathcal{N}_P$. Then $\sigma_T$ is the $(\partial \sigma_T, T)$–chamber, where $\sigma_T$ is the geometric cell in $P^*$ spanned by $T$. In particular, $\partial \sigma_T$ is a cell complex that is $GHS^m$, $m < n$, and since all 2–cells of $P^*$ are Euclidean, it follows that all 2–cells of $\partial \sigma_T$ are Euclidean. Hence, by induction and Corollary 6.4.3, it follows that for every $T \in \mathcal{N}_P$, $L^2_q b_1(\mathcal{U}(W, C_T)) = L^2_q b_1(\Sigma_{L_T}) = 0$. This and Proposition 3.3.1 imply that the $E^{0,1}_1$ and $E^{1,0}_1$ terms in the $E_1$ sheet of the spectral sequence in Proposition 6.4.5 are zero, which in turn implies that $L^2_q b_1(\partial \Phi_L) = 0$.

Consider the special case of Theorem 7.1.3 when $n = 2$. In this case, Theorem 7.1.3, along with Lemma 7.1.1, explicitly compute the $L^2_q$–Betti numbers for all $q$: they are always concentrated in a single dimension. We emphasize this in the following corollary.

**Corollary 7.1.4.** Suppose that the labeled nerve $L$ is the one-skeleton of a cell complex that is a $GHS^2$, where all 2–cells are Euclidean.

- If $q \in \bar{R}$, then $L^2_q H_*(\Sigma_L)$ is concentrated in dimension 0.
- If $q \notin R$ and $q \leq 1$, then $L^2_q H_*(\Sigma_L)$ is concentrated in dimension 1.
- If $q \geq 1$, then $L^2_q H_*(\Sigma_L)$ is concentrated in dimension 2.
Furthermore, 

\[
\chi_q(\Sigma_L) = 1 - \sum_{s \in S} \frac{q_s}{1 + q_s} + \sum_{s, t \in S \atop m_{st} = 2} \frac{q_s q_t}{1 + q_s + q_t + q_s q_t} + \sum_{s, t \in S \atop m_{st} = 3} \frac{q_s^3}{1 + 2q_s + 2q_s^2 + q_s^3} + \\
\sum_{s, t \in S \atop m_{st} = 4} \frac{q_s^2 q_t^2}{1 + q_s + q_t + 2q_s q_t + q_s^2 q_t + q_s q_t^2 + q_s^2 q_t^2} + \\
\sum_{s, t \in S \atop m_{st} = 6} \frac{q_s^3 q_t^3}{1 + q_s + q_t + 2q_s q_t + q_s^2 q_t + q_s q_t^2 + 2q_s^2 q_t^2 + q_s^2 q_t^3 + q_s^3 q_t^2 + q_s^3 q_t^3}.
\]

If we place some restrictions on either our labels or the cell complex, then the formulas in Theorem 7.1.3 become relatively simple, as illustrated by the following corollaries.

**Corollary 7.1.5.** Suppose that \( L \) is the one-skeleton of a cell complex that is a GHS\(^n\), \( n \geq 2 \), where all 2–cells are 2–simplices. Give \( L \) the labels \( m_{st} = 3 \). Then \( L_q^2 b_*(\Sigma_L) \) is concentrated in degree 2.

Furthermore, 

\[
L_q^2 b_2(\Sigma_L) = 1 - \frac{Vq}{1 + q} + \frac{Eq^3}{1 + 2q + 2q^2 + q^3},
\]

where \( V \) and \( E \) are the number of vertices and edges of \( L \), respectively.

Recall that an \( n \)–dimensional octahedron has \( 2n \) vertices and \( 2n(n - 1) \) edges.

**Corollary 7.1.6.** Suppose that \( L \) the one skeleton of an \( n \)–dimensional octahedron with \( n \geq 3 \) and the labels \( m_{st} = 3 \). Then \( L_q^2 b_*(\Sigma_L) \) is concentrated in degree 2. Furthermore, 

\[
L_q^2 b_2(\Sigma_L) = 1 - \frac{2nq}{1 + q} + \frac{2n(n - 1)q^3}{(1 + 2q + 2q^2 + q^3)}.
\]

**Corollary 7.1.7.** Let \( L = K_n(3) \) with \( n \geq 3 \). Then \( L_q^2 b_*(\Sigma_L) \) is concentrated in degree 2. Furthermore, 

\[
L_q^2 b_2(\Sigma_L) = 1 - \frac{nq}{1 + q} + \frac{n(n - 1)q^3}{2(1 + 2q + 2q^2 + q^3)}.
\]

**Remark 7.1.8.** Note that under the hypothesis of the above corollaries, all generators in \( S \) are conjugate, so in this case \( q = q \), where \( q \geq 1 \) is a positive real number.
If we assume that $W$ is right-angled, we have the following consequences of Theorem 7.1.3.

**Corollary 7.1.9.** Suppose that $L$ is the one-skeleton of a cell complex that is a $GHS^n$, $n \geq 2$, where all 2–cells are 2–cubes. Give $L$ the labels $m_{st} = 2$. Then $L^2_q b_*(\Sigma L)$ is concentrated in degree 2. Furthermore,

$$L^2_q b_2(\Sigma_L) = 1 - \sum_{s \in S} \frac{q_s}{1 + q_s} + \sum_{\{s,t\} \in S} \frac{q_s q_t}{1 + q_s + q_t + q_s q_t}.$$  

Analogous to the case where $L = K_n(3)$, let $C_n(2)$ denote the one-skeleton of an $n$–cube with edges labeled by 2. If we assume that $L = C_n(2)$ and that $q = q$, where $q$ is a positive real number, then we obtain simple formulas for the $L^2_q$–Betti numbers. Recall that an $n$–cube has $2^n$ vertices and $n2^{n-1}$ edges.

**Corollary 7.1.10.** Let $L = C_n(2)$ with $n \geq 2$. Then $L^2_q b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,

$$L^2_q b_2(\Sigma_L) = 1 - \frac{2^n q}{1 + q} + \frac{n2^{n-1} q^2}{1 + 2q + q^2}.$$  

We can also allow ourselves to remove some edges from $L = K_n(3)$. We denote by $K^l_n(3)$ the complete graph on $n$ vertices, labeled by 3’s and with $l$ edges removed. We have the following consequence of Corollary 7.1.7.

**Corollary 7.1.11.** Suppose that $L = K^l_n(3)$, where $n \geq 5$ and $l \leq n - 4$. Then $L^2_q b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,

$$L^2_q b_2(\Sigma_L) = 1 - \frac{2^n q}{1 + q} + \frac{n(n - 1) q^3}{2(1 + 2q + 2q^2 + q^3)} - \frac{l q^3}{1 + 2q + 2q^2 + q^3}.$$  

**Proof.** We first note that removing an edge from $K_n(3)$ splits the graph into two copies of $K_{n-1}(3)$ intersecting at $K_{n-2}(3)$. Since $n \geq 5$ and $q \geq 1$ we have the following Mayer–Vietoris sequence:

$$\cdots \rightarrow L^2_q H_1(\Sigma K_{n-2}) \rightarrow L^2_q H_1(\Sigma K_{n-1}) \oplus L^2_q H_1(\Sigma K_{n-1}) \rightarrow L^2_q H_1(\Sigma K_1) \rightarrow 0 \quad (\ast)$$
We first handle the case where \( L = K_5^1(3) \). Removing an edge from \( K_5(3) \) splits the graph into two copies of \( K_4(3) \) intersecting at \( K_3(3) \). Corollary 7.1.7 computes the \( L_q^2 \)–(co)homology of each of the pieces in this decomposition and applying the sequence \( (*) \) now proves the assertion for the case \( L = K_5^1(3) \).

The proof for \( L = K_n^l(3) \) is now by induction, the above computation serving as the base case. Suppose that the theorem is true for \( m < n \). Begin by removing an edge from \( K_n(3) \), splitting it as two copies of \( K_{n-1}(3) \) intersecting at \( K_{n-2}(3) \). Now, we remove the remaining \( l - 1 \leq n - 5 \) edges from each of the graphs in the splitting, the worst case scenario being that we remove \( l - 1 \) edges from \( K_{n-2}(3) \) (which in turn removes \( l - 1 \) edges from each copy of \( K_{n-1}(3) \)). Nevertheless, the inductive hypothesis is satisfied for each of the \( K_{n-1} \)'s in the splitting no matter how the remaining edges are removed. Applying a Mayer–Vietoris sequence analogous to \( (*) \) now shows that the theorem holds for \( L = K_n^l(3) \).

With the help of ruins (see Section 4.2), we are also able to make computations when we change some labels on \( L = K_n(3) \).

**Theorem 7.1.12.** Let \( L = K_n \), the complete graph on \( n \) vertices with \( n \geq 5 \). Let \( k \leq n - 4 \), and suppose that we label \( k \) edges of \( L \) with \( m_{st} \in \mathbb{N} - \{1, 3\} \) and label the remaining edges by 3. Then \( L_q^2 b_q(\Sigma_L) \) is concentrated in degree 2.

**Proof.** The proof is by induction on \( n \). First consider the case where \( L = K_5 \) with one label \( m_{st} \in \mathbb{N} - \{1, 3\} \). Then by Corollary 7.1.7, \( L^2 b_1(\Sigma(S - s)) = L^2 b_1(\Sigma_{K_4(3)}) = 0 \). According to sequence (4.1), it remains to show that \( L_q^2 H_1(\Omega_{S(s)}, \partial) = 0 \). We turn our attention to sequence (4.1) with \( U = S, T = \{s, t\}, U' = S - t, \) and \( T' = \{s\} \). By Proposition 4.2.1, \( L_q^2 H_1(\Omega_{ST}, \partial) = 0 \), the point being that the relative chain complex of \( (\Omega_{ST}, \partial \Omega_{ST}) \) has no one-dimensional cells. So, by weak exactness, it remains to show that \( L_q^2 H_1(\Omega_{U'T'}, \partial) = 0 \). We consider the following weak exact sequence:

\[
\cdots \rightarrow L_q^2 H_1(\Sigma(S - \{s, t\})) \rightarrow L_q^2 H_1(\Sigma(S - t)) \rightarrow L_q^2 H_1(\Omega_{U'T'}, \partial) \rightarrow \cdots
\]

Note that

\[
L_q^2 b_0(\Sigma(S - \{s, t\})) = L_q^2 b_0(\Sigma_{K_3(3)}) = 0
\]
and

\[ L_q^2 b_1(\Sigma(S-t)) = L_q^2 b_1(\Sigma_{K_4(3)}) = 0 \]

by Theorem 3.3.4 and Corollary 7.1.7, respectively. By weak exactness, \( L_q^2 H_1(\Omega_{U'T'}, \partial) = 0 \), and hence \( L_q^2 H_1(\Omega_{S(s)}, \partial) = 0 \), thus proving the assertion for \( L = K_5 \).

Now, suppose that the theorem is true for \( L = K_m, \ m < n \). We wish to show the theorem is true for \( L = K_n \). Begin by choosing an edge \( e \) with vertices \( s \) and \( t \) and label different from 3. We now observe that \( L^2 b_1(\Sigma(S-s)) = L^2 b_1(\Sigma_{K_{n-1}}) = 0 \) by the inductive hypothesis, since \( K_{n-1} \) now has at most \( n-5 \) edges with a label different from 3. Similarly, the inductive hypothesis implies \( L_q^2 b_1(\Sigma(S-t)) = 0 \) and \( L_q^2 b_0(\Sigma(S-\{s,t\})) = 0 \). Hence the weak exact sequences used in the proof for the case \( L = K_5 \) allow us to conclude that \( L_q^2 b_1(\Sigma_L) = L_q^2 b_1(\Sigma(S)) = 0 \).

\[ \blacksquare \]

Remark 7.1.13. Note that in conjunction with Theorem 3.3.4 and Corollary 7.1.4, the above argument gives an alternate proof of Corollary 7.1.7.

7.2 Connection to the Weighted Singer Conjecture

We note that Theorem 7.1.3 provides convincing evidence for the validity of a weighted version of Theorem 5.0.2 when \( L \) is a triangulation of the \((n-1)\)–sphere. Suppose that the labeled nerve \( L' \) is the one-skeleton of a cellulation of a \( GHS^{n-1} \), \( n \geq 3 \), where all 2–cells are Euclidean. Build \( L' \) to a triangulation that is a \( GHS^{n-1} \) by coning on each empty cell and labeling new edges by 2. In other words, perform the following sequence of right-angled cones. First begin by coning on each empty 2–cell, then on each empty 3–cell, and so on, until each empty cell has been coned off (if \( n = 3 \), this process stops when each empty 2–cell has been coned off).

Theorem 7.2.1. Suppose that the nerve \( L \) a \( GHS^{n-1} \), \( n \geq 3 \), obtained via the above construction and suppose that \( q \geq 1 \). Then

\[ L_q^2 b_k(\Sigma_L) = 0 \text{ for } k \leq 1. \]
Proof. The proof of the theorem follows the strategy of Lemma 7.1.1: one performs careful book-keeping using Mayer–Vietoris sequences when constructing \( L \) from \( L' \). Theorem 7.1.3 tells us that \( L' \) originally satisfies \( L^2_q b_1(\Sigma_{L'}) = 0 \). To construct \( L \) from \( L' \), we first began by coning empty 2-cells, then successively coning higher dimensional cells, labeling new edges by 2. If at each step of this process we employ a Mayer–Vietoris sequence, then Theorem 7.1.3, in conjunction with the fact that right-angled cones will not develop new homology below dimension 2, implies that \( L^2_q b_1(\Sigma_L) = 0 \). 

7.3 Quasi-Lánner groups

A 2–spherical Coxeter group \( W \) is quasi-Lánner if it acts properly (but not co-compactly) on hyperbolic space \( \mathbb{H}^n \) by reflections with fundamental chamber an \( n \)–simplex of finite volume. For brevity, we say that \( W \) is of type \( QL_n \). Quasi-Lánner groups have been classified and only exist in dimensions 3 through 10. For a complete list, see [13, §6.9]. We note that the Coxeter group with corresponding nerve \( L = K_4(3) \) is on the list.

All non-spherical proper special subgroups of a quasi-Lánner group are Euclidean and on the list appearing in [13, pg. 34]. Moreover, if \( W \) is of type \( QL_n \), then the only proper infinite special subgroups are those \( W_T \) with \( |T| = n - 1 \). Hence, by Proposition 2.9.1, if \( W \) is of type \( QL_n \), then \( vcd W = n - 1 \). With this observation, we prove the following theorem.

Theorem 7.3.1. Suppose that \( W \) is of type \( QL_n \). Then \( L^2_q b_k(\Sigma_L) = 0 \) whenever \( k \geq n - 1 \) and \( q \leq 1 \), or \( k \leq 1 \) and \( q \geq 1 \).

Proof. We first suppose that \( q = 1 \). Since \( W \) is of type \( QL_n \), we can realize a finite volume \( n \)–simplex in hyperbolic space \( \mathbb{H}^n \), with \( W \) acting by reflections along codimension-one faces (note that this simplex has some ideal vertices). By a theorem of Cheeger–Gromov [5], \( L^2_1 H_k(\Sigma_L) \cong L^2_1 \mathcal{H}^k(\mathbb{H}^n) \), where \( L^2_1 \mathcal{H}^k \) denotes the \( L^2 \) de Rham cohomology. By a theorem of Dodziuk [11], \( L^2_1 \mathcal{H}^k(\mathbb{H}^n) = 0 \) for all \( k \geq 0 \) if \( n \) is
odd, and is concentrated in dimension $\frac{n}{2}$ if $n$ is even. In particular, $L_1^2 b_{n-1}(\Sigma_L) = 0$.

The result for $q \leq 1$ now follows by Lemma 3.3.8 and the fact that vcd $W = n - 1$.

Now, suppose that $q \geq 1$. Consider the fattened Davis complex $\Phi_L$ with respect to $P = \Delta^n$, the standard $n$–simplex (see Remark 6.1.1 and Figure 7.1).

![Figure 7.1: $K^f$ when $L = K_4(3)$](image)

Weighted Poincaré duality implies that

$$L_q^2 H_1(\Phi_L, \partial \Phi_L) \cong L_{q-1}^2 H_{n-1}(\Phi_L) \cong L_{q-1}^2 H_{n-1}(\Sigma_L) = 0,$$

so by the long exact sequence for the pair $(\Phi_L, \partial \Phi_L)$ it remains to show $L_q^2 H_1(\partial \Phi_L) = 0$. Proposition 6.3.2 implies that each $C_T$ appearing in $\partial K^f$ corresponds to a set $T \in \mathcal{N}_P$ with $W_T$ a Euclidean reflection group. In particular, Corollary 6.4.3 and Theorem 3.3.4 imply that $L_q^2 b_1(\mathcal{U}(W, C_T)) = 0$. Hence the $E_1^{0,1}$ term in the $E_1$ sheet of the spectral sequence of Proposition 6.4.5 is zero. By Proposition 3.3.1, the first row of the $E_1$ sheet is also zero, and in particular $E_1^{1,0}$ is zero. Therefore $L_q^2 b_1(\partial \Phi_L) = 0$.

Of important note is the case when $W$ is $QL_3$. In this special case, Theorem 7.3.1 explicitly computes the $L_q^2$–Betti numbers for all $q$: they are always concentrated in a single dimension.

**Corollary 7.3.2.** Suppose that $W$ is of type $QL_3$. Then

- If $q \in \bar{R}$, then $L_q^2 H_*(\Sigma_L)$ is concentrated in dimension 0.
• If \( q \notin \mathbb{R} \) and \( q \leq 1 \), then \( L^2_qH_*(\Sigma_L) \) is concentrated in dimension 1.

• If \( q \geq 1 \), then \( L^2_qH_*(\Sigma_L) \) is concentrated in dimension 2.

Since the \( L^2_q \)–(co)homology is always concentrated in a single dimension, one can use Proposition 3.3.2, along with Theorem 2.2.1, to obtain explicit formulas for the \( L^2_q \)–Betti numbers.

### 7.4 Other 2–spherical groups

We now perform computations for other 2–spherical groups, removing the restriction that the nerve \( L \) is a graph. Given a Coxeter system \( (W, S) \), we make a particular choice of \( P \) for the construction of \( \Phi_L \), namely \( P = \Delta^{(|S| - 1)} \), the standard \((|S| - 1)\)–simplex (see Remark 6.1.1).

While one could argue the following lemma using the spectral sequence, we use a simple Mayer–Vietoris sequence argument to illustrate the technique behind the machinery.

**Lemma 7.4.1.** Suppose that \( (W, S) \) is infinite 2–spherical with \( |S| = 5 \) and \( \text{vcd} W \leq 3 \). Furthermore, suppose that every infinite special subgroup \( W_T \), with \( |T| = 3, 4 \), is Euclidean or \( QL_3 \), and that \( L^2_q b_3(\Sigma_L) = 0 \). Then \( L^2_q b_k(\Sigma_L) = 0 \) for \( k < 2 \).

**Proof.** We wish to reduce the proof to showing that \( L^2_q b_1(\partial \Phi_L) = 0 \). If \( \text{vcd} W = 2 \), then this is accomplished by Lemma 6.4.2 (ii). If \( \text{vcd} W = 3 \), then according to Lemma 6.4.2 (i), we reduce the proof to showing \( L^2_q b_1(\partial \Phi_L) = 0 \) if we show that \( L^2_q b_3(\Sigma_L) = 0 \). By Lemma 3.3.8, we reach this conclusion since by assumption \( L^2_q b_3(\Sigma_L) = 0 \). So, to complete the proof, we must show that \( L^2_q b_1(\partial \Phi_L) = 0 \).

Let \( \mathcal{N}_P^{(j)} = \{ T \in \mathcal{N}_P \mid \text{Card}(T) = j \} \) and set

\[
A_j = \bigcup_{T \in \mathcal{N}_P^{(j)}} \mathcal{U}(W, C_T).
\]

Note that \( |S| = 5 \) and all proper non-spherical subsets \( T \) have order 3 or 4, so by Proposition 6.3.2, \( \partial \Phi_L = A_3 \cup A_4 \). Figure 7.2 illustrates the chamber for \( \partial \Phi_L \) for the case where \( L = K_5(3) \).
By Proposition 6.3.1 (i),

\[ A_3 \cap A_4 = \bigcup_{U \in \mathcal{N}_P^3} \mathcal{U}(W, C_U) \cap \mathcal{U}(W, C_V) \]

By Corollary 6.4.3,

\[ L^2_q b_k(A_j) = \sum L^2_q b_k(\mathcal{U}(W, C_T)) \]
\[ = \sum L^2_q b_k(\Phi_{L_T}) \]

and

\[ L^2_q b_k(A_3 \cap A_4) = \sum L^2_q b_k(\mathcal{U}(W, C_T)) \]
\[ = \sum L^2_q b_k(\Phi_{L_T}) \]

Here \( L_T \) is the subcomplex of \( L \) corresponding to the infinite subgroup \( W_T \), which is either Euclidean or of type \( QL_3 \). By Theorem 3.3.4 and Theorem 7.3.1,
\( L^2_q b_k(\Phi_{LT}) = 0 \) for \( k < 2 \). Hence
\[
L^2_q H_k(A_j) = 0 \text{ for } j = 3, 4 \text{ and } L^2_q H_k(A_3 \cap A_4) = 0 \text{ for } k < 2. \quad (\diamond)
\]

Now, consider the Mayer–Vietoris sequence:
\[
\cdots \to L^2_q H_k(A_3 \cap A_4) \to L^2_q H_k(A_3) \oplus L^2_q H_k(A_4) \to L^2_q H_k(\partial \Phi_L) \to \cdots
\]

Inputting (\diamond) into this sequence yields
\[
L^2_q H_k(\partial \Phi_L) = 0 \text{ for } k < 2.
\]

Lemma 6.4.2 now concludes that \( L^2_q b_1(\Sigma_L) = 0 \).

**Theorem 7.4.2.** Suppose that \((W, S)\) is infinite 2–spherical with \(|S| \geq 5\). Suppose furthermore that:

1. For every \( T \subseteq S \) with \(|T| \geq 5\), \( \text{vcd } W_T \leq |T| - 2 \).
2. \( L^2_1 b_{|S|-2}(\Sigma_L) = 0 \).
3. Every infinite subgroup \( W_T \), with \(|T| = 3, 4\), is Euclidean or \( QL_3 \).

Then \( L^2_q b_k(\Sigma_L) = 0 \) for \( k < 2 \).

**Proof.** The statement for \( L^2_q b_0(\Sigma_L) \) follows from Proposition 3.3.1. So, we turn our attention to showing \( L^2_q b_1(\Sigma_L) = 0 \). The proof of the theorem is now by induction on \(|S|\). Lemma 7.4.1 serving as the base case. By Lemma 3.3.8, since \( \text{vcd } W \leq |S| - 2 \), it follows that \( L^2_{q-1} b_{|S|-2}(\Sigma_L) = 0 \). Furthermore, \( \Phi_L \) has dimension \(|S| - 1\), so by Lemma 6.4.2 it now suffices to show that \( L^2_q b_1(\partial \Phi_L) = 0 \). By assumption, every non-spherical special subgroup \( W_U \) with \(|U| = 3, 4\) is Euclidean or \( QL_3 \). Thus every non-spherical special subgroup \( W_U \), with \( 4 < |U| < |S| \) satisfies the inductive hypothesis. Therefore by induction, Theorem 3.3.4, and Theorem 7.3.1, for any \( T \in \mathcal{N}_P \), we have that \( L^2_q b_1(\Sigma_{LT}) = 0 \) (Here \( LT \) is the subcomplex of \( L \) corresponding to the special subgroup \( W_T \)).

Hence by Corollary 6.4.3 (i), for every \( T \in \mathcal{N}_P \),
\[ L_2^1 b_1(U(W, C_T)) = L_2^1 b_1(\Sigma_{L_T}) = 0. \]

This implies that the \( E_1^{0,1} \) term in the \( E_1 \) sheet of the spectral sequence of Proposition 6.4.5 is zero. By Proposition 3.3.1, the first row of the \( E_1 \) sheet is also zero, and in particular \( E_1^{1,0} \) is zero. Therefore \( L_2^1 b_1(\partial L) = 0. \)

With the help of Theorem 5.0.2, we drop condition 2 in Theorem 7.4.2.

**Corollary 7.4.3.** Suppose that \((W, S)\) is infinite 2–spherical with \(|S| \geq 5\). Suppose furthermore that:

1. For every \(T \subseteq S\) with \(|T| \geq 5\), \(vcdW_T \leq |T| - 2\).

2. Every infinite subgroup \(W_T\), with \(|T| = 3, 4\), is Euclidean or QL_3.

Then \( L_2^1 b_k(\Sigma_L) = 0 \) for \( k < 2 \).

**Proof.** Note that condition 2 in Theorem 7.4.2 is vacuously satisfied if \( vcdW \leq |S| - 3 \), so we suppose that \( vcdW = |S| - 2 \). We must show that \( L_2^1 b_{|S| - 2}(\Sigma_L) = 0 \), and to do this, we use an argument analogous to the one in Lemma 7.1.1. We first begin by coning empty 2–simplices of \( L \), and then empty 3–simplices, and so on, until all empty simplices have been coned off. We then label all new edges by 2. In this way we obtain a newly promoted nerve \( L' \) which is a triangulation of \( S^{|S| - 2} \), and in particular, \( \Sigma_{L'} \) is an \((|S| - 1)\)–manifold. By Theorem 5.0.2, \( L_2^1 b_{|S| - 2}(\Sigma_{L'}) = 0 \), and using the arguments of Lemma 7.1.1, we can conclude that \( L_2^1 b_{|S| - 2}(\Sigma_L) = 0 \). \[ \]

As a corollary to Theorem 7.4.2, we also obtain a specialized version of Conjecture 5.0.1 where \( n = 4 \) and \( W \) is 2–spherical.

**Corollary 7.4.4.** Suppose that \((W, S)\) is 2–spherical with \(|S| \geq 6\) and that the nerve \( L \) is a triangulation of \( S^3 \). Furthermore, suppose that every infinite special subgroup \( W_T \), with \(|T| = 3, 4\), is Euclidean or QL_3. Then

\[ L_4^2 b_k(\Sigma_L) = 0 \] for \( k < 2 \).
\textit{Proof.} Since \( L \) is a triangulation of \( S^3 \), it follows that \( \text{vcd} W = 4 \). In particular, \( W \) satisfies the hypothesis of Theorem 7.4.2.

\textbf{Remark 7.4.5.} Figure 7.3 gives examples of Coxeter diagrams whose corresponding Coxeter system \((W,S)\) has \(|S| = 6\) and satisfies the hypothesis of Corollary 7.4.4 (if two vertices are not connected, then the implied label between them is 2). The author does not know whether there exist examples whenever \(|S| \geq 7\).

\[ \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1, \quad \frac{1}{t} + \frac{1}{u} + \frac{1}{v} = 1, \quad m = 2, 3, 4 \]

- If \( m = 3 \), then \( s, r, u, t \neq 6 \) and either \( s, r \neq 4 \) or \( u, t \neq 4 \).
- If \( m = 4 \), then \( s, r, u, t \neq 4, 6 \).

Figure 7.3: 2–spherical Coxeter diagrams satisfying the hypothesis of Corollary 7.4.4
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