May 2017

Associated Hypothesis in Linear Models with Unbalanced Data

Rica Katharina Wedowski
University of Wisconsin-Milwaukee

ASSOCIATED HYPOTHESIS IN LINEAR MODELS WITH UNBALANCED DATA

by

Rica Wedowski

A Thesis Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Master of Science

in Mathematics

at

The University of Wisconsin-Milwaukee

May 2017

ABSTRACT

ASSOCIATED HYPOTHESIS IN LINEAR MODELS WITH UNBALANCED DATA

by

Rica Wedowski

The University of Wisconsin-Milwaukee, 2017
Under the Supervision of Professor Jay Beder

In a two-way linear model one can test six different hypotheses regarding the effects in this model. Those hypotheses can be ranked from less specific to more specific. Therefore the more specific hypotheses are nested in the less specific ones. To test those nested hypotheses sequential sums of squares are used. Searle sees a problem with these since they test an associated hypothesis that has the same sums of squares but involve the sample sizes. Hypotheses should be generic and not dependent on the data. The proof he uses in his book *Linear Models for Unbalanced Data* is not easy to understand. Therefore this thesis verifies his equations for the associated hypothesis for the nested hypotheses *Only A present* given *No interaction* with an unpublished theorem of Beder. It also shows a way to derive Searle's equations for the associated hypothesis from this theorem.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGMENTS

I would like to thank my adviser Professor Jay Beder for his help on this thesis and my committee members Professor Allen Bell and Professor Kevin McLeod. I'd would also like to thank my parents, my sister, my boyfriend and my friends for their support.

1 Introduction

In [2] Shayle R. Searle says that using the sequential sums of squares for testing a nested hypothesis in a linear model leads to some problems. He shows that the actual tested hypothesis is an associated hypothesis involving the number of samples. He sees them as not usable, because the hypothesis should be independent from the data used.

This thesis goes through the definition of a linear model and how to formulate constraints for a linear model. We view a linear model $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ as a transformation from the parameter space $\mathbb{R}^p$ into the observation space $\mathbb{R}^N$. We then can define a constraint for this model as restricting the parameter $\boldsymbol{\beta}$ on a subspace of $\mathbb{R}^p$. This can be written as $\boldsymbol{\beta} \in W$ or as $\boldsymbol{\beta} \perp U$ where $U \subset \mathbb{R}^p$ and $U^\perp = W \subset \mathbb{R}^p$. This can be used to test the restricted model. We set the restriction as the null-hypothesis and then fit the unrestricted model and the restricted model. For testing we compare the sums of squares of both fits. We can equally test nested hypotheses. Saying that H_2 implies H_1, we test H_2 given that H_1 is true by comparing both restricted models with each other using a sequential sum of squares.

Then we will use the above described way of formulating the hypothesis to see how Searle gets his equations for the associated hypothesis, using a Theorem of Beder [1]. This is shown for H_2 : *Only A present* given H_1 : *No interaction* in a 2×3 model. From those two hypotheses we can derive a unique subspace W^* such that the sum of squares of the hypothesis $\boldsymbol{\beta} \in W^*$ is equal to the sequential sums of squares of the nested hypotheses. Therefore we can also formulate a vector space $U^* \perp W^*$. The thesis shows that the two contrast vectors we can get from Searle's equations span the subspace U^*. Finally we use some data from [2] to show an example of how to derive such equations for the associated hypothesis, more precisely the two contrast vectors from the vector space U^*.

2 Background Knowledge

In this chapter we follow the exposition in [1] to describe some important background knowledge to understand the purpose of this thesis.

2.1 Linear models

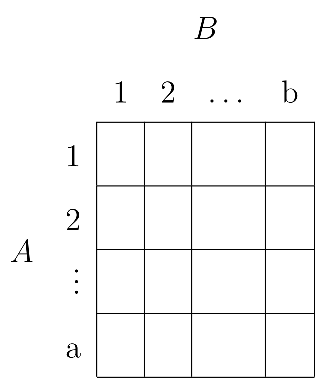
A linear model has the form

$$E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}. \tag{2.1}$$

$\mathbf{Y}$ is the $N \times 1$ vector of the observations, $\boldsymbol{\beta}$ is the vector of the unknown parameters. Its length is denoted by p. $\mathbf{X}$ is the $N \times p$ design matrix and can be viewed as a linear transformation from the parameter space $\mathbb{R}^p$ to the observation space $\mathbb{R}^N$.

2.1.1 Two-way-factorial model

A two-way factorial model involves two factors, A and B. Such a model can be written as a $a \times b$ model, where a denotes the number of levels in the A factor and b the number of levels in the B factor. The combinations of those two factors can be seen as the numbered cells in such a diagram:



In a two-way factorial experiment the number of observations taken for each cell (i,j) is n_{ij}. We will set $\sum_{i,j} n_{ij} = N$, the total number of observations. The dimension of the parameter space is $p = a * b$. When letting $\mu_{ij} = E(Y_{ijk}), k = 1, \ldots, n_{ij}$ the linear model can be written as

$$E \begin{pmatrix} \mathbf{Y}_{11} \\ \mathbf{Y}_{12} \\ \vdots \\ \mathbf{Y}_{ab} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n_{11}} & & & \\ & \mathbf{1}_{n_{12}} & & \\ & & \ddots & \\ & & & \mathbf{1}_{n_{ab}} \end{pmatrix} \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{ab} \end{pmatrix} \tag{2.2}$$

where $\mathbf{Y}_{ij} = \left(Y_{ij1}, Y_{ij2}, \ldots, Y_{ijn_{ij}} \right)'$ and $\mathbf{1}_{n_{ij}}$ is the $n_{ij} \times 1$ vector of ones for $i = 1, \ldots, a$ and $j = 1, \ldots, b$. (The blank entries in the matrix are zeros.) Note that X is full rank. Here and elsewhere, the cells are listed in lexicographic order (i.e., row by row).

2.1.2 Linear constraint and hypothesis

A linear constraint is a condition set on the parameter vector $\boldsymbol{\beta}$. This can be either a hypothesis or some overall assumption on the model. A model with a linear constraint of any kind is called a constrained or reduced model. Considering a linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}$ with $\boldsymbol{\beta} \in \mathbb{R}^p$, a linear constraint about $\boldsymbol{\beta}$ looks like

$$\begin{cases} \mathbf{c}_1' \boldsymbol{\beta} = 0 \\ \quad \vdots \\ \mathbf{c}_d' \boldsymbol{\beta} = 0 \end{cases} \tag{2.3}$$

If we have $\sum_j c_{ij} = 0$ for the coefficients of a vector $\mathbf{c}_i$, we call $\mathbf{c}_i$ a *contrast vector*. A linear constraint can be written in different ways:

Lemma 2.1 ([1, Lemma 1.2]). *The following are equivalent:*

a There exist $\mathbf{c}_1, \ldots, \mathbf{c}_d$ *such that* $\boldsymbol{\beta}$ *satisfies (2.3)* *b There exists a matrix* $\mathbf{C}$ *such that*

$$\mathbf{C}'\boldsymbol{\beta} = 0 \tag{2.4}$$

c There exists a subspace $U \subset \mathbb{R}^p$ *such that*

$$\boldsymbol{\beta} \perp U \tag{2.5}$$

d There exists a subspace $W \subset \mathbb{R}^p$ *such that*

$$\boldsymbol{\beta} \in W \tag{2.6}$$

e There exists a matrix $\mathbf{W}$ *and a vector* $\boldsymbol{\beta}_0$ *such that*

$$\boldsymbol{\beta} = \mathbf{W}\boldsymbol{\beta}_0 \tag{2.7}$$

It can be shown that $\mathbf{c}_1, \ldots, \mathbf{c}_d$ are the columns of $\mathbf{C}$ and that they also span U. The columns $\mathbf{w}_1, \ldots, \mathbf{w}_k$ of $\mathbf{W}$ span the subspace W. The following theorem shows that a constrained model is also a linear model:

Theorem 2.1 ([1, Theorem 1.7])**.** *If* $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ *is a linear model and if* $\boldsymbol{\beta}$ *is subject to a linear constraint* $\boldsymbol{\beta} \in W$*, then the constrained model is also a linear model*

$$E(\mathbf{Y}) = \mathbf{X}_0\boldsymbol{\beta}_0 \tag{2.8}$$

where $\mathbf{X}_0 = \mathbf{X}\mathbf{W}$ *and* $\mathbf{W}$ *and* $\boldsymbol{\beta}_0$ *are as in (2.7). We have* $\mathbf{X}(W) = R(\mathbf{X}_0)$*. If* $W \cap N(\mathbf{X}) = (0)$ *then we may choose* $\mathbf{X}_0$ *to have full rank.*

Here $N(\mathbf{X})$ is the null space of $\mathbf{X}$.

2.1.3 Hypotheses in a two-factor design

For the two-factor design there exist six relevant hypotheses. Those are shown in Figure 2.1. The three underlined are the most easy to describe and will be further explained.

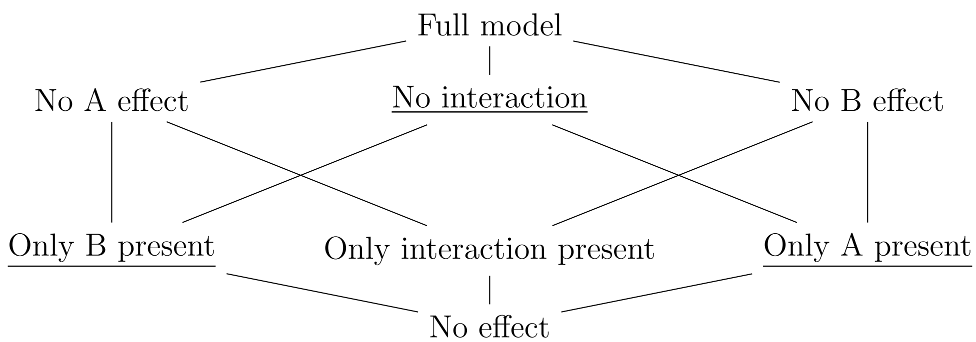


Figure 2.1: Six important hypothesis and their relationships

H_0**: Only A present.** If the hypothesis *Only A present* holds, the cell means of each row are the same. So the array should look like this:

		B			
		1	2	...	b
	1	μ_1	μ_1	$\dots$	μ_1
	2	μ_2	μ_2	$\dots$	μ_2
A	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	a	μ_a	μ_a	$\dots$	μ_a

This hypothesis is defined by $a(b-1)$ independent equations:

$$\begin{gathered}\mu_{11} = \mu_{12} = \ldots = \mu_{1b} \\ \mu_{21} = \mu_{22} = \ldots = \mu_{2b} \\ \vdots \\ \mu_{a1} = \mu_{a2} = \ldots = \mu_{ab}\end{gathered} \tag{2.9}$$

Each equation can be written in the form $\mathbf{c}_i\boldsymbol{\mu} = 0$ with $i = 1, \ldots, a(b-1)$. For example, $\mu_{11} = \mu_{12}$ is $\mathbf{c}_1\boldsymbol{\mu} = 0$ where $\mathbf{c}_1 = (1, -1, 0, 0, 0, 0)'$. All $\mathbf{c}_i$ are contrast vectors and together they span the subspace of $\mathbb{R}^p$ denoted by U_{Aonly}. This is a subspace as in (2.5). From this follows that $\dim(U_{Aonly}) = a(b-1)$.

H_0: **Only B present.** Analogous to "Only A present", this hypothesis holds if the cell means in each column are equal. This can be described by $(a-1)b$ equations.

$$\begin{aligned} \mu_{11} &= \mu_{21} = \ldots = \mu_{a1} \\ \mu_{12} &= \mu_{22} = \ldots = \mu_{b2} \\ &\vdots \\ \mu_{1a} &= \mu_{2a} = \ldots = \mu_{ab} \end{aligned} \tag{2.10}$$

The ensuing subspace is denoted by U_{Bonly} and $\dim(U_{Bonly}) = (a-1)b$.

H_0: **Additivity (No interaction between A and B).**

Definition 2.1 ([1, Definition 2.1]). *We say that factors A and B are additive if a change in the level of A produces an equal change in expected response at every level of B. When the factors are additive, we say that they have no interaction.*

In other words, to get the cell means in a row i' we just need to add a constant to all cell means in another row i. This holds for all pairs of rows in this model. So to derive the equations for this hypothesis, it suffices to compare the first row with all other rows since all other equations are redundant. For example, for the *ith* row in the *jth* column,

	1	...	j	...
1	μ_{11}	...	μ_{1j}	...
⋮	⋮		⋮	
i	μ_{i1}	...	μ_{ij}	...
⋮	⋮		⋮	

H_0 holds if $\mu_{ij} - \mu_{1j} = \mu_{i1} - \mu_{11}$.

Writing this for all other rows and columns, we get the following $(a-1)(b-1)$ equations to describe this hypothesis:

$$\mu_{ij} - \mu_{1j} = \mu_{i1} - \mu_{11},\ i = 2, \ldots, a, j = 2, \ldots, b \tag{2.11}$$

or in terms of formula (2.3),

$$\mu_{ij} - \mu_{1j} - \mu_{i1} + \mu_{11} = 0,\ i = 2, \ldots, a, j = 2, \ldots, b \tag{2.12}$$

We call the the resulting subspace U_{12} and $\dim(U_{12}) = (a-1)(b-1)$.

For the other hypotheses we can formulate these equations:

- No A effect ($\boldsymbol{\beta} \perp U_1$)

$$\mu_{1\cdot} = \mu_{2\cdot} = \ldots = \mu_{a\cdot}$$
$$\dim(U_1) = a - 1$$

  with $\mu_{i\cdot} = \sum_j \mu_{ij}$ for $i = 1, \ldots, a$.

- No B effect ($\boldsymbol{\beta} \perp U_2$)

$$\mu_{\cdot 1} = \mu_{\cdot 2} = \ldots = \mu_{\cdot b}$$
$$\dim(U_2) = b - 1$$

  with $\mu_{\cdot j} = \sum_i \mu_{ij}$ for $j = 1, \ldots, b$.

- No effect

$$\mu_{ij} \text{ equal for all i,j} \tag{2.13}$$

They are further described in [1].

As Figure 2.1 indicates, some hypotheses are more specific then others. For example *Only A present* can be gotten by combining *No interaction* with the hypothesis *No B present.*

2.2 Least squares

After observing some data we try to fit the linear model. To estimate the parameter $\boldsymbol{\beta}$ we use the method of least squares. We let $\hat{\mathbf{Y}}$ represent the fitted values and $\mathbf{Y}$ the vector of observed values. Both vectors are from the $\mathbb{R}^N$ and in this case we numerate the elements by $1, ..., N$. This method tries to minimized the quantity $\sum_{j=1}^{N}(Y_j - \hat{Y}_j)^2$ or written with the euclidean norm $SSE := \|\mathbf{Y} - \hat{\mathbf{Y}}\|^2$. The resulting estimator for $\boldsymbol{\beta}$ is then

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \tag{2.14}$$

The fitted values can then be calculated by

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}} \tag{2.15}$$

$$= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \tag{2.16}$$

Equation (2.15) gives us that $\hat{\mathbf{Y}} \in V := R(X)$, the columnspace of $\mathbf{X}$. Then we can set

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \tag{2.17}$$

and see that $\mathbf{P}$ projects $\mathbf{Y}$ on $\hat{\mathbf{Y}}$. Or in other words $\mathbf{P}$ is an orthogonal projection from $\mathbb{R}^N$ onto V.

2.3 Testing a linear hypothesis

To test a hypothesis H_0 we have first to fit the initial model $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ and then fit the model restricted by H_0. After this we compare the SSEs of both models; if those values are close we can not reject H_0. In other words, if the ratio SSE_R/SSE is sufficient large we will reject H_0. We use a F-Test as a decision criteria. With SSE_R we denote the *Sum of Squares* of the restricted model. To fit both models we use the method of least squares as described in section 2.2. In order to fit the restricted model we have to write the hypothesis in the form $H_0 : \boldsymbol{\beta} \in W$. Since W is a subspace of $\mathbb{R}^p$, $V_0 = \mathbf{X}(W)$ is a subspace of $V = R(\mathbf{X})$. Using Theorem 2.1 we can write the restricted model as $E(\mathbf{Y}) = \mathbf{X}_0\boldsymbol{\beta}_0$ and then we get $V_0 = \mathbf{X}(W) = R(\mathbf{X_0})$. The fitted values for the restricted model therefore lie in V_0. We denote them by $\hat{\mathbf{Y}}_0$, the orthogonal projection from $\mathbf{Y}$ into V_0. For the *Sum of Squares* we get

$$SSE_R = \|\mathbf{Y} - \hat{\mathbf{Y}}_0\|^2$$

Since SSE/SSE_R does not have a F distribution, we use $SS(H_0) = \|\hat{\mathbf{Y}} - \hat{\mathbf{Y}}_0\|^2$ and the ratio $SS(H_0)/SSE$. This can be done because

$$\begin{aligned} &\|\mathbf{Y} - \hat{\mathbf{Y}}_0\|^2 = \|\mathbf{Y} - \hat{\mathbf{Y}}\|^2 + \|\hat{\mathbf{Y}} - \hat{\mathbf{Y}}_0\|^2 \\ \Leftrightarrow\ & SSE_R = SSE + SS(H_0) \\ \Leftrightarrow\ & \frac{SSE_R}{SSE} = 1 + \frac{SS(H_0)}{SSE} \end{aligned}$$

and so SSE_R/SSE is large only if $SS(H_0)/SSE$ is large. So we will reject H_0 on a significance level α if

$$F = \frac{SS(H_0)(r - r_0)}{SSE(N - r)} > F_{\alpha, r-r_0, N-r} \tag{2.18}$$

where $r = \dim(V) = \operatorname{rank}(\mathbf{X})$, $r_0 = \dim(V_0) = \operatorname{rank}(\mathbf{X}_0)$ and $F_{\alpha,r-r_0,N-r}$ is the $(1-\alpha)$-quantile of the $F(r - r_0, N - r)$ distribution. For the degrees of freedom of H_0 we have the following corollary:

Corollary 2.1 ([1, Corollary 4.2]). *If* $\mathbf{X}$ *is full rank, and if* H_0 *is the statement* $\boldsymbol{\beta} \perp U$*, then* $\mathrm{df}(H_0) = \dim(U)$.

The quantity $SS(H_0)$ is known as the *adjusted sum of squares* for testing H_0.

2.3.1 Nested hypothesis and sequential sums of squares

An hypothesis H_1 is *nested* in another hypothesis H_2 if H_1 implies H_2. This means, if we write H_i in the form of $\boldsymbol{\beta} \in W_i$, H_1 is nested in H_2 if and only if $W_1 \subset W_2$. Then we can write the *sequential sums of squares*:

$$SS(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2) = \|\hat{\mathbf{Y}}_2 - \hat{\mathbf{Y}}_1\|^2 \tag{2.19}$$

or

$$SS(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2) = SS(\boldsymbol{\beta} \in W_1) - SS(\boldsymbol{\beta} \in W_2) \tag{2.20}$$

And we get the *sequential degrees of freedom*:

$$\mathrm{df}(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2) = \dim(W_2) - \dim(W_1) \tag{2.21}$$

$$\mathrm{df}(\boldsymbol{\beta} \perp U_1 | \boldsymbol{\beta} \perp U_2) = \dim(U_1) - \dim(U_2) \tag{2.22}$$

2.3.2 Associated hypothesis

Searle claims that using the sequential sums of squares for testing such nested hypotheses actually tests an associated hypothesis. An *associated hypothesis* H^* is the hypothesis whose adjusted sum of squares $SS(\boldsymbol{\beta} \in W^*)$ equals the sequential sum of squares $SS(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2)$ of the nested hypothesis H_1 given H_2. The associated hypotheses, Searle finds in [2, page 112], are dependent on the sample size n_{ij} in each cell. He says this is why those hypothesis are not usable, since it should be generic and independent of the data that is used. His equations for the associated hypotheses are described in Table 2.1. The hypotheses

Source of variance	df	Associated Hypothesis
Mean	1	$H : \frac{1}{N}\sum_i \sum_j n_{ij}\mu_{ij} = 0$
Rows, adjusted for mean	$a-1$	$H : \rho_i'$ *all equal* for $\rho_i' = \frac{1}{n_{i.}}\sum_j n_{ij}\mu_{ij}$
Columns, adjusted for rows	$b-1$	$H : \gamma_j' = \frac{1}{n_{.j}}\sum_i n_{ij}\rho_i' \forall j$ for $\gamma_j' = \frac{1}{n_{.j}}\sum_i n_{ij}\mu_{ij}$

Table 2.1: Searle's associated hypothesis

he looked at were first comparing the whole model with the hypothesis that all cell means are equal (*Means*). Then he tests this hypothesis, given that only the row effect is present (*Rows, adjusted for mean*). After this he compares the hypothesis that only the row effect is present with the hypothesis that there is no interaction between columns in rows, but both effects might be present (*Columns, adjusted for rows*). In other terms, he tests *only A effect* given *no interaction.*

Searle's proof of the formulas in Table 2.1 is a bit obscure. Using the notation for the sequential sums of squares from equation (2.19) it is hard to see how the number of samples comes into those equations for the associated hypothesis. We will use the following theorem of Beder to derive the associated hypothesis from the sequential hypothesis:

Theorem 2.2 ([1, Theorem 4.4])**.** *Consider the model* $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$*, where* $\mathbf{X}$ $(N \times p)$ *has full rank, and let* $W_1 \subset W_2 \subset \mathbb{R}^p$*. Then there is a unique subspace* W^* *satisfying* $SS(\boldsymbol{\beta} \in W^*) = SS(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2)$*, and we have*

$$\mathrm{df}(\boldsymbol{\beta} \in W^*) = \mathrm{df}(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2). \tag{2.23}$$

where $\mathrm{df}(\boldsymbol{\beta} \in W_1 | \boldsymbol{\beta} \in W_2)$ *is given by (2.21). The subspace is given by* $W^* = R(\mathbf{TP}^*)$*, where* $\mathbf{T} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ *and* $\mathbf{P}^*$ *is defined as follows. Let* $V_I = \mathbf{X}(W_i)$*, let* $V = R(\mathbf{X})$ *be the range (columnspace) of* $\mathbf{X}$*, and let* $\mathbf{P}$ *and* $\mathbf{P}_i$ *be the orthogonal projections of* $\mathbb{R}^N$ *on* V *and* *on* V_i*, respectively. Then* $\mathbf{P}^* = \mathbf{P} - \mathbf{P}_2 + \mathbf{P}_1$.

If we want to use the notation $\boldsymbol{\beta} \perp U^*$ for the associated hypothesis we can apply this corollary:

Corollary 2.2 ([1, Corollary 4.7])**.** *Given the assumptions and notation of Theorem 2.2, the subspace* U^* *is given by* $U^* = N(\mathbf{P}^*\mathbf{T}')$.

In the next chapter we are going to show how the equation for the associated hypothesis Searle gives in [2] for *Columns, adjusted for Rows* can be verified by Theorem 2.2 in a 2×3 model.

3 Associated Hypothesis

3.1 Associated hypothesis in a 2×3 case

The goal of this chapter is to verify the equation for the associated hypothesis Searle claims in his book with help of Theorem 2.2 in a 2×3 model. Here we have $a = 2$, $b = 3$ and so $p = 2 * 3 = 6$. Our observation space has the dimension $N = \sum_{i=1}^{2} \sum_{j=1}^{3} n_{ij}$. Therefore we can look at the following array for the cell means:

$$\begin{array}{c|c|c||c} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{1\cdot} \\ \hline \mu_{21} & \mu_{22} & \mu_{23} & \mu_{2\cdot} \\ \hline\hline \mu_{\cdot 1} & \mu_{\cdot 2} & \mu_{\cdot 3} & \mu_{\cdot\cdot} \end{array} \tag{3.1}$$

And for the number of observations in each cell applies:

$$\begin{array}{c|c|c||l} n_{11} & n_{12} & n_{13} & n_{1\cdot} \\ \hline n_{21} & n_{22} & n_{23} & n_{2\cdot} \\ \hline\hline n_{\cdot 1} & n_{\cdot 2} & n_{\cdot 3} & n_{\cdot\cdot} = N \end{array} \tag{3.2}$$

The case we are looking at is, as Searle calls it, *Columns, adjusted for rows.* He says that an equation for the associated hypothesis for this source of variation looks like

$$H : \gamma_j^{'} = \frac{1}{n_{\cdot j}} \sum_{i=1}^{2} n_{ij} \rho_i^{'} \quad for\, j = 1, 2 \tag{3.3}$$

with for each j

$$\gamma_j^{'} = \frac{1}{n_{\cdot j}} \sum_{i=1}^{2} n_{ij} \mu_{ij} \tag{3.4}$$

and for each i

$$\rho_i^{'} = \frac{1}{n_{i\cdot}} \sum_{j=1}^{3} n_{ij}\mu_{ij} \tag{3.5}$$

In the notation of this thesis we are looking for the hypotheses H_1*:Only A present (No B present and no interaction)* given H_2*: No interaction*. It can be seen that H_1 is the more specific hypothesis and so for the subspaces W_1 and W_2 of $\mathbb{R}^6$ belonging to H_1 and H_2 we have $W1 \subset W2 \subset \mathbb{R}^6$.

Theorem 2.2 says that the unique subspace W^* belonging to the associated hypothesis equals to the range of $\mathbf{TP}^*$ or, using Corollary 2.2, that U^* with $\boldsymbol{\beta} \perp U^*$ is the nullspace of $\mathbf{P}^*\mathbf{T}^{'}$. If Searle's equation for the associated hypothesis can be derived from Theorem 2.2, then we can find contrast vectors $\mathbf{c}_i$ using his equations that span the vector space U^*. The number of contrast vectors we need to derive equals the dimension of U^*, so the degrees of freedom he gives for the associated hypothesis have to equal $\dim(U^*)$.

Searle says that the degrees of freedoms for *Columns, adjusted for rows* are $(b-1) = 3-1 = 2$. So we can get two contrast vectors from his associated hypothesis. We have to take (3.3) and replace γ' and ρ' with the formulas (3.4) and (3.5).

$$\begin{aligned}
&\gamma_1^{'} = \frac{1}{n_{\cdot 1}} \sum_{i=1}^{2} n_{i1}\rho_i^{'} \\
\Leftrightarrow &\frac{1}{n_{\cdot 1}} \sum_{i=1}^{2} n_{i1}\mu_{i1} = \frac{1}{n_{\cdot 1}} \sum_{i=1}^{2} n_{i1} \frac{1}{n_{i\cdot}} \sum_{k=1}^{3} n_{ik}\mu_{ik} \\
\Leftrightarrow &\frac{1}{n_{\cdot 1}}[n_{11}\mu_{11} + n_{21}\mu_{21}] = \frac{1}{n_{\cdot 1}}[\frac{1}{n_{1\cdot}}(n_{11}^2\mu_{11} + n_{11}n_{12}\mu_{12} + n_{11}n_{13}\mu_{13}) + \frac{1}{n_{2\cdot}}(n_{21}^2\mu_{21} + n_{21}n_{22}\mu_{22} + n_{21}n_{23}\mu_{23})] \\
\Leftrightarrow &0 = \frac{1}{n_{1\cdot}}[-n_{11}(n_{12} + n_{13})\mu_{11} + n_{11}n_{12}\mu_{12} + n_{11}n_{13}\mu_{13}] + \frac{1}{n_{2\cdot}}[-n_{21}(n_{22} + n_{23})\mu_{21} + n_{21}n_{22}\mu_{22} + n_{21}n_{23}\mu_{23}]
\end{aligned}$$

$$\begin{aligned}
&\gamma_2^{'} = \frac{1}{n_{\cdot 2}} \sum_{i=1}^{2} n_{i2}\rho_i^{'} \\
\Leftrightarrow &\frac{1}{n_{\cdot 2}} \sum_{i=1}^{2} n_{i2}\mu_{i2} = \frac{1}{n_{\cdot 2}} \sum_{i=1}^{2} n_{i2} \frac{1}{n_{i\cdot}} \sum_{k=1}^{3} n_{ik}\mu_{ik} \\
\Leftrightarrow &\frac{1}{n_{\cdot 2}}[n_{12}\mu_{12} + n_{22}\mu_{22}] = \frac{1}{n_{\cdot 2}}[\frac{1}{n_{1\cdot}}(n_{11}n_{12}\mu_{11} + n_{12}^2\mu_{12} + n_{12}n_{13}\mu_{13}) + \frac{1}{n_{2\cdot}}(n_{21}n_{22}\mu_{21} + n_{22}^2\mu_{22} + n_{22}n_{23}\mu_{23})] \\
\Leftrightarrow &0 = \frac{1}{n_{1\cdot}}[n_{11}n_{12}\mu_{11} - n_{12}(n_{11} + n_{13})\mu_{12} + n_{12}n_{13}\mu_{13}] + \frac{1}{n_{2\cdot}}[n_{21}n_{22}\mu_{21} - n_{22}(n_{21} + n_{23})\mu_{22} + n_{22}n_{23}\mu_{23}]
\end{aligned}$$

The two contrast vectors are

$$\mathbf{c}_1 = \begin{bmatrix} -\frac{n_{11}(n_{12}+n_{13})}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{12}}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{13}}{n_{11}+n_{12}+n_{13}} \\ -\frac{n_{21}(n_{22}+n_{23})}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{22}}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{23}}{n_{21}+n_{22}+n_{23}} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} \frac{n_{11}n_{12}}{n_{11}+n_{12}+n_{13}} \\ -\frac{n_{12}(n_{11}+n_{13})}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{12}n_{13}}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{21}n_{22}}{n_{21}+n_{22}+n_{23}} \\ -\frac{n_{22}(n_{21}+n_{23})}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{22}n_{23}}{n_{21}+n_{22}+n_{23}} \end{bmatrix} \tag{3.6}$$

It can easily be seen that for both vectors $\mathbf{c}_i$

$$\sum_{j=1}^{6} c_{ij} = 0$$

is true. The next step would be to show that both vectors $\mathbf{c}_1, \mathbf{c}_2$ are in the nullspace of $\mathbf{P}^*\mathbf{T}'$ or in other words that

$$\mathbf{P}^*\mathbf{T}'\mathbf{c}_i = \mathbf{0}, \textit{for } i = 1, 2 \tag{3.7}$$

where $\mathbf{0}$ equals a vector of zeros. If this is the case, those vectors span U^* and they can be used to formulate the associated hypothesis for testing H_1 : *Only A effect* given H_2*: No interaction*. First we need the matrix $\mathbf{P}^*$ and the matrix $\mathbf{T}$.

Getting $\mathbf{T}$ is very easy. After Theorem 2.2 the formula for it is $\mathbf{T} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. The design matrix $\mathbf{X}$ in a linear model looks like

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_{n_{11}} & & & & & \\ & \mathbf{1}_{n_{12}} & & & & \\ & & \mathbf{1}_{n_{13}} & & & \\ & & & \mathbf{1}_{n_{21}} & & \\ & & & & \mathbf{1}_{n_{22}} & \\ & & & & & \mathbf{1}_{n_{23}} \end{bmatrix} \tag{3.8}$$

So we get

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n_{11} & & & & & \\ & n_{12} & & & & \\ & & n_{13} & & & \\ & & & n_{21} & & \\ & & & & n_{22} & \\ & & & & & n_{23} \end{bmatrix}$$

$$\Rightarrow (\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} \frac{1}{n_{11}} & & & & & \\ & \frac{1}{n_{12}} & & & & \\ & & \frac{1}{n_{13}} & & & \\ & & & \frac{1}{n_{21}} & & \\ & & & & \frac{1}{n_{22}} & \\ & & & & & \frac{1}{n_{23}} \end{bmatrix}$$

and then finally

$$\mathbf{T} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} \frac{1}{n_{11}}\mathbf{1}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{1}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{1}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{1}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{1}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} \tag{3.9}$$

The next step is to calculate $\mathbf{P}^* = \mathbf{P} - \mathbf{P}_2 + \mathbf{P}_1$. To get $\mathbf{P}$ we need to project the observation space $\mathbb{R}^N$ on the columnspace of $\mathbf{X}$, here denoted by $V = R(\mathbf{X})$. We use equation (2.17) for P given in section 2.2:

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \tag{3.10}$$

Using (3.8) for $\mathbf{X}$ we get:

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} \frac{1}{n_{11}}\mathbf{J}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{J}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{J}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{J}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{J}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{J}_{n_{23}} \end{bmatrix} \tag{3.11}$$

With $\mathbf{J}_n$ denoting the $n \times n$ matrix of ones.

Before deriving the matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, we need the vector spaces W_1 and W_2 which belong to the two hypotheses.

H_2**: No interaction** For the hypothesis *No interaction* the following equations have to be true:

$$\mu_{22} - \mu_{12} = \mu_{21} - \mu_{11} \Leftrightarrow \mu_{22} = \mu_{12} + \mu_{21} - \mu_{11} \tag{3.12}$$

$$\mu_{23} - \mu_{13} = \mu_{21} - \mu_{11} \Leftrightarrow \mu_{23} = \mu_{13} + \mu_{21} - \mu_{11} \tag{3.13}$$

Using these equations, the subspace W_2 can be written as follows:

$$W_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \tag{3.14}$$

So W_2 is the columnspace of the matrix

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}. \tag{3.15}$$

$\mathbf{P}_2$ is the hardest to get. To calculate $\mathbf{P}_2$ we can use the formula (3.10), but replacing $\mathbf{X}$ with $\mathbf{X}_2$. First we need to calculate $\mathbf{X}_2$, the image of W_2 using $\mathbf{X}$. This gives us

$$\mathbf{X}_2 = \mathbf{X}\mathbf{W}_2 = \begin{bmatrix} \mathbf{1}_{n_{11}} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{n_{12}} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_{13}} & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n_{21}} \\ -\mathbf{1}_{n_{22}} & \mathbf{1}_{n_{22}} & 0 & \mathbf{1}_{n_{22}} \\ -\mathbf{1}_{n_{23}} & 0 & \mathbf{1}_{n_{23}} & \mathbf{1}_{n_{23}} \end{bmatrix}$$

After this, calculating $\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'$ is not easy since getting the inverse of

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{bmatrix} n_{11}+n_{22}+n_{23} & -n_{22} & -n_{23} & -n_{22}-n_{23} \\ -n_{22} & n_{12}+n_{22} & 0 & n_{22} \\ -n_{23} & 0 & n_{13}+n_{23} & n_{23} \\ -n_{22}-n_{23} & n_{22} & n_{23} & n_{21}+n_{22}+n_{23} \end{bmatrix}$$

is hard to do manually. Using a software for symbolic calculations [1] gives us

$$\mathbf{P}_2 = \frac{1}{d_2}\begin{bmatrix} A & B \end{bmatrix} \tag{3.16}$$

with

$$A = \begin{bmatrix} (n_{13}n_{21}n_{\cdot 2}+(n_{13}+n_{21})n_{23}n_{\cdot 2}+n_{12}n_{22}n_{\cdot 3})\mathbf{J}_{n_{11}} & n_{21}n_{22}n_{\cdot 3}\mathbf{J}_{n_{11}n_{12}} & n_{21}n_{23}n_{\cdot 2}\mathbf{J}_{n_{11}n_{13}} \\ n_{21}n_{22}n_{\cdot 3}\mathbf{J}_{n_{12}n_{11}} & (n_{13}n_{22}n_{\cdot 1}+(n_{13}+n_{22})n_{23}n_{\cdot 1}+n_{11}n_{21}n_{\cdot 3})\mathbf{J}_{n_{12}} & n_{22}n_{23}n_{\cdot 1}\mathbf{J}_{n_{12}n_{13}} \\ n_{21}n_{23}n_{\cdot 2}\mathbf{J}_{n_{13}n_{11}} & n_{22}n_{23}n_{\cdot 1}\mathbf{J}_{n_{13}n_{12}} & (n_{12}n_{22}n_{\cdot 1}+(n_{12}+n_{22})n_{23}n_{\cdot 1}+n_{11}n_{21}n_{\cdot 2})\mathbf{J}_{n_{13}} \\ (n_{12}n_{22}n_{\cdot 3}+n_{13}n_{23}n_{\cdot 2})\mathbf{J}_{n_{21}n_{11}} & -n_{11}n_{22}n_{\cdot 3}\mathbf{J}_{n_{21}n_{12}} & -n_{11}n_{23}n_{\cdot 2}\mathbf{J}_{n_{21}n_{13}} \\ -n_{12}n_{21}n_{\cdot 3}\mathbf{J}_{n_{22}n_{11}} & (n_{11}n_{21}n_{\cdot 3}+n_{13}n_{23}n_{\cdot 1})\mathbf{J}_{n_{22}n_{12}} & -n_{12}n_{23}n_{\cdot 1}\mathbf{J}_{n_{22}n_{13}} \\ -n_{13}n_{21}n_{\cdot 2}\mathbf{J}_{n_{23}n_{11}} & -n_{13}n_{22}n_{\cdot 1}\mathbf{J}_{n_{23}n_{12}} & (n_{11}n_{21}n_{\cdot 2}+n_{12}n_{22}n_{\cdot 1})\mathbf{J}_{n_{23}n_{13}} \end{bmatrix}$$

$$B = \begin{bmatrix} (n_{12}n_{22}n_{\cdot 3}+n_{13}n_{23}n_{\cdot 2})\mathbf{J}_{n_{11}n_{21}} & -n_{12}n_{21}n_{\cdot 3}\mathbf{J}_{n_{11}n_{22}} & -n_{13}n_{21}n_{\cdot 2}\mathbf{J}_{n_{11}n_{23}} \\ -n_{11}n_{22}n_{\cdot 3}\mathbf{J}_{n_{12}n_{21}} & (n_{11}n_{21}n_{\cdot 3}+n_{13}n_{23}n_{\cdot 1})\mathbf{J}_{n_{12}n_{22}} & -n_{13}n_{22}n_{\cdot 1}\mathbf{J}_{n_{12}n_{23}} \\ -n_{11}n_{23}n_{\cdot 2}\mathbf{J}_{n_{13}n_{21}} & -n_{12}n_{23}n_{\cdot 1}\mathbf{J}_{n_{13}n_{22}} & (n_{11}n_{21}n_{\cdot 2}+n_{12}n_{22}n_{\cdot 1})\mathbf{J}_{n_{13}n_{23}} \\ (n_{11}n_{13}n_{\cdot 2}+(n_{11}+n_{13})n_{23}n_{\cdot 2}+n_{12}n_{22}n_{\cdot 3})\mathbf{J}_{n_{21}} & n_{11}n_{12}n_{\cdot 3}\mathbf{J}_{n_{21}n_{22}} & n_{11}n_{13}n_{\cdot 2}\mathbf{J}_{n_{21}n_{23}} \\ n_{11}n_{12}n_{\cdot 3}\mathbf{J}_{n_{22}n_{21}} & (n_{11}n_{12}n_{\cdot 3}+(n_{11}+n_{12})n_{21}n_{\cdot 3}+n_{13}n_{23}n_{\cdot 1})\mathbf{J}_{n_{22}} & n_{12}n_{13}n_{\cdot 1}\mathbf{J}_{n_{22}n_{23}} \\ n_{11}n_{13}n_{\cdot 2}\mathbf{J}_{n_{23}n_{21}} & n_{12}n_{13}n_{\cdot 1}\mathbf{J}_{n_{23}n_{22}} & (n_{11}n_{13}n_{\cdot 2}+(n_{11}+n_{13})n_{21}n_{\cdot 2}+n_{12}n_{22}n_{\cdot})\mathbf{J}_{n_{23}} \end{bmatrix}$$

and

$$d_2 = n_{\cdot 1}n_{\cdot 2}n_{13}n_{23} + n_{\cdot 1}n_{12}n_{22}n_{\cdot 3} + n_{11}n_{21}n_{\cdot 2}n_{\cdot 3}$$

H_1**: Only A effect (No B effect, No interaction)** The hypothesis *Only A effect* is a specialization of H_2*: No interaction* since we not only assume the missing interaction between both effects, but also that no B effect exists. Hence the resulting vector space W_2

[1] sympy, python library for symbolic calculations

has to be a subspace of W_1. The equations that describe this hypothesis are

$$\mu_{11} = \mu_{12} = \mu_{13} \tag{3.17}$$

$$\mu_{21} = \mu_{22} = \mu_{23}. \tag{3.18}$$

From this we can derive the following subspace

$$W_1 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}) \tag{3.19}$$

the columnspace of

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{3.20}$$

Analogous to calculate $\mathbf{P}_2$ we can again use (3.10) to get $\mathbf{P}_1$ by replacing $\mathbf{X}$ with $\mathbf{X}_1$.

For $\mathbf{X}_1$ we get

$$\mathbf{X}_1 = \mathbf{X}\mathbf{W}_1 = \begin{bmatrix} \mathbf{1}_{n_{11}} & 0 \\ \mathbf{1}_{n_{12}} & 0 \\ \mathbf{1}_{n_{13}} & 0 \\ 0 & \mathbf{1}_{n_{21}} \\ 0 & \mathbf{1}_{n_{22}} \\ 0 & \mathbf{1}_{n_{23}} \end{bmatrix} \tag{3.21}$$

This time calculating $\mathbf{P}_1$ can be easily done manually:

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{bmatrix} n_{1.} & 0 \\ 0 & n_{2.} \end{bmatrix} \Rightarrow (\mathbf{X}_1'\mathbf{X}_1)^{-1} = \begin{bmatrix} \frac{1}{n_{1.}} & 0 \\ 0 & \frac{1}{n_{2.}} \end{bmatrix}$$

$$\Rightarrow \mathbf{P}_1 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1' \tag{3.22}$$

$$= \begin{bmatrix} \frac{1}{n_{1.}}\mathbf{J}_{n_{11}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{11}n_{12}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{11}n_{13}} & 0 & 0 & 0 \\ \frac{1}{n_{1.}}\mathbf{J}_{n_{12}n_{11}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{12}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{12}n_{13}} & 0 & 0 & 0 \\ \frac{1}{n_{1.}}\mathbf{J}_{n_{13}n_{11}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{13}n_{12}} & \frac{1}{n_{1.}}\mathbf{J}_{n_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_{2.}}\mathbf{J}_{n_{21}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{21}n_{22}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{21}n_{23}} \\ 0 & 0 & 0 & \frac{1}{n_{2.}}\mathbf{J}_{n_{22}n_{21}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{22}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{22}n_{23}} \\ 0 & 0 & 0 & \frac{1}{n_{2.}}\mathbf{J}_{n_{23}n_{21}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{23}n_{22}} & \frac{1}{n_{2.}}\mathbf{J}_{n_{23}} \end{bmatrix} \tag{3.23}$$

$$= \begin{bmatrix} \frac{1}{n_{1.}}\mathbf{J}_{n_{1.}} & 0 \\ 0 & \frac{1}{n_{2.}}\mathbf{J}_{n_{2.}} \end{bmatrix} \tag{3.24}$$

With $\mathbf{T}$, $\mathbf{P}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ we now can verify whether $\mathbf{c}_1$ and $\mathbf{c}_2$ span the subspace $U^* =$

$N(\mathbf{P}^*\mathbf{T}')$. To calculate $\mathbf{P}^*\mathbf{T}'$ we get

$$\mathbf{P}^*\mathbf{T}' = (\mathbf{P} - \mathbf{P}_2 + \mathbf{P}_1)\mathbf{T}' = \mathbf{P}\mathbf{T}' - \mathbf{P}_2\mathbf{T}' + \mathbf{P}_1\mathbf{T}'$$

and so we have to show for both vectors $\mathbf{c}_i$, $i = 1, 2$

$$\mathbf{P}^*\mathbf{T}'\mathbf{c}_i = (\mathbf{P}\mathbf{T}' - \mathbf{P}_2\mathbf{T}' + \mathbf{P}_1\mathbf{T}')\mathbf{c}_i = \mathbf{P}\mathbf{T}'\mathbf{c}_i - \mathbf{P}_2\mathbf{T}'\mathbf{c}_i + \mathbf{P}_1\mathbf{T}'\mathbf{c}_i = 0 \tag{3.25}$$

First we get

$$\mathbf{P}\mathbf{T}' = \begin{bmatrix} \frac{1}{n_{11}}\mathbf{J}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{J}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{J}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{J}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{J}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{J}_{n_{23}} \end{bmatrix} * \begin{bmatrix} \frac{1}{n_{11}}\mathbf{1}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{1}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{1}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{1}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{1}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n_{11}}\mathbf{1}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{1}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{1}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{1}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{1}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

$$\Rightarrow \mathbf{PT}'\mathbf{c}_1 = \begin{bmatrix} \frac{1}{n_{11}}\mathbf{1}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{1}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{1}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{1}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{1}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} * \begin{bmatrix} -\frac{n_{11}(n_{12}+n_{13})}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{12}}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{13}}{n_{11}+n_{12}+n_{13}} \\ -\frac{n_{21}(n_{22}+n_{23})}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{22}}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{23}}{n_{21}+n_{22}+n_{23}} \end{bmatrix} = \begin{bmatrix} -\frac{n_{12}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ -\frac{n_{22}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

Analogous we can calculate

$$\mathbf{P}_1\mathbf{T}' = \begin{bmatrix} \frac{1}{n_{1\cdot}}\mathbf{J}_{n_{1\cdot}} & 0 \\ 0 & \frac{1}{n_{2\cdot}}\mathbf{J}_{n_{2\cdot}} \end{bmatrix} * \begin{bmatrix} \frac{1}{n_{11}}\mathbf{1}_{n_{11}} & & & & & \\ & \frac{1}{n_{12}}\mathbf{1}_{n_{12}} & & & & \\ & & \frac{1}{n_{13}}\mathbf{1}_{n_{13}} & & & \\ & & & \frac{1}{n_{21}}\mathbf{1}_{n_{21}} & & \\ & & & & \frac{1}{n_{22}}\mathbf{1}_{n_{22}} & \\ & & & & & \frac{1}{n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & 0 & 0 & 0 \\ \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & 0 & 0 & 0 \\ \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

$$\Rightarrow \mathbf{P}_1\mathbf{T}'\mathbf{c}_1 = \begin{bmatrix} \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{11}} & 0 & 0 & 0 \\ \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{12}} & 0 & 0 & 0 \\ \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & \frac{1}{n_{1\cdot}}\mathbf{1}_{n_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{21}} \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{22}} \\ 0 & 0 & 0 & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} & \frac{1}{n_{2\cdot}}\mathbf{1}_{n_{23}} \end{bmatrix} * \begin{bmatrix} -\frac{n_{11}(n_{12}+n_{13})}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{12}}{n_{11}+n_{12}+n_{13}} \\ \frac{n_{11}n_{13}}{n_{11}+n_{12}+n_{13}} \\ -\frac{n_{21}(n_{22}+n_{23})}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{22}}{n_{21}+n_{22}+n_{23}} \\ \frac{n_{21}n_{23}}{n_{21}+n_{22}+n_{23}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And finally

$$\mathbf{P}_2\mathbf{T}'\mathbf{c}_1 = \begin{bmatrix} -\frac{n_{12}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ -\frac{n_{22}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix}$$

With those three results we can easily see, that

$$\mathbf{P}\mathbf{T}'\mathbf{c}_1 - \mathbf{P}_2\mathbf{T}'\mathbf{c}_1 + \mathbf{P}_1\mathbf{T}'\mathbf{c}_1 = \begin{bmatrix} -\frac{n_{12}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ -\frac{n_{22}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} - \begin{bmatrix} -\frac{n_{12}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{11}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ -\frac{n_{22}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{21}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

This shows that c_1 lies in U^*. Next we have to do those calculations with c_2. This is analogous to the above steps and we get

$$\mathbf{P}\mathbf{T}'\mathbf{c}_2 - \mathbf{P}_2\mathbf{T}'\mathbf{c}_2 + \mathbf{P}_1\mathbf{T}'\mathbf{c}_2 = \begin{bmatrix} \frac{n_{12}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ -\frac{n_{11}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{12}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ \frac{n_{22}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ -\frac{n_{21}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{22}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} - \begin{bmatrix} \frac{n_{12}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{11}} \\ -\frac{n_{11}+n_{13}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{12}} \\ \frac{n_{12}}{n_{11}+n_{12}+n_{13}}\mathbf{1}_{n_{13}} \\ \frac{n_{22}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{21}} \\ -\frac{n_{21}+n_{23}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{22}} \\ \frac{n_{22}}{n_{21}+n_{22}+n_{23}}\mathbf{1}_{n_{23}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

So we have two vectors that lie in U^*, and since $\dim(U^*) = \mathrm{df}(H^*) = \dim(W_2) - \dim(W_1) = 4 - 2 = 2$, we know they span the vector space. This verifies Searle's equation for the associated hypothesis for *Columns, adjusted for rows* with the help of Theorem 2.2 and Corollary 2.2 in a linear 2×3 model.

3.2 Deriving the associated hypothesis

After verifying Searle's equation for the associated hypothesis we want to see how to derive this equation from Theorem 2.2. This means we have to find U^*, so that we can use this to write equations in the form $\mathbf{c}'\boldsymbol{\beta} = 0$. Collorary 2.2 says that U^* is the nullspace of $\mathbf{P}^*\mathbf{T}'$. First we will see how to do this at the following example:

	Variety		
Soil	1	2	3
1	6	13	14
	10	15	22
	11		
2	12	31	18
	15		9
	19		12
	18		

Table 3.1: Number of days to first germination of three varieties of carrot seed grown in two different potting soils. [2, Table 4.1]

Example 3.1. In [2] Searle gives some illustrative data for a linear 2×3 model and describes how the associated hypothesis can look in this case[2]. The data is given in Table 3.1. We get the following number of observations in each cell:

$$\begin{array}{c|c|c||c} n_{11}=3 & n_{12}=2 & n_{13}=2 & n_{1\cdot}=7 \\ \hline n_{21}=4 & n_{22}=1 & n_{23}=3 & n_{2\cdot}=8 \\ \hline\hline n_{\cdot 1}=7 & n_{\cdot 2}=3 & n_{\cdot 3}=5 & n_{\cdot\cdot}=N=15 \end{array} \tag{3.26}$$

To derive the associated hypothesis we first need to calculate $\mathbf{T}$. We use (3.9) for this and insert the values for all n_{ij} from (3.26). We get:

$$\mathbf{T} = \begin{bmatrix} \frac{1}{3}\mathbf{1}_3 & & & & & \\ & \frac{1}{2}\mathbf{1}_2 & & & & \\ & & \frac{1}{2}\mathbf{1}_2 & & & \\ & & & \frac{1}{4}\mathbf{1}_4 & & \\ & & & & \mathbf{1}_1 & \\ & & & & & \frac{1}{3}\mathbf{1}_3 \end{bmatrix}$$

Next we have to calculate $\mathbf{P}^* = \mathbf{P} - \mathbf{P}_2 + \mathbf{P}_1$. Again, we can use (3.11), (3.22) and (3.16) for $\mathbf{P}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ and insert the values from (3.26):

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3}\mathbf{J}_3 & & & & & \\ & \frac{1}{2}\mathbf{J}_2 & & & & \\ & & \frac{1}{2}\mathbf{J}_2 & & & \\ & & & \frac{1}{4}\mathbf{J}_4 & & \\ & & & & \mathbf{J}_1 & \\ & & & & & \frac{1}{3}\mathbf{J}_3 \end{bmatrix}$$

[2] [2, Table 4.10, page 114]

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{7}\mathbf{J}_7 & 0 \\ 0 & \frac{1}{8}\mathbf{J}_8 \end{bmatrix}$$

$$\mathbf{P}_2 = \frac{1}{376}\begin{bmatrix} 88\mathbf{J}_3 & 20\mathbf{J}_{3,2} & 36\mathbf{J}_{3,2} & 28\mathbf{J}_{3,4} & -40\mathbf{J}_{3,1} & -24\mathbf{J}_{3,3} \\ 20\mathbf{J}_{2,3} & 137\mathbf{J}_2 & 21\mathbf{J}_{2,2} & -15\mathbf{J}_{2,4} & 102\mathbf{J}_{2,1} & -14\mathbf{J}_{2,3} \\ 36\mathbf{J}_{2,3} & 21\mathbf{J}_{2,2} & 113\mathbf{J}_2 & -27\mathbf{J}_{2,4} & -42\mathbf{J}_{2,1} & 50\mathbf{J}_{2,3} \\ 28\mathbf{J}_{4,3} & -15\mathbf{J}_{4,2} & -27\mathbf{J}_{4,2} & 73\mathbf{J}_4 & 30\mathbf{J}_{4,1} & 18\mathbf{J}_{4,3} \\ -40\mathbf{J}_{1,3} & 102\mathbf{J}_{1,2} & -42\mathbf{J}_{1,2} & 30\mathbf{J}_{1,4} & 172\mathbf{J}_1 & 28\mathbf{J}_{1,3} \\ -24\mathbf{J}_{3,3} & -14\mathbf{J}_{3,2} & 50\mathbf{J}_{3,2} & 18\mathbf{J}_{3,4} & 28\mathbf{J}_{3,1} & 92\mathbf{J}_3 \end{bmatrix}$$

Then we get for $\mathbf{P}^*$:

$$\begin{aligned} \mathbf{P}^* &= \mathbf{P} - \mathbf{P}_2 + \mathbf{P}_1 \\ &= \begin{bmatrix} \frac{239}{987}\mathbf{J}_3 & \frac{59}{658}\mathbf{J}_{3,2} & \frac{31}{658}\mathbf{J}_{3,2} & -\frac{7}{94}\mathbf{J}_{3,4} & \frac{5}{47}\mathbf{J}_{3,1} & \frac{3}{47}\mathbf{J}_{3,3} \\ \frac{59}{658}\mathbf{J}_{2,3} & \frac{733}{2632}\mathbf{J}_2 & \frac{229}{2632}\mathbf{J}_{2,2} & \frac{15}{376}\mathbf{J}_{2,4} & -\frac{51}{188}\mathbf{J}_{2,1} & \frac{7}{188}\mathbf{J}_{2,3} \\ \frac{31}{658}\mathbf{J}_{2,3} & \frac{229}{2632}\mathbf{J}_{2,2} & \frac{901}{2632}\mathbf{J}_2 & \frac{27}{376}\mathbf{J}_{2,4} & \frac{21}{188}\mathbf{J}_{2,1} & -\frac{25}{188}\mathbf{J}_{2,3} \\ -\frac{7}{94}\mathbf{J}_{4,3} & \frac{15}{376}\mathbf{J}_{4,2} & \frac{27}{376}\mathbf{J}_{4,2} & \frac{17}{94}\mathbf{J}_4 & \frac{17}{376}\mathbf{J}_{4,1} & \frac{29}{376}\mathbf{J}_{4,3} \\ \frac{5}{47}\mathbf{J}_{1,3} & -\frac{51}{188}\mathbf{J}_{1,2} & \frac{21}{188}\mathbf{J}_{1,2} & \frac{17}{376}\mathbf{J}_{1,4} & \frac{251}{376}\mathbf{J}_1 & \frac{19}{376}\mathbf{J}_{1,3} \\ \frac{3}{47}\mathbf{J}_{3,3} & \frac{7}{188}\mathbf{J}_{3,2} & -\frac{25}{188}\mathbf{J}_{3,2} & \frac{29}{376}\mathbf{J}_{3,4} & \frac{19}{376}\mathbf{J}_{3,1} & \frac{241}{1128}\mathbf{J}_3 \end{bmatrix} \end{aligned}$$

Finally we have

$$\mathbf{P}^*\mathbf{T}' = \begin{bmatrix} \frac{239}{987} & \frac{59}{658} & \frac{31}{658} & -\frac{7}{94} & \frac{5}{47} & \frac{3}{47} \\ \frac{239}{987} & \frac{59}{658} & \frac{31}{658} & -\frac{7}{94} & \frac{5}{47} & \frac{3}{47} \\ \frac{239}{987} & \frac{59}{658} & \frac{31}{658} & -\frac{7}{94} & \frac{5}{47} & \frac{3}{47} \\ \frac{59}{658} & \frac{733}{2632} & \frac{229}{2632} & \frac{15}{376} & -\frac{51}{188} & \frac{7}{188} \\ \frac{59}{658} & \frac{733}{2632} & \frac{229}{2632} & \frac{15}{376} & -\frac{51}{188} & \frac{7}{188} \\ \frac{31}{658} & \frac{229}{2632} & \frac{901}{2632} & \frac{27}{376} & \frac{21}{188} & -\frac{25}{188} \\ \frac{31}{658} & \frac{229}{2632} & \frac{901}{2632} & \frac{27}{376} & \frac{21}{188} & -\frac{25}{188} \\ -\frac{7}{94} & \frac{15}{376} & \frac{27}{376} & \frac{17}{94} & \frac{17}{376} & \frac{29}{376} \\ -\frac{7}{94} & \frac{15}{376} & \frac{27}{376} & \frac{17}{94} & \frac{17}{376} & \frac{29}{376} \\ -\frac{7}{94} & \frac{15}{376} & \frac{27}{376} & \frac{17}{94} & \frac{17}{376} & \frac{29}{376} \\ -\frac{7}{94} & \frac{15}{376} & \frac{27}{376} & \frac{17}{94} & \frac{17}{376} & \frac{29}{376} \\ \frac{5}{47} & -\frac{51}{188} & \frac{21}{188} & \frac{17}{376} & \frac{251}{376} & \frac{19}{376} \\ \frac{3}{47} & \frac{7}{188} & -\frac{25}{188} & \frac{29}{376} & \frac{19}{376} & \frac{241}{1128} \\ \frac{3}{47} & \frac{7}{188} & -\frac{25}{188} & \frac{29}{376} & \frac{19}{376} & \frac{241}{1128} \\ \frac{3}{47} & \frac{7}{188} & -\frac{25}{188} & \frac{29}{376} & \frac{19}{376} & \frac{241}{1128} \end{bmatrix} \tag{3.27}$$

To find the nullspace of $\mathbf{P}^*\mathbf{T}'$ we want to find all vectors $\mathbf{c}$ where $\mathbf{P}^*\mathbf{T}'\mathbf{c} = 0$. For this we write the matrix in reduced row echelon form:

$$\mathbf{P}^*\mathbf{T}' = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{9}{7} & \frac{5}{7} \\ 0 & 1 & 0 & 0 & -\frac{23}{14} & -\frac{1}{42} \\ 0 & 0 & 1 & 0 & \frac{5}{14} & -\frac{29}{42} \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.28}$$

We get the following equations:

$$\begin{cases} c_1 = -\frac{9}{7}c_5 - \frac{5}{7}c_6 \\ c_2 = \frac{23}{14}c_5 + \frac{1}{42}c_6 \\ c_3 = -\frac{5}{14}c_5 + \frac{29}{42}c_6 \\ c_4 = -c_5 - c_6 \end{cases} \tag{3.29}$$

So we have two degrees of freedom and therefore we can find two linearly independent vectors $\mathbf{c}_1 = (c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16})'$ and $\mathbf{c}_2 = (c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26})'$. To calculate $\mathbf{c}_1$ we set $c_{15} = \frac{1}{2}$ and $c_{16} = \frac{3}{2}$. Then we can calculate the other coefficients:

$$\begin{cases} c_{11} = -\frac{9}{14} - \frac{15}{14} = -\frac{12}{7} \\ c_{12} = \frac{23}{28} + \frac{1}{28} = \frac{6}{7} \\ c_{13} = -\frac{5}{28} + \frac{29}{28} = \frac{6}{7} \\ c_{14} = -\frac{1}{2} - \frac{3}{2} = -2 \end{cases} \tag{3.30}$$

Analogously we set $c_{25} = -\frac{7}{8}$ and $c_{26} = \frac{3}{8}$ and calculate there remaining coefficients of $\mathbf{c}_2$:

$$\begin{cases} c_{21} = \frac{9}{8} - \frac{15}{56} = \frac{6}{7} \\ c_{22} = -\frac{23}{16} + \frac{1}{112} = -\frac{10}{7} \\ c_{23} = \frac{5}{16} + \frac{29}{112} = \frac{4}{7} \\ c_{24} = \frac{7}{8} - \frac{3}{8} = \frac{1}{2} \end{cases} \tag{3.31}$$

After this we can write equations in form $\mathbf{c}_i'\boldsymbol{\beta} = 0$ for $i = 1, 2$ and get:

$$\begin{cases} \mathbf{c}_1'\boldsymbol{\beta} = 0 \\ \mathbf{c}_2'\boldsymbol{\beta} = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} -\frac{12}{7}\mu_{11} + \frac{6}{7}\mu_{12} + \frac{6}{7}\mu_{13} - 2\mu_{21} + \frac{1}{2}\mu_{22} + \frac{3}{2}\mu_{23} = 0 \\ \frac{6}{7}\mu_{11} - \frac{10}{7}\mu_{12} + \frac{4}{7}\mu_{13} - \frac{1}{2}\mu_{21} - \frac{7}{8}\mu_{22} + \frac{3}{8}\mu_{23} = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 3\mu_{11} + 4\mu_{21} = \frac{3}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13}) + \frac{1}{2}(4\mu_{21} + \mu_{22} + 3\mu_{23}) \\ 2\mu_{12} + \mu_{22} = \frac{2}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13}) + \frac{1}{8}(4\mu_{21} + \mu_{22} + 3\mu_{23}) \end{cases}$$
$$\Leftrightarrow \begin{cases} \frac{1}{7}(3\mu_{11} + 4\mu_{21}) = \frac{1}{7}(\frac{3}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13}) + \frac{1}{2}(4\mu_{21} + \mu_{22} + 3\mu_{23})) \\ \frac{1}{3}(2\mu_{12} + \mu_{22}) = \frac{1}{3}(\frac{2}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13}) + \frac{1}{8}(4\mu_{21} + \mu_{22} + 3\mu_{23})) \end{cases}$$

When we set $\rho_1' = \frac{1}{7}(3\mu_{11} + 2\mu_{12} + 2\mu_{13})$, $\rho_2' = \frac{1}{8}(4\mu_{21} + \mu_{22} + 3\mu_{23})$, $\gamma_1' = \frac{1}{7}(3\mu_{11} + 4\mu_{21})$ and

$\gamma_2' = \frac{1}{3}(2\mu_{12}+\mu_{22})$, we can write this hypothesis in the form Searle uses in [2]:

$$\begin{cases} \gamma_1' = \frac{1}{7}(3\rho_1'+4\rho_2') \\ \gamma_2' = \frac{1}{3}(2\rho_1'+\rho_2') \end{cases}$$

If we want to derive the associated hypothesis in a general 2×3 case, we need to follow the steps in Example 3.1. Here calculating the nullspace of $\mathbf{P}^*\mathbf{T}'$ can be challenging and expensive. This is why some further algorithm for this should be considered.

4 Conclusion

In this thesis we could verify the equations for the associated hypothesis Searle gave in [2] for *columns, adjusted for rows* using the Theorem 2.2 from Beder in [1] in a linear 2×3 model. We derived two contrast vectors from Searle's equations and showed that they span the subspace U^*. The sum of squares of $\boldsymbol{\beta} \perp U^*$ equals the sequential sum of squares of the hypothesis *Only A present* given *No interaction*.

While verifying the associated hypothesis we can see that the matrix $\mathbf{P}_2$ is very complicated. Therefore $\mathbf{P}^*$ is also complicated and calculating the nullspace from $\mathbf{P}^*\mathbf{T}'$ is challenging, but this is necessary to derive equations for the associated hypothesis from Theorem 2.2. Example 3.1 shows the typical steps that one may use to derive the equations. Further one should examine if a more efficient algorithm can be found to calculate a basis for U^* in the general 2×3 case.

Furthermore calculating $\mathbf{P}_2$ manually was very difficult, especially calculating $(\mathbf{X}_2'\mathbf{X}_2)^{-1}$. Therefore we used a software library for symbolic calculations, but this cannot be used for verifying the associated hypothesis in a general $a\times b$ model. Therefore some more mathematics have to be found that help to do this.

Bibliography

[1] Beder, J. *Linear Models and Design.* unpublished, 2016.

[2] Searle, S. R. *Linear Models for Unbalanced Data.* John Wiley and Sons, 1987.